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EXPONENTIAL STABILITY FOR POROUS THERMOELASTIC SYSTEMS WITH GURTIN-PIPKIN FLUX

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Abstract. In this article, we study the stability of a porous thermoelastic system with Gurtin-Pipkin flux. Under suitable assumptions for the derivative of the heat flux relaxation kernel, we establish the existence and uniqueness of solution by applying the semigroup theory, and prove the exponential stability of system without considering the wave velocity by the means of estimates of the resolvent operator norm.

1. Introduction

In this work, we consider the porous thermoelastic transmission system with Gurtin-Pipkin flux,

$$\begin{gathered} \rho u_{tt} - \mu u_{xx} - b\varphi_x - \gamma u_{xxt} = 0 \quad \text{in } (0,1)\times\mathbb{R}^+, \\ J\varphi_{tt} - \delta\varphi_{xx} + bu_x + \xi\varphi + \beta\theta_x = 0 \quad \text{in } (0,1)\times\mathbb{R}^+, \\ c\theta_t + q_x + \beta\varphi_{xt} = 0 \quad \text{in } (0,1)\times\mathbb{R}^+, \end{gathered} \tag{1.1}$$

where $\mathbb{R}^+ = [0,\infty)$ and

$$q = -\int_{-\infty}^{t} g(t-s)\theta_x(x,s)\,ds. \tag{1.2}$$

This system of equations was firstly derived by Gurtin and Pipkin [1]. The initial and boundary conditions for system (1.1) are as follows,

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } (0,1), \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) \quad \text{in } (0,1), \\ \theta(x,t) = \theta_0(x,-t) \quad \text{in } (0,1)\times(-\infty,0], \\ u(0,t) = u(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = 0 \quad \text{in } \mathbb{R}^+, \\ \theta(0,t) = \theta(1,t) = 0 \quad \text{in } \mathbb{R}. \end{gathered} \tag{1.3}$$

Here, u is transversal displacement, φ is the volume fraction, θ temperature, and q is the heat flux. We assume the coefficients $\rho, J, c, \mu, b, \delta, \gamma, \xi$ are positive constants such that $\mu\xi \geq b^2$. The heat conductivity relaxation kernel $g > 0$ and the parameter

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β denotes a non-zero coupling coefficient. Note that the coupling coefficient does not play an important role in the analysis.

The system we are studying highlights the heat flux q, which can be used in some materials to describe how memory effects can dominate. As far as we know, q comes in many forms and is used in a variety of systems, such as thermoelastic systems, Timoshenko systems, Bresse systems and so on. When heat flux q is expressed by Fourier's law or Cattaneo's law, a large number of scholars have studied the existence and asymptotic behavior of solutions for related systems.

When the heat flux q is in terms of Fourier's law, we have

$$q = -k\theta_x. \tag{1.4}$$

Casas and Quintanilla [2] studied the thermoelastic system

$$\begin{gathered} \rho u_{tt} - \mu u_{xx} - b\varphi_x + \beta\theta_x = 0 \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\ J\varphi_{tt} - \alpha\varphi_{xx} + bu_x + \xi\varphi - m\theta + \tau\varphi_t = 0 \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\ c\theta_t - k\theta_{xx} + \beta u_{tx} + m\varphi_t = 0 \quad \text{in } (0,\pi) \times \mathbb{R}^+. \end{gathered} \tag{1.5}$$

Using the semigroup method, they demonstrated that the system was exponentially stable under a combination of porous dissipation and thermal effect. Apalara [3] considered the porous thermoelastic system with memory terms, mainly, the memory term was used to replace the porous dissipation term in (1.5). He used the energy method to obtain stable results of various forms of solution through different memory effects. Al-Mahdi et al. [4] considered the new kernel $g'(t) \leq -\gamma(t)G(g(t))$ and established new general decay results in the case of infinite memory. Magaña and Quintanilla [5] introduced a strong damping mechanism,

$$\begin{gathered} \rho u_{tt} - \mu u_{xx} - b\varphi_x + \beta\theta_x - \gamma u_{xxt} = 0 \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\ J\varphi_{tt} - \delta\varphi_{xx} + bu_x + \xi\varphi - m\theta = 0 \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\ c\theta_t - k\theta_{xx} + \beta u_{tx} + m\varphi_t = 0 \quad \text{in } (0,\pi) \times \mathbb{R}^+. \end{gathered} \tag{1.6}$$

They used the same method as in [2] to prove that the system decays slowly in the presence thermal effect. In addition, they introduced microtemperature and found out the exponential decay of this system. Also, we can see [6]. Pamplona et al. [7] obtained the conclusion of (1.6) using higher-order energy methods. Djebabla et al. [8] studied porous thermoelastic system with time delay, they used the energy method combined with multiplicative technique and showed the polynomial decay estimate. We can also refer to [9, 10] to study more thermoelastic systems.

In Timoshenko systems, Rivera and Racke [11] considered the system

$$\begin{gathered} \varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0,L) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 \quad \text{in } (0,L) \times \mathbb{R}^+, \\ \rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0 \quad \text{in } (0,L) \times \mathbb{R}^+. \end{gathered} \tag{1.7}$$

They demonstrated exponential stability through the damping effect of heat conduction. For the angle of rotation system with memory term

$$
\begin{gathered}
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \beta\theta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\
\rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) - \beta\theta + \int_0^t g(t-s)\psi_{xx}(s)\,ds = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\
\rho_3\theta_t - \kappa\theta_{xx} + \beta(\varphi_{xt} + \psi_t) = 0 \quad \text{in } (0, L) \times \mathbb{R}^+.
\end{gathered} \tag{1.8}
$$

When $\beta = 1$, Messaoudi and Fareh [12] established the general decay result by constructing energy functional for equal wave velocities, that is,

$$
\mathcal{X} = \frac{k}{\rho_1} - \frac{\alpha}{\rho_2} = 0. \tag{1.9}
$$

Later, they used the same method to consider the case of $\mathcal{X} \neq 0$ in [13] and also obtained the general decay result. When $\beta > 0$ and $\beta \neq 1$, Almeida Júnior et al. [14] studied (1.8) at $g = 0$. Considering the case of $\mathcal{X} \neq 0$, they found that related semigroups had different polynomial decay rates under different boundary conditions. The semigroup decays optimally at the rate of $1/\sqrt{t}$ for fully Dirichlet boundary conditions and at the rate of $1/\sqrt[4]{t}$ for Dirichlet-Neumann-Dirichlet boundary conditions. In the presence of memory term, Apalara [15] extended the above system for any $\beta > 0$, and obtained a general stability result independent of wave velocity by using the energy method under Neumann-Dirichlet-Dirichlet boundary conditions. General forms of Bresse system can also be coupled to thermal effect,

$$
\begin{gathered}
\rho_1\varphi_{tt} = k(\varphi_x + \psi + l\omega)_x + lk_0(\omega_x - l\varphi) = 0 \quad \text{in } (0, \pi) \times \mathbb{R}^+, \\
\rho_2\psi_{tt} = b\psi_{xx} - k(\varphi_x + \psi + l\omega) - \gamma\theta_x \quad \text{in } (0, \pi) \times \mathbb{R}^+, \\
\rho_1\omega_{tt} = k_0(\omega_x - l\varphi)_x - lk(\varphi_x + \psi + l\omega) \quad \text{in } (0, \pi) \times \mathbb{R}^+, \\
\rho_3\theta_t = q_x - \gamma\psi_{tx} \quad \text{in } (0, \pi) \times \mathbb{R}^+.
\end{gathered} \tag{1.10}
$$

Fatori and Rivera [16] showed the Bresse-Fourier system was exponentially stable if and only if $k = k_0$ and (1.9) holds. For a discussion of the type III thermoelastic Bresse system readers may refer to [17] which considered the effect of memory item.

When the heat flux q is in terms of Cattaneo's law, q satisfies

$$
\tau_0 q_t + q + \kappa\theta_x = 0. \tag{1.11}
$$

Fareh and Messaoudi [18] investigated the porous thermoelastic system with unnecessary positive definite energy

$$
\begin{gathered}
\rho u_{tt} - \mu u_{xx} - b\varphi_x = 0 \quad \text{in } (0, 1) \times \mathbb{R}^+, \\
J\varphi_{tt} - \alpha\varphi_{xx} + bu_x + \xi\varphi + \beta\theta_x = 0 \quad \text{in } (0, 1) \times \mathbb{R}^+, \\
c\theta_t + q_x + \beta\varphi_{tx} + \delta\theta = 0 \quad \text{in } (0, 1) \times \mathbb{R}^+, \\
\tau_0 q_t + q + \kappa\theta_x = 0 \quad \text{in } (0, 1) \times \mathbb{R}^+.
\end{gathered} \tag{1.12}
$$

Under Dirichlet-Neumann-Dirichlet boundary conditions, they assumed that $\mu\xi = b^2$ and introduced the stability number

$$
\mathcal{X} = \beta^2 - \Big(\frac{c\alpha\mu}{\rho} - \frac{\alpha\kappa}{\rho_0}\Big) - \Big(\frac{J}{\alpha} - \frac{\rho}{\mu}\Big). \tag{1.13}
$$

When $\mathcal{X} = 0$, they got exponential stability of this system, and when $\mathcal{X} \neq 0$, this system was polynomially stable. Fernàndez-Sare and Racke [19] considered the Timoshenko system

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\ \tau_0 q_t + q + \kappa\theta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+. \end{aligned} \tag{1.14}$$

They showed that the solution of the above system was not exponentially stable, even if the condition (1.9) be satisfied. Santos et al. [20] considered (1.14), they introduced a new stability number

$$\mathcal{X} = \Big(\tau - \frac{\kappa\rho_1}{\rho_3}\Big) - \Big(\rho_2 - \frac{\alpha\rho_1}{\kappa}\Big) - \Big(\frac{\tau\delta^2\rho_1}{\kappa\rho_3}\Big), \tag{1.15}$$

and established an exponential stability for $\mathcal{X} = 0$. Also, they discussed the case of $\mathcal{X} \neq 0$, obtained the optimal polynomial decay. For the related Timoshenko system with frictional damping, we can also refer to [21]. The new Bresse system established by coupling with (1.11) through (1.10) was studied by Keddi et al. [22], they used the same method as [18] to get the exponential decay result of system.

For heat flux q of Gurtin-Pipkin type, we refer the readers to [23]. Here, we briefly describe a few systems. Pata and Vuk [24] considered the linear thermoelastic system

$$\begin{aligned} u_{tt} - u_{xx} + \theta_x = 0 \quad \text{in } (0, l) \times \mathbb{R}^+, \\ \theta_t - \int_{-\infty}^t g(t-s)\theta_{xx}(x,s)\,ds + u_{tx} = 0 \quad \text{in } (0, l) \times \mathbb{R}^+. \end{aligned} \tag{1.16}$$

They used the semigroup method to achieve that the solution of system had an exponential decay result. And in the latest literature, Fareh [25] studied the porous thermoelastic system with porous damping

$$\begin{aligned} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x \quad \text{in } (0, \pi) \times \mathbb{R}^+, \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + \delta\theta - \tau\varphi_t \quad \text{in } (0, \pi) \times \mathbb{R}^+, \\ c\theta_t = \int_{-\infty}^t g(t-s)\theta_{xx}(x,s)\,ds - \beta u_{xt} - \delta\varphi_t \quad \text{in } (0, \pi) \times \mathbb{R}^+, \end{aligned} \tag{1.17}$$

and showed that the exponential decay of solution in the presence of the more general convolution integral form and the porous dissipation coefficient τ. Dell'Oro and Pata [26] considered the coupled Timoshenko system

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0, l) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x = 0 \quad \text{in } (0, l) \times \mathbb{R}^+, \\ \rho_3\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{tx} = 0 \quad \text{in } (0, l) \times \mathbb{R}^+. \end{aligned} \tag{1.18}$$

When (1.2) was applied in (1.10), Dell'Oro [27] studied the asymptotic stability of the system. They defined the stability number

$$\mathcal{X}_g = \Big(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\Big)\Big(\frac{\rho_1}{k} - \frac{\rho_2}{b}\Big) - \frac{1}{g(0)k_1}\frac{\rho_1\gamma^2}{\rho_3 bk}. \tag{1.19}$$

and showed that this system was exponentially stable if and only if $\mathcal{X}_g = 0$ and $k = k_0$.

In this article, we study the asymptotic behavior of a porous thermoelastic system with Gurtin-Pipkin flux under strong damping which improves the conclusions of [7, 28]. Thus, we know that in some materials, memory items can dominate. The rest of this article is structured as follows: In section 2, we give some preliminaries and reset the system (1.1)-(1.3) to an abstract Cauchy problem. In section 3, we give the well-posedness of the system. In in section 4, we give the main conclusion that the solution of the system is exponentially stable.

2. Preliminaries

In this section, we give the definitions and assumptions for proving the conclusion of this article. Here $(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual scalar product and the norm in $L^2(0,1)$, respectively. $\|\cdot\|_{-1}$ denotes the norm of the space $H^{-1}(0,1)$ which is the conjugate space of $H_0^1(0,1)$ and $\langle\cdot,\cdot\rangle$ denotes the conjugate pairs. We set the spaces

$$\begin{gathered}
L_*^2(0,1) = \big\{\psi \in L^2(0,1) : \int_0^1 \psi(x)\,dx = 0\big\},\\
H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1),
\end{gathered}$$

$$\begin{aligned}
\mathcal{M} &= L_k^2((0,\infty); H_0^1(0,1))\\
&= \{\zeta(x,s) \in L^2\left((0,\infty); H_0^1(0,1)\right) : \int_0^\infty k(s)\int_0^1 \zeta_x^2(x,s)\,dx\,ds < +\infty\}.
\end{aligned}$$

The space $\mathcal{M}$ is endowed with the inner product and norm:

$$\langle\zeta,\xi\rangle_{\mathcal{M}} = \int_0^\infty k(s)\big(\zeta_x(s),\xi_x(s)\big)\,ds, \quad \|\zeta\|_{\mathcal{M}}^2 = \int_0^\infty k(s)\|\zeta_x(s)\|^2\,ds.$$

Meanwhile, we define the space

$$\mathcal{K} = \{\zeta|\zeta_s \in \mathcal{M} : \lim_{s\to 0}\|\zeta_x(s)\| = 0\}.$$

Now, we define the state space

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times L^2(0,1) \times \mathcal{M}$$

endowed with the inner product

$$\begin{aligned}
\langle Z, Z^*\rangle_{\mathcal{H}} = \mu\int_0^1 u_x u_x^*\,dx &+ \xi\int_0^1 ww^*\,dx + b\int_0^1 wu_x^*\,dx + b\int_0^1 w^* u_x\,dx\\
&+ \rho\int_0^1 vv^*\,dx + J\int_0^1 zz^*\,dx + c\int_0^1 \theta\theta^*\,dx\\
&+ \delta\int_0^1 w_x w_x^*\,dx + \int_0^\infty\int_0^1 k(s)\zeta_x(s)\zeta_x^*(s)\,dx\,ds,
\end{aligned} \tag{2.1}$$

for any $Z = (u,v,w,z,\theta,\zeta)^T \in \mathcal{H}$, $Z^* = (u^*,v^*,w^*,z^*,\theta^*,\zeta^*)^T \in \mathcal{H}$.

As in [24, 25], we introduce some new variables

$$\begin{gathered}
\theta^t(x,s) = \theta(x,t-s), \quad s \ge 0,\\
\eta^t(x,s) = \int_0^s \theta^t(x,\tau)\,d\tau, \quad s \ge 0,
\end{gathered}$$

which denote the past history and the summed past history of θ up to t, respectively. We denote

$$\eta_0(x,s)=\int_0^s \theta_0(x,\tau)\,d\tau, \quad s\geq 0.$$

We can easily show that

$$\eta_t^t(x,s)=\theta(x,t)-\eta_s^t(x,s). \tag{2.2}$$

Further, we assume that $g(\infty)=0$ and $\eta^t(x,0)=\lim_{s\to 0}\eta^t(x,s)=0$, then

$$-\int_{-\infty}^t g(t-s)\theta_{xx}(x,s)\,ds=\int_0^\infty g'(s)\eta_{xx}^t(x,s)\,ds. \tag{2.3}$$

Setting $k(s)=-g'(s)$, combining (2.2) and (2.3), system (1.1)-(1.3) can be written as

$$\begin{gathered}
\rho u_{tt}-\mu u_{xx}-b\varphi_x-\gamma u_{xxt}=0 \quad \text{in } (0,1)\times\mathbb{R}^+,\\
J\varphi_{tt}-\delta\varphi_{xx}+bu_x+\xi\varphi+\beta\theta_x=0 \quad \text{in } (0,1)\times\mathbb{R}^+,\\
c\theta_t-\int_0^\infty k(s)\eta_{xx}^t(x,s)\,ds+\beta\varphi_{xt}=0 \quad \text{in } (0,1)\times\mathbb{R}^+,\\
\eta_t^t(x,s)=\theta(x,t)-\eta_s^t(x,s) \quad \text{in } (0,1)\times\mathbb{R}^+\times\mathbb{R}^+,
\end{gathered} \tag{2.4}$$

supplemented with the initial and boundary conditions

$$\begin{gathered}
u(x,0)=u_0(x), u_t(x,0)=u_1(x), \varphi(x,0)=\varphi_0(x) \quad \text{in } (0,1),\\
\varphi_t(x,0)=\varphi_1(x), \theta(x,0)=\theta_0(x) \quad \text{in } (0,1),\\
\eta^0(x,s)=\eta_0(x,s) \quad \text{in } (0,1)\times\mathbb{R}^+,\\
u(0,t)=u(1,t)=\varphi_x(0,t)=\varphi_x(1,t)=0 \quad \text{in } \mathbb{R}^+,\\
\theta(0,t)=\theta(1,t)=\eta^t(0,s)=\eta^t(1,s)=0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^+.
\end{gathered} \tag{2.5}$$

Remark 2.1 ([29])**.** From $(2.4)_2$ and the boundary conditions, we easily verify that

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)\,dx+\frac{\xi}{J}\int_0^1 \varphi(x,t)\,dx=0.$$

By solving this ordinary differential equation and using the initial data of φ, we obtain

$$\int_0^1 \varphi(x,t)\,dx=\Big(\int_0^1 \varphi_0(x)\,dx\Big)\cos\Big(\sqrt{\frac{\xi}{J}}t\Big)+\sqrt{\frac{J}{\xi}}\Big(\int_0^1 \varphi_1(x)\,dx\Big)\sin\Big(\sqrt{\frac{\xi}{J}}t\Big).$$

We introduce

$$\bar\varphi(x,t)=\varphi(x,t)-\Big(\int_0^1 \varphi_0(x)\,dx\Big)\cos\Big(\sqrt{\frac{\xi}{J}}t\Big)-\sqrt{\frac{J}{\xi}}\Big(\int_0^1 \varphi_1(x)\,dx\Big)\sin\Big(\sqrt{\frac{\xi}{J}}t\Big),$$

then

$$\begin{gathered}
\bar\varphi_x(x,t)=\varphi_x(x,t) \quad \text{in } (0,1)\times\mathbb{R}^+,\\
\bar\varphi_{xx}(x,t)=\varphi_{xx}(x,t) \quad \text{in } (0,1)\times\mathbb{R}^+,
\end{gathered}$$

and

$$\begin{gathered}
\bar\varphi_x(0,t)=\varphi_x(0,t)=0 \quad \text{in } \mathbb{R}^+,\\
\bar\varphi_x(1,t)=\varphi_x(1,t)=0 \quad \text{in } \mathbb{R}^+.
\end{gathered}$$

Furthermore, we find that $(u, \bar{\varphi}, \theta, \eta)$ satisfies the same boundary conditions as $(2.5)_4$, $(2.5)_5$ and

$$\int_0^1 \bar{\varphi}(x,t)\,dx = 0.$$

Hence, the Poincaré inequality is applicable for $\bar{\varphi}$ provided that $\bar{\varphi} \in H^1(0,1)$. For the rest of the paper, we will use $\bar{\varphi}$ instead of φ. For convenience, we still denote φ in the followings.

The relevant Poincaré inequality is

$$\int_0^1 \psi^2\,dx \le C_p \int_0^1 \psi_x^2\,dx, \quad \forall \psi \in H_*^1(0,1).$$

To prove our results more easily, we make some hypotheses:

(H1) The relaxation function $k : \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing of class $C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ such that

$$k(s) \ge 0, \quad k'(s) \le 0, \quad s \ge 0,$$

(H2) k is summable on $\mathbb{R}^+$, we have

$$\int_0^\infty k(s)\,ds = k_0 > 0, \quad \int_0^\infty sk(s)\,ds = k_1 > 0,$$

(H3) There exists a positive constant ν such that

$$k'(s) \le -\nu k(s), \quad s \ge 0.$$

Let $U = (u, v, \varphi, w, \theta, \eta^t)^T$, where $v = u_t$ and $w = \varphi_t$, then system (2.4)-(2.5) is equivalent to the abstract Cauchy problem

$$\begin{gathered} \frac{d}{dt}U(t) = \mathcal{A}U(t), \\ U(0) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \eta_0)^T, \end{gathered} \tag{2.6}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \varphi \\ w \\ \theta \\ \eta^t \end{pmatrix} = \begin{pmatrix} v \\ \frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\varphi_x + \frac{\gamma}{\rho}v_{xx} \\ w \\ \frac{\delta}{J}\varphi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\varphi - \frac{\beta}{J}\theta_x \\ \frac{1}{c}\int_0^\infty k(s)\eta_{xx}^t(s)\,ds - \frac{\beta}{c}w_x \\ \theta - \eta_s^t \end{pmatrix}, \tag{2.7}$$

with domain

$$\begin{aligned} D(\mathcal{A}) = \big\{ & U \in \mathcal{H} : u \in H^2(0,1) \cap H_0^1(0,1), v \in H_0^1(0,1), \varphi \in H^2(0,1) \cap H_*^1(0,1) \\ & w \in H_*^1(0,1), \theta \in H_0^1(0,1), \int_0^\infty k(s)\eta^t(s)\,ds \in H^2(0,1), \eta^t \in \mathcal{K}, \eta^t(0) = 0 \big\}. \end{aligned}$$

We introduce the related energy functional

$$\begin{aligned} E(t) = & \frac{1}{2}\int_0^1 \left(\mu|u_x|^2 + \xi|\varphi|^2 + 2bu_x\varphi + \rho|v|^2 + J|w|^2 + c|\theta|^2 + \delta|\varphi_x|^2\right)\,dx \\ & + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2. \end{aligned} \tag{2.8}$$

Note that from the assumption $\mu\xi \geq b^2$,

$$\begin{aligned}\int_0^1 \big(\mu|u_x|^2 + \xi|\varphi|^2 + 2bu_x\varphi\big)\,dx &= \frac{\mu}{2}\|u_x + \frac{b}{\mu}\varphi\|^2 + \frac{\xi}{2}\|\varphi + \frac{b}{\xi}u_x\|^2 \\ &\quad + \frac{1}{2}(\mu - \frac{b^2}{\xi})\|u_x\|^2 + \frac{1}{2}(\xi - \frac{b^2}{\mu})\|\varphi\|^2 \geq 0.\end{aligned} \tag{2.9}$$

3. Well-posedness

In this section, we give the existence and the uniqueness of solution for system (2.6). We use the semigroup method to prove this conclusion, which involves the Lax-Milgram theorem, the Lumer-Phillips theorem and Hille-Yosida theorem. Among them, the content of the Lumer-Phillips theorem is as follows.

Lemma 3.1 ([30])**.** *A densely defined linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup of contractions on $\mathcal{H}$ if and only if $\mathcal{A}$ is m-dissipative, i.e., it satisfies*

- (i) $\Re e(\langle \mathcal{A}U, U\rangle_{\mathcal{H}}) \leq 0$, $U \in D(\mathcal{A})$;
- (ii) $\exists \lambda > 0$, $\lambda I - \mathcal{A}$ *is surjective.*

In the reflexive Banach space $\mathcal{H}$, we know that operator $\mathcal{A}$ is densely defined from (ii) of Lemma 3.1.

Theorem 3.2. *Assume* (H1)–(H3) *and that for each $U(0) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \eta_0)^T$ in $\mathcal{H}$, system* (2.6) *has a unique solution $U \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $U(0) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \eta_0)^T \in D(\mathcal{A})$ then the solution U satisfies*

$$U \in C\big(\mathbb{R}^+; D(\mathcal{A})\big) \cap C^1\big(\mathbb{R}^+; \mathcal{H}\big).$$

Proof. We first prove that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. Firstly, for all $U = (u, v, \varphi, w, \theta, \eta^t)^T \in D(\mathcal{A})$, we have

$$\begin{aligned}&\Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \\ &= \mu\int_0^1 v_x u_x\,dx + \xi\int_0^1 w\varphi\,dx + b\int_0^1 wu_x\,dx + b\int_0^1 \varphi v_x\,dx \\ &\quad + \int_0^1 (\mu u_{xx} + b\varphi_x + \gamma v_{xx})v\,dx + \int_0^1 \big(\delta\varphi_{xx} - bu_x - \xi\varphi - \beta\vartheta_x\big)w\,dx \\ &\quad + \int_0^1 \Big(\int_0^\infty k(s)\eta^t_{xx}(s)\,ds - \beta w_x\Big)\theta\,dx \\ &\quad + \delta\int_0^1 w_x\varphi_x\,dx + \int_0^1\int_0^\infty k(s)\left(\theta_x - \eta^t_{xs}\right)\eta^t_x(s)\,ds\,dx \\ &= -\gamma\|v_x\|^2 - \frac{1}{2}\int_0^\infty k(s)\frac{d}{ds}\|\eta^t_x(s)\|^2\,ds\end{aligned} \tag{3.1}$$

Integrating by parts the second term on the right-hand side, we have

$$\frac{1}{2}\int_0^\infty k(s)\frac{d}{ds}\|\eta^t_x(s)\|^2\,ds = \frac{1}{2}k(s)\|\eta^t_x(s)\|^2\Big|_0^\infty - \frac{1}{2}\int_0^\infty k'(s)\|\eta^t_x(s)\|^2\,ds. \tag{3.2}$$

From the definition of $\eta^t(x, s)$, we have $\eta^t_x(s)|_{s=0} = 0$. Hence, (3.2) can be rewritten

$$\frac{1}{2}\int_0^\infty k(s)\frac{d}{ds}\|\eta^t_x(s)\|^2\,ds = \frac{1}{2}\lim_{s\to\infty} k(s)\|\eta^t_x(s)\|^2 - \frac{1}{2}\int_0^\infty k'(s)\|\eta^t_x(s)\|^2\,ds.$$

From (H1) and the boundedness on the left, we know that the second term on the right is equal to zero. Therefore, we obtain

$$\Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le -\gamma\|v_x\|^2 + \frac{1}{2}\int_0^\infty k'(s)\|\eta_x^t(s)\|^2\,ds \le 0. \tag{3.3}$$

Thus $\mathcal{A}$ is dissipative.

Secondly, we show that $I-\mathcal{A}$ is surjective. For any $F=(f_1,f_2,f_3,f_4,f_5,f_6)\in\mathcal{H}$, we seek $U=(u,v,\varphi,w,\theta,\eta^t)^T\in D(\mathcal{A})$ satisfying

$$(I-\mathcal{A})U = F, \tag{3.4}$$

equivalently, we obtain

$$u - v = f_1, \tag{3.5}$$

$$\rho v - \mu u_{xx} - b\varphi_x - \gamma v_{xx} = \rho f_2, \tag{3.6}$$

$$\varphi - w = f_3, \tag{3.7}$$

$$Jw - \delta\varphi_{xx} + bu_x + \xi\varphi + \beta\theta_x = Jf_4, \tag{3.8}$$

$$c\theta - \int_0^\infty k(s)\eta_{xx}^t(x,s)\,ds + \beta w_x = cf_5, \tag{3.9}$$

$$\eta^t - \theta + \eta_s^t(x,s) = f_6. \tag{3.10}$$

From (3.10), we can find an exact solution

$$\eta^t(s) = (1-e^{-s})\theta + \int_0^s e^{\tau - s} f_6(\tau)\,d\tau. \tag{3.11}$$

Substituting (3.5), (3.7), (3.11) into (3.6), (3.8), (3.9), we have the system

$$\begin{gathered}
\rho u - (\mu+\gamma)u_{xx} - b\varphi_x = \rho(f_1+f_2) - \gamma f_{1xx} \in H^{-1}(0,1),\\
(J+\xi)\varphi - \delta\varphi_{xx} + bu_x + \beta\theta_x = J(f_3+f_4) \in L^2(0,1),\\
c\theta - \int_0^\infty k(s)(1-e^{-s})\,ds\theta_{xx} + \beta\varphi_x\\
= \int_0^\infty k(s)\Big(\int_0^s e^{\tau-s} f_{6xx}(\tau)\,d\tau\Big)\,ds + \beta f_{3x} + cf_5 \in H^{-1}(0,1),
\end{gathered} \tag{3.12}$$

where

$$\int_0^\infty k(s)(1-e^{-s})\,ds > 0.$$

Indeed, let $\psi\in H_0^1(0,1)$ such that $\|\psi_x\|\le 1$, and by applying some formulas, we have

$$|\langle \gamma f_{1xx}, \psi\rangle| = |\langle \gamma f_{1x}, \psi_x\rangle| \le \gamma\|f_{1x}\| < \infty,$$

and

$$\begin{aligned}
&\big|\Big\langle \int_0^\infty k(s)\Big(\int_0^s e^{\tau-s} f_{6xx}(\tau)\,d\tau\Big)\,ds, \psi\Big\rangle\big|\\
&= \big|\Big\langle \int_0^\infty k(s)\Big(\int_0^s e^{\tau-s} f_{6x}(\tau)\,d\tau\Big)\,ds, \psi_x\Big\rangle\big|\\
&\le \int_0^\infty k(s)e^{-s}\Big(\int_0^s e^{\tau}\|f_{6x}(\tau)\|\,d\tau\Big)\,ds\\
&\le \int_0^\infty e^{\tau}\|f_{6x}(\tau)\|\int_\tau^\infty k(s)e^{-s}\,ds\,d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\infty k(\tau)e^{\tau}\|f_{6x}(\tau)\| \int_\tau^\infty e^{-s}\,ds\,d\tau \\
&= \int_0^\infty k(\tau)\|f_{6x}(\tau)\|\,d\tau < \infty.
\end{aligned}$$

The first, second, and third equations of (3.12) are multiplied by $u_1 \in H_0^1(0,1)$, $\varphi_1 \in H_*^1(0,1)$, and $\theta_1 \in H_0^1(0,1)$ respectively, integrate over $(0,1)$ and add them, we have the variational formulation

$$B\big((u,\varphi,\theta),(u_1,\varphi_1,\theta_1)\big) = L(u_1,\varphi_1,\theta_1), \tag{3.13}$$

where $B : \left[H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1)\right]^2 \to R$ is a bounded bilinear form defined by

$$\begin{aligned}
&B\left((u,\varphi,\theta),(u_1,\varphi_1,\theta_1)\right) \\
&= \rho\int_0^1 uu_1\,dx + (\mu+\gamma)\int_0^1 u_x u_{1x}\,dx - b\int_0^1 \varphi_x u_1\,dx + (J+\xi)\int_0^1 \varphi\varphi_1\,dx \\
&\quad + \delta\int_0^1 \varphi_x\varphi_{1x}\,dx + b\int_0^1 u_x\varphi_1\,dx + \beta\int_0^1 \theta_x\varphi_1\,dx + c\int_0^1 \theta\theta_1\,dx \\
&\quad + \int_0^\infty k(s)(1-e^{-s})\,ds\int_0^1 \theta_x\theta_{1x}\,dx + \beta\int_0^1 \varphi_x\theta_1\,dx,
\end{aligned}$$

and $L : H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1) \to R$ is the linear functional

$$\begin{aligned}
L\left(u_1,\varphi_1,\theta_1\right) &= \rho\int_0^1 (f_1+f_2)u_1\,dx + \gamma\int_0^1 f_{1x}u_{1x}\,dx + J\int_0^1 (f_3+f_4)\varphi_1\,dx \\
&\quad + \beta\int_0^1 f_{3x}\theta_1\,dx + c\int_0^1 f_5\theta_1\,dx \\
&\quad + \int_0^1 \theta_1\int_0^\infty k(s)\Big(\int_0^s e^{\tau-s}f_{6xx}(\tau)\,d\tau\Big)\,ds.
\end{aligned}$$

Utilizing Poincaré inequality, we obtain

$$\begin{aligned}
&B\left((u,\varphi,\theta),(u,\varphi,\theta)\right) \\
&= \rho\int_0^1 u^2\,dx + (\mu+\gamma)\int_0^1 u_x^2\,dx + 2b\int_0^1 u_x\varphi\,dx + (J+\xi)\int_0^1 \varphi^2\,dx \\
&\quad + \delta\int_0^1 \varphi_x^2\,dx + c\int_0^1 \theta^2\,dx + \int_0^\infty k(s)(1-e^{-s})\,ds\int_0^1 \theta_x^2\,dx \\
&\geq \alpha\|(u,\varphi,\theta)\|^2,
\end{aligned}$$

for some constant $\alpha > 0$. Thus, $B(\cdot,\cdot)$ is coercive. According to the Lax-Milgram theorem, (3.13) has a unique solution

$$(u,\varphi,\theta) \in H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1).$$

If we take $(u_1,\varphi_1,\theta_1) = (u_1,0,0)$ in (3.13), we have

$$(\mu+\gamma)\int_0^1 u_x u_{1x}\,dx = b\int_0^1 (\varphi_x - u)u_1\,dx + \rho\int_0^1 (f_1+f_2)u_1\,dx + \gamma\int_0^1 f_{1x}u_{1x}\,dx,$$

which means that $u \in H^2(0,1)\cap H_0^1(0,1)$.

Similarly, if we take $(u_1, \varphi_1, \theta_1) = (0, \varphi_1, 0)$ in (3.13), we have

$$\delta \int_0^1 \varphi_x \varphi_{1x} = J \int_0^1 (f_3 + f_4 - \varphi)\varphi_1 \, dx - \xi \int_0^1 \varphi\varphi_1 \, dx - b \int_0^1 u_x \varphi_1 \, dx - \beta \int_0^1 \theta_x \varphi_1 \, dx$$

which means that $\varphi \in H^2(0,1) \cap H_0^1(0,1)$.

Moveover, from (3.5), (3.7) and (3.9), we observe that

$$v \in H_0^1(0,1), \quad w \in H_*^1(0,1), \quad \int_0^\infty k(s)\eta^t(s) \, ds \in H^2(0,1).$$

Inserting (**??**) in (3.10), we obtain

$$\eta_s^t(s) = e^{-s}\theta + f_6(s) - \int_0^s e^{y-s} f_6(y) dy,$$

thus, we have $\eta^t \in \mathcal{K}$ and $\eta^t(0) = 0$. Hence, there exists a unique solution $U \in D(\mathcal{A})$. Consequently, $\mathcal{A}$ is a maximal monotone operator, i.e., the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. Finally, the conclusion of Theorem 3.2 can be obtained by applying the Hille-Yosida theorem. □

4. Stability

In this section, we give the stability result of system (2.6) by means of estimates of the resolvent operator norm.

Lemma 4.1 ([27]). *Let $\mathcal{A}$ be the infinitesimal generator of a contraction semigroup $S(t)$ acting on space $\mathcal{H}$. Then, the following statements are equivalent:*

- (i) *$S(t)$ is exponentially stable;*
- (ii) *There exists $\varepsilon > 0$, such that*
$$\inf_{\lambda \in \mathbb{R}} \|(i\lambda - \mathcal{A})U\|_{\mathcal{H}} \geq \varepsilon \|U\|_{\mathcal{H}}, \quad \forall U \in D(\mathcal{A});$$
- (iii) *The imaginary axis $i\mathbb{R}$ is contained in the resolvent set $\rho(\mathcal{A})$ of the operator $\mathcal{A}$ and*
$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{H})} < \infty.$$

Theorem 4.2. *Assume that* (H1)–(H3) *are satisfied and $U(0) \in D(\mathcal{A})$. Then the energy of system* (2.6) *is exponentially stable.*

Proof. We prove (ii) by a contradiction argument. Suppose that the claim is false, then there exist two sequences $\{\lambda_n\} \subset R$ and $\{U_n\} \subset D(\mathcal{A})$, with

$$\|U_n\|_{\mathcal{H}} = 1, \tag{4.1}$$

such that

$$\|i\lambda_n U_n - \mathcal{A}U_n\|_{\mathcal{H}} \to 0. \tag{4.2}$$

Equivalently, we have

$$i\lambda_n u_n - v_n \to 0 \quad \text{in } H_0^1(0,1), \tag{4.3}$$

$$i\rho\lambda_n v_n - \mu D^2 u_n - bD\varphi_n - \gamma D^2 v_n \to 0 \quad \text{in } L^2(0,1), \tag{4.4}$$

$$i\lambda_n \varphi_n - w_n \to 0 \quad \text{in } H_*^1(0,1), \tag{4.5}$$

$$iJ\lambda_n w_n - \delta D^2 \varphi_n + bDu_n + \xi\varphi_n + \beta D\theta_n \to 0 \quad \text{in } L_*^2(0,1), \tag{4.6}$$

$$ic\lambda_n \theta_n - \int_0^\infty k(s) D^2 \eta_n^t(s) \, ds + \beta D w_n \to 0 \quad \text{in } L^2(0,1), \tag{4.7}$$

$$i\lambda_n\eta_n^t-\theta_n+D_s\eta_n^t(s)\to 0 \quad \text{in } \mathcal{M}, \tag{4.8}$$

in which we denote $D=\frac{\partial}{\partial x}$ and $D_s=\frac{\partial}{\partial s}$. We just have to show that each component of U_n goes to 0 in the norm of $\mathcal{H}$. We will prove it in two cases.

Case 1. Assuming $\lambda_n\not\to 0$, that is sequence λ_n satisfies

$$\inf_{n\in\mathbb{N}}|\lambda_n|>0.$$

Now, taking the inner product of both sides of (4.2) with U_n, and then taking the real part, we obtain

$$\Re e\langle(i\lambda_n-\mathcal{A})U_n,U_n\rangle_{\mathcal{H}}=\gamma\|Dv_n\|^2-\frac{1}{2}\int_0^\infty k'(s)\|D\eta_n^t(s)\|^2\,ds\to 0, \tag{4.9}$$

from (H1), (H3) and Poincaré inequality, we obtain

$$\|v_n\|\to 0, \tag{4.10}$$

$$\|\eta_n^t\|_{\mathcal{M}}^2\le -\frac{1}{2\nu}\int_0^\infty k'(s)\|D\eta_n^t(s)\|^2\,ds\to 0. \tag{4.11}$$

Moreover, from (4.3), we find that $u_n\to 0$ in $L^2(0,1)$. The injection $L^2\hookrightarrow H^{-1}$ is continuous, hence (4.7) holds in H^{-1} instead of L^2. Regarding the second item of (4.7), we have

$$\begin{aligned}
\Big\|\int_0^\infty k(s)D^2\eta_n^t(s)\,ds\Big\|_{-1} &\le \Big\|\int_0^\infty k(s)D\eta_n^t(s)\,ds\Big\| \\
&= \Big(\int_0^1\Big(\int_0^\infty \sqrt{k(s)}\sqrt{k(s)}D\eta_n^t(s)\,ds\Big)^2dx\Big)^{1/2} \\
&\le \Big(\int_0^\infty k(s)\,ds\int_0^\infty k(s)\|D\eta_n^t(s)\|^2\,ds\Big)^{1/2} \\
&= \sqrt{k_0}\|\eta_n^t\|_{\mathcal{M}}\to 0,
\end{aligned}$$

this means $\|ic\lambda_n\theta_n+\beta Dw_n\|_{-1}\to 0$. Since

$$\|Dw_n\|_{-1}=\sup_{\|D\psi\|\le 1}|\langle Dw_n,\psi\rangle|\le\|w_n\|<\infty,$$

it follows that $\lambda_n\theta_n$ is bounded in H^{-1} and

$$\|ic\lambda_n\theta_n\|_{-1}\le C_1,$$

for a positive constant C_1 independent of $n\in\mathbb{N}$.

Introducing $\hat\theta_n$ such that

$$\begin{gathered}
D^2\hat\theta_n=-\theta_n,\\
\hat\theta_n(0)=\hat\theta_n(1)=0
\end{gathered}$$

and rewriting (4.8), we have

$$i\lambda_n\eta_n^t-\theta_n+D_s\eta_n^t(s)=\zeta_n, \tag{4.12}$$

with $\zeta_n\to 0$ in $\mathcal{M}$, we can find an exact solution

$$\eta_n^t(s)=\frac{1}{i\lambda_n}(1-e^{-i\lambda_n s})\theta_n+\int_0^s e^{-i\lambda_n(s-r)}\zeta_n(r)dr.$$

Setting

$$a_n = \int_0^\infty k(s)(1 - e^{-i\lambda_n s})\, ds,$$
$$b_n = i\lambda_n \int_0^\infty k(s) \int_0^s e^{-i\lambda_n (s-r)} \langle D\zeta_n(r), D\hat{\theta}_n\rangle_{\mathcal{M}}\, dr\, ds,$$

we obtain

$$a_n \|\theta_n\|^2 + b_n = i\lambda_n \langle \eta_n^t, \hat{\theta}_n\rangle_{\mathcal{M}}. \tag{4.13}$$

By calculations, we find that

$$i\lambda_n \langle \eta_n^t, \hat{\theta}_n\rangle_{\mathcal{M}} \le C|\lambda_n| \|\theta_n\|_{-1} \int_0^\infty k(s)\|D\eta_n^t(s)\|\, ds \le C\|\eta_n^t\|_{\mathcal{M}} \to 0.$$

On the one hand, integrating both sides of (H3) over $(s, s+r)$, we have

$$k(s+r) \le e^{-\nu r} k(s), \quad \forall r \ge 0 \text{ and } s > 0;$$

therefore,

$$\begin{aligned}
|b_n| &\le |\lambda_n| \|\theta_n\|_{-1} \int_0^\infty \sqrt{k(s)} \int_0^s e^{-\frac{\nu(s-r)}{2}} \sqrt{k(r)} \|D\zeta_n(r)\|\, dr\, ds \\
&\le C \int_0^\infty \sqrt{k(s)} e^{-\nu r/2} \int_0^s e^{\nu r/2} \sqrt{k(r)} \|D\zeta_n(r)\| dr\, ds \\
&\le C \int_0^\infty e^{\nu r/2} \sqrt{k(r)} \|D\zeta_n(r)\| \int_r^\infty e^{-\nu r/2} \sqrt{k(s)}\, ds\, dr \\
&\le C \int_0^\infty e^{\nu r/2} k(r) \|D\zeta_n(r)\| \int_r^\infty e^{-\nu r/2}\, ds\, dr \\
&\le C \int_0^\infty k(r) \|D\zeta_n(r)\| dr \\
&\le C\|\zeta_n\|_{\mathcal{M}} \to 0.
\end{aligned}$$

On the other hand,

$$a_n \to \int_0^\infty k(s)\, ds > 0.$$

Back to (4.13), we obtain

$$\|\theta_n\| \to 0. \tag{4.14}$$

Next, we will prove that $\|w_n\| \to 0$. Setting

$$W_n = \int_0^x w_n(y) dy \ \in H_0^1(0,1).$$

Integrating both sides of (4.6) over $(0, x)$, we have

$$\sup_{n\in\mathbb{N}} |\lambda_n| \|W_n\| < \infty.$$

Now, taking the inner product of both sides of (4.7) with W_n, namely

$$ic\lambda_n(\theta_n, W_n) - \int_0^\infty k(s) \langle D^2\eta_n^t(s), W_n\rangle\, ds + \beta(Dw_n, W_n) \to 0. \tag{4.15}$$

Using the Cauchy-Schwartz inequality and (4.14), we obtain an estimate of the first term of (4.15),

$$|ic\lambda_n(\theta_n, W_n)| \le c|\lambda_n| \|W_n\| \|\theta_n\| \to 0.$$

By calculations, the second term of (4.15) satisfies

$$\big|\int_0^\infty k(s)\langle D^2\eta_n^t(s), W_n\rangle\,ds\big| \le \|w_n\|\int_0^\infty k(s)\|D\eta_n^t(s)\|^2\,ds \to 0.$$

Therefore, $|\beta(Dw_n, W_n)| = \beta\|w_n\|^2 \to 0$; this means that

$$\|w_n\| \to 0. \tag{4.16}$$

From (4.5), we can easily check that

$$\|\varphi_n\| \to 0. \tag{4.17}$$

Moreover, using Cauchy-Schwartz inequality, we obtain

$$2b\int_0^1 Du_n\varphi_n\,dx \to 0. \tag{4.18}$$

Taking the inner product of both sides of (4.4) and (4.6) with u_n and φ_n, respectively, we find that

$$\|Du_n\| \to 0, \tag{4.19}$$

$$\|D\varphi_n\| \to 0. \tag{4.20}$$

According to (4.10)-(4.11), (4.14), (4.16)-(4.20), we obtain that $\|U_n\|_{\mathcal{H}} \to 0$ which contradicts (4.1).

Case 2. Assuming $\lambda_n \to 0$, from (4.2), we have

$$\|v_n\| \to 0, \quad \|w_n\| \to 0, \tag{4.21}$$

and

$$\mu D^2u_n + bD\varphi_n + \gamma D^2v_n \to 0 \quad \text{in } L^2(0,1), \tag{4.22}$$

$$\delta D^2\varphi_n - bDu_n - \xi\varphi_n - \beta D\theta_n \to 0 \quad \text{in } H_*^1(0,1), \tag{4.23}$$

$$\theta_n - D_s\eta_n^t(s) \to 0 \quad \text{in } \mathcal{M}. \tag{4.24}$$

Taking the inner product of both sides of (4.24) with $s\hat{\theta}_n$, we have

$$\langle\theta_n, s\hat{\theta}_n\rangle_{\mathcal{M}} - \langle D_s\eta_n^t(s), s\hat{\theta}_n\rangle_{\mathcal{M}} \to 0.$$

Since

$$\begin{aligned}
|\langle D_s\eta_n^t(s), s\hat{\theta}_n\rangle_{\mathcal{M}}| &= \big|\int_0^\infty sk(s)\frac{d}{ds}\int_0^1 D\eta_n^t D\hat{\theta}_n\,dx\,ds\big| \\
&= \big|\int_0^\infty sk(s)\frac{d}{ds}\int_0^1 \eta_n^t\theta_n\,dx\,ds\big| \\
&= \big|\int_0^\infty k(s)\int_0^1 \eta_n^t\theta_n\,dx\,ds + \int_0^\infty sk'(s)\int_0^1 \eta_n^t\theta_n\,dx\,ds\big| \\
&\le \|\theta_n\|\Big[\int_0^\infty k(s)\|\eta_n^t(s)\|\,ds + \int_0^\infty sk'(s)\|\eta_n^t(s)\|\,ds\Big] \\
&\le \int_0^\infty k(s)\|\eta_n^t(s)\|\,ds + \int_0^\infty sk'(s)\|\eta_n^t(s)\|\,ds,
\end{aligned} \tag{4.25}$$

and

$$-\int_0^\infty s^2k'(s)\,ds = 2\int_0^\infty sk(s)\,ds = C_2 < \infty.$$

Applying Hölder and Poincaré inequalities, we obtain

$$\begin{aligned}\int_0^\infty k(s)\|\eta_n^t(s)\|\,ds &= \int_0^\infty \sqrt{k(s)}\sqrt{k(s)}\|\eta_n^t(s)\|\,ds\\ &\le \sqrt{\int_0^\infty k(s)\,ds}\sqrt{\int_0^\infty k(s)\|\eta_n^t(s)\|^2\,ds}\\ &\le C_p\sqrt{k_0}\sqrt{\int_0^\infty k(s)\|D\eta_n^t(s)\|^2\,ds}\\ &\le C\|\eta_n^t\|_{\mathcal{M}} \to 0,\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty sk'(s)\|\eta_n^t(s)\|\,ds &= \int_0^\infty s\sqrt{-k'(s)}\sqrt{-k'(s)}\|\eta_n^t(s)\|\,ds\\ &\le \Big(-\int_0^\infty s^2k'(s)\,ds\int_0^\infty -k'(s)\|\eta_n^t(s)\|^2\,ds\Big)^{1/2}\\ &\le \Big(\frac{-C_2}{C_P}\int_0^\infty -k'(s)\|D\eta_n^t(s)\|^2\,ds\Big)^{1/2} \to 0.\end{aligned}$$

Therefore, $|\langle D_s\eta_n^t(s), s\hat{\theta}_n\rangle_{\mathcal{M}}| \to 0$. In addition,

$$\langle \theta_n, s\hat{\theta}_n\rangle_{\mathcal{M}} = \int_0^\infty sk(s)(D\theta_n, D\hat{\theta}_n)\,ds = k_1\|\theta_n\|^2 \to 0,$$

this means

$$\|\theta_n\| \to 0. \tag{4.26}$$

Taking the inner product of both sides of (4.22) and (4.23) with u_n and φ_n, we have

$$\mu\|Du_n\|^2 + b(\varphi_n, Du_n) \to 0, \tag{4.27}$$

$$\delta\|D\varphi_n\|^2 + b(Du_n,\varphi_n) + \xi\|\varphi_n\|^2 \to 0. \tag{4.28}$$

Summing (4.27) and (4.28), we have

$$\mu\|Du_n\|^2 + 2b\Re e(Du_n,\varphi_n) + \xi\|\varphi_n\|^2 + \delta\|D\varphi_n\|^2 \to 0.$$

From (2.9), we infer that

$$\|D\varphi_n\| \to 0. \tag{4.29}$$

Using Poincaré inequality, we have

$$\|\varphi_n\| \to 0. \tag{4.30}$$

Back to (4.28), we obtain

$$\|Du_n\| \to 0. \tag{4.31}$$

Also, we obtain that

$$\|U_n\|_{\mathcal{H}} \to 0,$$

which contradicts (4.1). The proof of Theorem 4.2 is complete. □

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