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STABILITY ANALYSIS OF THE PEACEMAN-RACHFORD METHOD FOR PARABOLIC EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. We consider an efficient finite difference method solving of two-dimensional parabolic equations with nonlocal conditions. The specific feature of the investigated problem is that the nonlocal condition contains the values of solution's derivatives at different points. We prove the stability of this method in specific energy norm. The main stability condition is that all eigenvalues of the corresponding difference problem are positive. Results of computational experiments are presented.

1. Introduction

In this article, we solve a two-dimensional linear parabolic equation using finite difference method (FDM),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad (x,y) \in \Omega,\ t \in (0,T], \tag{1.1}$$

where $\Omega = (0,1) \times (0,1)$, with nonlocal boundary conditions

$$\gamma \frac{\partial u(0,y,t)}{\partial x} = \frac{\partial u(1,y,t)}{\partial x}, \quad \gamma \in (0,1], \tag{1.2}$$

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = 0, \tag{1.3}$$

and initial condition

$$u(x,y,t) = \varphi(x,y), \quad (x,y) \in \Omega. \tag{1.4}$$

Intensive research of differential problems with various nonlocal conditions started after Cannon and Kamynin had published their works [5, 16]. The authors, instead of classical boundary condition $u(0,t) = \mu(t)$, formulate nonlocal condition

$$\int_0^x u(x,t)dx = E(t),$$

where $E(t)$ is the known amount of heat in interval $(0,x)$ at time t.

To the authors' knowledge, boundary condition of type (1.2) first time was formulated in [14] for one-dimensional parabolic equation. It was observed that mathematical models with nonlocal conditions (1.2) are encountered in physical problems describing processes of particles' diffusion in turbulent plasma.

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For the one-dimensional problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, t > 0,$$
$$u(x,0) = u_0(x),$$
$$u(0,t) = 0, \quad \gamma \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x}, \quad \gamma \in (0,1],$$

the stability of finite difference schemes (FDS) is proved in rather complex energy norms (see [9, 10, 12, 13] and references therein). The structure of FDS spectrum is investigated in detail for this purpose. The equivalence between these energy norms and L_2-norm in vector space is proved in [10].

Subject of various type differential problems with nonlocal conditions like the uniqueness and existence of a solution, numerical methods and applications, spectrum analysis is widely investigated (see e.g. [2, 8, 18, 19, 21, 33] and references therein).

Problems with various types of nonlocal conditions are considered to be one of modern areas of research in theory of differential equations and numerical analysis. Mentioned research area is bound together with applications in science and technology. Nonlocal problems have special feature. The structure of spectrum of differential or difference operators with nonlocal conditions is more complicated and substantive compared to respective spectrum of operator with classical (Dirichlet or Neumann) boundary conditions (see e.g. [3, 17, 23, 28, 33] and references therein).

Problem (1.1)–(1.4) with $f = 0$ is solved in [11] using Crank-Nicolson difference method with approximation error $O(h^2 + \tau^2)$. Stability of used method is proved in special energy norm

$$\|u\|_D = (Du, u)^{1/2},$$

where D is symmetric positive definite matrix depending on eigenvectors and associated vectors of difference problem.

It is well known that stability is one of the most important properties of numerical method both for theoretical research and practical applications. Another important point arises for the two- and multi-dimensional parabolic problems with nonlocal conditions: how to solve finite difference scheme efficiently at every time layer t_k, when the method is implicit.

The main task of this article is to construct efficient FDM for the two-dimensional parabolic equation (1.1) with nonlocal boundary condition (1.2). The example of FDM for problems with classical (Dirichlet or Neumann) boundary conditions is Peaceman-Rachford alternating direcion implicit (ADI) method.

In this article we modify Peaceman-Rachford ADI method by applying it to the special nonlocal condition (1.2) and investigate stability conditions. For this purpose we use results about spectrum structure of one-dimensional difference problem with nonlocal conditions.

To the authors' knowledge, ADI method for the parabolic equation with (1.2) type nonlocal condition has not been investigated earlier. For the two-dimensional parabolic and elliptic equations with other type nonlocal conditions ADI methods theoretically and practically were investigated in [7, 27, 30, 34].

This article is organized as follows. In Section 2 we formulate Peaceman-Rachford ADI method for differential problem (1.1)-(1.4). We prove that approximation error is $O(h^2 + \tau^2)$. In Section 3 we investigate stability of the method in a special energy norm. We provide numerical examples for the ADI method in Section 4. Remarks and generalizations are provided in Section 5.

2. Numerical method and approximation error

First, we define

$$U_{ij}^n := U(x_i, y_j, t^n),$$

where $x_i = ih, i = \overline{0,N}$; $y_j = jh, j = \overline{0,N}$; $h = 1/N$; $t^n = n\tau$, $n = \overline{0,M}$; $\tau = T/M$; $N, M \in \mathbb{Z}$.

We denote differences approximating derivatives of solution as

$$\delta_x U_{ij}^n := \frac{U_{i+1,j}^n - U_{ij}^n}{h}, \quad \delta_{\bar{x}} U_{ij}^n := \frac{U_{ij}^n - U_{i-1,j}^n}{h}, \quad \delta_t U_{ij}^n := \frac{U_{ij}^{n+1} - U_{ij}^n}{h},$$
$$\delta_x^2 U_{ij}^n := \frac{U_{i-1,j}^n - 2U_{ij}^n + U_{i+1,j}^n}{h^2}, \quad \delta_y^2 U_{ij}^n := \frac{U_{i,j-1}^n - 2U_{ij}^n + U_{i,j+1}^n}{h^2}.$$

We formulate Crank-Nicolson difference method for the problem (1.1)–(1.4). We emphasize that approximation accuracy of nonlocal condition (1.2) is $O(h^2 + \tau^2)$.

$$\begin{gathered}\frac{U_{ij}^{n+1} - U_{ij}^n}{\tau} = \frac{1}{2}\left(\delta_x^2 U_{ij}^{n+1} + \delta_y^2 U_{ij}^{n+1}\right) + \frac{1}{2}\left(\delta_x^2 U_{ij}^n + \delta_y^2 U_{ij}^n\right) + f_{ij}^{n+1/2}, \\ i, j = \overline{1, N-1},\end{gathered} \tag{2.1}$$

$$\begin{aligned}\frac{U_{Nj}^{n+1} - U_{Nj}^n}{\tau} &= \frac{1}{2}\Big(\frac{2}{h}\left(\gamma\delta_x U_{0j}^{n+1} - \delta_{\bar{x}} U_{Nj}^{n+1}\right) + \delta_y^2 U_{Nj}^{n+1}\Big) \\ &\quad + \frac{1}{2}\Big(\frac{2}{h}\left(\gamma\delta_x U_{0j}^n - \delta_{\bar{x}} U_{Nj}^n\right) + \delta_y^2 U_{Nj}^n\Big) \\ &\quad + \gamma f_{0j}^{n+1/2} + f_{Nj}^{n+1/2}, \quad j = \overline{1, N-1},\end{aligned} \tag{2.2}$$

$$U_{0j}^{n+1} = U_{i0}^{n+1} = U_{iN}^{n+1} = 0, \quad i, j = \overline{0, N}, \tag{2.3}$$

$$U_{ij}^0 = \varphi_{ij}, \quad i, j = \overline{0, N}. \tag{2.4}$$

Equation (2.1) is written in standard form changing second derivatives with differences. It approximates differential equation (1.1) at all inner Ω points $i, j = \overline{1, N-1}$ with accuracy $O(h^2 + \tau^2)$, under the assumption that the solution of differential problem is sufficiently smooth.

Equation (2.2), approximating nonlocal condition (1.2), is derived in the following way. First, condition (1.2) is rewritten as

$$\gamma\Big(\delta_x u_{0j}^{n+1/2} - \frac{h}{2}\frac{\partial^2 u_{0j}^{n+1/2}}{\partial x^2} + O(h^2)\Big) = \delta_{\bar{x}} U_{Nj}^{n+1/2} + \frac{h}{2}\frac{\partial^2 u_{Nj}^{n+1/2}}{\partial x^2} + O(h^2). \tag{2.5}$$

We make an assumption that differential equation (1.1) is valid not only in inner Ω points, but also on boundaries, when $x = 0$ and $x = 1$ (i.e. $i = 0$ and $i = N$). This assumption is usual for FDM if boundary condition has solution's derivative with respect to spatial variable [24]. In accordance with this assumption, we replace second order derivatives with respect to x in (2.5) by their expression from differential equation (1.1). Further step is approximation of differential expression: derivatives $\partial u/\partial t$ and $\partial^2 u/\partial y^2$ are replaced by differences with accuracy $O(h^2+\tau^2)$. We also use the fact that $U_ij^{n+1/2} = \frac{1}{2}\left(U_{ij}^{n+1} + U_{ij}^n\right) + O(\tau^2)$. We obtain equation an approximating nonlocal condition (1.2) with accuracy $O(h^2+\tau^2)$ by eliminating approximation errors. After some trivial transformations we obtain difference equation of form (2.2).

Now we rewrite equations (2.1) and (2.2) in the form

$$\frac{U_{ij}^{n+1}-U_{ij}^{n}}{\tau}=\frac{1}{2}\left(\tilde{\delta}_x^2 U_{ij}^{n+1}+\delta_y^2 U_{ij}^{n+1}\right)+\frac{1}{2}\left(\tilde{\delta}_x^2 U_{ij}^{n}+\delta_y^2 U_{ij}^{n}\right)+\tilde{f}_{ij}^{n+1/2}, \tag{2.6}$$

where

$$\tilde{\delta}_x^2 U_{ij}^n := \begin{cases}\delta_x^2 U_{ij}^n, & i=\overline{1,N-1},\\ \frac{2}{h}\left(\gamma\delta_x U_{0j}^n-\delta_{\bar{x}}U_{Nj}^n\right), & i=N,\end{cases} \tag{2.7}$$

$$\tilde{f}_{ij}^{n+1/2} := \begin{cases} f_{ij}^{n+1/2}, & i=\overline{1,N-1},\\ \gamma f_{0j}^{n+1/2}+f_{Nj}^{n+1/2}, & i=N.\end{cases} \tag{2.8}$$

Remark 2.1. As noted above, difference equation (2.2) approximates nonlocal condition (1.2) with accuracy $O(h^2+\tau^2)$, assuming equation (1.1) is valid and on boundary $x=0$ and $x=1$. The advantage of this approach is that difference equation (2.2) formally (but only formally) has the same structure, as difference equation (2.1). Actually, term $\tilde{\delta}_x^2 U_{Nj}^n$ is not an approximation of $\partial^2 U/\partial x^2$ at the point (x_N, y_j, t^n). But, this term is a kind of generalized difference analogue of derivative. Expanding U_{1j}^n and $U_{N-1,j}^n$ in the Taylor series, when $|\partial^3 u/\partial x^3|\le C$ and nonlocal condition (1.2) is valid, we have

$$\begin{aligned}\tilde{\delta}_x^2 U_{Nj}^n &= \frac{2}{h}\Big(\gamma\frac{U_{1j}^n-U_{0j}^n}{h}-\frac{U_{Nj}^n-U_{N-1,j}^n}{h}\Big)\\ &=\frac{2}{h}\Big(\gamma\frac{\partial U_{0j}^n}{\partial x}+\gamma\frac{h}{2}\frac{\partial^2 U_{0j}^n}{\partial x^2}-\frac{\partial U_{Nj}^n}{\partial x}+\frac{h}{2}\frac{\partial^2 U_{Nj}^n}{\partial x^2}+O(h^2)\Big)\\ &=\gamma\frac{\partial^2 U_{0j}^n}{\partial x^2}+\frac{\partial^2 U_{Nj}^n}{\partial x^2}+O(h).\end{aligned} \tag{2.9}$$

Similarly, we can write another form of this expression

$$\tilde{\delta}_x^2 U_{Nj}^n=\gamma\frac{\partial^2 \widetilde{U}_{0j}^n}{\partial x^2}+\frac{\partial^2 \bar{U}_{Nj}^n}{\partial x^2}, \tag{2.10}$$

where $\widetilde{U}_{0j}^n=U(\tilde{x}_0,y_j,t^n)$, $\tilde{x}_0\in[0,h]$; $\bar{U}_{Nj}^n=U(\bar{x}_N,y_j,t^n)$, $\bar{x}_N\in[1-h,1]$. This remark will be used for evaluation of ADI method's aproximation error.

Now, taking into account approximation form of nonlocal condition (1.2) in Crank-Nicolson method, we construct a following alternating direction method for differential problem (1.1)–(1.4) (Peaceman-Rachford ADI method)

$$\frac{U_{ij}^{n+1/2}-U_{ij}^n}{\tau/2}=\delta_x^2 U_{ij}^{n+1/2}+\delta_y^2 U_{ij}^n+f_{ij}^{n+1/2},\quad i,j=\overline{1,N-1}, \tag{2.11a}$$

$$\begin{aligned}\frac{U_{Nj}^{n+1/2}-U_{Nj}^n}{\tau/2}&=\frac{2}{h}\left(\gamma\delta_x U_{0j}^{n+1/2}-\delta_{\bar{x}}U_{Nj}^{n+1/2}\right)+\delta_y^2 U_{Nj}^n\\ &\quad+\gamma f_{0j}^{n+1/2}+f_{Nj}^{n+1/2},\quad j=\overline{1,N-1},\end{aligned} \tag{2.11b}$$

$$U_{0j}^{n+1/2}=0,\quad j=\overline{1,N-1}, \tag{2.11c}$$

$$U_{0j}^n=U_{i0}^n=U_{iN}^n=0, \tag{2.11d}$$

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\tau/2}=\delta_x^2U_{ij}^{n+1/2}+\delta_y^2U_{ij}^{n+1}+f_{ij}^{n+1/2},\quad i,j=\overline{1,N-1},\tag{2.12a}$$

$$\begin{aligned}\frac{U_{Nj}^{n+1}-U_{Nj}^{n+1/2}}{\tau/2}&=\frac{2}{h}\left(\gamma\delta_xU_{0j}^{n+1/2}-\delta_{\bar{x}}U_{Nj}^{n+1/2}\right)+\delta_y^2U_{Nj}^{n+1}\\&\quad+\gamma f_{0j}^{n+1/2}+f_{Nj}^{n+1/2},\quad j=\overline{1,N-1},\end{aligned}\tag{2.12b}$$

$$U_{i0}^{n+1}=U_{iN}^{n+1}=0,\quad j=\overline{1,N-1},\tag{2.12c}$$

$$U_{0j}^{n+1/2}=U_{i0}^{n+1/2}=U_{iN}^{n+1/2}=0.\tag{2.12d}$$

The method, described for formulas (2.11) and (2.12), can be rewritten in another form taking into account (2.7) and (2.8). We will use the following form in further theoretical investigation

$$\frac{U_{ij}^{n+1/2}-U_{ij}^{n}}{\tau/2}=\tilde{\delta}_x^2U_{ij}^{n+1/2}+\delta_y^2U_{ij}^{n}+\tilde{f}_{ij}^{n+1/2},\tag{2.13}$$

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\tau/2}=\tilde{\delta}_x^2U_{ij}^{n+1/2}+\delta_y^2U_{ij}^{n+1}+\tilde{f}_{ij}^{n+1/2},\tag{2.14}$$

for $i=\overline{1,N}$, $j=\overline{1,N-1}$. One needs to solve N-th order system $N-1$ times (for every fixed $j=\overline{1,N-1}$) with unknowns $U_{1j}^{n+1/2},U_{2j}^{n+1/2},\dots,U_{Nj}^{n+1/2}$ to find $U_{ij}^{n+1/2}$ from system (2.13). Similarly, one can find U_{ij}^{n+1} from system (2.14).

The matrix of system (2.14) is tridiagonal and boundary conditions are of Dirichlet type. The matrix of system (2.13) differs from diagonal matrix only by one element ($U_{1j}^{n+1/2}$ coefficient in the $N-th$ equation is not equal to zero). Systems of such type usually arise in solving one-dimensional boundary value problem with periodic boundary condition [26]. The number of arithmetic operations to solve systems (2.13) and (2.14) is proportional to N. Therefore, number of arithmetic operations to find U_{ij}^{n+1}, when U_{ij}^{n} are known, using algorithm (2.13), (2.14) is proportional to N^2 (that is proportional to the number of grid points). Such algorithms are called efficient (or economical, see e.g. [24]).

Each one of the systems (2.13) and (2.14) separately approximates differential problem (1.1)–(1.4) with approximation error $O(h^2+\tau)$.

Now, we prove that the system (2.13)–(2.14) approximates differential problem with accuracy $O(h^2+\tau^2)$. For this purpose we use the method described in [24, Ch. 9, §1] for parabolic equations with boundary conditions of Dirichlet type.

We eliminate intermediate values $U_{ij}^{n+1/2}$ from the system (2.13)–(2.14). For this purpose, we substract eq. (2.13) from (2.14) for every index pair (i,j). For every i value we have

$$U_{ij}^{n+1/2}=\frac{U_{ij}^{n+1}+U_{ij}^{n}}{2}-\frac{\tau}{4}\delta_y^2\left(U_{ij}^{n+1}-U_{ij}^{n}\right),\quad i=\overline{1,N},j=\overline{1,N-1}.\tag{2.15}$$

By substituting this expression into system (2.13), after some transformations, we obtain

$$\begin{aligned}\frac{U_{ij}^{n+1}-U_{ij}^{n}}{\tau} &= \frac{1}{2}\left(\tilde{\delta}_x^2 U_{ij}^{n+1}+\delta_y^2 U_{ij}^{n+1}\right)+\frac{1}{2}\left(\tilde{\delta}_x^2 U_{ij}^{n}+\delta_y^2 U_{ij}^{n}\right)\\ &\quad -\frac{\tau}{4}\tilde{\delta}_x^2\delta_y^2\frac{U_{ij}^{n+1}-U_{ij}^{n}}{\tau}+\tilde{f}_{ij}^{n+1/2},\quad i=\overline{1,N},\ j=\overline{1,N-1}.\end{aligned}\tag{2.16}$$

Systems (2.13)–(2.14) and (2.16) are equivalent.

Now, we can evaluate approximation error of the ADI method (2.13)–(2.14). Suppose one more assumption is valid (except for the standard differential problem's smoothness conditions, which ensure approximation order $O(h^2+\tau^2)$ of Crank-Nicolson method)

$$\left|\frac{\partial^5 u}{\partial x^2\partial y^2\partial t}\right|\le C.$$

Then, the corresponding differences are also bounded, and

$$\left|\delta_x^2\delta_y^2\frac{U_{ij}^{n+1}-U_{ij}^{n}}{\tau}\right|\le C$$

regardless of h and τ values.The following inequality is also valid (according to (2.10))

$$\left|\tilde{\delta}_x^2\delta_y^2\frac{U_{ij}^{n+1}-U_{ij}^{n}}{\tau}\right|\le 2C.$$

Therefore, the ADI method written in the form of (2.16) differs from Crank-Nicolson method only in $O(\tau^2)$ term. The approximation error of both methods is of the $O(h^2+\tau^2)$ order.

3. Stability of difference scheme

In this section we investigate the stability of the ADI method (2.13)–(2.14) with boundary conditions (2.11c), (2.12c). First, we rewrite the ADI method in the matrix form. We define $N\times N$ matrix $\mathbf{\Lambda}_x$ and $(N-1)\times(N-1)$ matrix $\mathbf{\Lambda}_y$ by

$$\mathbf{\Lambda}_x=\begin{pmatrix}2&-1&0&\cdots&0&0\\-1&2&-1&\ddots&0&0\\0&-1&2&\ddots&0&0\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\0&0&0&-1&2&-1\\-2&0&0&\cdots&-1&2\end{pmatrix},\quad \mathbf{\Lambda}_y=\begin{pmatrix}2&-1&0&\cdots&0&0\\-1&2&-1&\ddots&0&0\\0&-1&2&\ddots&0&0\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\0&0&0&-1&2&-1\\0&0&0&\cdots&-1&2\end{pmatrix}$$

We denote $\mathbf{I}_x$ as an $N \times N$ identity matrix and $\mathbf{I}_y$ as an $(N-1)\times(N-1)$ identity matrix. Now, we construct block matrices

$$\begin{gathered}\mathbf{A}_1 = \begin{pmatrix} \mathbf{\Lambda}_x & & & & \\ & \mathbf{\Lambda}_x & & & \\ & & \ddots & & \\ & & & \mathbf{\Lambda}_x & \\ & & & & \mathbf{\Lambda}_x \end{pmatrix}, \\ \mathbf{A}_2 = \begin{pmatrix} 2\mathbf{I}_x & -\mathbf{I}_x & & \dots & & \\ -\mathbf{I}_x & 2\mathbf{I}_x & -\mathbf{I}_x & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\mathbf{I}_x & 2\mathbf{I}_x & -\mathbf{I}_x \\ & & & \dots & -\mathbf{I}_x & 2\mathbf{I}_x \end{pmatrix}.\end{gathered} \tag{3.1}$$

Every row and column of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ has $N-1$ blocks of order $N \times N$.

We use Kronecker (tensor) product to investigate the eigenvalues and eigenvectors of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$.

Definition 3.1 ([32]). Let $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ be two rectangular matrices of order $m \times n$ and $p \times q$, accordingly. Matrix $\mathbf{C}$ of order $mp \times nq$ is called the Kronecker (tensor) product of matrices $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & \ddots & \ddots & a_{2n}\mathbf{B} \\ \dots & \ddots & \ddots & \dots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{nm}\mathbf{B} \end{pmatrix}.$$

We rewrite the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ using Kronecker product

$$\mathbf{A}_1 = \mathbf{I}_y \otimes \mathbf{\Lambda}_x, \quad \mathbf{A}_2 = \mathbf{\Lambda}_y \otimes \mathbf{I}_x. \tag{3.2}$$

Furthermore, we directly check that these matrices commute

$$\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1 = \mathbf{\Lambda}_y \otimes \mathbf{\Lambda}_x. \tag{3.3}$$

Now, we rewrite the systems of difference equations (2.13) and (2.14) with boundary conditions (2.11c) and (2.12c) in the matrix form

$$\Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_1\Big) U^{n+1/2} = \Big(\mathbf{I} - \frac{\tau}{2}\mathbf{A}_2\Big) U^n + \frac{\tau}{2}\tilde{f}^{n+1/2}, \tag{3.4}$$

$$\Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_2\Big) U^{n+1} = \Big(\mathbf{I} - \frac{\tau}{2}\mathbf{A}_1\Big) U^{n+1/2} + \frac{\tau}{2}\tilde{f}^{n+1/2}, \tag{3.5}$$

where $\mathbf{I}$ is $(N-1)N \times (N-1)N$ identity matrix and

$$U^n = \left\{U_{ij}^n\right\}, \ \tilde{f}^{n+1/2} = \{\tilde{f}_{ij}^{n+1/2}\}, \quad i = \overline{1, N}, j = \overline{1, N-1}.$$

Expressing $U^{n+1/2}$ from (3.4) and substituting into (3.5), we have

$$U^{n+1} = \mathbf{S}U^n + \frac{\tau}{2}\mathbf{S}_1\tilde{f}^{n+1/2}, \tag{3.6}$$

where

$$\mathbf{S} = \Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_2\Big)^{-1} \Big(\mathbf{I} - \frac{\tau}{2}\mathbf{A}_1\Big) \Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_1\Big)^{-1} \Big(\mathbf{I} - \frac{\tau}{2}\mathbf{A}_2\Big), \tag{3.7}$$

$$\mathbf{S}_1 = \Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_2\Big)^{-1} + \Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_2\Big)^{-1} \Big(\mathbf{I} - \frac{\tau}{2}\mathbf{A}_1\Big) \Big(\mathbf{I} + \frac{\tau}{2}\mathbf{A}_2\Big)^{-1}. \tag{3.8}$$

We further define norms of any matrix and any vector to investigate stability of difference scheme (3.6). We also formulate theoretical proposition that we need to investigate stability of the difference method.

Proposition 3.2 ([1, Th. 7.8]). *Let $\rho(\mathbf{A})$ be a spectral radius of an arbitrary square matrix $\mathbf{A}$. If $\varepsilon > 0$ is given, then there exists a matrix norm $\|\mathbf{A}\|_*$ for which*

$$\|\mathbf{A}\|_* \le \rho(\mathbf{A}) + \varepsilon.$$

Practically, we use a corollary of this proposition.

Corollary 3.3. *For any square matrix $\mathbf{A}$, there exists a matrix norm $\|\mathbf{A}\|_* < 1$ if and only if $\rho(\mathbf{A}) < 1$.*

Now, we investigate when the condition

$$\rho(\mathbf{S}) < 1 \tag{3.9}$$

is valid for scheme (3.6). We find eigenvalues and eigenvectors of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, as $\gamma \in (0,1)$. We formulate two eigenvalue problems (for the one-dimensional difference operator)

$$\mathbf{\Lambda}_x \mathbf{V} = \mu \mathbf{V}, \tag{3.10}$$

where $\mathbf{V} = \{V_i\}$, $i = \overline{1,N}$, and

$$\mathbf{\Lambda}_y \mathbf{W} = \eta \mathbf{W}, \tag{3.11}$$

where $\mathbf{W} = \{W_j\}$, $j = \overline{1,N-1}$.

Suppose N is even. Then, the eigenvalues of the problem (3.10) are of the form (see [11])

$$\begin{gathered}
\mu_1 = \frac{4}{h^2}\sin^2\frac{\psi h}{2},\\
\mu_{2k} = \frac{4}{h^2}\sin^2\big(\pi k - \frac{\psi h}{2}\big), \quad k = \overline{1, N/2},\\
\mu_{2k+1} = \frac{4}{h^2}\sin^2\big(\pi k + \frac{\psi h}{2}\big), \quad k = \overline{1, (N/2)-1},
\end{gathered} \tag{3.12}$$

where $\psi = \arccos\gamma$, $0 < \psi < 1$. The corresponding eigenvectors $V^{(k)} := \{V_i^{(k)}\}$, $i = \overline{1,N}$ of the problem (3.10) are defined as

$$\begin{gathered}
V_i^{(1)} = \sin(\psi i h),\\
V_i^{(2k)} = \sin\left((2\pi k - \psi) i h\right), \ k = \overline{1, N/2},\\
V_i^{(2k+1)} = \sin\left((2\pi k + \psi) i h\right), \ k = \overline{1, (N/2)-1}.
\end{gathered} \tag{3.13}$$

All the eigenvalues $\mu_1, \dots, \mu_N$ are positive and distinct and all the eigenvectors $V^{(1)}, \dots, V^{(N)}$ are linearly independent. If N is odd, expressions (3.12) and (3.13) are the same, except index k, which, in this case, is $k = \overline{1, (N-1)/2}$.

The eigenvalues and eigenvectors of difference problem (3.11) are of the form (see [24])

$$\eta_l = \frac{4}{h^2}\sin^2\frac{\pi l h}{2}, \quad l = \overline{1, N-1}, \tag{3.14}$$

$$W^{(l)} = \left\{W_j^{(l)}\right\} = \{\sin l\pi j h\}, \quad j,l = \overline{1, N-1}. \tag{3.15}$$

All the eigenvalues (3.14) are positive and distinct and the eigenvectors (3.15) are linearly independent.

Lemma 3.4. *The matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ have a common system of eigenvectors.*

Proof. Using eigenvectors $V^{(k)}$, $k=\overline{1,N}$ of the matrix $\mathbf{\Lambda}_x$ and eigenvectors $W^{(l)}$, $l=\overline{1,N-1}$ we construct new vector

$$U^{(k,l)} = W^{(l)} \otimes V^{(k)} = \Big(W_1^{(l)}V^{(k)}, W_2^{(l)}V^{(k)}, \dots, W_{N-1}^{(l)}V^{(k)}\Big)'. \tag{3.16}$$

Using properties of tensor product (see [32]) we have

$$(\mathbf{I}_y \otimes \mathbf{\Lambda}_x + \mathbf{\Lambda}_y \otimes \mathbf{I}_x)(\mathbf{W} \otimes \mathbf{V}) = (\mu+\eta)(\mathbf{W} \otimes \mathbf{V}).$$

It means that the eigenvalues and the eigenvectors of a matrix $\mathbf{A}_1+\mathbf{A}_2$ are $\mu_k+\eta_l$ and $W^{(l)} \otimes V^{(k)}$, respectively.

Further, from the definitions of $\mathbf{A}_1$ and $U^{(k,l)}$ directly follows

$$\mathbf{A}_1 U^{(k,l)} = \mu_k U^{(k,l)}. \tag{3.17}$$

Using expressions (3.16) and (3.17) we have

$$\mathbf{A}_2 U^{(k,l)} = \eta_l U^{(k,l)}.$$

The eigenvectors $W^{(l)} \otimes V^{(k)}$ of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are the same. □

Corollary 3.5 ([32])**.** *Since the eigenvectors $V^{(k)}$, $k=\overline{1,N}$ of the matrix $\mathbf{\Lambda}_x$ and the eigenvectors $W^{(l)}$, $l=\overline{1,N-1}$ of the matrix $\mathbf{\Lambda}_y$ are linearly independent, then $U^{(k,l)} = W^{(l)} \otimes V^{(k)}$ is the linearly independent system of vectors.*

Now, we define the norms of the matrices and vectors that we will use for investigation of stability of scheme (3.6). We generate the matrix $\mathbf{P}$, which columns are linearly independent eigenvectors $U^{(k,l)}$ of the matrix $\mathbf{A}_1$ (or $\mathbf{A}_2$).

We define the norm of any $m \times m$ matrix $\mathbf{A}$ as

$$\|\mathbf{A}\|_* := \|\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\|_2, \tag{3.18}$$

where $\|\mathbf{A}\|_2 = (\max_{1\le i\le m} \lambda_i(\mathbf{A}^*\mathbf{A}))^{1/2}$ is the classical matrix norm and $\mathbf{A}^*$ is the adjoint matrix. We define the compatible vector norm by the formula

$$\|\mathbf{U}\|_* = \|\mathbf{P}^{-1}\mathbf{U}\|_2 = \Big(\sum_{i=1}^{m} |\tilde{U}_i|^2\Big)^{1/2}, \tag{3.19}$$

where $\tilde{U}_i$, $i=\overline{1,m}$ are the coordinates of the vector $\mathbf{P}^{-1}\mathbf{U}$. Indeed

$$\|\mathbf{A}\mathbf{U}\|_* = \|\mathbf{P}^{-1}\mathbf{A}\mathbf{U}\|_2 = \|\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{U}\|_2 \le \|\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\|_2\|\mathbf{P}^{-1}\mathbf{U}\|_2 = \|\mathbf{A}\|_*\|\mathbf{U}\|_*.$$

For the special (symmetric or nonsymmetric) $(m\times m)$ matrix $\mathbf{S}$ we have relation

$$\|\mathbf{S}\|_* = \|\mathbf{P}^{-1}\mathbf{S}\mathbf{P}\|_2 = \|\mathbf{J}\|_2 = \max_{1\le i\le m} \|\mu_i(\mathbf{S})\| = \rho(\mathbf{S}), \tag{3.20}$$

where $\mathbf{J} = \operatorname{diag}(\mu^1,\dots,\mu^m)$, μ^i, $i=\overline{1,m}$ are the eigenvalues and $\rho(\mathbf{S})$ is the spectral radius of matrix $\mathbf{S}$.

One should not interpret (3.20) as the norm of any nonsymmetric matrix. This formula means that the norm of any matrix could be defined (see (3.18)) in a way that for particularly chosen matrix $\mathbf{S}$ equality $\|\mathbf{S}\|_* = \rho(\mathbf{S})$ is valid.

We rewrite the vector norm (3.19) in an other form

$$\|\mathbf{U}\|_* = \big(\mathbf{P}^{-1}\mathbf{U}, \mathbf{P}^{-1}\mathbf{U}\big)_2^{1/2} = \big((\mathbf{P}\mathbf{P}^*)^{-1}\mathbf{U}, \mathbf{U}\big)_2^{1/2} = (\mathbf{D}\mathbf{U}, \mathbf{U})_2^{1/2}, \tag{3.21}$$

where $\mathbf{D}$ is a positive definite matrix. The norm, defined as (3.21), is usually called energy norm, generated by the matrix $\mathbf{D}$.

Now, we return to the stability analysis of the scheme (3.6). We have

$$\|U^{n+1}\|_* = \|\mathbf{S}\|_*\|U^n\|_* + \frac{\tau}{2}\|\mathbf{S}_1\|_*\|\tilde{f}^{n+1/2}\|_*.$$

Since the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ have the common system of eigenvectors, then, taking into account expressions (3.7) and (3.8), we have

$$\|\mathbf{S}\|_* = \rho(\mathbf{S}) = \max_{k,l}\Big|\frac{\left(1-\frac{\tau}{2}\mu_k\right)\left(1-\frac{\tau}{2}\eta_l\right)}{\left(1+\frac{\tau}{2}\mu_k\right)\left(1+\frac{\tau}{2}\eta_l\right)}\Big| := q < 1,$$

where $\mu_k > 0$ and $\eta_l > 0$.

Similarly,

$$\|\mathbf{S}_1\|_* = \rho(\mathbf{S}_1) = \max_{k,l}\Big|\frac{1}{1+\frac{\tau}{2}\eta_l} + \frac{1-\frac{\tau}{2}\mu_k}{\left(1+\frac{\tau}{2}\mu_k\right)\left(1+\frac{\tau}{2}\eta_l\right)}\Big| < 1,$$

where $\mu_k > 0$ and $\eta_l > 0$. Therefore, we obtain the classical estimate for solution of the difference scheme

$$\|U^{n+1}\|_* \le q\|U^n\|_* + \frac{\tau}{2}\|\tilde{f}^{n+1/2}\|_*, \quad 0 < q < 1, \tag{3.22}$$

which implies the stability of difference method in norm $\|\mathbf{U}\|_*$. So, the following theorem is valid.

Theorem 3.6. *If* $0 < \gamma < 1$*, then the ADI method* (2.13)–(2.14) *for the differential problem* (1.1)–(1.4) *is stable in the norm* $\|\mathbf{U}\|_*$.

Proof. Proof of theorem follows from the above stated investigation. Really, if $0 < \gamma < 1$, then all the eigenvalues $\mu_1, \dots, \mu_N$ are positive [11]. The eigenvalues $\eta_1, \dots, \eta_N$ are positive regardless of γ value. So, $\rho(\mathbf{S}) < 1$ and $\rho(\mathbf{S}_1) < 1$. □

Remark 3.7. In this article, the stability of ADI method, with nonsymmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, is based on the assumption that eigenvalues of these matrices are positive. To the authors' knowledge, first time for the parabolic equations with nonlocal conditions this assumption to prove the stability of ADI method was used in [27]. However, several attempts to find suffiecient conditions for the stability or convergence of the ADI method with nonsymmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ were in the past (not for nonlocal boudary problems). One of these results is described in [24, Ch. 10, § 4.5]. It is proved that $\|\mathbf{S}\| < 1$, if $(\mathbf{A}_\alpha u, u) \ge \delta_\alpha(u,u)$, $\delta_\alpha > 0$, $\alpha = 1, 2$. It is easy to see that these assumptions are inappropriate for investigating the stability of ADI method presented in this article. All the eigenvalues of the matrix $\mathbf{A}_1$, defined as (3.1), are positive. Nevertheless, if $u = \{u_i\}$, $u_i \equiv 1$, then

$$(\mathbf{A}_1 u, u) = \sum_{i,j=1}^{N} \alpha_{ij} = 0.$$

Therefore, assumption $(\mathbf{A}_\alpha u, u) \ge \delta_\alpha(u,u)$, $\delta_\alpha > 0$ is not fulfilled.

4. Numerical experiments

Numerical experiments are performed to illustrate and confirm theoretical results. We consider examples, where exact solution U of the problem (1.1)–(1.4) is not known. We compare two difference solutions U and U^*. U^* is obtained twice reducing h and τ values. From theoretical investigation we know that approximation error is $O(h^2+\tau^2)$. So, it is presumed that if h and τ are reduced twice, maximum relative error

$$\Delta U := \Big|\frac{\max_{0\le i,j\le N} U_{ij}^n - \max_{0\le i,j\le N} U_{ij}^{*n}}{\max_{0\le i,j\le N} U_{ij}^n}\Big|,$$

should be decreasing four times in each step.

Case 1. $\gamma = 0$ (classical boundary condition), $f(x,y,t)=0$, $\varphi = y(1-y^2)(\frac{\pi}{2}\sin\frac{\pi}{2}x)$.

h	τ	$T=1$		$T=5$	
		ΔU	ΔU ratio	ΔU	ΔU ratio
0.1	0.1	$1.07 \cdot 10^{-6}$		$1.25 \cdot 10^{-6}$	
0.05	0.05	$2.81 \cdot 10^{-7}$	3.81	$3.29 \cdot 10^{-7}$	3.8
0.025	0.025	$7.26 \cdot 10^{-8}$	3.87	$8.55 \cdot 10^{-8}$	3.85
0.0125	0.0125	$1.85 \cdot 10^{-8}$	3.92	$2.20 \cdot 10^{-8}$	3.89
0.00625	0.00625	$4.66 \cdot 10^{-9}$	3.97	$5.58 \cdot 10^{-9}$	3.94

Case 2. $\gamma = 0.5$, $f(x,y,t)=0$, $\varphi = y(1-y^2)(\frac{\pi}{2}\sin\frac{\pi}{2}x + \gamma\frac{x^2}{2})$.

h	τ	$T=1$		$T=5$	
		ΔU	ΔU ratio	ΔU	ΔU ratio
0.1	0.1	$2.83 \cdot 10^{-6}$		$2.06 \cdot 10^{-6}$	
0.05	0.05	$7.47 \cdot 10^{-7}$	3.79	$5.47 \cdot 10^{-7}$	3.78
0.025	0.025	$1.96 \cdot 10^{-7}$	3.81	$1.43 \cdot 10^{7}$	3.82
0.0125	0.0125	$5.09 \cdot 10^{-8}$	3.85	$3.72 \cdot 10^{-8}$	3.85
0.00625	0.00625	$1.31 \cdot 10^{-8}$	3.88	$9.62 \cdot 10^{-9}$	3.87

Case 3. $\gamma = 1$, $f(x,y,t)=0$, $\varphi = y(1-y^2)(\frac{\pi}{2}\sin\frac{\pi}{2}x + \gamma\frac{x^2}{2})$.

h	τ	$T=1$		$T=5$	
		ΔU	ΔU ratio	ΔU	ΔU ratio
0.1	0.1	$3.54 \cdot 10^{-6}$		$2.88 \cdot 10^{-6}$	
0.05	0.05	$1.01 \cdot 10^{-6}$	3.51	$8.14 \cdot 10^{-7}$	3.54
0.025	0.025	$2.80 \cdot 10^{-7}$	3.60	$2.27 \cdot 10^{-7}$	3.59
0.0125	0.0125	$7.51 \cdot 10^{-8}$	3.73	$6.19 \cdot 10^{-8}$	3.66
0.00625	0.00625	$1.93 \cdot 10^{-8}$	3.89	$1.64 \cdot 10^{-8}$	3.78

Case 4. Unstable example $\boldsymbol{\gamma = 2}$, $f(x,y,t)=0$, $\varphi = y(1-y^2)(\frac{\pi}{2}\sin\frac{\pi}{2}x + \gamma\frac{x^2}{2})$.

h	τ	ΔU		
		$T=1$	$T=2$	$T=5$
0.025	0.025	$1.80 \cdot 10^{-4}$	$1.80 \cdot 10^{-3}$	$3.01 \cdot 10^{-2}$
0.025	0.0125	$2.29 \cdot 10^{-2}$	$7.16 \cdot 10^{2}$	$1.62 \cdot 10^{3}$
0.025	0.00625	$3.17 \cdot 10^{2}$	$4.76 \cdot 10^{4}$	$6.12 \cdot 10^{4}$
0.025	0.003125	$9.15 \cdot 10^{4}$	$2.03 \cdot 10^{5}$	$2.23 \cdot 10^{6}$

Remark 4.1. In this section we include examples with various γ values, starting from classical boundary conditions case ($\gamma = 0$) and some examples of nonlocal boundary conditions ($\gamma = 0.5$ and $\gamma = 1$). We also provide examples of experiments with different T values and show that stability does not depend on T choice. It follows from practical experiment that the constructed ADI method is stable and approximation error is of order $O(h^2+\tau^2)$.

In Case 4 the example of unstable scheme with $\gamma = 2$ is provided. It is known that if $\gamma > 1$, then the difference scheme for one-dimensional parabolic equation with nonlocal condition (1.2) is not stable with respect to the initial data [13]. It means that the second equation of the ADI method (2.13)–(2.14) is unstable. If the scheme is unstable, then ΔU grows while increasing the number of layers on the t axis. One can see from the experiment that either if τ decreases ($h = \text{const}$, $T = \text{const}$), or T increases ($h = \text{const}$, $\tau = \text{const}$), then the relative error ΔU grows indefinitely.

5. Remarks and generalizations

Stability of alternating direction method is proved in a special, quite complex vector norm $\|\mathbf{U}\|_*$ (see (3.19)). The equivalence of the same norm to the L_2-norm for the one-dimensional parabolic equation with the nonlocal condition, similar to (1.2), is proved in [10]. To the authors' knowledge, there was no investigation of stability, norms and spectrum for two-dimensional parabolic nonlocal problems of type (1.1)–(1.4). Nevertheless, introduced and investigated norm has important advantage. It follows from Proposition 3.2 and Corollary 3.3 that if difference method is not stable in norm $\|\mathbf{U}\|_*$, than $\rho(\mathbf{S}) > 1$, and difference scheme is not stable in every other norm. This fact, for problems with nonlocal conditions, has been noticed and commented a lot previously (see, [13, 4, 29]).

In this article, we theoretically investigated stability of alternating direction method in a case, when parameter γ in nonlocal condition is from interval $(0, 1)$. With these γ values eigenvectors of both matrix $\mathbf{S}$ and matrix $\mathbf{A}_1 + \mathbf{A}_2$ are linearly independent, that is eigenvectors form a basis in vector space $\mathbf{H}_{(N-1)N}$. However, if $\gamma = 1$, then matrix $\mathbf{\Lambda}_x$, the ADI method's matrix $\mathbf{S}$ and Crank-Nicolson matrix $\mathbf{A}_1 + \mathbf{A}_2$ all have multiple eigenvalues corresponding to only one eigenvector. Therefore, these eigenvectors do not form basis in $\mathbf{H}_{(N-1)N}$ (see [11]). This means that one cannot define norms $\|\mathbf{A}\|_*$ and $\|\mathbf{U}\|_*$ using formulas (3.18) and (3.19), because $\mathbf{P}^{-1}$ does not exist. Nevertheless, results of numerical experiments show that in sense of approximation accuracy variants $\gamma \in (0, 1)$ and $\gamma = 1$ are both suitable. Not diving into the theoretical proofs, we explain this situation in two aspects.

First, notice that Lemma 3.4 is valid for the case $\gamma = 1$ (see [32]). Therefore, according to Proposition 3.2 and Corollary 3.3, the same way as in case $\gamma \in (0, 1)$, we have inequality $\rho(\mathbf{S}) < 1$. From this inequality follows that norm $\|\mathbf{S}\|_* < 1$ can be defined (without providing the method). In other words, in the case $\gamma = 1$ the ADI method is also stable. In this article we do not specify exact $\|\mathbf{S}\|_*$ and $\|\mathbf{U}\|_*$ expressions in the case $\gamma = 1$. This can be done at least in two ways. Just notice that the aim is not equality $\|\mathbf{S}\|_* = \rho(\mathbf{S})$, but $\|\mathbf{S}\|_* = \rho(\mathbf{S}) + \varepsilon$, $\varepsilon < 0$. One of the techniques is described in [25, Ch. 2, §3]. Another technique can be found in [15].

Second, in one-dimensional case problem (1.1)–(1.4) for both cases $\gamma \in (0, 1)$ and $\gamma = 1$ has the same important property, related to concept of strong regularity of boundary conditions [20, 22]. Let $\gamma = 1$. Consider eigenvalue problem for nonlocal problem (1.1)–(1.4) in one-dimensional case. Define system of functions $\{\varphi_k(x)\}$, $k = 1, 2, \dots$ for this eigenvalue problem, which consists of eigenvectors and associated vectors. Similar system $\{\psi_l(x)\}$, $l = 1, 2, \dots$ consists of eigenvectors and associated vectors of adjoint eigenvalue problem. It is proved in [11] that both these systems are biorthonormal

$$(\varphi_k, \psi_l) = \int_0^1 \varphi_k(x)\psi_l(x)\,dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

In other words, the system of root functions of differential problem forms Risz basis in $L_2(0,1)$, what is typical for problem with $\gamma \in (0,1)$ and, in general case, for problems with strongly regular boundary conditions. In more detail this property for two-dimensional hyperbolic equation with nonlocal condition is discussed in [22].

Furthermore, while investigating alternating direction method in this article, only stability is proved and nothing is said about convergence of difference method. We notice that similar situation occurs also in other papers, where difference methods for differential equations are considered (see e.g. [11] and references therein). Often, as well as in this article, approximation of differential problem and stability of difference scheme with nonlocal conditions are considered in different vector norms. Additional investigation is required to prove equivalence of these norms. It is proved in one-dimensional case (see [10]) that the norm $\|\mathbf{U}\|_*$ is equivalent to vector L_2-norm for all $\gamma \in (0,1)$ and $\gamma = 1$ values. This implies the convergence of the ADI method in L_2-norm. To the authors' knowledge, there are no investigations for the two-dimensional case. It is obvious that matrices $\mathbf{\Lambda}_x$, $\mathbf{\Lambda}_y$, $\mathbf{A}_1$ and $\mathbf{A}_2$, defined in this article, are M-matrices [32]. Therefore, convergence of difference methods can be proved, by using properties of M-matrices [31, 6].

In the authors' opinion, both of the ways to consider convergence of difference schemes with nonlocal conditions are worth separate investigation.

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