*Electronic Journal of Differential Equations*, Vol. 2020 (2020), No. 95, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# ENERGY DECAY FOR VARIABLE COEFFICIENT VISCOELASTIC WAVE EQUATION WITH ACOUSTIC BOUNDARY CONDITIONS IN DOMAINS WITH NONLOCALLY REACTING BOUNDARY

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Abstract. In this article, we study a variable coefficients viscoelastic wave equation with acoustic boundary conditions in domains with nonlocally reacting boundary. By constructing suitable Lyapunov functionals and using the energy compensation method, we prove that under suitable conditions on the initial data and the relaxation function, the energy of the system has an explicit and general decay rate.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, $\Gamma_0$ and $\Gamma_1$ are closed and disjoint with $\operatorname{meas}(\Gamma_0) > 0$. In this paper we consider the viscoelastic wave equation of variable coefficients with the acoustic boundary conditions

$$
\begin{gathered}
u'' - Lu + \int_0^t g(t-\tau)Lu(\tau)d\tau + \rho(u') = 0 \quad \text{in } \Omega \times (0,\infty), \\
u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
\frac{\partial u}{\partial \nu_L} - \int_0^t g(t-\tau)\frac{\partial u}{\partial \nu_L}(\tau)d\tau = z' \quad \text{on } \Gamma_1 \times (0,\infty), \\
fz'' - p^2\Delta_\Gamma z + qz' + hz = -u' \quad \text{on } \Gamma_1 \times (0,\infty), \\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega, \\
z(x,0) = z_0(x), \quad z'(x,0) = z_1(x) \quad \text{on } \Gamma_1,
\end{gathered} \tag{1.1}
$$

where

$$
Lu = \operatorname{div}(A(x)\nabla u) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big), \quad \frac{\partial u}{\partial \nu_L} = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i.
$$

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2010 *Mathematics Subject Classification.* 35L70, 35B35.
*Key words and phrases.* Variable coefficients; viscoelastic wave equation; acoustic boundary conditions; nonlocally reacting boundary.

Submitted December 19, 2019. Published September 17, 2020.

The symbol $'$ denotes the derivative with respect to time $t$, $\nu_L = A\nu$, where $\nu = (\nu_1, \dots, \nu_n)$ represents the outward unit normal vector to $\Gamma$, and $\Delta_\Gamma$ is the Laplace-Beltrami operator. In addition, $p$ is a positive constant, $\rho : R \to R$, $g : R^+ \to R^+$ and $f$, $q$, $h : \overline{\Gamma_1} \to R$ are functions.

When $g = 0$ and $p = 0$, the boundary conditions $(1.1)_3$ and $(1.1)_4$ are the classical acoustic boundary conditions introduced by Morse and Ingard [23] and developed by Beale and Rosencrans [2, 3] via the assumption that each point on the boundary reacts to the excess pressure of the acoustic wave like a resistive harmonic oscillator or spring and each point of the boundary does not affect each other. The models usually are related to the problems of noise control and suppression in practical applications and have been studied by many authors, see [6, 7, 8] and the references therein. Límaco et al. [16] investigated a nonlinear wave equation of Carrier type and established the existence of regular weak solution. Gao, Liang and Xiao [11] obtained the uniform stability of a nonlinear acoustic wave system with an internal localized damping term $\omega(x)u_t$. For the case $f = 0$, which means the material of surface is much lighter than the fluid medium, Hao and He [14, 15] studied two variable-coefficient wave equations with the acoustic boundary conditions, and they obtained the exponential decay result and general decay result respectively.

On the other hand, when $g = 0$ and $p > 0$, the boundary conditions $(1.1)_3$ and $(1.1)_4$ are called acoustic boundary conditions to non-locally reacting boundary (see [9]), which models the surface $\Gamma_1$ reacts to the excess pressure as an elastic membrane. Later, Frota et al [10] studied the following semilinear wave equation

$$u'' - \Delta u + \alpha u' + \rho(u') = F.$$

They proved the existence, uniqueness of solution by Galerkin's method and obtained an exponential decay result. Moreover they also improved their previous results since estimates they made can be adapted to the problem treated in [9]. Frota and Vicente [26] took into account the dissipative term $q(z')$ in stead of $qz'$ and put a nonlinear internal localized damping term in the wave equation to achieve uniform stability successfully. Recently, Ha [13] considered the following wave equation of variable coefficients

$$u'' - Lu + \rho(u') = 0,$$

where

$$Lu = \operatorname{div}(A(x)\nabla u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big).$$

Under suitable conditions on $\rho$, he improved his previous result [12] in which he focused on the case $A = I$, and obtained the general decay result. Liu [18] studied a variable coefficient wave equation with an acoustic undamped boundary condition and deduced the polynomial energy decay estimates by the Riemannian geometry method introduced by Yao [28].

In addition, the integral-differential term in (1.1) gives the memory effect to the problem, due to the mechanical response influenced by the history of the materials themselves. The study involving the wave equation with viscoelastic term and the acoustic boundary conditions can be found in [5, 19, 20]. For instance, Park and Park [25] studied the viscoelastic wave system

$$u'' - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$
\begin{gathered}
u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
\frac{\partial u}{\partial \nu} - \int_0^t g(t-\tau)\frac{\partial u}{\partial \nu}(\tau)d\tau = z' \quad \text{on } \Gamma_1 \times (0,\infty), \\
u' + qz' + hz = 0 \quad \text{on } \Gamma_1 \times (0,\infty),
\end{gathered}
$$

and deduced the energy decay rates under the assumption $\int_0^\infty g(s)ds < \frac{1}{2}$. Later, without this assumption condition on the relaxation function $g$, Liu [17] generalized the work to an arbitrary decay rate which does not necessarily decay exponentially or polynomially. In presence of variable-coefficient matrices $A(x)$, which reflects the inhomogeneous nature of the material in applications, Boukhatem and Benabderrahmane [4] considered the damped semilinear viscoelastic wave system

$$
\begin{gathered}
u'' - Lu + \int_0^t g(t-\tau)Lu(\tau)d\tau = |u|^{p-2}u \quad \text{in } \Omega \times (0,\infty), \\
u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
\frac{\partial u}{\partial \nu_L} - \int_0^t g(t-\tau)\frac{\partial u}{\partial \nu_L}(\tau)d\tau = h(x)z' \quad \text{on } \Gamma_1 \times (0,\infty), \\
u' + qz' + hz = 0 \quad \text{on } \Gamma_1 \times (0,\infty).
\end{gathered}
$$

Instead of using the Riemannian geometry method, they obtained the local existence of solution by combining the Faedo-Galerkin approximations and the contraction mapping theorem. Furthermore, they proved the solution exists globally in time and established a uniform decay result. From the previous works with memory effect and the acoustic boundary conditions, we can see that most authors considered the porous case ($f = 0$).

Motivated by the previous works, our goal of this paper is to prove the general decay estimates for problem (1.1). We consider the case $f > 0$, i.e., non-porous case and $\Gamma_1$ is non-locally reacting. To the best of our knowledge, it is hardly seen in current literature on the study of variable-coefficient viscoelastic wave equation with acoustic boundary conditions to nonlocally reacting boundary. Therefore, the model is novel and the study on the asymptotic behavior of solutions for (1.1) is interesting and significant. Also, problem (1.1) in this paper is a improvement of [13], because we consider the viscoelastic damping effect and the assumptions on $\rho$ allows a wider class of functions. Different from the method in [13], our strategy was to use the techniques of [21, 22, 27] with some necessary modifications due to the nature of problem (1.1). The main idea is to construct appropriate Lyapunov functionals and deduce the energy inequality which leads us to a general decay result.

The paper is organized as follows. In Section 2, we present some assumptions and materials needed in our work and give the main results of this paper. Then, some estimates are given and the general decay of energy for (1.1) is derived in Section 3.

## 2. Preliminaries

In this section, we present some assumptions and materials needed for our work. Throughout the paper $C_i$ ($i = 1, 2, \dots$) denote various positive constants which depend on the known constants. We consider the standard Sobolev spaces $L^q(\Omega)$ and $L^q(\Gamma_1)$ endowed with the usual inner products and norms. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{L^2(\Gamma_1)}$ and $\|\cdot\|_{L^q(\Gamma_1)}$ by $\|\cdot\|$, $\|\cdot\|_q$, $\|\cdot\|_{\Gamma_1}$ and $\|\cdot\|_{q,\Gamma_1}$, respectively.

Set $H(L,\Omega)=\{u\in H^1(\Omega); Lu\in L^2(\Omega)\}$ equipped with the norm

$$\|u\|_{H(L,\Omega)}=\big(\|u\|^2_{H^1(\Omega)}+\|Lu\|^2\big)^{1/2}.$$

Denoting $\gamma_0: H^1(\Omega)\to H^{1/2}(\Gamma)$ and $\gamma_1: H(L,\Omega)\to H^{-1/2}(\Gamma)$ the trace map of order 0 and the Neumann trace map on $H(L,\Omega)$, respectively, we have

$$\gamma_0(u)=u|_\Gamma \quad\text{and}\quad \gamma_1(u)=\Big(\frac{\partial u}{\partial \nu_L}\Big)_\Gamma.$$

Define $W=\{u\in V\cap H^3(\Omega); (\gamma_1(u))|_{\Gamma_1}\in H^1_0(\Gamma_1)\}$, where $V=\{u\in H^1(\Omega); \gamma_0(u)=0 \text{ on } \Gamma_0\}$ endowed with the norm

$$\|u\|_V=\Big(\sum_{i=1}^N\int_\Omega|\frac{\partial u}{\partial x_i}|^2dx\Big)^{1/2}.$$

By Poincaré's inequality and the continuity of the trace map, there exist positive constants $k_0$ and $k_1$ such that

$$\|u\|\le k_0\|\nabla u\| \quad\text{and}\quad \|\gamma_0(u)\|_{\Gamma_1}\le k_1\|\nabla u\|,\quad u\in V. \tag{2.1}$$

We consider the Sobolev space $H^m(\Gamma_1)$, $m=1,2$ with respect to the norm

$$\|z\|_{H^m(\Gamma_1)}=\Big(\sum_{i=0}^m\|\nabla^i z\|^2_{\Gamma_1}\Big)^{1/2},\quad m=1,2,$$

where $\nabla^i$ is the covariant derivative operator of order $i$. Let $H^1_0(\Gamma_1)$ be the closure of $C^\infty_0(\Gamma_1)$ in $H^1(\Gamma_1)$. The Poincaré's inequality holds in $H^1_0(\Gamma_1)$, thus there exists a constant $k_2$ such that

$$\|z\|_{\Gamma_1}\le k_2\|\nabla_\tau z\|_{\Gamma_1},\quad z\in H^1_0(\Gamma_1), \tag{2.2}$$

where $\nabla_\tau$ is the tangential gradient on $\Gamma_1$. Therefore on $H^1_0(\Gamma_1)$ we have the inner product and norm

$$(z,v)_{\Gamma_1}=\int_{\Gamma_1}\langle\nabla_\tau z(x),\nabla_\tau v(x)\rangle d\Gamma_1,\quad \|z\|_{\Gamma_1}=\|\nabla_\tau z\|_{\Gamma_1},$$

which is equivalent to the usual norm endowed by $H^1(\Gamma_1)$. Next, we consider $H^1_0(\Gamma_1)\cap H^2(\Gamma_1)$ endowed with the norm

$$\|z\|_{H^1_0(\Gamma_1)\cap H^2(\Gamma_1)}=\|\Delta_\Gamma z\|_{\Gamma_1},$$

here $\Delta_\Gamma z=\operatorname{div}\nabla_\tau z$, which is equivalent to the usual norm endowed by $H^2(\Gamma_1)$.

We will use the following assumptions:

(A1) The matrix $A(x)=(a_{ij}(x))$, with entires $a_{ij}(x)\in C^1(\bar\Omega)$, is symmetric and there exists a positive constant $a_0$ such that for all $x\in\bar\Omega$ and $\zeta=(\zeta_1,\zeta_2,\dots,\zeta_n)\in\mathbb{R}^n$, we have

$$\sum_{i,j=1}^n a_{ij}(x)\zeta_j\zeta_i\ge a_0|\zeta|^2.$$

(A2) The relaxation function $g : R^+ \to R^+$ is a bounded $C^1$ function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0,$$
$$g'(t) \le -\xi(t)g(t), \quad t \ge 0,$$

in which $\xi : [0,\infty) \to [0,\infty)$ is a positive nonincreasing $C^1$ function satisfying

$$\int_0^\infty \xi(s)ds = \infty.$$

(A3) $\rho : R \to R$ is a nondecreasing $C^1$ function and there exist positive constants $\epsilon$, $c_1$, $c_2 > 0$ and an increasing function $H_1 : R_+ \to R_+$ of class $C^1(R_+) \cap C^2(R^+)$ satisfying $H_1(0) = 0$, and $H_1$ is linear or $H_1'(0) = 0$ and $H_1''(t) > 0$ on $(0, \epsilon]$ such that

$$c_1|s| \le |\rho(s)| \le c_2|s| \quad \text{if } |s| \ge \epsilon,$$
$$s^2 + \rho^2(s) \le H_1^{-1}(s\rho(s)) \quad \text{if } |s| \le \epsilon.$$

(A4) The positive functions $f$, $q$, $h$ are essentially bounded and there exist positive constants $f_i, q_i, h_i$ $(i = 0, 1)$ such that

$$f_0 \le f \le f_1, \quad q_0 \le q \le q_1, \quad h_0 \le h \le h_1, \quad x \in \Gamma_1.$$

To simplify calculation in our analysis, we introduce the following notation

$$(g \diamond u)(t) = \int_0^t g(t-\tau)a(u(t) - u(\tau), u(t) - u(\tau))d\tau,$$

where

$$a(u(t), v(t)) = \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial v(t)}{\partial x_i}dx = \int_\Omega A\nabla u(t)\nabla v(t)dx.$$

**Lemma 2.1.** *For $g \in C^1(0,T)$ and $u \in C^1(0,T;V)$, we have*

$$\begin{aligned} &\int_0^t g(t-\tau)a(u(\tau), u'(t))d\tau \\ &= \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)a(u(t), u(t)) \\ &\quad - \frac{1}{2}\frac{d}{dt}\Big((g \diamond u)(t) - \int_0^t g(\tau)d\tau a(u(t), u(t))\Big). \end{aligned} \tag{2.3}$$

Similar to [10], a well posedness theorem can be derived by using Faedo-Galerkin method and we omit the proof.

**Theorem 2.2.** *Suppose that assumptions* (A1)–(A4) *hold and the initial data satisfies*

$$(u_0, u_1, z_0) \in W \times V \times (H_0^1(\Gamma_1) \cap H^2(\Gamma_1)) \tag{2.4}$$

*and the compatibility condition*

$$\frac{\partial u_0}{\partial \nu} = z_1 \quad \text{in } L^2(\Gamma_1). \tag{2.5}$$

*Then, there exists a unique solution $(u, z)$ to* (1.1) *satisfying*

$$u \in L^\infty_{\rm loc}(0,\infty;V), \quad u' \in L^\infty_{\rm loc}(0,\infty;V), \quad u'' \in L^\infty_{\rm loc}(0,\infty;L^2(\Omega)),$$

$$z \in L^\infty_{\rm loc}(0,\infty; H^1_0(\Gamma_1)\cap H^2(\Gamma_1)), \quad z' \in L^\infty_{\rm loc}(0,\infty; H^1_0(\Gamma_1)),$$
$$z'' \in L^\infty_{\rm loc}(0,\infty; L^2(\Gamma_1)).$$

We denote the modified energy functional $E(t)$ associated with problem (1.1) by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u'\|^2 + \frac{1}{2}\Big(1-\int_0^t g(\tau)d\tau\Big)a(u(t),u(t)) + \frac{1}{2}(g\diamond u)(t) \\ &+\frac{1}{2}\|f^{1/2}z'\|^2_{\Gamma_1} + \frac{p^2}{2}\|\nabla_\tau z\|^2_{\Gamma_1} + \frac{1}{2}\|h^{1/2}z\|^2_{\Gamma_1}. \end{aligned} \tag{2.6}$$

Multiplying the first equation in (1.1) by $u_t$ and the fourth equation by $z_t$, integrating over $\Omega$ and $\Gamma_1$ respectively, using integration by parts and (2.3), we obtain the following lemma.

**Lemma 2.3.** *Suppose that assumptions* (A1)–(A4), (2.4) *and* (2.5) *hold. Then $E(t)$ is nonincreasing and satisfies*

$$E'(t) = -\|q^{1/2}z'\|^2_{\Gamma_1} - \frac{1}{2}g(t)a(u(t),u(t)) + \frac{1}{2}(g'\diamond u)(t) - \int_\Omega u'\rho(u')dx. \tag{2.7}$$

Now we can state the main result of this paper.

**Theorem 2.4.** *Suppose that assumptions* (A1)–(A4), (2.4) *and* (2.5) *hold. Then there exist positive constants $\epsilon_0, t_0, \mu_1, \mu_2$ and nonnegative constant $\mu_3$ such that the solution of system* (1.1) *satisfies*

$$E(t) \le \mu_1 H^{-1}\big(\mu_2\int_0^t \xi(s)ds + \mu_3\big), \quad t\ge t_0, \tag{2.8}$$

*where*

$$H(r) = \int_r^1 \frac{1}{H_0(s)}ds \quad \text{and} \quad H_0(r) = rH_1'(\epsilon_0 r).$$

*Here, $H$ is strictly decreasing and convex on $(0,1]$, with $\lim_{r\to 0} H(r) = +\infty$.*

## 3. Decay estimate

In this section we give the proof of our main result. To do this, we define the functional

$$L(t) := E(t) + \varepsilon\psi(t) + \eta\phi(t), \tag{3.1}$$

where $\varepsilon$ and $\eta$ are positive constants to be chosen later and

$$\psi(t) := \int_\Omega uu'dx + \int_{\Gamma_1} fzz'd\Gamma + \int_{\Gamma_1} uz\,d\Gamma, \tag{3.2}$$

$$\phi(t) := -\int_\Omega u'\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx. \tag{3.3}$$

It is easy to obtain the following result, i.e. the functional $L$ is equivalent to the energy functional $E$.

**Lemma 3.1.** *Suppose that assumptions* (A1)–(A4), (2.4) *and* (2.5) *hold. Then for $\varepsilon,\eta>0$ small enough, there exist two positive constants $\lambda_1$ and $\lambda_2$ such that*

$$\lambda_1 E(t) \le L(t) \le \lambda_2 E(t).$$

**Lemma 3.2.** *Let assumptions* (A1)–(A4), (2.4) *and* (2.5) *hold. Then there exists some constant* $C_1$ *such that the functional* $\psi(t)$ *satisfies*

$$\begin{aligned} \psi'(t) \le \|u'\|^2 + C_1\|z'\|^2_{\Gamma_1} - \|h^{1/2}z\|^2_{\Gamma_1} - \frac{l}{4}a(u(t),u(t)) \\ -\frac{p^2}{2}\|\nabla_\tau z\|^2_{\Gamma_1} + \frac{1-l}{2l}(g\diamond u)(t) + \frac{1}{4\alpha_1}\int_\Omega \rho^2(u')dx. \end{aligned} \tag{3.4}$$

*Proof.* By differentiating $\psi$ and using (1.1), we obtain

$$\begin{aligned} \psi'(t) &= \|u'\|^2 + \|f^{1/2}z'\|^2_{\Gamma_1} - p^2\|\nabla_\tau z\|^2_{\Gamma_1} - \|h^{1/2}z\|^2_{\Gamma_1} - a(u(t),u(t)) \\ &\quad + \int_{\Gamma_1} uz'd\Gamma - \int_\Omega u\rho(u')dx - \int_{\Gamma_1} zu'd\Gamma - \int_{\Gamma_1} qzz'd\Gamma \\ &\quad + \frac{d}{dt}\int_{\Gamma_1} uzd\Gamma + \int_0^t g(t-\tau)\int_\Omega A\nabla u(t)\nabla u(\tau)\,dx\,d\tau. \end{aligned} \tag{3.5}$$

Now we estimate the last term on the right-hand side of (3.5). By (A2), Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned} &\int_0^t g(t-\tau)\int_\Omega A\nabla u(t)\nabla u(\tau)\,dx\,d\tau \\ &\le \frac{1}{2}a(u(t),u(t)) + \frac{1}{2}\int_\Omega A\Big(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)| + \nabla u(t))d\tau\Big)^2 dx \\ &\le \frac{1}{2}\big(1+(1+\lambda)(1-l)^2\big)a(u(t),u(t)) + \frac{1}{2}\Big(1+\frac{1}{\lambda}\Big)(1-l)(g\diamond u)(t). \end{aligned} \tag{3.6}$$

Using (2.1), (2.2), (2.3), (A1), (A4) and Cauchy's inequality, we arrive at

$$\Big|\int_{\Gamma_1} uz'd\Gamma\Big| \le \frac{\alpha_1 k_1^2}{a_0}a(u(t),u(t)) + \frac{1}{4\alpha_1}\|z'\|^2_{\Gamma_1}, \tag{3.7}$$

$$\int_\Omega u\rho(u')dx \le \frac{\alpha_1 k_0^2}{a_0}a(u(t),u(t)) + \frac{1}{4\alpha_1}\int_\Omega \rho^2(u')dx, \tag{3.8}$$

$$-\int_{\Gamma_1} zu'd\Gamma \le -\frac{d}{dt}\int_{\Gamma_1} uzd\Gamma + \frac{\alpha_1 k_1^2}{a_0}a(u(t),u(t)) + \frac{1}{4\alpha_1}\|z'\|^2_{\Gamma_1}, \tag{3.9}$$

$$\int_{\Gamma_1} qzz'd\Gamma \le \alpha_2 k_2^2 q_1^2\|\nabla_\tau z\|^2_{\Gamma_1} + \frac{1}{4\alpha_2}\|z'\|^2_{\Gamma_1}. \tag{3.10}$$

Substituting (3.6)–(3.10) into (3.5) and taking

$$\lambda = \frac{l}{1-l}, \quad \alpha_1 = \frac{a_0 l}{4(2k_1^2+k_0^2)}, \quad \alpha_2 = \frac{p^2}{2k_2^2q_1^2},$$

we obtain (3.4) with $C_1 = f_1 + \frac{1}{2\alpha_1} + \frac{1}{4\alpha_2}$. This completes the proof. □

**Lemma 3.3.** *Suppose that assumptions* (A1)–(A4), (2.4) *and* (2.5) *hold, then there exist two positive constants* $C_2, C_3$ *such that the functional* $\phi(t)$ *satisfies*

$$\begin{aligned} \phi'(t) \le \Big(\mu - \int_0^t g(\tau)d\tau\Big)\|u'\|^2 + \mu(1+2(1-l)^2)a(u(t),u(t)) + \|z'\|^2_{\Gamma_1} \\ + C_2(1-l)(g\diamond u)(t) + \mu\int_\Omega \rho^2(u')dx - C_3(g'\diamond u)(t). \end{aligned} \tag{3.11}$$

*Proof.* Differentiating $\phi$ and using (1.1), we obtain

$$\begin{aligned}
\phi'(t) =& \int_\Omega A\nabla u \int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\,dx \\
&- \int_\Omega \Big(\int_0^t g(t-\tau)A\nabla u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx \\
&- \int_{\Gamma_1} z' \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau d\Gamma \\
&+ \int_\Omega \rho(u') \int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&- \int_\Omega u' \int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\,dx - \int_0^t g(\tau)d\tau \|u'\|^2 \\
:=& I_1 + I_2 + I_3 + I_4 + I_5 - \int_0^t g(\tau)d\tau\|u'\|^2.
\end{aligned} \tag{3.12}$$

Now, we estimate the terms on the right-hand side of (3.12). By (2.1), (2.2), (A2) and Cauchy's inequality, we obtain for any $\mu > 0$

$$\begin{gathered}
|I_1| \le \mu a(u(t),u(t)) + \frac{1}{4\mu}(1-l)(g\diamond u)(t), \\
|I_2| \le 2\mu(1-l)^2 a(u(t),u(t)) + \Big(2\mu + \frac{1}{4\mu}\Big)(1-l)(g\diamond u)(t), \\
|I_3| \le \|z'\|^2_{\Gamma_1} + \frac{k_1^2}{4a_0}(1-l)(g\diamond u)(t), \\
|I_4| \le \mu \int_\Omega \rho^2(u')dx + \frac{k_0^2}{4\mu a_0}(1-l)(g\diamond u)(t), \\
|I_5| \le \mu\|u'\|^2 - \frac{k_0^2 g(0)}{4\mu a_0}(g'\diamond u)(t).
\end{gathered}$$

Taking into account these estimates, (3.12) yields (3.11) with

$$C_2 = 2\mu + \frac{1}{2\mu} + \frac{k_0^2}{4\mu a_0} + \frac{k_1^2}{4a_0}, \quad C_3 = \frac{k_0^2 g(0)}{4\mu a_0}.$$

This completes the proof. □

Next we prove our main result.

*Proof of Theorem 2.4.* For a fixed positive number $t_0$, we define $g_0 := \int_0^{t_0} g(\tau)d\tau$. Since $g$ is nonincreasing and $g(0) > 0$, we have $\int_0^t g(\tau)d\tau \ge g_0, t \ge t_0$. Then combining (A4), (2.7), (3.1), (3.4) and (3.11), we deduce that

$$\begin{aligned}
L'(t) \le& -\big(\eta(g_0-\mu)-\varepsilon\big)\|u'\|^2 - \big(\frac{l\varepsilon}{4} - \eta\mu(1+2(1-l)^2)\big)a(u(t),u(t)) \\
&-(q_0 - C_1\varepsilon - \eta)\|z'\|^2_{\Gamma_1} - \frac{p^2\varepsilon}{2}\|\nabla_\tau z\|^2_{\Gamma_1} + \Big(\frac12 - C_3\eta\Big)(g'\diamond u)(t) \\
&+\Big(\frac{\varepsilon}{2l} + C_2\eta\Big)(1-l)(g\diamond u)(t) - \varepsilon\|h^{1/2}z\|^2_{\Gamma_1} - \frac12 g(t)a(u(t),u(t)) \\
&- \int_\Omega u'\rho(u')dx + \Big(\frac{\varepsilon}{4\alpha_1} + \eta\mu\Big)\int_\Omega \big(u'^2 + \rho^2(u')\big)dx.
\end{aligned} \tag{3.13}$$

At this point, we choose $\mu > 0$ such that

$$g_0 - \mu > \frac{g_0}{2}, \quad \frac{4\mu}{l}\big(1 + 2(1-l)^2\big) < \frac{g_0}{4}.$$

Then, (3.13) yields

$$\begin{aligned} L'(t) \le &-\Big(\frac{g_0\eta}{2} - \varepsilon\Big)\|u'\|^2 - \frac{l}{4}\Big(\varepsilon - \frac{g_0\eta}{4}\Big)a(u(t), u(t)) - \frac{p^2\varepsilon}{2}\|\nabla_\tau z\|^2_{\Gamma_1} \\ &-(q_0 - C_1\varepsilon - \eta)\|z'\|^2_{\Gamma_1} + \Big(\frac{\varepsilon}{2l} + C_2\eta\Big)(1-l)(g \diamond u)(t) \\ &+\Big(\frac{1}{2} - C_3\eta\Big)(g' \diamond u)(t) - \varepsilon\|h^{1/2}z\|^2_{\Gamma_1} - \frac{1}{2}g(t)a(u(t), u(t)) \\ &- \int_\Omega u'\rho(u')dx + \Big(\frac{\varepsilon}{4\alpha_1} + \eta\mu\Big)\int_\Omega \big(u'^2 + \rho^2(u')\big)dx. \end{aligned} \tag{3.14}$$

Taking $\varepsilon$ and $\eta$ small enough such that Lemma 3.1 remains valid, we pick

$$\frac{g_0\eta}{4} < \varepsilon < \frac{g_0\eta}{2}, \quad q_0 - C_1\varepsilon - \eta > 0, \quad \frac{1}{2} - C_3\eta > 0.$$

Hence, we have

$$\frac{g_0\eta}{2} - \varepsilon > 0 \quad \text{and} \quad \frac{l}{4}\Big(\varepsilon - \frac{g_0\eta}{4}\Big) > 0.$$

Whence, it follows from (A2), (2.6), (2.7) that

$$L'(t) \le -C_4E(t) + C_5(g \diamond u)(t) + C_6\int_\Omega \big(u'^2 + \rho^2(u')\big)dx \tag{3.15}$$

where $C_4$ is a positive constant and

$$C_5 := \Big(\frac{\varepsilon}{2l} + C_2\eta\Big)(1-l), \quad C_6 := \frac{\varepsilon}{4\alpha_1} + \eta\mu.$$

Multiplying (3.15) by $\xi(t)$ and applying (A2), (2.7), we have

$$\begin{aligned} \xi(t)L'(t) &\le -C_4\xi(t)E(t) + C_5\xi(t)(g \diamond u)(t) + C_6\xi(t)\int_\Omega \big(u'^2 + \rho^2(u')\big)dx \\ &\le -C_4\xi(t)E(t) - C_5(g' \diamond u)(t) + C_6\xi(t)\int_\Omega \big(u'^2 + \rho^2(u')\big)dx \\ &\le -C_4\xi(t)E(t) - 2C_5E'(t) + C_6\xi(t)\int_\Omega \big(u'^2 + \rho^2(u')\big)dx. \end{aligned} \tag{3.16}$$

Exploiting the fact that $\xi$ is a nonincreasing continuous function and defining

$$F(t) := \xi(t)L(t) + 2C_5E(t),$$

we see from Lemma 3.1 and (3.16) that $F(t) \sim E(t)$, and

$$F'(t) \le -C_4\xi(t)E(t) + C_6\xi(t)\int_\Omega \big(u'^2 + \rho^2(u')\big)dx. \tag{3.17}$$

To obtain our desired result, we shall estimate the last term on the right-hand side of (3.17). For this purpose, we adapt the arguments in [24].

**Case 1.** $H_1$ is linear on $[0, \epsilon]$. Then, by (A2), (A3) and (2.7), we deduce that there exists some positive constant $C_7$ such that

$$F'(t) \le -C_4\xi(t)E(t) + C_7\int_\Omega u'\rho(u')dx \le -C_4\xi(t)E(t) - C_7E'(t),$$

which together with (3.17) give, as $J(t) := F(t) + C_7E(t)$ and

$$J'(t) \le -C_4\xi(t)E(t).$$

Hence, using that $J(t) \sim E(t)$, we easily obtain for $t \ge t_0$,

$$E(t) \le C_8 e^{-C_4\int_0^t \xi(s)ds} := C_8H^{-1}\Big(C_4\int_0^t \xi(s)ds\Big). \tag{3.18}$$

**Case 2.** $H_1'(0) = 0$ and $H_1'' > 0$ on $(0, \epsilon]$. In this case, we choose $0 < \epsilon_1 < \epsilon$ such that

$$s\rho(s) \le \min\{\epsilon, H_1(s)\}, \quad s \le \epsilon_1,$$

Then, it is easy to show that

$$c_1|s| \le |\rho(s) \le c_2|s| \quad \text{if } |s| \ge \epsilon_1,$$
$$s^2 + \rho^2(s) \le H_1^{-1}(s\rho(s)) \quad \text{if } |s| \le \epsilon_1.$$

Next we consider a partition of $\Omega$,

$$\Omega_1 = \{x \in \Omega : |u'| \le \epsilon_1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |u'| > \epsilon_1\}.$$

To estimate the last term on the right side of (3.17), we set

$$S(t) := \frac{1}{|\Omega_1|}\int_{\Omega_1} u'\rho(u')dx.$$

By Jensen's inequality, we obtain

$$H_1^{-1}(S(t)) \ge C_9\int_{\Omega_1} H_1^{-1}(u'\rho(u'))dx.$$

From this and (2.7), we have

$$\begin{aligned}
\xi(t)\int_\Omega (u'^2 + \rho^2(u'))dx &= \xi(t)\int_{\Omega_1} (u'^2 + \rho^2(u'))dx + \xi(t)\int_{\Omega_2} (u'^2 + \rho^2(u'))dx \\
&\le \xi(t)\int_{\Omega_1} H_1^{-1}\big(u'\rho(u')\big)dx - C_{10}E'(t) \\
&\le \frac{1}{C_9}\xi(t)H_1^{-1}(S(t)) - C_{10}E'(t).
\end{aligned}$$

Therefore, (3.17) yields

$$F'(t) \le -C_4\xi(t)E(t) + C_{11}\xi(t)H_1^{-1}(S(t)) - C_6C_{10}E'(t), \tag{3.19}$$

which gives

$$R_0'(t) \le -C_4\xi(t)E(t) + C_{11}\xi(t)H_1^{-1}(S(t)), \tag{3.20}$$

where $R_0(t) := F(t) + C_6C_{10}E(t)$, and $R_0(t) \sim E(t)$ because of Lemma 3.1.

Now, for $\epsilon_0 < \epsilon$ and $c_0 > 0$, we define

$$R_1(t) := H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)R_0(t) + c_0E(t).$$

Then, it is easy to show that for $a_1$, $a_2 > 0$,

$$a_1R_1(t) \le E(t) \le a_2R_1(t).$$

Recalling that $E'(t) \le 0$, $H_1'(r) > 0$, $H_1''(r) > 0$ on $(0, \epsilon]$, and using (3.20), we obtain

$$R_1'(t) = \epsilon_0\frac{E'(t)}{E(0)}H_1''\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)R_0(t) + H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)R_0'(t) + c_0E'(t)$$

$$\le -C_4\xi(t)E(t)H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)+C_{11}\xi(t)H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)H_1^{-1}(S(t))+c_0E'(t).$$

On the other hand, thanks to the argument given in [1], we have

$$H_1^*(s)=s(H_1')^{-1}(s)-H_1((H_1')^{-1}(s)),\quad \text{if } s\in(0,H_1'(\epsilon)],$$

where $H_1^*$ is the Legendre transform of the convex function $H_1$ defined by

$$H_1^*(s):=\sup_{t\in R_+}(st-H_1(t)).$$

Then, the fact that $H_1'(0)=0$ and $H$, $(H_1')^{-1}$ are increasing functions yields

$$H_1^*(s)\le s(H_1')^{-1}(s),\quad \text{if } s\in(0,H_1'(\epsilon)]. \tag{3.21}$$

Using Young's inequality, we obtain

$$AB\le H_1^*(A)+H_1(B)\quad \text{if } A\in(0,H_1'(\epsilon)],\ B\in(0,\epsilon]. \tag{3.22}$$

Taking $A=H_1'(\epsilon_0\frac{E(t)}{E(0)})$ and $B=H_1^{-1}(S(t))$, from (2.7), (3.20), (3.21) and (3.22) it follows that

$$\begin{aligned}
R_1'(t)&\le -C_4\xi(t)E(t)H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)+C_{11}\xi(t)H_1^*\Big(H_1'\big(\epsilon_0\frac{E(t)}{E(0)}\big)\Big)\\
&\quad +C_{11}\xi(t)S(t)+c_0E'(t)\\
&\le -C_4\xi(t)E(t)H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)+C_{11}\epsilon_0\xi(t)\frac{E(t)}{E(0)}H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)\\
&\quad -C_{12}E'(t)+c_0E'(t),
\end{aligned}$$

where $C_{12}:=\frac{C_{11}\xi(0)}{|\Omega_1|}$. Choosing $\epsilon_0$ small enough such that

$$C_{13}:=C_4E(0)-C_{11}\epsilon_0>0$$

and taking $c_0>C_{12}$, we arrive at

$$R_1'(t)\le -C_{13}\xi(t)\frac{E(t)}{E(0)}H_1'\Big(\epsilon_0\frac{E(t)}{E(0)}\Big)=-C_{13}\xi(t)H_0\Big(\frac{E(t)}{E(0)}\Big), \tag{3.23}$$

where $H_0(r)=rH_1'(\epsilon_0 r)$. By the strict convexity of $H_1$ on $(0,\epsilon]$, we can see that $H_0'(t)$ and $H_0(t)>0$ on $(0,1]$. Thus, setting

$$R(t):=\frac{a_1R_1(t)}{E(0)},$$

which satisfies $R(t)\sim E(t)$, and using (3.23), we have

$$R'(t)\le -\frac{a_1C_{13}}{E(0)}\xi(t)H_0\Big(\frac{E(t)}{E(0)}\Big)=-\mu_2\xi(t)H_0(R(t)).$$

A simple integration over $(t_0,t)$ yields

$$R(t)\le H^{-1}(\mu_2\int_{t_0}^t\xi(s)ds+\mu_3),\quad t\ge t_0. \tag{3.24}$$

Combining (3.18) and (3.24), we obtain the desired result. The proof is complet. $\square$

**Acknowledgements.** The authors are very grateful to Dr. Zhaosheng Feng and the referees for their careful reading and valuable comments and suggestions, which improved this article. This work is partially supported by the NNSF of China (11871315, 61374089), and by the Natural Science Foundation of Shanxi Province of China (201801D121003, 201901D111021).

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