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EXISTENCE OF GLOBAL WEAK SOLUTIONS FOR A TWO-DIMENSIONAL KELLER-SEGEL-NAVIER-STOKES SYSTEM WITH POROUS MEDIUM DIFFUSION AND ROTATIONAL FLUX

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Abstract. This article concerns a two-dimensional Keller-Segel-Navier-Stokes system with porous medium diffusion and rotational flux describing the coral fertilization. Based on the Gagliardo-Nerenberg inequality and an energy-type argument, we show that, in the context of the nonlinear diffusions of sperm and eggs with index $m > 1$ and $l > 0$, the corresponding initial-boundary value problem possesses at least one global bounded weak solution.

1. Introduction

Broadcast spawning is a reproduction strategy observed in many benthic invertebrates, for example corals, sea urchins and sea anemones. To spawn, adult males and females synchronously release sperm and eggs into the surrounding flow. For successful and efficient fertilization to take place, concentrated parcels of sperm and egg must come into close proximity. Laboratory studies and numerical simulations demonstrate that a variety of physical, biological and chemical factors such as structured stirring by ambient flow, sperm motility and taxis, play an important role in this process [4, 18].

For the coral broadcast spawning problem, field measurements of the fertilization rates are often greater than 90% [13, 22]. However, the details of the relevant physical and biological aspects of the problem that result in high coral fertilization rates have not been well understood. Mathematical study and models become necessary to improve our understanding of the fertilization of sperm and egg gametes of benthic invertebrates.

As sperm and eggs are initially separated by the surrounding sea water, egg gametes release a chemical enzyme which attracts spermatozoids. The fertilization process is affected by the structure string of flow field, sperm motility and taxis. To better understand this phenomena, mathematical models describing this fertilization process should merge reactions, chemotaxis, diffusion, and transport of fluid velocities. To consider the effect of chemotaxis on the coral fertilization process,

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Kiselev-Ryzhik [11, 12] introduced a two-component chemotaxis-fluid model,

$$\begin{gathered} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) - n^q, \\ 0 = \Delta c + n, \end{gathered} \tag{1.1}$$

where c is the unknown concentration of the enzyme secreted by eggs, n is the unknown density of both sperm and eggs, and u is a given smooth solenoidal fluid vector field. The results in [11] show that if $q > 2$, the increasing of χ (the chemotactic sensitivity) can enhance the coral fertilization. We can find that only the cells transport through the fluid is taken into consideration in the model (1.1), while the chemicals is ignored. A more realistic chemotaxis-fluid model involving the advection motion of both cells and chemicals was considered in [1], i.e. the model with the parabolic-equation $c_t + u \cdot \nabla c = \Delta c - c + n$ instead of the second elliptic-equation in (1.1). Moreover, a chemotaxis-fluid model with unknown (Navier-)Stokes fluid velocity was studied in [5] and its generalization is given by

$$\begin{gathered} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n \nabla c) + \mu n(1-n), \\ c_t + u \cdot \nabla c = \Delta c - c + n, \\ u_t + \kappa (u \cdot \nabla) u = \Delta u - \nabla P + n \nabla \phi, \\ \nabla \cdot u = 0. \end{gathered} \tag{1.2}$$

We highlight that such chemotaxis-fluid models possess some regularity properties similar to the fluid-free counterparts, i.e. the well-known Keller-Segel type models, which have been extensively studied in the past fifty years. For instance, when $m = 1$, the associated initial-boundary value problem in bounded planar domains admits global classical solutions for any $\mu > 0$ [25], global classical solutions with appropriately small initial data for $\mu = 0$ [17]; and the three-dimensional initial-boundary problem possesses global classical solutions for $\mu > 23$ [24]. These restrictions on μ can be relaxed when the linear diffusion of the cell density is replaced by nonlinear porous medium diffusions [10, 16, 21, 26, 29].

In the process of coral fertilization, the chemoattractant is an enzyme released by eggs to attract sperm, and the densities of sperm and eggs are different. To better reflect the biological reality, very recently, the following four-component Keller-Segel-Navier-Stokes model was proposed by researchers,

$$\begin{gathered} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n S(x,n,c) \nabla c) - \mu n \rho, \quad x \in \Omega, \ t > 0, \\ \rho_t + u \cdot \nabla \rho = \Delta \rho - \mu n \rho, \quad x \in \Omega, \ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + \rho, \quad x \in \Omega, \ t > 0, \\ u_t + \kappa (u \cdot \nabla) u = \Delta u - \nabla P + (n + \rho) \nabla \phi, \quad x \in \Omega, \ t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, \ t > 0. \end{gathered} \tag{1.3}$$

This model is a further refinement of the model (1.2) in that it distinguishes sperm from eggs, and only sperm cells show chemotaxis to the enzyme released by eggs. It was shown that the associated initial-boundary value problem with scalar chemotactic sensitivity (i.e. $S(x,n,c) = \mathbb{I}$), in the two-dimensional setting, possesses a unique global classical solution which tends towards a spatially homogeneous equilibrium in the large time limit [6]. Furthermore, the global bounded solutions to the three-dimensional Keller-Segel-Stokes counterpart with tensor-valued chemotactic sensitivity $S(x,n,c)$ satisfying $|S| \le \chi(1+n)^{-\alpha}$ were further established under the condition that $\alpha \ge 1/3$ in [14], and $\alpha > 0$ in [15].

In this article, we consider the following four-component Keller-Segel-Navier-Stokes system with porous medium diffusion and tensor-valued chemotactic sensitivity,

$$
\begin{gathered}
n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(nS(x,n,c)\nabla c) - \mu n\rho, \quad x\in\Omega,\ t>0,\\
\rho_t + u\cdot\nabla\rho = \Delta\rho^l - \mu n\rho, \quad x\in\Omega,\ t>0,\\
c_t + u\cdot\nabla c = \Delta c - c + \rho, \quad x\in\Omega,\ t>0,\\
u_t + (u\cdot\nabla)u = \Delta u - \nabla P + (n+\rho)\nabla\phi, \quad x\in\Omega,\ t>0,\\
\nabla\cdot u = 0, \quad x\in\Omega,\ t>0,\\
(\nabla n^m - nS(x,n,c)\nabla c)\cdot\nu = \nabla c\cdot\nu = \nabla\rho\cdot\nu = 0, \quad u=0, \quad x\in\partial\Omega,\ t>0,\\
n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad \rho(x,0)=\rho_0(x), \quad u(x,0)=u_0(x), \quad x\in\Omega,
\end{gathered}
\tag{1.4}
$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary, n and ρ denote the population densities of the unfertilized sperm and eggs, c is the concentration of the chemical signal which is an enzyme released by eggs, u is the divergence free sea fluid velocity, P and ϕ denote the associated pressure of the fluid and the gravitational potential, S represents the tensor-valued chemotactic sensitivity, and the parameters $\mu>0$, $m>1$, and $l>0$.

Our goal is to study the existence and boundedness of global solutions to problem (1.4). To formulate our main result, we assume that the tensor-valued chemotactic sensitivity $S(x,n,c)$ satisfies

$$
S\in C^2((\overline{\Omega}\times[0,\infty)^2);\mathbb{R}^{2\times 2}) \tag{1.5}
$$

and for some $\chi>0$,

$$
|S(x,n,c)|\le\chi \quad \text{for } (x,n,c)\in\Omega\times[0,\infty)^2. \tag{1.6}
$$

Moreover, we assume that the gravitational potential ϕ satisfies

$$
\phi\in W^{2,\infty}(\Omega) \tag{1.7}
$$

and the initial data (n_0,c_0,v_0,u_0) fulfills

$$
\begin{gathered}
n_0\in C^0(\overline{\Omega}) \quad \text{with } n_0\ge 0,\ n_0\not\equiv 0 \text{ in } \Omega,\\
c_0\in W^{2,\infty}(\Omega) \quad \text{with } c_0\ge 0 \text{ in } \overline{\Omega},\\
\rho_0\in C_0(\overline{\Omega}) \quad \text{with } \rho_0\ge 0 \text{ in } \Omega,\\
u_0\in D(A^\beta) \quad \text{with some } \beta\in(1/2,1),
\end{gathered}
\tag{1.8}
$$

where $A:=-\mathcal{P}\Delta$ denotes the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, defined on its domain $D(A):=W^{2,2}(\Omega;\mathbb{R}^2)\cap W_0^{1,2}(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega;\mathbb{R}^2)$ with $L^2_\sigma(\Omega;\mathbb{R}^2):=\{\varphi\in L^2(\Omega;\mathbb{R}^2)|\nabla\cdot\varphi=0\}$, and with $\mathcal{P}$ representing the Helmholtz projection of $L^2(\Omega;\mathbb{R}^2)$ onto $L^2_\sigma(\Omega;\mathbb{R}^2)$ (see [6]).

Theorem 1.1. *Let $\mu>0$, $m>1$, $l>0$, $\Omega\subset\mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\Omega$, and the assumptions* (1.5)-(1.7) *hold. Then for every initial data (n_0,c_0,ρ_0,u_0) satisfying* (1.8)*, problem* (1.4) *possesses a global weak solution (n,c,v,u,P) in the sense of Definition 4.1, which is uniformly bounded in the sense that there exists a positive constant C such that for $t>0$,*

$$
\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\rho(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C. \tag{1.9}
$$

This article is organized as follows. In Section 2, we present some preliminary results. In Section 3, we prove the global existence of classical solutions to the regularized problem. In Section 4, we prove Theorem 1.1.

2. Preliminaries

We first consider a regularized version of problem (1.4),

$$\begin{gathered}
n_{\epsilon t}+u_\epsilon\cdot\nabla n_\epsilon=\Delta(n_\epsilon+\epsilon)^m-\nabla\cdot(n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon)-\mu n_\epsilon\rho_\epsilon,\quad x\in\Omega,\ t>0,\\
\rho_{\epsilon t}+u_\epsilon\cdot\nabla\rho_\epsilon=\Delta(\rho_\epsilon+\epsilon)^l-\mu n_\epsilon\rho_\epsilon,\quad x\in\Omega,\ t>0,\\
c_{\epsilon t}+u_\epsilon\cdot\nabla c_\epsilon=\Delta c_\epsilon-c_\epsilon+\rho_\epsilon,\quad x\in\Omega,\ t>0,\\
u_{\epsilon t}+u_\epsilon\cdot\nabla u_\epsilon=\Delta u_\epsilon-\nabla P_\epsilon+(n_\epsilon+\rho_\epsilon)\nabla\phi,\quad x\in\Omega,\ t>0,\\
\nabla\cdot u_\epsilon=0,\quad x\in\Omega,\ t>0,\\
\nabla n_\epsilon\cdot\nu=\nabla c_\epsilon\cdot\nu=\nabla\rho_\epsilon\cdot\nu=0,\quad u_\epsilon=0,\quad x\in\partial\Omega,\ t>0,\\
n_\epsilon(x,0)=n_0(x),\quad c_\epsilon(x,0)=c_0(x),\quad x\in\Omega,\\
\rho_\epsilon(x,0)=\rho_0(x),\quad u_\epsilon(x,0)=u_0(x),\quad x\in\Omega,
\end{gathered}\tag{2.1}$$

for $\epsilon\in(0,1)$ and

$$S_\epsilon(x,n,c)=\varrho_\epsilon(x)S(x,n,c),\quad x\in\Omega,\ n\ge 0,\ c\ge 0.\tag{2.2}$$

Here, $(\varrho_\epsilon)_{\epsilon\in(0,1)}\in C_0^\infty(\Omega)$ is a family of standard cut-off functions satisfying

$$\varrho_\epsilon\in(0,1)\quad\text{in }\Omega,\quad \varrho_\epsilon\nearrow 1\quad\text{as }\epsilon\searrow 0.\tag{2.3}$$

In view of (1.6), we then derive that

$$|S_\epsilon(x,n,c)|\le\chi,\quad x\in\Omega,n\ge 0,\ c\ge 0.\tag{2.4}$$

Next, we state the existence and extensibility criterion for local solutions to this regularized problem. The proof can be obtained with minor modifications of the proof of [28, Lemma 2.1], so we omit it here.

Lemma 2.1. *Let $m>1$, $l>0$, $\mu>0$, $\Omega\subset\mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\Omega$, and suppose the conditions* (1.5)-(1.8) *hold. Then, for any $\epsilon\in(0,1)$, there exist $T_{\max,\epsilon}\in(0,\infty]$ and a classical solution $(n_\epsilon,c_\epsilon,\rho_\epsilon,u_\epsilon,P_\epsilon)$ to problem* (2.1) *in $\Omega\times[0,T_{\max,\epsilon})$. Here, $T_{\max,\epsilon}$ denotes the maximal existence time, and*

$$\begin{gathered}
n_\epsilon\in C^0(\overline{\Omega}\times[0,T_{\max,\epsilon}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\epsilon})),\\
c_\epsilon\in C^0(\overline{\Omega}\times[0,T_{\max,\epsilon}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\epsilon}))\cap\cap_{p>1}L^\infty([0,T_{\max,\epsilon});W^{1,p}(\Omega)),\\
\rho_\epsilon\in C^0(\overline{\Omega}\times[0,T_{\max,\epsilon}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\epsilon})),\\
u_\epsilon\in C^0(\overline{\Omega}\times[0,T_{\max,\epsilon}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\epsilon})),\\
P_\epsilon\in C^{1,0}(\overline{\Omega}\times(0,T_{\max,\epsilon}))
\end{gathered}$$

for any $p>2$. Furthermore, n_ϵ,c_ϵ and ρ_ϵ are nonnegative in $\Omega\times(0,T_{\max,\epsilon})$, and if $T_{\max,\epsilon}<\infty$, then

$$\begin{aligned}
&\limsup_{t\nearrow T_{\max,\epsilon}}\Big\{\|n_\epsilon(\cdot,t)\|_{L^\infty(\Omega)}+\|c_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|\rho_\epsilon(\cdot,t)\|_{L^\infty(\Omega)}\\
&+\|A^\beta u_\epsilon(\cdot,t)\|_{L^2(\Omega)}\Big\}=\infty
\end{aligned}\tag{2.5}$$

for $\beta\in(1/2,1)$ given in (1.8).

The well-known Gagliardo-Nirenberg interpolation inequality will be frequently used throughout this paper. For readers' convenience, we state its version in two-dimensional case [7, 20].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, let j, k be any integers satisfying $0 \leq j < k$, and let $1 \leq q, r \leq \infty$, $p \in \mathbb{R}^+$, $\alpha \in [\frac{j}{k}, 1]$ such that*
$$\frac{1}{p} = \frac{j}{2} + (\frac{1}{q} - \frac{k}{2})\alpha + \frac{1-\alpha}{r}.$$
Then for any function $u(x) \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exists two positive constants c_1, c_2 depending only on Ω, j, k, q, r, n such that
$$\|D^j u\|_{L^p} \leq c_1 \|D^k u\|_{L^q}^{\alpha} \|u\|_{L^r}^{1-\alpha} + c_2 \|u\|_{L^r}.$$

Lemma 2.3 ([23])**.** *Let $T > 0$, $\tau \in (0, T)$, $a > 0$, $b > 0$. Assume that $h(t)$ is a nonnegative function belonging to $L^1_{\rm loc}[0, T)$, and $y : [0, T) \to [0, \infty)$ is absolutely continuous function satisfying*
$$\begin{gathered} y'(t) + ay(t) \leq h(t) \quad \text{for a.e. } t \in [0, T), \\ \int_t^{t+\tau} h(s)ds \leq b \quad \text{for } t \in [0, T - \tau). \end{gathered}$$
Then
$$y(t) \leq \max\{y(0) + b, \frac{b}{a\tau} + 2b\} \quad \text{for } t \in [0, T).$$

3. Existence of global solution for the regularized problem

In this section, we concentrate on proving the existence of classical global solutions to the regularized problem (2.1). Let us first state some basic estimates on $n_\epsilon, \rho_\epsilon$ and c_ϵ.

Lemma 3.1. *The solution of problem (2.1) satisfies*
$$\int_\Omega n_\epsilon \leq \int_\Omega n_0 \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.1}$$
$$\|\rho_\epsilon\|_{L^\infty(\Omega)} \leq \|\rho_0\|_{L^\infty(\Omega)} \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.2}$$
moreover, we have
$$\|c_\epsilon\|_{L^\infty(\Omega)} \leq \max\{\|c_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^\infty(\Omega)}\} \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.3}$$
$$\int_t^{t+\tau}\int_\Omega |\nabla c_\epsilon|^2 \leq \frac{1}{2}(\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega| + \max\{\|c_0\|^2_{L^2(\Omega)}, \|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|\}) \tag{3.4}$$
for $t \in (0, T_{\max,\epsilon} - \tau)$, where $\tau = \min\{1, \frac{1}{2}T_{\max,\epsilon}\}$.

Proof. Integrating the first equation in problem (2.1) over Ω and using integration by parts and the divergence free of the fluid ($\nabla \cdot u_\epsilon = 0$), we obtain
$$\frac{d}{dt}\int_\Omega n_\epsilon = -\mu \int_\Omega n_\epsilon \rho_\epsilon \leq 0 \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.5}$$
which implies (3.1). Multiplying the second equation in problem (2.1) by ρ_ϵ^{p-1} ($p \geq 2$) and integrating by parts then yields
$$\frac{1}{p}\frac{d}{dt}\int_\Omega \rho_\epsilon^p + l(p-1)\int_\Omega (\rho_\epsilon + \epsilon)^{l-1}\rho_\epsilon^{p-2}|\nabla \rho_\epsilon|^2 = -\mu \int_\Omega n_\epsilon \rho_\epsilon^p \leq 0 \tag{3.6}$$

for $t \in (0, T_{\max,\epsilon})$, which implies

$$\|\rho_\epsilon\|_{L^p(\Omega)} \le \|\rho_0\|_{L^p(\Omega)} \quad \text{for } t \in (0, T_{\max,\epsilon}).$$

Letting $p \to \infty$, we have

$$\|\rho_\epsilon\|_{L^\infty(\Omega)} \le \|\rho_0\|_{L^\infty(\Omega)} \quad \text{for } t \in (0, T_{\max,\epsilon}),$$

i.e., (3.2) is valid. Similarly, we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega c_\epsilon^p + (p-1)\int_\Omega c_\epsilon^{p-2}|\nabla c_\epsilon|^2 + \int_\Omega c_\epsilon^p &\le \frac{p-1}{p}\int_\Omega c_\epsilon^p + \frac{1}{p}\int_\Omega \rho_\epsilon^p \\ &\le \frac{p-1}{p}\int_\Omega c_\epsilon^p + \frac{1}{p}\|\rho_\epsilon\|_{L^\infty}^p|\Omega| \end{aligned} \tag{3.7}$$

for $t \in (0, T_{\max,\epsilon})$. This, along with (3.2), yields

$$\frac{d}{dt}\int_\Omega c_\epsilon^p + p(p-1)\int_\Omega c_\epsilon^{p-2}|\nabla c_\epsilon|^2 + \int_\Omega c_\epsilon^p \le \|\rho_0\|_{L^\infty}^p|\Omega| \quad \text{for } t \in (0, T_{\max,\epsilon}). \tag{3.8}$$

Using the Gronwall inequality, we have

$$\|c_\epsilon\|_{L^p(\Omega)}^p \le e^{-t}\|c_0\|_{L^p(\Omega)}^p + \|\rho_0\|_{L^\infty}^p|\Omega|(1-e^{-t}) \le \max\{\|c_0\|_{L^p(\Omega)}^p, \|\rho_0\|_{L^\infty(\Omega)}^p|\Omega|\}$$

for $t \in (0, T_{\max,\epsilon})$, and then

$$\|c_\epsilon\|_{L^p(\Omega)} \le \max\{\|c_0\|_{L^p(\Omega)}, \|\rho_0\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{p}}\} \quad \text{for } t \in (0, T_{\max,\epsilon}). \tag{3.9}$$

Letting $p \to \infty$, we obtain

$$\|c_\epsilon\|_{L^\infty(\Omega)} \le \max\{\|c_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^\infty(\Omega)}\} \quad \text{for } t \in (0, T_{\max,\epsilon}),$$

thus, (3.3) is valid. Letting $p = 2$ in (3.8) and integrating over $(t, t+\tau)$ with $\tau = \min\{1, \frac{1}{2}T_{\max,\epsilon}\}$, we further derive that

$$\begin{aligned} \int_t^{t+\tau}\int_\Omega |\nabla c_\epsilon|^2 &\le \frac{1}{2}(\|\rho_0\|_{L^\infty(\Omega)}^2|\Omega|\tau + \|c_\epsilon(t)\|_{L^2(\Omega)}^2) \\ &\le \frac{1}{2}(\|\rho_0\|_{L^\infty(\Omega)}^2|\Omega| + \max\{\|c_0\|_{L^2(\Omega)}^2, \|\rho_0\|_{L^\infty(\Omega)}^2|\Omega|\}) \end{aligned}$$

for $t \in (0, T_{\max,\epsilon} - \tau)$, where we used (3.9) and $\tau \le 1$. Hence, (3.4) is valid. □

Based on the above elementary estimates, we can use the standard testing procedure to establish the following estimates on n_ϵ, which shall be used in Lemma 3.3 to derive the uniform bound on $\int_\Omega |u_\epsilon|^2$ and $\int_t^{t+\tau}\int_\Omega |\nabla u_\epsilon|^2$.

Lemma 3.2. *Let $m > 1$. Then there exists a constant $C > 0$ independent of ϵ such that the solution of problem (2.1) satisfies*

$$\int_\Omega (n_\epsilon + \epsilon)^{m-1} \le C \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.10}$$

$$\int_t^{t+\tau}\int_\Omega (n_\epsilon + \epsilon)^{2m-4}|\nabla n_\epsilon|^2 \le C \quad \text{for } t \in (0, T_{\max,\epsilon} - \tau) \tag{3.11}$$

with $\tau = \min\{1, \frac{1}{2}T_{\max,\epsilon}\}$.

Proof. Testing the first equation of (2.1) by $(n_\epsilon+\epsilon)^{m-2}$, integrating by parts over Ω and using the fact that $\nabla\cdot u_\epsilon=0$, we have

$$
\begin{aligned}
&\frac{1}{m-1}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^{m-1}+m(m-2)\int_\Omega (n_\epsilon+\epsilon)^{2m-4}|\nabla n_\epsilon|^2\\
&=-\int_\Omega (n_\epsilon+\epsilon)^{m-2}\nabla\cdot(n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon)-\mu\int_\Omega (n_\epsilon+\epsilon)^{m-2}n_\epsilon\rho_\epsilon\\
&=(m-2)\int_\Omega (n_\epsilon+\epsilon)^{m-3}n_\epsilon\nabla n_\epsilon\cdot S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon-\mu\int_\Omega (n_\epsilon+\epsilon)^{m-2}n_\epsilon\rho_\epsilon
\end{aligned}
\tag{3.12}
$$

for $t\in(0,T_{\max,\epsilon})$. We next divide the proof into three cases.

Case 1: $m>2$**.** Using (2.4) and Young's inequality, from (3.12) we derive

$$
\begin{aligned}
&\frac{1}{m-1}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^{m-1}+m(m-2)\int_\Omega (n_\epsilon+\epsilon)^{2m-4}|\nabla n_\epsilon|^2\\
&\le \chi(m-2)\int_\Omega (n_\epsilon+\epsilon)^{m-2}|\nabla n_\epsilon||\nabla c_\epsilon|-\mu\int_\Omega (n_\epsilon+\epsilon)^{m-2}n_\epsilon\rho_\epsilon\\
&\le \frac{m(m-2)}{4}\int_\Omega (n_\epsilon+\epsilon)^{2m-4}|\nabla n_\epsilon|^2+\frac{\chi^2(m-2)}{m}\int_\Omega |\nabla c_\epsilon|^2
\end{aligned}
\tag{3.13}
$$

for $t\in(0,T_{\max,\epsilon})$, which entails

$$
\begin{aligned}
&\frac{1}{m-1}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^{m-1}+\frac{3m(m-2)}{4}\int_\Omega (n_\epsilon+\epsilon)^{2m-4}|\nabla n_\epsilon|^2\\
&\le \frac{\chi^2(m-2)}{m}\int_\Omega |\nabla c_\epsilon|^2\quad \text{for } t\in(0,T_{\max,\epsilon}).
\end{aligned}
\tag{3.14}
$$

By the Hölder inequality and the Gagliardo-Nirenberg interpolation inequality, we can find two positive constants c_1 and c_2 independent of ϵ such that

$$
\begin{aligned}
&\int_\Omega (n_\epsilon+\epsilon)^{m-1}\\
&=\|(n_\epsilon+\epsilon)^{m-1}\|_{L^1(\Omega)}\\
&\le c_1\|(n_\epsilon+\epsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{1}{m-1}}\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{m-2}{m-1}}+c_1\|(n_\epsilon+\epsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}\\
&=c_1\|n_\epsilon+\epsilon\|_{L^1(\Omega)}\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{m-2}{m-1}}+c_1\|n_\epsilon+\epsilon\|_{L^1(\Omega)}^{m-1}\\
&\le c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{m-2}{m-1}}+c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)^{m-1}\\
&\le c_2\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{m-2}{m-1}}+c_2
\end{aligned}
$$

where we use $\epsilon\in(0,1)$ and (3.1). Since $\frac{m-2}{m-1}\in(0,1)$, by Young's inequality we have

$$
\begin{aligned}
\int_\Omega (n_\epsilon+\epsilon)^{m-1}&\le c_2\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{m-2}{m-1}}+c_2\\
&\le \frac{m(m-2)}{4(m-1)^2}\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^2+c_3\\
&=\frac{m(m-2)}{4}\int_\Omega (n_\epsilon+\epsilon)^{2(m-2)}|\nabla n_\epsilon|^2+c_3\quad \text{for } t\in(0,T_{\max,\epsilon}).
\end{aligned}
$$

This and (3.14) yield

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^{m-1} + \frac{m(m-1)(m-2)}{2}\int_\Omega |\nabla (n_\epsilon+\epsilon)^{m-1}|^2\\
&+(m-1)\int_\Omega (n_\epsilon+\epsilon)^{m-1}\\
&\le \frac{\chi^2(m-1)(m-2)}{m}\int_\Omega |\nabla c_\epsilon|^2 + c_4 \quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned}\tag{3.15}$$

where $c_4=c_3(m-1)$ is independent of ϵ. Let $y(t):=\int_\Omega (n_\epsilon(\cdot,t)+\epsilon)^{m-1}$, $h(t):=\frac{\chi^2(m-1)(m-2)}{m}\int_\Omega|\nabla c_\epsilon|^2+c_4$, then we have

$$y'(t)+(m-1)y(t)\le h(t)\quad \text{for } t\in(0,T_{\max,\epsilon}).$$

Note that

$$\begin{aligned}
\int_t^{t+\tau} h(s)ds \le{}& \frac{\chi^2(m-1)(m-2)}{2m}\big(\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|\\
&+\max\{\|c_0\|^2_{L^2(\Omega)},\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|\}\big)+c_4\tau := b,
\end{aligned}$$

for $t\in(0,T_{\max,\epsilon}-\tau)$ by (3.4). An application of Lemma 2.3 gives

$$\int_\Omega (n_\epsilon+\epsilon)^{m-1}\le \max\{\int_\Omega (n_0+1)^{m-1}+b,\ \frac{b}{(m-1)\tau}+2b\}\quad \text{for } t\in(0,T_{\max,\epsilon}).$$

Integrating (3.15) over $(t,\,t+\tau)$ yields

$$\begin{aligned}
&\frac{m(m-1)(m-2)}{2}\int_t^{t+\tau}\int_\Omega (n_\epsilon+\epsilon)^{2m-4}|\nabla n_\epsilon|^2\\
&\le b+\max\{\int_\Omega (n_0+1)^{m-1}+b,\ \frac{b}{(m-1)\tau}+2b\}\quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned}$$

therefore, (3.10) and (3.11) hold.

Case 2 : $m=2$. We only need to show that there exists a positive constant C independent of ϵ such that

$$\int_t^{t+\tau}\int_\Omega |\nabla n|^2\le C\quad \text{for } t\in(0,T_{\max,\epsilon}).$$

To this end, we test the first equation in problem (2.1) by $\ln(n_\epsilon+\epsilon)+1$ and use (2.4) and that $\nabla\cdot u_\epsilon=0$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)\ln(n_\epsilon+\epsilon)\\
&=\int_\Omega [\ln(n_\epsilon+\epsilon)+1]\Delta(n_\epsilon+\epsilon)^2-\int_\Omega[\ln(n_\epsilon+\epsilon)+1]\nabla\cdot(n_\epsilon S_\epsilon\nabla c_\epsilon)\\
&\quad-\mu\int_\Omega n_\epsilon[\ln(n_\epsilon+\epsilon)+1]\rho_\epsilon\\
&\le -2\int_\Omega|\nabla n_\epsilon|^2+\chi\int_\Omega\frac{n_\epsilon}{n_\epsilon+\epsilon}|\nabla n_\epsilon||\nabla c_\epsilon|-\mu\int_\Omega n_\epsilon[\ln(n_\epsilon+\epsilon)+1]\rho_\epsilon
\end{aligned}\tag{3.16}$$

for $t\in(0,T_{\max,\epsilon})$. Since $\eta(\ln\eta+1)\ge -e^{-2}$ for any $\eta>0$, using (3.2), we have

$$-\mu\int_\Omega n_\epsilon[\ln(n_\epsilon+\epsilon)+1]\rho_\epsilon\le -\mu\int_\Omega n_\epsilon(\ln n_\epsilon+1)\rho_\epsilon$$

$$
\begin{aligned}
&\le \frac{\mu}{e^2}\int_\Omega \rho_\epsilon \\
&\le \frac{\mu}{e^2}\|\rho_0\|_{L^\infty(\Omega)}|\Omega| \quad \text{for } t\in(0,T_{\max,\epsilon}).
\end{aligned}
$$

Moreover, applying Young's inequality, we have

$$
\begin{aligned}
\chi\int_\Omega \frac{n_\epsilon}{n_\epsilon+\epsilon}|\nabla n_\epsilon||\nabla c_\epsilon| &\le \chi\int_\Omega |\nabla n_\epsilon||\nabla c_\epsilon| \\
&\le \int_\Omega |\nabla n_\epsilon|^2+\frac{\chi^2}{4}\int_\Omega|\nabla c_\epsilon|^2 \quad \text{for } t\in(0,T_{\max,\epsilon}).
\end{aligned}
$$

This and (3.16) yield

$$
\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)\ln(n_\epsilon+\epsilon)+\int_\Omega|\nabla n_\epsilon|^2 \le \frac{\chi^2}{4}\int_\Omega|\nabla c_\epsilon|^2+\frac{\mu}{e^2}\|\rho_0\|_{L^\infty(\Omega)}|\Omega| \tag{3.17}
$$

for $t\in(0,T_{\max,\epsilon})$.

Noting that $x\ln x < x^2$ for $x>0$, using the Gagliardo-Nirenberg inequality, (3.1), and $\epsilon\in(0,1)$, we have

$$
\begin{aligned}
&\int_\Omega (n_\epsilon+\epsilon)\ln(n_\epsilon+\epsilon) \\
&\le \int_\Omega (n_\epsilon+\epsilon)^2 \\
&\le c_5\|n_\epsilon+\epsilon\|_{L^1(\Omega)}\|\nabla n_\epsilon\|_{L^2(\Omega)}+c_5\|n_\epsilon+\epsilon\|^2_{L^1(\Omega)} \\
&\le c_5(\|n_\epsilon\|_{L^1(\Omega)}+\epsilon|\Omega|)\|\nabla n_\epsilon\|_{L^2(\Omega)}+c_5(\|n_\epsilon\|_{L^1(\Omega)}+\epsilon|\Omega|)^2 \\
&\le c_5(\|n_0\|_{L^1(\Omega)}+|\Omega|)\|\nabla n_\epsilon\|_{L^2(\Omega)}+c_5(\|n_0\|_{L^1(\Omega)}+|\Omega|)^2 \\
&\le \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)}+c_6 \quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned} \tag{3.18}
$$

where c_5 and c_6 are positive constants independent of ϵ. Combining (3.17) and (3.18), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega[(n_\epsilon+\epsilon)\ln(n_\epsilon+\epsilon)+e^{-1}]+\int_\Omega[(n_\epsilon+\epsilon)\ln(n_\epsilon+\epsilon)+e^{-1}]+\frac{1}{2}\int_\Omega|\nabla n_\epsilon|^2 \\
&< \frac{\chi^2}{4}\int_\Omega|\nabla c_\epsilon|^2+c_7 \quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned} \tag{3.19}
$$

where $c_7=e^{-1}|\Omega|+\frac{\mu}{e^2}\|\rho_0\|_{L^\infty(\Omega)}|\Omega|+c_6$. Setting $y(t):=\int_\Omega[(n_\epsilon+\epsilon)\ln(n_\epsilon+\epsilon)+e^{-1}]$, $h(t)=\frac{\chi^2}{4}\int_\Omega|\nabla c_\epsilon|^2+c_7$, then (3.19) can be rewritten as

$$
y'(t)+y(t)+\frac{1}{2}\int_\Omega|\nabla n_\epsilon|^2\le h(t) \quad \text{for } t\in(0,T_{\max,\epsilon}). \tag{3.20}
$$

Since $x\ln x\ge -e^{-1}$ for $x>0$, we have $y(t)\ge 0$; moveover, in view of (3.4), we have

$$
\int_t^{t+\tau} h(s)ds \le \frac{\chi^2}{8}(\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|+\max\{\|c_0\|^2_{L^2(\Omega)},\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|\})+c_7\tau
$$

for $t \in (0, T_{\max,\epsilon} - \tau)$. Then, it follows from (3.20) and Lemma 2.3 that there exists a constant $c_8 > 0$ such that

$$\int_\Omega (n_\epsilon + \epsilon)\ln(n_\epsilon + \epsilon) \le c_8 \quad \text{for } t \in (0, T_{\max,\epsilon}).$$

Integrating (3.20) over the time interval $(t, t+\tau)$, we further obtain $c_9 > 0$ such that

$$\int_t^{t+\tau} \int_\Omega |\nabla n_\epsilon|^2 \le c_9 \quad \text{for } t \in (0, T_{\max,\epsilon} - \tau)$$

with $\tau = \min\{1, \frac{1}{2}T_{\max,\epsilon}\}$. This proves the desired assertion.

Case 3: $1 < m < 2$**.** Noting that $m-1 \in (0,1)$ and $\epsilon \in (0,1)$, using the Hölder inequality and (3.1), we have

$$\int_\Omega (n_\epsilon + \epsilon)^{m-1} \le \Big(\int_\Omega (n_\epsilon + \epsilon) \Big)^{m-1} |\Omega|^{2-m} \le \Big(\int_\Omega (n_0 + 1) \Big)^{m-1} |\Omega|^{2-m} \tag{3.21}$$

for $t \in (0, T_{\max,\epsilon})$, which implies (3.10). Then, we only need to show (3.11). To this end, we first multiply both sides of (3.12) by (-1), and then apply the Hölder inequality and the Young inequality to deduce that

$$\begin{aligned}
&\frac{1}{1-m}\frac{d}{dt}\int_\Omega (n_\epsilon + \epsilon)^{m-1} + m(2-m)\int_\Omega (n_\epsilon + \epsilon)^{2m-4}|\nabla n_\epsilon|^2 \\
&= (2-m)\int_\Omega (n_\epsilon + \epsilon)^{m-3} n_\epsilon \nabla n_\epsilon \cdot S_\epsilon(x, n_\epsilon, c_\epsilon)\nabla c_\epsilon + \mu \int_\Omega (n_\epsilon + \epsilon)^{m-2} n_\epsilon \rho_\epsilon \\
&\le \chi(2-m)\int_\Omega (n_\epsilon + \epsilon)^{m-2}|\nabla n_\epsilon||\nabla c_\epsilon| + \mu\|\rho_\epsilon\|_{L^\infty(\Omega)}\int_\Omega (n_\epsilon + \epsilon)^{m-1} \\
&\le \frac{m(2-m)}{4}\int_\Omega (n_\epsilon + \epsilon)^{2m-4}|\nabla n_\epsilon|^2 + \frac{\chi^2(2-m)}{m}\int_\Omega |\nabla c_\epsilon|^2 \\
&\quad + \mu\|\rho_\epsilon\|_{L^\infty(\Omega)}\int_\Omega (n_\epsilon + \epsilon)^{m-1} \quad \text{for } t \in (0, T_{\max,\epsilon}).
\end{aligned}$$

This, along with (3.21) and (3.2), lead to

$$\begin{aligned}
&\frac{1}{1-m}\frac{d}{dt}\int_\Omega (n_\epsilon + \epsilon)^{m-1} + \frac{3m(2-m)}{4}\int_\Omega (n_\epsilon + \epsilon)^{2m-4}|\nabla n_\epsilon|^2 \\
&\le \frac{\chi^2(2-m)}{m}\int_\Omega |\nabla c_\epsilon|^2 + c_{10} \quad \text{for } t \in (0, T_{\max,\epsilon}),
\end{aligned}$$

where

$$c_{10} := \mu\|\rho_0\|_{L^\infty(\Omega)}\Big(\int_\Omega (n_0 + 1)\Big)^{m-1}|\Omega|^{2-m}.$$

Integrating over the time interval $(t,\, t+\tau)$, then entails

$$\begin{aligned}
&\frac{3m(2-m)}{4}\int_t^{t+\tau}\int_\Omega (n_\epsilon + \epsilon)^{2m-4}|\nabla n_\epsilon|^2 \\
&\le \frac{\chi^2(2-m)}{m}\int_t^{t+\tau}\int_\Omega |\nabla c_\epsilon|^2 + c_{10}\tau + \frac{1}{m-1}\int_\Omega (n_\epsilon(t+\tau) + \epsilon)^{m-1}
\end{aligned}$$

for $t \in (0, T_{\max,\epsilon} - \tau)$. Following this, (3.21) and (3.4), we obtain (3.11). $\square$

Lemma 3.3. *Let $m > 1$. Then there exists a constant $C > 0$ independent of ϵ such that the solution of problem (2.1) satisfies*

$$\int_\Omega |u_\epsilon|^2 \le C \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.22}$$

$$\int_t^{t+\tau} \int_\Omega |\nabla u_\epsilon|^2 \le C \quad \text{for } t \in (0, T_{\max,\epsilon} - \tau), \tag{3.23}$$

where $\tau = \min\{1, \frac{1}{2}T_{\max,\epsilon}\}$.

Proof. Note first that $\nabla \cdot u_\epsilon = 0$ and $u_\epsilon|_{\partial\Omega} = 0$, then

$$\int_\Omega (u_\epsilon \cdot \nabla) u_\epsilon \cdot u_\epsilon = \frac{1}{2}\int_\Omega u_\epsilon \cdot \nabla |u_\epsilon|^2 = 0.$$

Testing the fourth equation in the problem (2.1) with u_ϵ shows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\epsilon|^2 + \int_\Omega |\nabla u_\epsilon|^2 = \int_\Omega n_\epsilon u_\epsilon \cdot \nabla\phi + \int_\Omega \rho_\epsilon u_\epsilon \cdot \nabla\phi \quad \text{for } t \in (0, T_{\max,\epsilon}). \tag{3.24}$$

Using the boundedness of ϕ assumed in (1.7) and the Hölder inequality, we have

$$\int_\Omega n_\epsilon u_\epsilon \cdot \nabla\phi \le \|\nabla\phi\|_{L^\infty(\Omega)} \|n_\epsilon\|_{L^q(\Omega)} \|u_\epsilon\|_{L^{q'}(\Omega)} \tag{3.25}$$

with $q' = \frac{q}{q-1}$ and $q > 1$ to be determined later. Since the space dimension is two, by means of the continuous embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{q'}(\Omega)$ and the Poincaré inequality in $W_0^{1,2}(\Omega)$, one can find $C_1 > 0$, $C_2 > 0$ such that

$$\|u_\epsilon\|_{L^{q'}(\Omega)} \le C_1 \|u_\epsilon\|_{W_0^{1,2}(\Omega)} \le C_2 \|\nabla u_\epsilon\|_{L^2(\Omega)}. \tag{3.26}$$

Thus, by Young's inequality and (3.25), we have

$$\int_\Omega n_\epsilon u_\epsilon \cdot \nabla\phi \le \frac{1}{4}\|\nabla u_\epsilon\|^2_{L^2(\Omega)} + C_3 \|n_\epsilon\|^2_{L^q(\Omega)} \quad \text{for } t \in (0, T_{\max,\epsilon}) \tag{3.27}$$

with $C_3 = C_2^2 \|\nabla\phi\|^2_{L^\infty(\Omega)}$. Similarly, we have

$$\int_\Omega \rho_\epsilon u_\epsilon \cdot \nabla\phi \le \frac{1}{4}\|\nabla u_\epsilon\|^2_{L^2(\Omega)} + C_3 \|\rho_\epsilon\|^2_{L^q(\Omega)} \quad \text{for } t \in (0, T_{\max,\epsilon}). \tag{3.28}$$

Substituting (3.27)-(3.28) into (3.24) yields

$$\frac{d}{dt}\int_\Omega |u_\epsilon|^2 + \int_\Omega |\nabla u_\epsilon|^2 \le 2C_3 \|n_\epsilon\|^2_{L^q(\Omega)} + 2C_3 \|\rho_\epsilon\|^2_{L^q(\Omega)} \quad \text{for } t \in (0, T_{\max,\epsilon}). \tag{3.29}$$

Note that (3.26) implies

$$\frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2 \ge \frac{C_1^2}{2C_2^2}\|u_\epsilon\|^2_{W_0^{1,2}(\Omega)} \ge \frac{C_1^2}{2C_2^2}\int_\Omega |u_\epsilon|^2,$$

and therefore we have

$$\frac{d}{dt}\int_\Omega |u_\epsilon|^2 + \frac{C_1^2}{2C_2^2}\int_\Omega |u_\epsilon|^2 + \frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2 \le 2C_3 \|n_\epsilon\|^2_{L^q(\Omega)} + 2C_3 \|\rho_\epsilon\|^2_{L^q(\Omega)} \tag{3.30}$$

for $t \in (0, T_{\max,\epsilon})$. Setting

$$y(t) := \int_\Omega |u_\epsilon|^2, \quad h(t) := 2C_3 \|n_\epsilon\|^2_{L^q(\Omega)} + 2C_3 \|\rho_\epsilon\|^2_{L^q(\Omega)}, \quad g(t) := \frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2,$$

we have

$$y'(t)+\frac{C_1^2}{2C_2^2}y(t)+g(t)\le h(t)\quad\text{for } t\in(0,T_{\max,\epsilon}).$$

If there exists $C>0$ such that

$$\int_t^{t+\tau}h(s)ds\le C\quad\text{for } t\in(0,T_{\max,\epsilon}-\tau),\tag{3.31}$$

then we can once more use Lemma 2.3 to derive (3.22) and (3.23).

Now, we merely need to show that the assumption (3.31) is indeed valid. To this end, we first apply the Gagiardo-Nirenberg inequality to find $c_1>0$ such that

$$\begin{aligned}
\|n_\epsilon\|_{L^q(\Omega)}^2 &\le \|(n_\epsilon+\epsilon)^{m-1}\|_{L^{\frac{q}{m-1}}(\Omega)}^{\frac{2}{m-1}}\\
&\le c_1\|(n_\epsilon+\epsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{2}{q(m-1)}}\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{2(q-1)}{q(m-1)}}\\
&\quad+c_1\|(n_\epsilon+\epsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{2}{m-1}}
\end{aligned}$$

for $t\in(0,T_{\max,\epsilon})$. Note that

$$\|(n_\epsilon+\epsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{1}{(m-1)}}=\|(n_\epsilon+\epsilon)\|_{L^1(\Omega)}\le\|n_0\|_{L^1(\Omega)}+|\Omega|$$

for $t\in(0,T_{\max,\epsilon})$, then we can find $c_2>0$ such that

$$\|n_\epsilon\|_{L^q(\Omega)}^2\le c_2(\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{2(q-1)}{q(m-1)}}+1)\quad\text{for } t\in(0,T_{\max,\epsilon}).$$

When $m\ge 2$, for any $q>1$, it is easily checked that the exponent

$$\frac{2(q-1)}{q(m-1)}<\frac{2}{m-1}\le 2.$$

Using Young's inequality, we have

$$\|n_\epsilon\|_{L^q(\Omega)}^2\le c_2(\|\nabla(n_\epsilon+\epsilon)^{m-1}\|_{L^2(\Omega)}^2+2)\quad\text{for } t\in(0,T_{\max,\epsilon}).\tag{3.32}$$

When $m\in(1,2)$, let $q=\frac{1}{2-m}$, then $q>1$ and $\frac{2(q-1)}{q(m-1)}=2$, thus (3.32) still holds. Using (3.32), (3.11), and (3.2), we obtain

$$\begin{aligned}
\int_t^{t+\tau}h(s)ds &= 2C_3\int_t^{t+\tau}(\|n_\epsilon(\cdot,s)\|_{L^q(\Omega)}^2+\|\rho_\epsilon(\cdot,s)\|_{L^q(\Omega)}^2)ds\\
&\le 2C_3\int_t^{t+\tau}c_2(\|\nabla(n_\epsilon+\epsilon)^{m-1}(\cdot,s)\|_{L^2(\Omega)}^2+2)ds\\
&\quad+2C_3\int_t^{t+\tau}\|\rho_\epsilon(\cdot,s)\|_{L^\infty(\Omega)}^2|\Omega|^{\frac{2}{q}}ds\\
&\le 2C_3c_2(m-1)^2\int_t^{t+\tau}\int_\Omega(n_\epsilon+\epsilon)^{2m-4}|\nabla n_\epsilon|^2\,dx\,ds\\
&\quad+2C_3\|\rho_0\|_{L^\infty(\Omega)}^2|\Omega|^{\frac{2}{q}}\tau\le C_4\quad\text{for } t\in(0,T_{\max,\epsilon}-\tau).
\end{aligned}$$

This proves (3.31) and therefore completes the proof. $\square$

Corollary 3.4. *Let $m>1$. There exists a positive constant C independent of ϵ, such that*
$$\int_\Omega |\nabla c_\epsilon|^2 \le C \quad \text{for } t\in(0,T_{\max,\epsilon}), \tag{3.33}$$
$$\int_t^{t+\tau}\int_\Omega |\Delta c_\epsilon|^2 \le C \quad \text{for } t\in(0,T_{\max,\epsilon}-\tau), \tag{3.34}$$
where $\tau=\min\{1,T_{\max,\epsilon}/2\}$.

Proof. We first claim that there exists $C_1>0$ independent of ϵ such that
$$\frac{d}{dt}\int_\Omega |\nabla c_\epsilon|^2+\int_\Omega |\Delta c_\epsilon|^2+2\int_\Omega |\nabla c_\epsilon|^2 \le C_1\Big\{\int_\Omega |\nabla u_\epsilon|^2+1\Big\} \quad t\in(0,T_{\max,\epsilon}). \tag{3.35}$$
The proof is similar to [6, Lemma 3.2], we omit the details here. Setting $h(t):=C\{\int_\Omega|\nabla u_\epsilon|^2+1\}$, (3.23) implies that there exists $C_2>0$ independent of ϵ such that
$$\int_t^{t+\tau} h(s)ds \le C_2 \quad \text{for all } t\in(0,T_{\max,\epsilon}-\tau).$$
Thus, we can, once more, utilize Lemma 2.3 to achieve (3.33) and (3.34). $\square$

Lemma 3.5. *Let $m>1$. Then the solution of* (2.1) *satisfies*
$$\int_\Omega (n_\epsilon+\epsilon)^m \le C \quad \text{for } t\in(0,T_{\max,\epsilon}), \tag{3.36}$$
$$\int_t^{t+\tau}\int_\Omega (n_\epsilon+\epsilon)^{2m} \le C \quad \text{for } t\in(0,T_{\max,\epsilon}-\tau) \tag{3.37}$$
where $\tau=\min\{1,\frac12 T_{\max,\epsilon}\}$.

Proof. Testing $(2.1)_1$ against $(n_\epsilon+\epsilon)^{m-1}$, and using $\nabla\cdot u_\epsilon=0$, (2.4), and Young's inequality, we have
$$\begin{aligned}
&\frac1m\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^m+m(m-1)\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2+\mu\int_\Omega (n_\epsilon+\epsilon)^{m-1}n_\epsilon\rho_\epsilon\\
&=(m-1)\int_\Omega (n_\epsilon+\epsilon)^{m-2}n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla n_\epsilon\cdot\nabla c_\epsilon\\
&\le \chi(m-1)\int_\Omega (n_\epsilon+\epsilon)^{m-1}|\nabla n_\epsilon||\nabla c_\epsilon|\\
&\le \frac{m(m-1)}{2}\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2+\frac{\chi^2(m-1)}{2m}\int_\Omega (n_\epsilon+\epsilon)|\nabla c_\epsilon|^2
\end{aligned}$$
for $t\in(0,T_{\max,\epsilon})$. Applying Young's inequality to the right most term gives
$$\begin{aligned}
&\frac1m\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^m+\frac{m(m-1)}{2}\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2\\
&\le \frac{\chi^2(m-1)}{2m}\int_\Omega (n_\epsilon+\epsilon)|\nabla c_\epsilon|^2\\
&\le \eta\int_\Omega (n_\epsilon+\epsilon)^{2m}+C_1(\eta)\int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}} \quad \text{for } t\in(0,T_{\max,\epsilon})
\end{aligned} \tag{3.38}$$
with $\eta>0$ to be determined later, and
$$C_1(\eta)=\frac{2m-1}{2m}\Big[\frac{\chi^2(m-1)}{2m}\Big]^{\frac{2m}{2m-1}}(2m\eta)^{-\frac{1}{2m-1}}.$$

By using Young's inequality with the above η we have

$$\int_\Omega (n_\epsilon+\epsilon)^m \le \eta\int_\Omega (n_\epsilon+\epsilon)^{2m} + \frac{1}{4\eta}|\Omega| \quad \text{for } t\in(0,T_{\max,\epsilon}). \tag{3.39}$$

By the Gagliardo-Nirenberg inequality and $\|n_\epsilon+\epsilon\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)} + |\Omega|$ which is implied by (3.1), and $\epsilon\in(0,1)$, we can find $c_1>0$ independent of ϵ such that

$$\begin{aligned}
&\int_\Omega (n_\epsilon+\epsilon)^{2m}\\
&= \|(n_\epsilon+\epsilon)^{\frac{2m-1}{2}}\|_{L^{\frac{4m}{2m-1}}(\Omega)}^{\frac{4m}{2m-1}}\\
&\le c_1\|(n_\epsilon+\epsilon)^{\frac{2m-1}{2}}\|_{L^{\frac{2}{2m-1}}(\Omega)}^{\frac{2}{2m-1}} \|\nabla(n_\epsilon+\epsilon)^{\frac{2m-1}{2}}\|_{L^2(\Omega)}^2\\
&\quad + c_1\|(n_\epsilon+\epsilon)^{\frac{2m-1}{2}}\|_{L^{\frac{2}{2m-1}}(\Omega)}^{\frac{4m}{2m-1}}\\
&= c_1\|n_\epsilon+\epsilon\|_{L^1(\Omega)}\|\nabla(n_\epsilon+\epsilon)^{\frac{2m-1}{2}}\|_{L^2(\Omega)}^2 + c_1\|n_\epsilon+\epsilon\|_{L^1(\Omega)}^{2m}\\
&\le c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)\|\nabla(n_\epsilon+\epsilon)^{\frac{2m-1}{2}}\|_{L^2(\Omega)}^2 + c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)_{L^1(\Omega)}^{2m}\\
&= c_2\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2 + c_3 \quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned} \tag{3.40}$$

where $c_2 = \frac{(2m-1)^2}{4}c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)$ and $c_3 = c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)_{L^1(\Omega)}^{2m}$ are positive constants independent of ϵ. Adding the term $\int_\Omega (n_\epsilon+\epsilon)^m$ on both sides of (3.38), and then using (3.39) and (3.40), yields

$$\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^m + \int_\Omega (n_\epsilon+\epsilon)^m + \frac{m(m-1)}{2}\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2\\
&\le 2\eta\int_\Omega (n_\epsilon+\epsilon)^{2m} + C_1(\eta)\int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}} + \frac{1}{4\eta}|\Omega|\\
&\le 2\eta c_2\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2 + 2\eta c_3 + C_1(\eta)\int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}} + \frac{1}{4\eta}|\Omega|
\end{aligned}$$

for $t\in(0,T_{\max,\epsilon})$. Letting $\eta = \frac{m(m-1)}{8c_2}$, we have

$$\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^m + \int_\Omega (n_\epsilon+\epsilon)^m + \frac{m(m-1)}{4}\int_\Omega (n_\epsilon+\epsilon)^{2m-3}|\nabla n_\epsilon|^2\\
&\le \frac{c_3 m(m-1)}{4c_2} + c_4\int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}} + \frac{2c_2}{m(m-1)}|\Omega| \quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned} \tag{3.41}$$

where

$$c_4 = C_1(\eta) = \frac{2m-1}{2m}[\frac{\chi^2(m-1)}{2m}]^{\frac{2m}{2m-1}}(\frac{m^2(m-1)}{4c_2})^{-\frac{1}{2m-1}}.$$

We claim that if there exists $C_2>0$ independent of ϵ such that

$$\int_t^{t+\tau}\int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}}\,dx\,ds \le C_2 \quad \text{for } t\in(0,T_{\max,\epsilon}), \tag{3.42}$$

then we can apply Lemma 2.3 to infer that there exists $C_3>0$ independent of ϵ such that

$$\int_\Omega (n_\epsilon+\epsilon)^m \le C_3 \quad \text{for } t\in(0,T_{\max,\epsilon}).$$

This proves (3.36), and then by integrating of (3.41) over $(t,\, t+\tau)$ and using (3.42) and (3.40) once again, we get (3.37).

Now, we turn to show that (3.42) is valid. Indeed, we first note that for any $l_0 \in (\frac{1}{m-1}, \frac{2m}{m-1})$, by using the boundedness of c_ϵ given in (3.3) and the Hölder inequality, one can easily get

$$\|c_\epsilon(\cdot,t)\|_{L^{l_0}(\Omega)} \le \|c_\epsilon(\cdot,t)\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{l_0}} \le \max\{\|c_0\|_{L^\infty}, \|\rho_0\|_{L^\infty}\}|\Omega|^{\frac{1}{l_0}} =: K_1 \tag{3.43}$$

for $t \in (0, T_{\max,\epsilon})$.

Then, by using the Gagliardo-Nirenberg inequality we have

$$\begin{aligned} \int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}} &= \|\nabla c_\epsilon\|^{\frac{4m}{2m-1}}_{L^{\frac{4m}{2m-1}}(\Omega)} \\ &\le c_5\|\Delta c_\epsilon\|^{\frac{4m}{2m-1}a}_{L^2(\Omega)}\|c_\epsilon\|^{\frac{4m}{2m-1}(1-a)}_{L^{l_0}(\Omega)} + c_5\|c_\epsilon\|^{\frac{4m}{2m-1}}_{L^{l_0}(\Omega)} \\ &\le c_6\|\Delta c_\epsilon\|^{\frac{4m}{2m-1}a}_{L^2(\Omega)} + c_7 \quad \text{for } t \in (0, T_{\max,\epsilon}) \end{aligned} \tag{3.44}$$

with

$$a = \frac{\frac{1}{2} + \frac{1}{l_0} - \frac{2m-1}{4m}}{\frac{1}{2} + \frac{1}{l_0}} \in (\frac{1}{2}, 1), \quad c_6 = c_5 K_1^{\frac{4m}{2m-1}(1-a)}, \quad c_7 = c_5 K_1^{\frac{4m}{2m-1}}.$$

Since $l_0 > \frac{1}{m-1}$, simple computation shows that

$$\frac{4m}{2m-1}a = \frac{4m}{2m-1}\,\frac{\frac{1}{2} + \frac{1}{l_0} - \frac{2m-1}{4m}}{\frac{1}{2} + \frac{1}{l_0}} < \frac{4m}{2m-1}\,\frac{\frac{1}{2} + m - 1 - \frac{2m-1}{4m}}{\frac{1}{2} + m - 1} = 2.$$

Hence, by using Young's inequality we have

$$\begin{aligned} \int_\Omega |\nabla c_\epsilon|^{\frac{4m}{2m-1}} &\le c_6\|\Delta c_\epsilon\|^{\frac{4m}{2m-1}a}_{L^2(\Omega)} + c_7 \\ &\le c_6(\|\Delta c_\epsilon\|^2_{L^2(\Omega)} + 1) + c_7 \\ &= c_6\int_\Omega |\Delta c_\epsilon|^2 dx + c_6 + c_7 \quad \text{for } t \in (0, T_{\max,\epsilon}). \end{aligned} \tag{3.45}$$

The above together with (3.34) give (3.42). □

Thanks the assumption $m > 1$, the following estimate is a direct consequence of (3.37) and Young's inequality.

Corollary 3.6. *Let $m > 1$. Then the solution of* (2.1) *satisfies*

$$\int_t^{t+\tau}\int_\Omega n_\epsilon^2 \le C \quad \textit{for } t \in (0, T_{\max,\epsilon} - \tau) \tag{3.46}$$

where $\tau = \min\{1, T_{\max,\epsilon}/2\}$.

With this spatio-temporal L^2 estimate in mind, following the same procedure as the proof of [6, Lemma 3.6] we can achieve the following estimate on u_ϵ.

Lemma 3.7. *Let $m > 1$. There exists a positive constant C independent of ϵ such that*

$$\int_\Omega |\nabla u_\epsilon|^2 \le C \quad \textit{for } t \in (0, T_{\max,\epsilon}). \tag{3.47}$$

By means of the continuous embedding that $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for any $p > 1$, (3.47) further implies the uniform boundedness of $\|u_\epsilon\|_{L^p(\Omega)}$.

Corollary 3.8. *Let $p > 1$. Then there exists a positive constant $C(p)$ such that*
$$\|u_\epsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p) \quad \textit{for } t \in (0, T_{\max,\epsilon}). \tag{3.48}$$

We next use (3.48) to enhance the regularity of c_ϵ obtained in Corollary 3.4 to the space $W^{1,2m}(\Omega)$.

Lemma 3.9. *Let $m > 1$. There exists a positive constant C independent of ϵ, such that the solution of* (2.1) *satisfies*
$$\|\nabla c_\epsilon\|_{L^{2m}(\Omega)} \le C \quad \textit{for } t \in (0, T_{\max,\epsilon}). \tag{3.49}$$

Proof. By the fact that $\nabla c_\epsilon \cdot \nabla \Delta c_\epsilon = \frac{1}{2}\Delta|\nabla c_\epsilon|^2 - |D^2 c_\epsilon|^2$, direct computations show that
$$\begin{aligned}
&\frac{1}{2m}\frac{d}{dt}\|\nabla c_\epsilon\|^{2m}_{L^{2m}(\Omega)}\\
&= \int_\Omega |\nabla c_\epsilon|^{2m-2}\nabla c_\epsilon \cdot \nabla(\Delta c_\epsilon - c_\epsilon + \rho_\epsilon - u_\epsilon \cdot \nabla c_\epsilon)\\
&= \frac{1}{2}\int_\Omega |\nabla c_\epsilon|^{2m-2}\Delta|\nabla c_\epsilon|^2 - \int_\Omega |\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|^2 - \int_\Omega |\nabla c_\epsilon|^{2m}\\
&\quad - \int_\Omega \rho_\epsilon \nabla\cdot(|\nabla c_\epsilon|^{2m-2}\nabla c_\epsilon) + \int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)\nabla\cdot(|\nabla c_\epsilon|^{2m-2}\nabla c_\epsilon)\\
&= -\frac{2(m-1)}{m^2}\int_\Omega |\nabla|\nabla c_\epsilon|^m|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla c_\epsilon|^{2m-2}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu} - \int_\Omega |\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|^2\\
&\quad - \int_\Omega \rho_\epsilon|\nabla c_\epsilon|^{2m-2}\Delta c_\epsilon - \int_\Omega \rho_\epsilon\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^{2m-2} + \int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2m-2}\Delta c_\epsilon\\
&\quad + \int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^{2m-2} - \int_\Omega |\nabla c_\epsilon|^{2m} \quad \text{for } t\in(0,T_{\max,\epsilon}).
\end{aligned}\tag{3.50}$$
Here, in view of $|\Delta c_\epsilon| \le \sqrt{2}|D^2 c_\epsilon|$ and Young's inequality, we have
$$\begin{aligned}
-\int_\Omega \rho_\epsilon|\nabla c_\epsilon|^{2m-2}\Delta c_\epsilon &\le \sqrt{2}\int_\Omega \rho_\epsilon|\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|\\
&\le \frac{1}{4}\int_\Omega |\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|^2 + 2\int_\Omega \rho_\epsilon^2|\nabla c_\epsilon|^{2m-2}\\
&\quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned}\tag{3.51}$$
$$\begin{aligned}
&-\int_\Omega \rho_\epsilon\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^{2m-2}\\
&= -(m-1)\int_\Omega \rho_\epsilon|\nabla c_\epsilon|^{2(m-2)}\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2\\
&\le \frac{m-1}{8}\int_\Omega |\nabla c_\epsilon|^{2m-4}|\nabla|\nabla c_\epsilon|^2|^2 + 2(m-1)\int_\Omega \rho_\epsilon^2|\nabla c_\epsilon|^{2m-2}\\
&\le \frac{m-1}{2m^2}\int_\Omega |\nabla|\nabla c_\epsilon|^m|^2 + 2(m-1)\int_\Omega \rho_\epsilon^2|\nabla c_\epsilon|^{2m-2} \quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned}\tag{3.52}$$
$$\begin{aligned}
\int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2m-2}\Delta c_\epsilon &\le \sqrt{2}\int_\Omega |u_\epsilon||\nabla c_\epsilon|^{2m-1}|D^2 c_\epsilon|\\
&\le \frac{1}{4}\int_\Omega |\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|^2 + 2\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2m}\\
&\quad \text{for } t\in(0,T_{\max,\epsilon}),
\end{aligned}\tag{3.53}$$

$$
\begin{aligned}
&\int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)\nabla c_\epsilon \cdot \nabla |\nabla c_\epsilon|^{2m-2} \\
&= (m-1)\int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)|\nabla c_\epsilon|^{2(m-2)}\nabla c_\epsilon \cdot \nabla |\nabla c_\epsilon|^2 \\
&\le \frac{m-1}{8}\int_\Omega |\nabla c_\epsilon|^{2m-4}|\nabla |\nabla c_\epsilon|^2|^2 + 2(m-1)\int_\Omega |u_\epsilon \cdot \nabla c_\epsilon|^2 |\nabla c_\epsilon|^{2m-2} \\
&\le \frac{m-1}{2m^2}\int_\Omega |\nabla |\nabla c_\epsilon|^m|^2 + 2(m-1)\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2m} \quad \text{for } t \in (0, T_{\max,\epsilon}).
\end{aligned}
\tag{3.54}
$$

Following the same procedure in [9] the boundary integral in (3.50) can be controlled. In fact, in view of [19, Lemma 4.2] we have

$$
\int_{\partial\Omega} |\nabla c_\epsilon|^{2m-2}\frac{\partial |\nabla c_\epsilon|^2}{\partial \nu} \le C_\Omega \int_{\partial\Omega} |\nabla c_\epsilon|^{2m} = C_\Omega \||\nabla c_\epsilon|^m\|^2_{L^2(\partial\Omega)}, \tag{3.55}
$$

where C_Ω is a positive upper bound for the curvatures of $\partial\Omega$. Taking $r \in (0, 1/2)$, the compact embedding $W^{r+\frac12,2}(\Omega)(\hookrightarrow W^{r,2}(\partial\Omega)) \hookrightarrow L^2(\partial\Omega)$ (see [8, Proposition 4.22]) implies that there exists a positive constant C_1 such that

$$
\||\nabla c_\epsilon|^m\|^2_{L^2(\partial\Omega)} \le C_1 \||\nabla c_\epsilon|^m\|^2_{W^{r+\frac12,2}(\Omega)}. \tag{3.56}
$$

By the fractional Gagliardo-Nirenberg inequality and $\|\nabla c_\epsilon\|_{L^2(\Omega)}$ begin bounded (stated in (3.33)), we can find positive constants c_1 and C_2 such that

$$
\begin{aligned}
\||\nabla c_\epsilon|^m\|^2_{W^{r+\frac12,2}(\Omega)} &\le c_1\|\nabla |\nabla c_\epsilon|^m\|^a_{L^2(\Omega)}\||\nabla c_\epsilon|^m\|^{1-a}_{L^{\frac{2}{m}}(\Omega)} + c_1\||\nabla c_\epsilon|^m\|_{L^{\frac{2}{m}}(\Omega)} \\
&= c_1\|\nabla |\nabla c_\epsilon|^m\|^a_{L^2(\Omega)}\||\nabla c_\epsilon|\|^{m(1-a)}_{L^2(\Omega)} + c_1\||\nabla c_\epsilon|\|^m_{L^2(\Omega)} \\
&\le C_2\|\nabla |\nabla c_\epsilon|^m\|^a_{L^2(\Omega)} + C_2
\end{aligned}
\tag{3.57}
$$

with $a = \frac{2m+2r-1}{2m} \in (r+\frac12,\, 1)$. Combing (3.55)-(3.57) and once again using Young's inequality gives

$$
\begin{aligned}
\int_{\partial\Omega} |\nabla c_\epsilon|^{2m-2}\frac{\partial |\nabla c_\epsilon|^2}{\partial \nu} &\le C_\Omega C_1 C_2 \|\nabla |\nabla c_\epsilon|^m\|^a_{L^2(\Omega)} + C_\Omega C_1 C_2 \\
&\le \frac{m-1}{m^2}\int_\Omega |\nabla |\nabla c_\epsilon|^m|^2 + C_4.
\end{aligned}
\tag{3.58}
$$

This and (3.50)-(3.54) show that

$$
\begin{aligned}
&\frac{1}{2m}\frac{d}{dt}\|\nabla c_\epsilon\|^{2m}_{L^{2m}(\Omega)} + \frac{m-1}{2m^2}\int_\Omega |\nabla |\nabla c_\epsilon|^m|^2 + \frac12 \int_\Omega |\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|^2 + \int_\Omega |\nabla c_\epsilon|^{2m} \\
&\le 2m\int_\Omega \rho_\epsilon^2 |\nabla c_\epsilon|^{2m-2} + 2m\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2m} + C_5 \quad \text{for } t \in (0, T_{\max,\epsilon})
\end{aligned}
\tag{3.59}
$$

with $C_5 = \frac{C_4}{2}$. Another two applications of Young's inequality then yield

$$
2m\int_\Omega \rho_\epsilon^2|\nabla c_\epsilon|^{2m-2} \le \frac12\int_\Omega |\nabla c_\epsilon|^{2m} + C_6\int_\Omega \rho_\epsilon^{2m} \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.60}
$$

$$
2m\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2m} \le \int_\Omega |\nabla c_\epsilon|^{2m+1} + C_7\int_\Omega u_\epsilon^{4m+2} \quad \text{for } t \in (0, T_{\max,\epsilon}), \tag{3.61}
$$

with $C_6 = \frac{m}{m-1}(\frac{1}{2}m)^{-\frac{1}{m-1}}(2m)^m$ and $C_7 = (2m)^{2m+1}$. For the integral on the right-hand side of (3.61), by means of the Gagliardo-Nirenberg inequality, (3.33), as well as Young's inequality, we have

$$\begin{aligned}
&\int_\Omega |\nabla c_\epsilon|^{2m+1} \\
&= \||\nabla c_\epsilon|^m\|_{L^{\frac{2m+1}{m}}(\Omega)}^{\frac{2m+1}{m}} \\
&\le C_8(\|\nabla|\nabla c_\epsilon|^m\|_{L^2(\Omega)}^{\frac{2m-1}{2m+1}}\||\nabla c_\epsilon|^m\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2}{2m+1}} + \||\nabla c_\epsilon|^m\|_{L^{\frac{2}{m}}(\Omega)})^{\frac{2m+1}{m}} \\
&\le C_9(\|\nabla|\nabla c_\epsilon|^m\|_{L^2(\Omega)}^{\frac{2m-1}{m}}\||\nabla c_\epsilon|\|_{L^2(\Omega)}^{\frac{2m}{2m+1}} + \||\nabla c_\epsilon|\|_{L^2(\Omega)}^{2m+1}) \\
&\le \frac{m-1}{2m^2}\int_\Omega |\nabla|\nabla c_\epsilon|^m|^2 + C_{10}.
\end{aligned} \tag{3.62}$$

Inserting (3.62) into (3.61), we obtain that

$$2m\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2m} \le \frac{m-1}{2m^2}\int_\Omega |\nabla|\nabla c_\epsilon|^m|^2 + C_7\int_\Omega u_\epsilon^{4m+2} + C_{10} \tag{3.63}$$

for $t \in (0, T_{\max,\epsilon})$. Substituting (3.60) and (3.63) into (3.59), dropping the nonnegative term $\frac{1}{2}\int_\Omega |\nabla c_\epsilon|^{2m-2}|D^2 c_\epsilon|^2$, and using the uniform boundedness of ρ_ϵ((3.2)) as well as the time-independent L^p estimate of u_ϵ for any $p > 1$ stated in (3.48), yields

$$\begin{aligned}
&\frac{1}{2m}\frac{d}{dt}\|\nabla c_\epsilon\|_{L^{2m}(\Omega)}^{2m} + \frac{1}{2}\int_\Omega |\nabla c_\epsilon|^{2m} \\
&\le C_6\int_\Omega \rho_\epsilon^{2m} + C_7\int_\Omega u_\epsilon^{4m+2} + C_5 + C_{10} \\
&\le C_6\|\rho_\epsilon\|_{L^\infty(\Omega)}^2 m|\Omega| + C_7\|u_\epsilon\|_{L^{4m+2}(\Omega)}^{4m+2} + C_5 + C_{10} \le C_{11} \quad \text{for } t \in (0, T_{\max,\epsilon}),
\end{aligned}$$

which implies (3.49). $\square$

Lemma 3.10. *Let $m > 1$. Then for any $p > 1$, there exists a positive constant C independent of ϵ, such that*

$$\|n_\epsilon(\cdot,t)\|_{L^p(\Omega)} \le C \quad \textit{for } t \in (0, T_{\max,\epsilon}). \tag{3.64}$$

Proof. Without loss of generality, let $p > m$. Testing the first equation in (2.1) against $(n_\epsilon + \epsilon)^{p-1}$ and using (2.4), Young's inequality and that $\nabla \cdot u_\epsilon = 0$, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^p + m(p-1)\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 + \mu\int_\Omega (n_\epsilon+\epsilon)^{p-1}n_\epsilon\rho_\epsilon \\
&= (p-1)\int_\Omega (n_\epsilon+\epsilon)^{p-2}n_\epsilon\nabla n_\epsilon\cdot(S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon) \\
&\le \chi(p-1)\int_\Omega (n_\epsilon+\epsilon)^{p-1}|\nabla n_\epsilon||\nabla c_\epsilon| \\
&\le \frac{m(p-1)}{2}\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 + \frac{\chi^2(p-1)}{2m}\int_\Omega (n_\epsilon+\epsilon)^{p+1-m}|\nabla c_\epsilon|^2
\end{aligned}$$

for $t \in (0, T_{\max,\epsilon})$. This, along with the nonnegativity of n_ϵ and ρ_ϵ, yields

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^p + \frac{m(p-1)}{2}\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 \\ &\le \frac{\chi^2(p-1)}{2m}\int_\Omega (n_\epsilon+\epsilon)^{p+1-m}|\nabla c_\epsilon|^2 \quad \text{for } t \in (0, T_{\max,\epsilon}). \end{aligned} \tag{3.65}$$

We need to estimate the last term in (3.65). By Hölder's inequality and (3.49), there exists $C_1 > 0$ such that

$$\begin{aligned} &\frac{\chi^2(p-1)}{2m}\int_\Omega (n_\epsilon+\epsilon)^{p+1-m}|\nabla c_\epsilon|^2 \\ &\le \frac{\chi^2(p-1)}{2m}\Big\{\int_\Omega (n_\epsilon+\epsilon)^{\frac{m(p+1-m)}{m-1}}\Big\}^{\frac{m-1}{m}}\Big\{\int_\Omega |\nabla c_\epsilon|^{2m}\Big\}^{1/m} \\ &\le C_1\Big\{\int_\Omega (n_\epsilon+\epsilon)^{\frac{m(p+1-m)}{m-1}}\Big\}^{\frac{m-1}{m}} \quad \text{for } t \in (0, T_{\max,\epsilon}). \end{aligned} \tag{3.66}$$

By using the Gagliardo-Nirenberg inequality and (3.1) we can find some positive constants c_1 and c_2 such that

$$\begin{aligned} &\Big\{\int_\Omega (n_\epsilon+\epsilon)^{\frac{m(p+1-m)}{m-1}}\Big\}^{\frac{m-1}{m}} \\ &= \|(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}})\|^{\frac{2(p+1-m)}{m+p-1}}_{L^{\frac{2m(p+1-m)}{(m-1)(m+p-1)}}(\Omega)} \\ &\le c_1\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(mp-m^2+1)}{m(p+m-1)}}_{L^2(\Omega)}\|(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(m-1)}{m(m+p-1)}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\ &\quad + c_1\|(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p+1-m)}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\ &= c_1\|n_\epsilon+\epsilon\|^{\frac{m-1}{m}}_{L^1(\Omega)}\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(mp-m^2+1)}{m(p+1-m)}}_{L^2(\Omega)} + c_1\|n_\epsilon+\epsilon\|^{p+1-m}_{L^1(\Omega)} \\ &\le c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)^{\frac{m-1}{m}}\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(mp-m^2+1)}{m(p+1-m)}}_{L^2(\Omega)} \\ &\quad + c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)^{p+1-m} \\ &\le c_2\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(mp-m^2+1)}{m(p+1-m)}}_{L^2(\Omega)} + c_2 \quad \text{for } t \in (0, T_{\max,\epsilon}). \end{aligned} \tag{3.67}$$

Since $m > 1$ and $p > m$, it is simple to check that

$$\frac{mp-m^2+1}{m(p+1-m)} \in (0,1).$$

Hence, it follows from (3.66)-(3.67) and Young's inequality that

$$\begin{aligned} &\frac{\chi^2(p-1)}{2m}\int_\Omega (n_\epsilon+\epsilon)^{p+1-m}|\nabla c_\epsilon|^2 \\ &\le C_1\Big\{c_2\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(mp-m^2+1)}{m(p+1-m)}}_{L^2(\Omega)} + c_2\Big\} \\ &\le \frac{m(p-1)}{(m+p-1)^2}\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^2_{L^2(\Omega)} + C_2 \\ &= \frac{m(p-1)}{4}\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 + C_2 \end{aligned} \tag{3.68}$$

for $t \in (0, T_{\max,\epsilon})$. Inserting (3.68) into (3.65), we derive that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^p + \frac{m(p-1)}{4}\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 \le C_2 \tag{3.69}$$

for $t \in (0, T_{\max,\epsilon})$. Again, using the Galiardo-Nirenberg inequality, (3.1), and Young's inequality, we have

$$\begin{aligned}
&\int_\Omega (n_\epsilon+\epsilon)^p \\
&= \|(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2p}{m+p-1}}_{L^{\frac{2p}{m+p-1}}(\Omega)} \\
&\le c_1\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p-1)}{m+p-1}}_{L^2(\Omega)}\|(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\
&\quad + c_1\|(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2p}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)} \\
&= c_1\|n_\epsilon+\epsilon\|_{L^1(\Omega)}\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p-1)}{m+p-1}}_{L^2(\Omega)} + c_1\|n_\epsilon+\epsilon\|^p_{L^1(\Omega)} \\
&\le c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p-1)}{m+p-1}}_{L^2(\Omega)} + c_1(\|n_0\|_{L^1(\Omega)}+|\Omega|)^p \\
&\le \frac{m(p-1)}{(m+p-1)^2}\|\nabla(n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}\|^2_{L^2(\Omega)} + C_3 \\
&= \frac{m(p-1)}{4}\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 + C_3 \quad \text{for } t\in(0,T_{\max,\epsilon}).
\end{aligned} \tag{3.70}$$

(3.69)-(3.70) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^p + \int_\Omega (n_\epsilon+\epsilon)^p \le C_2 + C_3 \quad \text{for } t\in(0,T_{\max,\epsilon}),$$

and hence, the Gronwall inequality implies (3.64). □

With the boundedness of $\|\phi\|_{L^\infty(\Omega)}$, $\|\rho_\epsilon\|_{L^\infty(\Omega)}$, $\|\nabla u_\epsilon\|_{L^2(\Omega)}$, and $\|n_\epsilon\|_{L^p(\Omega)}$, we can further achieve the following boundedness results.

Lemma 3.11. *Let $m>1$ and $\beta\in(\frac{1}{2},1)$. Then we can find some positive constant C independent of ϵ, such that*

$$\|A^\beta u_\epsilon(\cdot,t)\|_{L^2(\Omega)} \le C \quad \textit{for } t\in(0,T_{\max,\epsilon}), \tag{3.71}$$

$$\|u_\epsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for } t\in(0,T_{\max,\epsilon}), \tag{3.72}$$

$$\|c_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \textit{for } t\in(0,T_{\max,\epsilon}), \tag{3.73}$$

$$\|n_\epsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for } t\in(0,T_{\max,\epsilon}). \tag{3.74}$$

Proof. Following the same arguments as in [25, Lemma 3.11], see also [6, Lemma 3.10], we can obtain (3.71). This, in conjunction with the continuous embedding $D(A^\beta)\hookrightarrow L^\infty(\Omega)$ (implied by $\beta>\frac{1}{2}$), then entails (3.72). Furthermore, we can follow the proof of [25, Lemma 3.12] to obtain (3.73), and by means of a Moser-type iteration applied to the first equation in (2.1), we can achieve (3.74). □

By means of all above regularity properties of $n_\epsilon, c_\epsilon, \rho_\epsilon$ and u_ϵ and the extensibility criterion (2.5) in Lemma 2.1, we can show that actually $T_{\max,\epsilon}=\infty$ in Lemma 2.1 and then the classical solution $(n_\epsilon,\, \rho_\epsilon,\, c_\epsilon,\, u_\epsilon,\, P_\epsilon)$ of the regularized problem (2.1) exists globally.

4. Proof of Theorem 1.1

In this section, we study the global weak solution to the problem (1.4). Here, the definition of weak solution to the problem (1.4) is in the following sense.

Definition 4.1. Let $m > 1$, $l > 0$, $\mu > 0$, and suppose that (n_0, c_0, ρ_0, u_0) fulfills (1.8). Then a quadruple of functions (n, c, ρ, u) is called a global weak solution of the initial-boundary value problem (1.4) in $\Omega \times (0, \infty)$, if

$$\begin{gathered} n \in L^\infty([0,\infty); L^\infty(\Omega)), \quad \nabla n^m \in L^2_{\rm loc}((0,\infty); L^2(\Omega)), \\ c \in L^\infty((0,\infty); W^{1,1}(\Omega)), \quad \rho \in L^\infty((0,\infty); L^\infty(\Omega)), \\ \nabla \rho^l \in L^2_{\rm loc}((0,\infty); L^2(\Omega)), \quad u \in L^\infty((0,\infty); W^{1,1}(\Omega)) \\ \text{such that } \nabla\cdot u = 0 \text{ in the distributional sense in } \Omega\times(0,\infty), \end{gathered} \tag{4.1}$$

and for any $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, the following hold:

$$\begin{aligned} \int_0^\infty \int_\Omega n\varphi_t + \int_\Omega n_0 \varphi(\cdot,0) &= \int_0^\infty \int_\Omega \nabla n^m \cdot \nabla\varphi - \int_0^\infty \int_\Omega n(S(x,n,c)\nabla c)\cdot\nabla\varphi \\ &\quad - \int_0^\infty \int_\Omega nu\cdot\nabla\varphi + \mu \int_0^\infty \int_\Omega n\rho\varphi, \end{aligned}$$

and

$$\int_0^\infty \int_\Omega \rho\varphi_t + \int_\Omega \rho_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla\rho^l\cdot\nabla\varphi - \int_0^\infty \int_\Omega \rho u\cdot\nabla\varphi + \mu\int_0^\infty\int_\Omega n\rho\varphi,$$

as well as

$$\int_0^\infty \int_\Omega c\varphi_t + \int_\Omega c_0\varphi(\cdot,0) = \int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi + \int_0^\infty\int_\Omega c\varphi - \int_0^\infty\int_\Omega \rho\varphi - \int_0^\infty\int_\Omega cu\cdot\nabla\varphi\,.$$

For any $\zeta \in C_0^\infty(\overline{\Omega}\times[0,\infty);\mathbb{R}^2)$ fulfilling $\nabla\cdot\zeta \equiv 0$, it holds

$$\int_0^\infty\int_\Omega u\cdot\zeta_t + \int_\Omega u_0\cdot\zeta(\cdot,0) + \int_0^\infty\int_\Omega u\otimes u\cdot\nabla\zeta = \int_0^\infty\int_\Omega \nabla u\cdot\nabla\zeta - \int_0^\infty\int_\Omega (n+\rho)\nabla\phi\cdot\zeta.$$

We shall invoke the global classical solutions to the regularized problem (2.1) to approximate the weak solution to the problem (1.4). To this end, we need some further regularity properties for the global classical solutions to the problem (2.1).

Lemma 4.2. *Let $m > 1$, $l > 0$, $\mu > 0$, $\beta \in (1/2, 1)$, and $(n_\epsilon, c_\epsilon, \rho_\epsilon, u_\epsilon, p_\epsilon)$ be the global classical solution to problem* (2.1) *established in Lemma 2.1. Then there exists some positive constant C independent of ϵ such that*

$$\|n_\epsilon\|_{L^\infty(\Omega\times(0,\infty))} \le C, \tag{4.2}$$

$$\|\rho_\epsilon\|_{L^\infty(\Omega\times(0,\infty))} \le C, \tag{4.3}$$

$$\|c_\epsilon\|_{L^\infty(0,\infty;W^{1,\infty}(\Omega))} \le C, \tag{4.4}$$

$$\|u_\epsilon\|_{L^\infty(0,\infty;L^\infty(\Omega))} \le C, \tag{4.5}$$

$$\|A^\beta u_\epsilon\|_{L^\infty(0,\infty;L^2(\Omega))} \le C. \tag{4.6}$$

Moreover, for any $T > 0$, $k > m-1$, and $q > \max\{0, l-1\}$, we have

$$\int_0^T \int_\Omega |\nabla(n_\epsilon+\epsilon)^k|^2 \le C(m,k,T), \tag{4.7}$$

$$\int_0^\infty \int_\Omega |\nabla(\rho_\epsilon+\epsilon)^q|^2 \le C(l,q,\rho_0,\Omega). \tag{4.8}$$

Proof. In the previous section, we have already proved that $T_{\max,\epsilon} = \infty$. Hence, the uniform estimates (4.2)-(4.6) follow directly from Lemma 3.11 and (3.2). To prove (4.7), we integrate (3.69) over $(0,T)$ to obtain

$$\frac{1}{p}\int_\Omega (n_\epsilon+\epsilon)^p(\cdot,T) + \frac{m(p-1)}{4}\int_0^T\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 \le C_2T + \frac{1}{p}\int_\Omega (n_0+1)^p$$

for any $p > m-1$, where we use $\epsilon \in (0,1)$. By denoting $k := \frac{m+p-1}{2}$, then $k > m-1$, we have

$$\begin{aligned}
\frac{m(p-1)}{4}\int_0^T\int_\Omega (n_\epsilon+\epsilon)^{m+p-3}|\nabla n_\epsilon|^2 &= \frac{m(p-1)}{(m+p-1)^2}\int_0^T\int_\Omega |\nabla (n_\epsilon+\epsilon)^{\frac{m+p-1}{2}}|^2 \\
&= \frac{m(2k-m)}{4k^2}\int_0^T\int_\Omega |\nabla (n_\epsilon+\epsilon)^k|^2 \\
&\le C_2T + \frac{1}{2k+1-m}\int_\Omega (n_0+1)^{2k+1-m},
\end{aligned}$$

thus, we have (4.7) as desired.

To prove (4.8), multiplying the second equation in (2.1) by $(\rho_\epsilon+\epsilon)^{p-1}$ for any $p > \max\{1, l-1\}$ yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (\rho_\epsilon+\epsilon)^p + \frac{l(p-1)}{(l+p-1)^2}\int_\Omega |\nabla(\rho_\epsilon+\epsilon)^{\frac{l+p-1}{2}}|^2 = -\mu\int_\Omega (n_\epsilon+\epsilon)^{p-1}n_\epsilon\rho_\epsilon \le 0$$

for $t > 0$. Integrating over $(0,T)$ gives

$$\frac{l(p-1)}{(l+p-1)^2}\int_0^T\int_\Omega |\nabla(\rho_\epsilon+\epsilon)^{\frac{l+p-1}{2}}|^2 \le \frac{1}{p}\int_\Omega (\rho_0+1)^p,$$

and letting $q = \frac{l+p-1}{2}$ shows that (4.8) holds. □

On the basis of Lemma 4.2, we can deduce some regularity properties of time derivatives of certain powers of n_ϵ and ρ_ϵ, which shall be used to pass to the limit in the first two equations of problem (2.1).

Lemma 4.3. *Let $m > 1$, $l > 0$, $\gamma > \max\{1, m-1\}$, and $\lambda > \max\{1, l-1\}$. Then for $T > 0$ there exists positive constant $C(T)$ such that*

$$\int_0^T \|\partial_t (n_\epsilon+\epsilon)^\gamma\|_{(W_0^{2,2}(\Omega))^*}dt \le C(T) \quad \text{for } \epsilon \in (0,1), \tag{4.9}$$

$$\int_0^T \|\partial_t (\rho_\epsilon+\epsilon)^\lambda\|_{(W_0^{2,2}(\Omega))^*}dt \le C(T) \quad \text{for } \epsilon \in (0,1). \tag{4.10}$$

Proof. The process is similar to the proof of [27, Lemma 3.22], see also [16, Lemma 3.3]. For any $\psi \in W_0^{2,2}(\Omega)$, by a standard testing procedure we have

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma\psi \\
&= -m\gamma(\gamma-1)\int_\Omega (n_\epsilon+\epsilon)^{m+\gamma-3}|\nabla n_\epsilon|^2\psi - m\gamma\int_\Omega (n_\epsilon+\epsilon)^{m+\gamma-2}\nabla n_\epsilon\cdot\nabla\psi \\
&\quad + \gamma(\gamma-1)\int_\Omega n_\epsilon(n_\epsilon+\epsilon)^{\gamma-2}\nabla n_\epsilon\cdot(S_\epsilon\nabla c_\epsilon)\psi + \gamma\int_\Omega n_\epsilon(n_\epsilon+\epsilon)^{\gamma-1}(S_\epsilon\nabla c_\epsilon)\cdot\nabla\psi \\
&\quad + \int_\Omega (n_\epsilon+\epsilon)^\gamma u_\epsilon\cdot\nabla\psi - \mu\gamma\int_\Omega (n_\epsilon+\epsilon)^{\gamma-1}n_\epsilon\rho_\epsilon\psi
\end{aligned} \tag{4.11}$$

$$=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

To estimate $J_1, \dots, J_6$ term by term, we first invoke Lemma 4.2 to fix some positive constants C_1, C_2, C_3 and C_4 independent of ϵ such that

$$n_\epsilon \le C_1, \quad |\nabla c_\epsilon| \le C_2, \quad |u_\epsilon| \le C_3, \quad |\rho_\epsilon| \le C_4 \quad \text{in } \Omega \times (0,\infty). \tag{4.12}$$

Then we have (actually, the terms J_1, J_2, J_3, J_4, J_5 are the same as that in [16, Lemma 3.3])

$$|J_1| \le \frac{4m\gamma(\gamma-1)}{(m+\gamma-1)^2} \|\psi\|_{L^\infty(\Omega)} \int_\Omega \big|\nabla (n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\big|^2, \tag{4.13}$$

$$|J_2| \le \frac{m\gamma}{2(m+\gamma-1)} \|\nabla\psi\|_{L^2(\Omega)} \Big(\int_\Omega \big|\nabla (n_\epsilon+\epsilon)^{\gamma+m-1}\big|^2 + 1\Big), \tag{4.14}$$

$$\begin{aligned} |J_3| &\le (\gamma-1)\|\nabla c_\epsilon\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(\Omega)} \int_\Omega |S_\epsilon| \big|\nabla (n_\epsilon+\epsilon)^\gamma\big| \\ &\le (\gamma-1)\chi C_2 \|\psi\|_{L^\infty(\Omega)} \int_\Omega |\nabla (n_\epsilon+\epsilon)^\gamma|, \end{aligned} \tag{4.15}$$

$$|J_4| \le \gamma \int_\Omega (n_\epsilon+\epsilon)^\gamma |S_\epsilon||\nabla c_\epsilon||\nabla\psi| \le \gamma\chi(C_1+1)^\gamma C_2 |\Omega|^{\frac12} \|\nabla\psi\|_{L^2(\Omega)}, \tag{4.16}$$

$$|J_5| \le \int_\Omega (n_\epsilon+\epsilon)^\gamma |u_\epsilon||\nabla\psi| \le (C_1+1)^\gamma C_3 |\Omega|^{\frac12} \|\nabla\psi\|_{L^2(\Omega)}, \tag{4.17}$$

$$|J_6| \le \mu\gamma \int_\Omega (n_\epsilon+\epsilon)^{\gamma-1} n_\epsilon \rho_\epsilon |\psi| \le \mu\gamma (C_1+1)^\gamma C_4 \|\psi\|_{L^\infty(\Omega)} |\Omega|. \tag{4.18}$$

From $W_0^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, combining (4.11) with (4.13)-(4.18) we infer that there exists $C_5 > 0$ independent of ϵ such that

$$\begin{aligned} \Big| \int_\Omega \frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma \psi \Big| &\le C_5 \Big(\int_\Omega \big|\nabla (n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\big|^2 + \int_\Omega \big|\nabla (n_\epsilon+\epsilon)^{\gamma+m-1}\big|^2 \\ &\quad + \int_\Omega \big|\nabla (n_\epsilon+\epsilon)^\gamma\big| + 1\Big) \|\psi\|_{W_0^{2,2}(\Omega)}. \end{aligned} \tag{4.19}$$

Since $\gamma > m-1$ which implies $\frac{\gamma+m-1}{2} > m-1$ and $\gamma+m-1 > m-1$, for any $T > 0$, by (4.7), we can find $C_6(T) > 0$ such that

$$\int_0^T \int_\Omega \big|\nabla (n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\big|^2 + \int_0^T \int_\Omega \big|\nabla (n_\epsilon+\epsilon)^{\gamma+m-1}\big|^2 + \int_0^T \int_\Omega \big|\nabla (n_\epsilon+\epsilon)^\gamma\big| \le C_6(T).$$

Hence, (4.19) implies

$$\int_0^T \|\frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma\|_{(W_0^{2,2}(\Omega))^*} dt \le C_5(C_6+T).$$

This proves (4.9). By similar procedure, (4.10) can be achieved. □

Similar to [16, Lemma 3.2] and [27, Lemmas 3.18 and 3.19], by using the standard parabolic regularity theory, we can further establish some uniform Hölder regularity properties of $c_\epsilon, \nabla c_\epsilon$ and u_ϵ.

Lemma 4.4. *Let $m > 1$, $l > 0$, $\mu > 0$. Then there exists $\sigma \in (0,1)$ and some positive constant C independent of ϵ such that*

$$\|c_\epsilon\|_{C^{\sigma,\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} \le C \quad \text{for } t \ge 0, \tag{4.20}$$

$$\|u_\epsilon\|_{C^{\sigma,\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} \le C \quad \text{for } t \ge 0. \tag{4.21}$$

Moreover, for each $t_0 > 0$, *we can find* $C(t_0)) > 0$ *such that*

$$\|\nabla c_\epsilon\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C(t_0) \quad \text{for } t \ge t_0. \tag{4.22}$$

With the help of these a priori estimates, we can extract suitable subsequences of global classical solutions of (2.1) in a standard manner to approximate the global weak solution of (1.4). The proof is similar to [16, Lemma 3.4], [3, Lemma 5.4], we omit the details here.

Lemma 4.5. *Let* $m > 1$, $l > 0$, $\mu > 0$. *Then there exist a quadruple of functions* (n, c, ρ, u) *satisfying*

$$\begin{gathered}
n \in L^\infty((0,\infty); L^\infty(\Omega)), \quad \nabla n^m \in L^2_{\rm loc}((0,\infty); L^2(\Omega)),\\
\rho \in L^\infty((0,\infty); L^\infty(\Omega)), \quad \nabla \rho^l \in L^2((0,\infty); L^2(\Omega)),\\
c \in L^\infty((0,\infty); W^{1,\infty}(\Omega)), \quad u \in L^\infty((0,\infty); W^{1,2}_{0,\sigma}(\Omega)),\\
\nabla \cdot u = 0 \text{ in the distributional sense in } \Omega \times (0,\infty),
\end{gathered}$$

and a subsequence $\{\epsilon_j\}_{j=1}^\infty$ *converging to zero as* $j \to \infty$ *such that*

$$n_{\epsilon_j} \rightharpoonup n, \ \rho_{\epsilon_j} \rightharpoonup \rho \quad \text{weakly * in } L^\infty(\Omega \times (0,\infty)), \tag{4.23}$$

$$\nabla n^m_{\epsilon_j} \rightharpoonup \nabla n^m, \quad \text{in } L^2_{\rm loc}((0,\infty); L^2(\Omega)), \tag{4.24}$$

$$\nabla \rho^l_{\epsilon_j} \rightharpoonup \nabla \rho^l \quad \text{in } L^2((0,\infty); L^2(\Omega)), \tag{4.25}$$

$$c_{\epsilon_j} \rightharpoonup c, \quad \nabla c_{\epsilon_j} \rightharpoonup \nabla c \quad \text{weakly * in } L^\infty((0,\infty); L^\infty(\Omega)), \tag{4.26}$$

$$n_{\epsilon_j} S_{\epsilon_j}(x, n_{\epsilon_j}, c_{\epsilon_j}) \to nS(x,n,c) \quad \text{strongly in } L^2_{\rm loc}([0,\infty); L^2(\Omega)), \tag{4.27}$$

$$u_{\epsilon_j} \rightharpoonup u \quad \text{weakly * in } L^\infty((0,\infty); D(A^\beta))), \tag{4.28}$$

$$c_{\epsilon_j} \to c, \quad \nabla c_{\epsilon_j} \to \nabla c, \quad u_{\epsilon_j} \to u \quad \text{in } C^0_{\rm loc}(\overline{\Omega} \times [0,\infty)), \tag{4.29}$$

as $j \to \infty$.

For the prove Theorem 1.1, the existence of a global weak solutions to problem (1.4) is a consequence of Lemmas 4.5 and 4.2.

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