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LOWER BOUNDS AT INFINITY FOR SOLUTIONS TO SECOND ORDER ELLIPTIC EQUATIONS

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Abstract. We study lower bounds at infinity for solutions to

$$|Pu| \le M|x|^{-\delta_1}|\nabla u| + M|x|^{-\delta_0}|u|$$

where P is a second order elliptic operator. Our results are of quantitative nature and generalize those obtained in [3, 6].

1. Introduction

Let $Pu = \operatorname{div}(A\nabla u)$ be a second order elliptic operator in divergence form, where A is symmetric, Lipschitz and uniformly elliptic, i.e.

$$\lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \lambda^{-1}|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n,$$

for some $\lambda \in (0, 1]$. In this paper we study lower bounds at infinity for (non-trivial) solutions of

$$|Pu| \le M|x|^{-\delta_1}|\nabla u| + M|x|^{-\delta_0}|u| \quad \text{in } \mathbb{R}^n$$

where $M > 0$ and $\delta_0, \delta_1 \in \mathbb{R}$.

The first result in this direction was obtained by Meshkov [7] for solutions of

$$|\Delta u| \le M|u| \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

He showed that if $u(x)\exp(C|x|^{4/3}) \in L^\infty(\mathbb{R}^n)$ for all $C > 0$ then u must vanish identically. The exponent $4/3$ is optimal as Meshkov also constructed a function $u : \mathbb{R}^2 \to \mathbb{C}$ satisfying (1.1) such that

$$|u(x)| \le C\exp(-|x|^{4/3}) \quad \forall x \in \mathbb{R}^2.$$

This result was later extended in [2] to solutions of

$$|\Delta u + Eu| \le M|x|^{-\delta_0}|u| \quad \forall x \in \mathbb{R}^n \tag{1.2}$$

where E is a real constant. It was proved that if $u(x)\exp(C|x|^\alpha) \in L^\infty(\mathbb{R}^n)$ for all $C > 0$ where $\alpha = \max\{1, \frac{4-2\delta_0}{3}\}$ then $u \equiv 0$. The optimality of the exponent α was shown by a variant of Meshkov's example.

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The first quantitative version of these results was obtained by Bourgain and Kenig [1]. They proved that if u satisfies (1.1) then there exists $C > 0$ such that

$$\int_{B(x,1)} u^2 \geq \exp(-C|x|^{4/3} \log |x|) \quad \forall |x| \geq 10.$$

Their method can be easily adapted to give the bound

$$\int_{B(x,|x|/2)} u^2 \geq \exp(-C|x|^{4/3}) \quad \forall |x| \geq 10,$$

from which Meshkov's qualitative result can be recovered.

Subsequently, [3, 6] extended these estimates to solutions of operators with lower order terms, and operators with variable second order coefficients. The main results of this paper are improvements of their results.

Theorem 1.1. *Let P be as above and u be a nontrivial solution of*

$$|Pu| \leq M|x|^{-\delta_1}|\nabla u| + M|x|^{-\delta_0}|u| \quad \text{in } \mathbb{R}^n \setminus B_1 \tag{1.3}$$

where $M > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$. Suppose that $\alpha \geq \max\{\frac{4-2\delta_0}{3}, 2-2\delta_1\}$ and $\alpha > 1-\lambda^2$. Then there exists $\varepsilon = \varepsilon(n, \alpha, \lambda) > 0$ such that if

$$|\nabla A(x)| \leq \frac{\varepsilon}{|x|} \quad \text{and} \quad |u(x)| \leq e^{M|x|^\alpha}$$

when $|x| > 1$, then there exists $C > 0$ so that if $|x| \geq 10$,

$$\int_{B(x,|x|/2)} u^2 \geq \exp(-C|x|^\alpha) \tag{1.4}$$

and

$$\int_{B(x,1)} u^2 \geq \exp(-C|x|^\alpha \log |x|). \tag{1.5}$$

We also have the following similar result for operators with differentiable potentials.

Theorem 1.2. *Let P be as above and V be a Lipschitz function satisfying*

$$|V| + |Ax \cdot \nabla V| \leq M|x|^{-\delta_2}.$$

Let u be a nontrivial solution of

$$|Pu + Vu| \leq M|x|^{-\delta_1}|\nabla u| + M|x|^{-\delta_0}|u| \quad \text{in } \mathbb{R}^n \setminus B_1.$$

Suppose that $\alpha \geq \max\{\frac{4-2\delta_0}{3}, 2-2\delta_1, \frac{2-\delta_2}{2}\}$ and $\alpha > 1-\lambda^2$. Then there exists $\varepsilon = \varepsilon(n, \alpha, \lambda) > 0$ such that if

$$|\nabla A(x)| \leq \frac{\varepsilon}{|x|} \quad \text{and} \quad |u(x)| \leq e^{M|x|^\alpha}$$

when $|x| > 1$ then there exists $C > 0$ so that if $|x| \geq 10$,

$$\int_{B(x,|x|/2)} u^2 \geq \exp(-C|x|^\alpha)$$

and

$$\int_{B(x,1)} u^2 \geq \exp(-C|x|^\alpha \log |x|).$$

When V is a constant, we can take $\delta_2 = 0$ to obtain the following generalization of (1.2).

Corollary 1.3. *Let E be a real number and u is a nontrivial solution of*

$$|Pu + Eu| \le M|x|^{-\delta_1}|\nabla u| + M|x|^{-\delta_0}|u| \quad \text{in } \mathbb{R}^n \setminus B_1.$$

Let $\alpha = \max\{1, \frac{4-2\delta_0}{3}, 2-2\delta_1\}$, then there exists $\varepsilon = \varepsilon(n,\alpha,\lambda) > 0$ such that if

$$|\nabla A(x)| \le \frac{\varepsilon}{|x|}$$

and

$$|u(x)| \le e^{M|x|^\alpha}$$

when $|x| > 1$ then there exists $C > 0$ so that if $|x| \ge 10$,

$$\int_{B(x,|x|/2)} u^2 \ge \exp(-C|x|^\alpha)$$

and

$$\int_{B(x,1)} u^2 \ge \exp(-C|x|^\alpha \log|x|).$$

In [6], similar estimates were obtained under the slightly stronger condition

$$|\nabla A(x)| \le \frac{C}{|x|^{1+\varepsilon}} \quad \text{for } |x| > 1.$$

This was proved by using a chain of balls argument with balls of different radii which lead to slightly weaker lower bounds of the form

$$\exp(-C|x|^\alpha(\log|x|)^{\gamma(x)}),$$

where $\gamma(x)$ is a function of $\log|x|$. Besides giving stronger, and perhaps optimal, results, our approach also works in the case $\alpha < 1$, which is not possible with the method of [6]. We are not sure if the condition $\alpha > 1-\lambda^2$ is necessary though.

Our proof, detailed in Section 3, starts by proving the following lower bound for u on annuli

$$\int_{\{R-1\le|x|\le R\}} u^2 \ge \exp(-CR^\alpha) \quad \forall R \ge 10.$$

Using this bound and a chain of balls argument with balls of the same size we obtain (1.4). The key ingredient in this step is a three-ball inequality (see (3.6) and (3.7)). Finally, we deduce (1.5) from (1.4) using another application of the three-ball inequality.

2. Carleman estimates

In this section, we collect some Carleman estimates that play the key role in the proofs of Theorems 1.1 and 1.2. Throughout this section, C denotes a constant that only depends on α, n, λ and Λ, whose value may change from line to line.

The first Carleman estimate is a generalization of an estimate in [7, Lemma 1] where it is proved for $P = \Delta$ and $\alpha \in (2/3, 2]$. For convenience, we will use the notations $r = |x|$ and $\|\xi\|^2 = \langle A(x)\xi, \xi\rangle$.

Proposition 2.1. *Let P be as in Theorem 1.1, and $\alpha > 1-\lambda^2$. Then there exist positive constants β_0, C_0 and ε depending on α, n, λ and Λ such that if*

$$|\nabla A(x)| \le \varepsilon/|x| \quad \text{for } |x| \ge 1,$$

then

$$\beta^3 \int r^{2\alpha-2} e^{2\beta r^\alpha} u^2 + \beta \int e^{2\beta r^\alpha} |\nabla u|^2 \le C_0 \int r^{2-\alpha} e^{2\beta r^\alpha} |Pu|^2, \qquad (2.1)$$

for $u \in C_c^\infty(\mathbb{R}^n \setminus B_1)$ *and* $\beta \geq \beta_0$.

Proof. Let $v = e^{\beta r^\alpha} u$ then $e^{\beta r^\alpha} Pu = e^{\beta r^\alpha} P(e^{-\beta r^\alpha} v) =: P_\beta v$. Since $e^{\beta r^\alpha}|\nabla u| \leq |\nabla v| + \alpha\beta r^{\alpha-1}|v|$, it follows that (2.1) is equivalent to (with possibly a diffenrent value of C_0)

$$\beta^3 \int r^{2\alpha-2} v^2 + \beta \int |\nabla v|^2 \leq C_0 \int r^{2-\alpha} |P_\beta v|^2. \tag{2.2}$$

We have

$$\begin{aligned} P_\beta v &= [\operatorname{div}(A\nabla v) + \alpha^2\beta^2 r^{2\alpha-4}\|x\|^2 v] \\ &\quad - \alpha\beta\big[2r^{\alpha-2}\langle Ax, \nabla v\rangle + \operatorname{div}(r^{\alpha-2}Ax)v\big]. \end{aligned}$$

Let

$$F = \operatorname{tr} A + (\alpha-2)\frac{\|x\|^2}{|x|^2} + \frac{|Ax|^2}{\|x\|^2} - \lambda$$

and

$$\begin{aligned} \widetilde{P}_\beta v &= [\operatorname{div}(A\nabla v) + \alpha^2\beta^2 r^{2\alpha-4}\|x\|^2 v] \\ &\quad - \alpha\beta[2r^{\alpha-2}\langle Ax, \nabla v\rangle + r^{\alpha-2}Fv] \\ &=: Mv - Nv. \end{aligned}$$

Since $|\operatorname{div}(r^{\alpha-2}Ax) - r^{\alpha-2}F| \leq C\alpha\beta r^{\alpha-2}$, if $\beta \geq 4C^2\alpha^2$, then

$$\int |P_\beta v - \widetilde{P}_\beta v|^2 r^{2-\alpha} \leq C^2\alpha^2\beta^2 \int r^{\alpha-2}v^2 \leq \frac{1}{4}\beta^3 \int r^{2\alpha-2}v^2.$$

Thus, it suffices to prove (2.2) with $P_\beta v$ replaced by $\widetilde{P}_\beta v$. We have

$$\begin{aligned} \int |\widetilde{P}_\beta v|^2 r^{2-\alpha} &\geq -2\int r^{2-\alpha} Mv Nv \\ &= -2\alpha^3\beta^3 \int r^{2\alpha-4}\|x\|^2[2\langle Ax, \nabla v\rangle v + Fv^2] \\ &\quad - 2\alpha\beta \int 2\operatorname{div}(A\nabla v)\langle Ax, \nabla v\rangle + F\operatorname{div}(A\nabla v)v \\ &=: I + II. \end{aligned} \tag{2.3}$$

Writing $v\nabla v = \frac{1}{2}\nabla(v^2)$ then integrating by parts, we obtain

$$\begin{aligned} I &= 2\alpha^3\beta^3 \int [\operatorname{div}(r^{2\alpha-4}\|x\|^2 Ax) - r^{2\alpha-4}F\|x\|^2]v^2 \\ &= 2\alpha^3\beta^3 \int [2(\alpha-2)\frac{\|x\|^2}{|x|^2} + 2\frac{|Ax|^2}{\|x\|^2} - F + \operatorname{tr} A]\|x\|^2 r^{2\alpha-4}v^2 \\ &\quad + 2\alpha^3\beta^3 \int [\partial_i a_{ij}x_j\|x\|^2 r^{2\alpha-4} + a_{ik}x_k\partial_i a_{jl}x_l x_j r^{2\alpha-4}]v^2 \\ &\geq 2\alpha^3\beta^3 \int [(\alpha-2)\frac{\|x\|^2}{|x|^2} + \frac{|Ax|^2}{\|x\|^2} + \lambda - C\epsilon]\|x\|^2 r^{2\alpha-4}v^2. \end{aligned} \tag{2.4}$$

To estimate the term II in (2.3), we first use the Rellich-Necas identity to write

$$2\operatorname{div}(A\nabla v)\langle Ax, \nabla v\rangle = 2a_{lk}x_k\partial_i(a_{ij}\partial_j v\partial_l v) - a_{lk}x_k a_{ij}\partial_l(\partial_i v\partial_j v).$$

Then integrating by parts, we obtain

$$II = 2\alpha\beta \int 2\partial_i(a_{lk}x_k)a_{ij}\partial_j v\partial_l v - \partial_l(a_{lk}x_k a_{ij})\partial_i v\partial_j v + A\nabla v \cdot \nabla(Fv)$$

$$
\begin{aligned}
&= 2\alpha\beta \int 2|A\nabla v|^2 + (F - \operatorname{tr} A)\|\nabla v\|^2 + vA\nabla v \cdot \nabla F \\
&\quad + 2\alpha\beta \int 2\partial_i a_{lk} x_k a_{ij} \partial_j v \partial_l v - \partial_l (a_{lk} a_{ij}) x_k \partial_i v \partial_j v.
\end{aligned}
$$

As $|\nabla F| \le C/|x|$, we have

$$
|vA\nabla v \cdot \nabla F| \le C\epsilon \|\nabla v\|^2 + \epsilon^{-1} r^{-2} v^2.
$$

Together with $|A\nabla v|^2 \ge \lambda \|\nabla v\|^2$, this implies

$$
II \ge 2\alpha\beta \int \Big((\alpha - 2)\frac{\|x\|^2}{|x|^2} + \frac{|Ax|^2}{\|x\|^2} + \lambda - C\epsilon \Big) \|\nabla v\|^2 - 2\alpha\beta\epsilon^{-1} \int r^{-2} v^2. \tag{2.5}
$$

We have

$$
\begin{aligned}
(\alpha - 2)\frac{\|x\|^2}{|x|^2} + \frac{|Ax|^2}{\|x\|^2} + \lambda &\ge (\alpha - 1)\frac{\|x\|^2}{|x|^2} + \lambda \\
&\ge \min\{\alpha\lambda, (\alpha - 1)\lambda^{-1} + \lambda\} =: \mu > 0.
\end{aligned}
$$

Thus, if $\epsilon = \frac{\mu}{2C}$ and $\beta \ge \frac{4C}{\alpha\mu}$, then from (2.4) and (2.5) we obtain

$$
\begin{aligned}
\int |\widetilde{P}_\beta v|^2 r^{2-\alpha} &\ge \mu\alpha^3\beta^3 \int r^{2\alpha-2} v^2 + \mu\alpha\beta \int \|\nabla v\|^2 - 2\alpha\beta\epsilon^{-1} \int r^{-2} v^2 \\
&\ge \frac{1}{2}\mu\alpha^3\beta^3 \int r^{2\alpha-2} v^2 + \mu\alpha\beta \int \|\nabla v\|^2.
\end{aligned}
$$

This completes the proof. □

Proposition 2.2. *Let P be as in Theorem 1.2, V be a Lipschitz function and $\alpha > 1 - \lambda^2$. Then there exist positive constants β_0, C_0 and ε depending on α, n, λ and Λ such that if A satisfies*

$$
|\nabla A(x)| \le \varepsilon/|x| \quad \text{for} \quad |x| \ge 1,
$$

then

$$
\begin{aligned}
&\beta^3 \int r^{2\alpha-2} e^{2\beta r^\alpha} u^2 + \beta \int e^{2\beta r^\alpha} |\nabla u|^2 \\
&\le C_0 \int r^{2-\alpha} e^{2\beta r^\alpha} |Pu|^2 + C_0\beta \int e^{2\beta r^\alpha} (|V| + |Ax \cdot \nabla V|) u^2.
\end{aligned}
$$

for $u \in C_c^\infty(\mathbb{R}^n \setminus B_1)$ and $\beta \ge \beta_0$.

Proof. The proof is very similar to that of the previous proposition, so we will only indicate the modifications needed. Now $\widetilde{P}_\beta v$ has one more term Vv which we incorporate into Mv, i.e.

$$
\begin{aligned}
\widetilde{P}_\beta v &= [\operatorname{div}(A\nabla v) + \alpha^2\beta^2 r^{2\alpha-4}\|x\|^2 v + Vv] \\
&\quad - \alpha\beta[2r^{\alpha-2}\langle Ax, \nabla v\rangle + r^{\alpha-2} F v] \\
&=: Mv - Nv.
\end{aligned}
$$

We then have $-2\int r^{2-\alpha} MvNv = I + II + III$ where I and II are as in (2.3) and

$$
\begin{aligned}
III &= -4\alpha\beta \int \langle Ax, \nabla v\rangle Vv - 2\alpha\beta \int FVv^2 \\
&= 2\alpha\beta \int [\operatorname{div}(VAx) - FV] v^2
\end{aligned}
$$

As $|\operatorname{div}(VAx) - FV| \le C(|V| + |Ax \cdot \nabla V|)$,

$$|III| \le C\beta \int (|V| + |Ax \cdot \nabla V|)v^2$$

and the proposition follows. □

The next Carleman estimate is rather standard and can be found in [4, 5].

Proposition 2.3. *Let $Pu = \operatorname{div}(A\nabla u)$ where A is elliptic, symmetric with $A(0) = I$ and $|\nabla A| \le 1/\sqrt{\lambda}$. There exist positive constants β_0, C_0, and $\rho \le \sqrt{\lambda}/4$ depending only on n and an increasing function w satisfying $\frac{1}{C_0} \le \frac{w(x)}{|x|} \le C_0$ such that*

$$\beta^3 \int w^{-1-2\beta}u^2 + \beta \int w^{1-2\beta}|\nabla u|^2 \le C_0 \int w^{2-2\beta}|Pu|^2 \tag{2.6}$$

for $u \in C_c^2(B_\rho \setminus \{0\})$ and $\beta \ge \beta_0$.

We also have a similar estimate for operators with differentiable potentials.

Proposition 2.4. *Let $Pu = \operatorname{div}(A\nabla u) + Vu$ where V is Lipschitz, A is elliptic, symmetric with $A(0) = I$ and $|\nabla A| \le 1/\sqrt{\lambda}$. There exist positive constants β_0, C_0 and $\rho \le \sqrt{\lambda}/4$ depending only on n and an increasing function w satisfying $\frac{1}{C_0} \le \frac{w(x)}{|x|} \le C_0$ such that*

$$\begin{aligned} &\beta^3 \int w^{-1-2\beta}u^2 + \beta \int w^{1-2\beta}|\nabla u|^2 \\ &\le C_0 \int w^{2-2\beta}|Pu|^2 + C_0\beta \int w^{1-2\beta}(|V| + |Ax \cdot \nabla V|)u^2 \end{aligned}$$

for $u \in C_c^2(B_\rho \setminus \{0\})$ and $\beta \ge \beta_0$.

3. Proof of main theorem

The proofs of Theorems 1.1 and 1.2 follow the same lines, using Propositions 2.2 and 2.4 for the first theorem and Propositions 2.1 and 2.3 for the second theorem. Throughout this section, C denotes a constant that only depends on $\alpha, \delta_1, \delta_2, \lambda, M$ and n, whose value may change from line to line. For $r_2 > r_1 > 0$, we let $\mathcal{A}_{r_1,r_2} = \{x \in \mathbb{R}^n : r_1 \le |x| \le r_2\}$. The ball of radius r centered at a is denoted by $B(a,r)$ and $B_r = B(0,r)$.

As indicated in the Introduction, we first prove a lower bound for the L^2-norm of u on annuli.

Lemma 3.1. *Let P, u and α be as in the statement of Theorem 1.1. Then there exists positive constant C_1 such that*

$$\int_{\mathcal{A}_{R-1,R}} u^2 \ge \exp(-C_1 R^\alpha) \quad \forall R \ge 10. \tag{3.1}$$

Proof. Let φ be a smooth cut-off function satisfying

$$\varphi(x) = \begin{cases} 1 & \text{if } \frac{5}{3} \le |x| \le R - \frac{2}{3} \\ 0 & \text{if } \frac{4}{3} \ge |x| \text{ or } |x| \ge R - \frac{1}{3} \end{cases}$$

and

$$|\nabla \varphi(x)| + |\nabla^2 \varphi(x)| \le C_n \quad \forall x.$$

Let $v = \varphi u$ and

$$E = \mathrm{sppt}(\nabla\varphi) \subset \mathcal{A}_{4/3,5/3} \cup \mathcal{A}_{R-2/3,R-1/3}.$$

We have

$$\begin{aligned}|Pv| &= |\varphi Pu + 2A\nabla u \cdot \nabla\varphi + \operatorname{div}(A\nabla\varphi)u| \\ &\le M\varphi\big(|x|^{-\delta_0}|u| + |x|^{-\delta_1}|\nabla u|\big) + C[|\nabla u| + |u|]\mathbf{1}_E \\ &\le M\left(|x|^{-\delta_0}|v| + |x|^{-\delta_1}|\nabla v|\right) + C[|\nabla u| + \big(1 + M|x|^{-\delta_1}\big)|u|]\mathbf{1}_E.\end{aligned}$$

Applying the Carleman estimate (2.1), we obtain for $\beta \ge \beta_0$,

$$\begin{aligned}&\beta^3 \int e^{2\beta r^\alpha} r^{2\alpha-2} v^2 + \beta \int e^{2\beta r^\alpha} |\nabla v|^2 \\ &\le C_0 \int e^{2\beta r^\alpha} r^{2-\alpha} |Pv|^2 \\ &\le 4C_0M^2 \int e^{2\beta r^\alpha}\left(r^{2-\alpha-2\delta_0} v^2 + r^{2-\alpha-2\delta_1}|\nabla v|^2\right) \\ &\quad + 4C_0C^2(M^2+1) \int_E e^{2\beta r^\alpha} [|\nabla u|^2 + \left(1 + r^{-2\delta_1}\right) u^2].\end{aligned}$$

Choose $\beta = \beta_0 + 8C_0M^2$, then the first term of the right-hand side is absorbed by the left-hand side since $r^{2-\alpha-2\delta_0} \le r^{2\alpha-2}$ and $r^{2-\alpha-2\delta_1} \le 1$ for $r \ge 1$. As $v = u$ on $\mathcal{A}_{3,7}$, we deduce that

$$e^{2\beta\cdot 3^\alpha} \int_{\mathcal{A}_{3,7}} u^2 \le \beta^3 \int e^{2\beta r^\alpha} r^{2\alpha-2} v^2 \le C \int_E e^{2\beta r^\alpha} [|\nabla u|^2 + \left(1 + r^{-2\delta_1}\right) u^2]. \quad (3.2)$$

Applying the standard Cacciopolli's inequality, we obtain

$$\int_E e^{2\beta r^\alpha} [|\nabla u|^2 + \left(1 + r^{-2\delta_1}\right) u^2] \le Ce^{2\beta\cdot 2^\alpha} \int_{\mathcal{A}_{1,2}} u^2 + CR^\delta e^{2\beta R^\alpha} \int_{\mathcal{A}_{R-1,R}} u^2,$$

where $\delta = \max\{-\delta_0, -2\delta_1, 0\}$. Combining the above inequality and (3.2), we obtain

$$e^{2\beta\cdot 3^\alpha} \int_{\mathcal{A}_{3,7}} u^2 \le Ce^{2\beta\cdot 2^\alpha} \int_{\mathcal{A}_{1,2}} u^2 + CR^\delta e^{2\beta R^\alpha} \int_{\mathcal{A}_{R-1,R}} u^2. \quad (3.3)$$

If

$$\beta \ge \frac{\log(2C\int_{\mathcal{A}_{1,2}} u^2 / \int_{\mathcal{A}_{3,7}} u^2)}{2(3^\alpha - 2^\alpha)},$$

then the first term of the right-hand side of (3.3) can be absorbed by the left-hand side, and we obtain

$$R^{-\delta} e^{-2\beta R^\alpha} \int_{\mathcal{A}_{3,7}} u^2 \le C \int_{\mathcal{A}_{R-1,R}} u^2 dx.$$

This completes the proof. □

Next, we show that (3.1) and the upper bound on u give the desired lower bounds.

Lemma 3.2. *Let P, u be as in the statement of Theorem 1.1 and $\tau = \frac{\sqrt{\lambda}\rho}{4}$ where ρ is the constant appears in the statement of Proposition 2.3. Assume that for some positive constants C_1 and M,*

$$\int_{\mathcal{A}_{R-1,R}} u^2 \ge e^{-C_1 R^\alpha} \quad \forall R \ge 10,$$

and

$$|u(x)| \le e^{M|x|^\alpha} \quad \forall |x| \ge 1.$$

Then there exists $C_2 > 0$ *such that if* $|x| = R \ge 10$*, then*

$$\int_{B(x,\tau R)} u^2 \ge e^{-C_2 R^\alpha} \tag{3.4}$$

and

$$\int_{B(x,1)} u^2 \ge e^{-C_2 R^\alpha \log R}. \tag{3.5}$$

Proof. We first prove a version of the standard three-ball inequality: there exists $C > 0$ such that for $a \in \mathcal{A}_{R-\tau R, R+\tau R}$ and $1/R \le \tau_0 \le \tau_1/\lambda \le \tau_2/2 \le 2\tau/\lambda$, we have either

$$\int_{B(a,\tau_1 R)} u^2 \le \Big(\frac{w(\sqrt{\lambda}\tau_2/2)}{w(\sqrt{\lambda}\tau_0/2)}\Big)^{CR^\alpha} \int_{B(a,\tau_0 R)} u^2 \tag{3.6}$$

or

$$\int_{B(a,\tau_1 R)} u^2 \le CR^\delta \Big(\int_{B(a,\tau_0 R)} u^2\Big)^\theta \Big(\int_{B(a,\tau_2 R)} u^2\Big)^{1-\theta}. \tag{3.7}$$

Here $\delta = \max\{1, 2-\delta_0, 2-2\delta_1\}$ and

$$\theta = \frac{\log(w(\sqrt{\lambda}\tau_2/2)) - \log(w(\tau_1/\sqrt{\lambda}))}{\log(w(\sqrt{\lambda}\tau_2/2)) - \log(w(\sqrt{\lambda}\tau_0/2))}.$$

To show this, we first make a change of variables. Let $S = A(a)^{-1/2}$, $A_R(x) = SA(a + S^{-1}Rx)S^t$ and $P_R v = \operatorname{div}(A_R \nabla v)$. Then for $u_R(x) = u(a + S^{-1}Rx)$, we have

$$P_R u_R(x) = R^2 P u(a + S^{-1}Rx).$$

Thus,

$$|P_R u_R| \le CR^{1-\delta_1}|\nabla u_R| + CR^{2-\delta_0}|u_R| \quad \text{ in } B_\rho.$$

Let $r_0 = \sqrt{\lambda}\tau_0$, $r_1 = \tau_1/\sqrt{\lambda}$, $r_2 = \sqrt{\lambda}\tau_2$ and φ be a smooth cut-off function satisfying

$$\varphi(x) = \begin{cases} 1 & \text{if } \frac{2}{3}r_0 \le |x| \le \frac{1}{2}r_2 \\ 0 & \text{if } \frac{1}{2}r_0 \ge |x| \text{ or } |x| \ge \frac{2}{3}r_2 \end{cases}$$

and

$$|\nabla^l \varphi(x)| \le C|x|^{-l}, \quad l = 1, 2.$$

Let $v = \varphi u_R$ and

$$E = \operatorname{sppt}(\nabla\varphi) \subset \mathcal{A}_{r_0/2, 2r_0/3} \cup \mathcal{A}_{r_2/2, 2r_2/3}.$$

We have

$$\begin{aligned} |P_R v| &= |\varphi P_R u_R + 2A_R \nabla u_R \cdot \nabla\varphi + \operatorname{div}(A_R\nabla\varphi)u_R| \\ &\le C\varphi\left(R^{2-\delta_0}|u_R| + R^{1-\delta_1}|\nabla u_R|\right) + C[|x|^{-1}|\nabla u_R| + |x|^{-2}|u_R|]\mathbf{1}_E \\ &\le C\left(R^{2-\delta_0}|v| + R^{1-\delta_1}|\nabla v|\right) + C[|x|^{-1}|\nabla u_R| + \left(|x|^{-2} + |x|^{-1}R^{1-\delta_1}\right)|u_R|]\mathbf{1}_E. \end{aligned}$$

Note that $A_R(0) = Id$ and $|\nabla A_R| \le \sqrt{\lambda}$ in B_ρ. Thus, we can apply Proposition 2.3 to P_R and v to obtain for $\beta \ge \beta_0$

$$\beta^3 \int w^{-1-2\beta} v^2 + \beta \int w^{1-2\beta} |\nabla v|^2$$

$$\begin{aligned}&\le C\int w^{2-2\beta}|P_R v|^2\\&\le C\int w^{2-2\beta}\left(R^{4-2\delta_0}v^2+R^{2-2\delta_1}|\nabla v|^2\right)\\&\quad+C\int_E w^{2-2\beta}[|x|^{-2}|\nabla u_R|^2+\left(|x|^{-4}+|x|^{-2}R^{2-2\delta_1}\right)u_R^2].\end{aligned}$$

If $\beta \ge CR^\alpha$ then the first term of the right-hand side is absorbed by the left-hand side, hence we obtain

$$\beta^3\int w^{-1-2\beta}v^2 \le C\int_E w^{2-2\beta}[|x|^{-2}|\nabla u_R|^2+\left(|x|^{-4}+|x|^{-2}R^{2-2\delta_1}\right)u_R^2].$$

Since $v=u$ on $\mathcal{A}_{r_0,r_1}$ the left-hand side is greater than

$$w^{-1-2\beta}(r_1)\int_{\mathcal{A}_{r_0,r_1}} u_R^2.$$

By the standard Cacciopolli's inequality, the right-hand side is smaller than

$$CR^\delta w^{-1-2\beta}(r_0/2)\int_{\mathcal{A}_{\frac{r_0}{3},r_0}} u_R^2 + CR^\delta w^{-1-2\beta}(r_2/2)\int_{\mathcal{A}_{\frac{r_2}{3},r_2}} u_R^2,$$

where $\delta=\max\{1,2-\delta_0,2-2\delta_1\}$. Thus, we obtain

$$\begin{aligned}&w^{-1-2\beta}(r_1)\int_{\mathcal{A}_{r_0,r_1}} u_R^2\\&\le CR^\delta w^{-1-2\beta}(r_0/2)\int_{\mathcal{A}_{\frac{r_0}{3},r_0}} u_R^2 + CR^\delta w^{-1-2\beta}(r_2/2)\int_{\mathcal{A}_{\frac{r_2}{3},r_2}} u_R^2.\end{aligned}$$

Adding $w^{-1-2\beta}(r_1)\int_{B_{r_0}} u_R^2$ to both sides gives

$$\begin{aligned}w^{-1-2\beta}(r_1)\int_{B_{r_1}} u_R^2 \le& CR^\delta w^{-1-2\beta}(r_0/2)\int_{B_{r_0}} u_R^2 &(3.8)\\&+CR^\delta w^{-1-2\beta}(r_2/2)\int_{B_{r_2}} u_R^2. &(3.9)\end{aligned}$$

If

$$\frac{\log(\int_{B_{r_2}} u_R^2/\int_{B_{r_0}} u_R^2)}{2\log(w(r_2/2)/w(r_0/2))} \le CR^\alpha$$

then (3.6) holds. Otherwise, choosing

$$\beta=\frac{\log(\int_{B_{r_2}} u_R^2/\int_{B_{r_0}} u_R^2)}{2\log(w(r_2/2)/w(r_0/2))}$$

in (3.8) gives

$$w^{-1-2\beta}(r_1)\int_{B_{r_1}} u_R^2 \le CR^\delta w^{-1-2\beta}(r_0/2)\int_{B_{r_0}} u_R^2,$$

which implies

$$\int_{B_{r_1}} u_R^2 \le CR^\delta\Big(\int_{B_{r_0}} u_R^2\Big)^\theta\Big(\int_{B_{r_2}} u_R^2\Big)^{1-\theta}.$$

Undoing the change of variables, noting that $a+S^{-1}RB_{r_j}\subset B(a,\tau_j R)$ for $j=0,2$ while $a+S^{-1}RB_{r_1}\supset B(a,\tau_1 R)$, we obtain (3.7).

Choose $\tau_0 = \tau$, $\tau_1 = 2\tau$, and $\tau_2 = 4\tau/\lambda$. It is easy to see that with this choice of parameters, both (3.6) and (3.7) implies that for some $C_3 > 0$,

$$\int_{B(a,2\tau R)} u^2 \le e^{C_3 R^\alpha} \Big(\int_{B(a,\tau R)} u^2 \Big)^\theta \quad \text{if } a \in \mathcal{A}_{R-\tau R, R+\tau R}. \tag{3.10}$$

Here we have used the upper bound $|u(x)| \le e^{M|x|^\alpha}$.

We next deduce (3.4) from (3.10). Let $|x| = R$ and $Q = \{a_1 = x, a_2, a_3, \dots, a_N\}$ be a $\frac{\tau R}{2}$-net on the sphere of radius R. By scaling and symmetry, it easy to check that this can be done with N independent of R and x. Note that any $a_j \in Q$ can be connected to $a_1 = x$ by a sequence of points in Q such that the distances between consecutive points smaller than τR. Thus, applying (3.10) repeatedly gives

$$\int_{B(a_j,\tau R)} u^2 \le e^{C_3 R^\alpha/(1-\theta)} \Big(\int_{B(x,\tau R)} u^2 \Big)^{\theta^N}.$$

Since $\cup_{j=1}^N B(a_j, \tau R) \supset \mathcal{A}_{R-1,R}$, summing over j gives

$$e^{-C_1 R^\alpha} \le \int_{\mathcal{A}_{R-1,R}} u^2 \le \sum_{j=1}^N \int_{B(a_j,\tau R)} u^2 \le N e^{C_3 R^\alpha/(1-\theta)} \Big(\int_{B(x,\tau R)} u^2 \Big)^{\theta^N}$$

from which (3.4) follows.

Finally, we prove the lower bound (3.5). Choose $\tau_0 = 1/R$, $\tau_1 = 2\tau$ and $\tau_2 = 4\tau/\lambda$. If (3.6) holds then (3.5) follows because

$$\frac{w(\sqrt{\lambda}\tau_2/2)}{w(\sqrt{\lambda}\tau_0/2)} = \frac{w(\rho/2)}{w(\sqrt{\lambda}/(2R))} \le CR.$$

On the other hand, if (3.7) holds, then it follows that

$$\int_{B(x,\tau R)} u^2 \le C R^\delta e^{M R^\alpha} \Big(\int_{B(x,1)} u^2 \Big)^\theta.$$

Since

$$\theta = \frac{\log(w(\rho/2)) - \log(w(\rho/4))}{\log(w(\rho/2)) - \log(w(\sqrt{\lambda}/(2R)))} \ge \frac{C}{\log R},$$

inequality (3.5) follows from (3.4). □

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