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FRACTIONAL-POWER APPROACH FOR THE STUDY OF ELLIPTIC SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH VARIABLE-OPERATOR COEFFICIENTS IN AN UNBOUNDED DOMAIN

FATIHA BOUTAOUS

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Abstract. In this article, we give new results on the study of elliptic complete abstract second order differential equations with variable operator coefficients under Dirichlet boundary conditions, and set in $\mathbb{R}_+$. In the framework of Hölderian spaces and under some compatibility conditions, we prove the main results on the existence, uniqueness and maximal regularity of the classical solution of this kind of problems which have not been studied in variable coefficients case. We use semigroups theory, fractional powers of linear operators, Dunford's functional calculus and interpolation theory. In this work, we consider some differentiability assumptions on the resolvents of square roots of linear operators.

1. Introduction

In a Complex Banach space X, we consider the complete second-order differential equation with variable operator coefficients,

$$u''(x) + B(x)u'(x) + A(x)u(x) - \lambda u(x) = f(x), \quad x \in (0, +\infty), \tag{1.1}$$

under the Dirichlet nonhomogeneous boundary conditions

$$u(0) = \varphi, \quad u(+\infty) = 0. \tag{1.2}$$

Here λ is a positive real number, φ is a given element in X, $f \in C^{\theta}_{\infty}([0,\infty); X)$, $0 < \theta < 1$, where $C^{\theta}_{\infty}([0,\infty); X)$ is the space of bounded and θ-Hölder continuous-vector valued functions $\phi : [0,\infty) \to X$ such that

$$\begin{gathered} \sup_{x \geq 0} \|\phi(x)\|_X < +\infty, \\ \exists C > 0 : \forall x,\ s \geq 0,\ \|\phi(x) - \phi(s)\|_X \leq C|x - s|^{\theta}, \\ \text{with } \lim_{x \to +\infty} \phi(x) = 0, \end{gathered}$$

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endowed with the norm

$$\|\phi\|_{C^\theta_\infty([0,\infty);X)} := \sup_{x\geq 0}\|\phi(x)\|_X + \sup_{x\neq s}\frac{\|\phi(x)-\phi(s)\|_X}{|x-s|^\theta}. \tag{1.3}$$

$(B(x))_{x\geq 0}$ is a family of bounded linear operators, and $(A(x))_{x\geq 0}$ is a family of closed linear operators in X, with domains $D(A(x))$ not necessarily dense in X. Set

$$A_\lambda(x) = A(x) - \lambda I, \quad \lambda > 0.$$

We seek for a classical solution $u(\cdot)$ to Problem (1.1)-(1.2), that is

$$\begin{gathered} u \in C^2_\infty([0,\infty),X),\ u(x)\in D(A_\lambda(x)) \text{ for every } x\geq 0;\\ x\mapsto A_\lambda(x)u(x)\in C_\infty([0,\infty),X);\\ \text{and } u \text{ satisfies Problem (1.1)-(1.2).} \end{gathered} \tag{1.4}$$

We suppose that the family of operators $(B(x))_{x\geq 0}$ satisfies

$$\exists C>0 : \forall x\in[0,\infty),\ \|B(x)\|_{L(X)}\leq C. \tag{1.5}$$

The term $B(x)u'(x)$ is considered as a "perturbation" in some sense.

We consider Problem (1.1)-(1.2) in an elliptic situation: the family of linear closed operators $(A_\lambda(x))_{x\geq 0}$ satisfies

$$\begin{gathered} \exists C>0 : \forall x\geq 0, \forall z\geq 0, \quad \exists (A_\lambda(x)-zI)^{-1}\in L(X)\\ \text{and } \|(A_\lambda(x)-zI)^{-1}\|_{L(X)}\leq C/(1+z), \end{gathered} \tag{1.6}$$

which holds in some sector $\Pi_{\theta_0,r_0}\subset\rho(A_\lambda(x))$ (the resolvent set of $A_\lambda(x)$)

$$\Pi_{\theta_0,r_0} = \{z\in\mathbb{C}\backslash\{0\} : |\arg(z)|\leq\theta_0\}\cup\{z\in\mathbb{C} : |z|\leq r_0\},$$

where θ_0 and r_0 are small positive numbers. According to Assumption (1.6), for every $x\geq 0$ and every $\lambda>0$, the square roots

$$K_\lambda(x) = -(-A_\lambda(x))^{1/2}$$

are well defined and generate analytic semigroups $(e^{yK_\lambda(x)})_{y>0}$ not necessarily strongly continuous in 0 (see Balakrishnan [2] for dense domains and Martinez-Sanz [16] for nondense domains).

Equation (1.1) has been studied by several authors via various approaches:

- The constant case of operators $A(x) = A$ and $B(x) = B$. For the bounded interval, when $B = 0$, Krein [13] provided an approach based on the fractional powers of linear operators, while in [7] the authors provided another approach based on the sum theory of linear operators. Moreover, a study for the general case when $B(x) = B$, can be found in [9]. Now, for the interval $[0,\infty)$, when $B(x) = 0$, Berroug has used in [3] a method based on direct calculus using Dunford's operational calculus, while in [4] he studied Equation (1.1) by the Krein's method in the frame work of Hölderian spaces. For the general case, a study in the frame work of L^p spaces can be found in [8].
- In the variable case of operators $A(x)$ and $B(x)$. For the bounded interval, when $B(x) = 0$, a direct method based on Dunford's operational calculus has been used in [14] under some Hypotheses on differentiability of resolvents of operators $A_\lambda(x)$. Moreover, the case of bounded operators $B(x)$ has been studied in [10] by using the previous method. However, in [5] the authors used the Krein's approach, under some natural differentiability

Assumptions on the resolvents of the square roots $K_\lambda(x)$ combining those of Yagi [20], Da Prato-Grisvard [7] and Acquistapace-Terreni [1].

This article extends and improves the studies done in [5] and [6] by considering the interval $[0,\infty)$, where we study the existence, the uniqueness and the maximal regularity of the classical solution of Problem (1.1)-(1.2). In particular, we give necessary and sufficient conditions to obtain a unique classical solution of Problem (1.1)-(1.2) satisfying maximal regularity.

The remainder of this article is structured as follows: In Section 2, the Assumptions of this work are given and the representation of the solution of Problem (1.1)-(1.2) is built using the analytic semigroups and the Dunford's operational calculus. Section 3 is devoted to the study of the regularity of the classical solution, and Section 4 includes the equation verified by the solution and its resolution. In section 5, our main results on the existence, the uniqueness and the maximal regularity of the classical solution are proved. Finally, in section 6, we provide an example to which our abstract results apply.

2. Assumptions and construction of the solution

2.1. **Assumptions.** We begin by recalling that, from Hypothesis (1.6) it is well known that there exists a sector $S_{\theta_1+\pi/2,r_1} \subset \rho(K_\lambda(x))$ defined by

$$S_{\theta_1+\pi/2,r_1} = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \le \theta_1 + \pi/2\} \cup \{z \in \mathbb{C} : |z| \le r_1\},$$

with a small $\theta_1 > 0$ and $r_1 > 0$. Let us consider, the curve

$$\Gamma = \{z = \rho e^{\pm i(\theta_1+\pi/2)} : \rho \ge r_1\} \cup \{z \in \mathbb{C} : |z| = r_1, |\arg(z)| \ge \theta_1 + \pi/2\}$$

oriented from $\infty e^{-i(\theta_1+\pi/2)}$ to $\infty e^{i(\theta_1+\pi/2)}$. Therefore, for all $x \ge 0$, $y > 0$ and positive integer ω, we have

$$e^{yK_\lambda(x)} = -\frac{1}{2i\pi}\int_\Gamma e^{yz}(K_\lambda(x) - zI)^{-1}dz,$$
$$(K_\lambda(x))^\omega e^{yK_\lambda(x)} = -\frac{1}{2i\pi}\int_\Gamma z^\omega e^{yz}(K_\lambda(x) - zI)^{-1}dz,$$

and for all $z \ge 0$, $x \ge 0$, we have

$$(K_\lambda(x) - zI)^{-1} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{s}(A_\lambda(x) - sI)^{-1}}{s + z^2}\, ds.$$

This last equality has an analytic continuation (in z) in the sector $S_{\theta_1+\pi/2,r_1}$. (see [19, (2.32) p. 37]). Before giving the remaining assumptions of this work, we recall the following basic result that will be useful in the sequel.

Lemma 2.1. *Under Hypothesis* (1.6)*, there exists a constant $C > 0$, such that*

(1) $\forall z \in S_{\theta_1+\pi/2,r_1}$, $\forall x \ge 0$, $\|(K_\lambda(x) - zI)^{-1}\|_{L(x)} \le \frac{C}{|z|}$.
(2) $\exists \delta > 0 : \forall x \ge 0, \forall y > 0$, $\|e^{yK_\lambda(x)}\|_{L(X)} \le Ce^{-\delta y}$.
(3) $\exists \delta > 0 : \forall \alpha \in \mathbb{N}, \forall x \ge 0, \forall y > 0$, $\|(K_\lambda(x))^\alpha e^{yK_\lambda(x)}\|_{L(X)} \le Cy^{-\alpha}e^{-\delta y}$.

The proof of statement (1) is based on analytic semigroup's properties. While the proof of (2) and (3) can be found in [17, Theorem 6.13, p. 74].

In addition to Assumptions (1.5)-(1.6), we assume that for all $z \in S_{\theta_1+\pi/2,r_1}$, the mapping $x \mapsto (K_\lambda(x) - zI)^{-1}$, defined on $[0,\infty)$, is in $C^2([0,\infty), L(X))$ and there exist $C > 0$, $\rho \in (1/2, 1]$ and $\alpha \in (0,1)$ such that for all $z \in S_{\theta_1+\pi/2,r_1}$ and all $x, s \in [0,+\infty)$,

$$\|\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\|_{L(X)} \le \frac{C}{|z|^\rho}\,, \tag{2.1}$$

$$\begin{gathered}\|\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1} - \frac{\partial}{\partial s}(K_\lambda(s)-zI)^{-1}\|_{L(X)} \le \frac{C|x-s|^\alpha}{|z|^\rho} \\ \text{with } \alpha+\rho-1>0\,,\end{gathered} \tag{2.2}$$

$$\|\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}\|_{L(X)} \le C|z|^{1-\rho}, \tag{2.3}$$

$$\|\frac{d^2}{dx^2}(K_\lambda(x))^{-1} - \frac{d^2}{ds^2}(K_\lambda(s))^{-1}\|_{L(X)} \le C|x-s|^\alpha. \tag{2.4}$$

$$B(0)(X) \subset \overline{D(K_\lambda(0))} = \overline{D(A(0))}, \tag{2.5}$$

$$\frac{d}{dx}(K_\lambda(x))^{-1}\big|_{x=0}(D(K_\lambda(0))) \subset \overline{D(K_\lambda(0))} = \overline{D(A(0))}\,, \tag{2.6}$$

To study the regularity of the solution we give an essential result.

Lemma 2.2. *Under Hypotheses* (1.6) *and* (2.1)*, we have:*

(1) *The map $x \mapsto e^{xK_\lambda(x)}\varphi$ belongs to the space $C([0,\infty); X)$ if and only if $\varphi \in \overline{D(K_\lambda(0))} = \overline{D(A(0))}$, in this case $\lim_{x\to 0} e^{xK_\lambda(x)}\varphi = \varphi$.*

(2) *The map $x \mapsto e^{xK_\lambda(0)}\varphi$ belongs to the space $C^\theta([0,\infty); X)$ if and only if $\varphi \in D_{K_\lambda(0)}(\theta,+\infty) = D_{A(0)}(\theta/2,+\infty)$, where*

$$D_{K_\lambda(0)}(\theta,+\infty) = \{\phi \in X : \sup_{r>0}\|r^\theta K_\lambda(0)(K_\lambda(0)-rI)^{-1}\phi\|_X < +\infty\}$$

is the known real interpolation space defined, for instance, in [11].

Proof. We just give the sketches of the proof.

(1) For the sake of the proof, we will write

$$e^{xK_\lambda(x)}\varphi = e^{xK_\lambda(0)}\varphi + (e^{xK_\lambda(x)} - e^{xK_\lambda(0)})\varphi.$$

Thanks to Sinestrari [18, Proposition 1.2-(i), p. 20], we obtain

$$x \mapsto e^{xK_\lambda(0)}\varphi \in C([0,\infty); X) \text{ if and only if } \varphi \in \overline{D(K_\lambda(0))} = \overline{D(A(0))}.$$

This last equality follows from the properties of fractional powers of sectorial operators, see Haase [12]. Moreover, from (2.1) we obtain

$$\begin{aligned}\|(e^{xK_\lambda(x)} - e^{xK_\lambda(0)})\varphi\|_X &= \|\int_0^x \frac{\partial}{\partial\xi}e^{xK_\lambda(\xi)}\varphi d\xi\|_X \\ &\le Cx^\rho\|\varphi\|_X \to 0, \quad \text{as } x \to 0.\end{aligned}$$

Hence $\lim_{x\to 0} e^{xK_\lambda(x)}\varphi = \varphi$.

Item (2) follows by a similar proof as the one in [1, Proposition 3.4-(iii), p. 26] and in [18, Theorem 3.1-(b) and (f), p. 39]. Finally, the equality $D_{K_\lambda(0)}(\theta,+\infty) = D_{A(0)}(\theta/2,+\infty)$ follows from the Lions-Peetre reiteration interpolation property given in [15]. $\square$

Remark 2.3. All the constants given above are independent of x and always, we have $\alpha+\rho-1 < \alpha$ and $\alpha+\rho-1 < \rho$. Moreover, we can replace z by $\sqrt{\lambda}+z$ in Assumptions (2.1), (2.2) and (2.3).

2.2. **Construction of the solution.** First, we recall briefly that in the case $B \equiv 0$, $A(x) = A$ is a complex scalar $z \in \mathbb{C} \backslash \mathbb{R}_+$, such that $\sqrt{-z}$ is the analytic determination defined by $\operatorname{Re}(\sqrt{-z}) > 0$. By using the method based on the variation of constant and Green's Kernel, the solution of Problem (1.1)-(1.2) is

$$u(x) = e^{-x\sqrt{-z}}\varphi - \int_0^{+\infty} k_{\sqrt{-z}}(x,s)f(s)\,ds,$$

where

$$k_{\sqrt{-z}}(x,s) = \begin{cases} \frac{e^{-\sqrt{-z}x}\sinh\sqrt{-z}s}{\sqrt{-z}}, & \text{if } 0 \leqslant s \leqslant x, \\ \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}x}{\sqrt{-z}}, & \text{if } \ x \leqslant s. \end{cases}$$

The later formula can be also written as

$$k_{\sqrt{-z}}(x,s) = \begin{cases} \frac{e^{-\sqrt{-z}(x-s)}}{2\sqrt{-z}}(1 - e^{-2\sqrt{-z}s}), & \text{if } 0 \leqslant s \leqslant x, \\ \frac{e^{-\sqrt{-z}(s-x)}}{2\sqrt{-z}}(1 - e^{-2\sqrt{-z}x}), & \text{if } x \leqslant s. \end{cases}$$

According to the Dunford's functional calculus and the definition of analytic semigroups generated by the square roots $K = -(-A)^{1/2}$, the solution of Problem (1.1)-(1.2) can be written as

$$\begin{aligned} u(x) &= -\frac{1}{2\pi i}\int_\Gamma e^{-\sqrt{-z}x}(K - zI)^{-1}\varphi\,dz \\ &\quad + \frac{1}{4\pi i}\int_\Gamma\int_0^x \frac{e^{-\sqrt{-z}(x-s)}}{\sqrt{-z}}(1 - e^{-2\sqrt{-z}s})(K - zI)^{-1}f(s)\,ds\,dz \\ &\quad + \frac{1}{4\pi i}\int_\Gamma\int_x^{+\infty} \frac{e^{-\sqrt{-z}(s-x)}}{\sqrt{-z}}(1 - e^{-2\sqrt{-z}x})(K - zI)^{-1}f(s)dsdz \\ &= e^{xK}\varphi - \frac{1}{2}\int_0^{+\infty} e^{(x+s)K}K^{-1}f(s)ds + \frac{1}{2}\int_0^x e^{(x-s)K}K^{-1}f(s)ds \\ &\quad + \frac{1}{2}\int_x^{+\infty} e^{(s-x)K}K^{-1}f(s)ds. \end{aligned}$$

It is worth noting that so far the operators $A(x)$ were considered constant. From now on, operators $A(x)$ will be variable and satisfy the natural ellipticity Hypothesis (1.6). We now consider the following representation of the solution u (with $K_\lambda(x) = -(-A_\lambda(x))^{1/2}$),

$$\begin{aligned} u(x) &= e^{xK_\lambda(x)}\varphi^* - \frac{1}{2}\int_0^{+\infty} e^{(x+s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\ &\quad + \frac{1}{2}\int_0^x e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\ &\quad + \frac{1}{2}\int_x^{\infty} e^{(s-x)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds, \end{aligned} \tag{2.7}$$

where φ^* and g^* are unknown elements to be determined in an adequate space ($g^* \in C^\beta_\infty([0,\infty);X)$, $(0 < \beta < 1)$) see ((1.3), in order to obtain a classical solution u of Problem (1.1)-(1.2), see (1.4). A formal calculus gives

$$u(0) = \varphi^* = \varphi \quad \text{and} \quad u(+\infty) = 0.$$

For this last condition, we can have

$$\begin{aligned}
&\int_0^x e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\
&= \int_0^{x/2} e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)ds + \int_{x/2}^x e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\
&= (I) + (II).
\end{aligned}$$

Therefore, by Lemma 2.1-(2), we obtain

$$\begin{aligned}
\|(I)\|_X &\le C\|g^*\|_{C^\beta_\infty([0,\infty);X)}\Big(\int_{x/2}^x e^{-\delta(x-s)}ds\Big) \\
&\le C\|g^*\|_{C^\beta_\infty([0,\infty);X)}\Big(e^{-\frac{\delta x}{2}} - e^{-\delta x}\Big) \to 0, \quad \text{as } x \to +\infty.
\end{aligned}$$

Since $g^* \in C^\beta_\infty([0,\infty);X)$ we have $\lim_{x\to+\infty}\sup_{s\in[\frac{x}{2},x]}\|g^*(s)\|_X = 0$, which leads to

$$\begin{aligned}
\|(II)\|_X &\le C\sup_{s\in[\frac{x}{2},x]}\|g^*(s)\|_X\Big(\int_{x/2}^x e^{-\delta(x-s)}ds\Big) \\
&\le C\sup_{s\in[\frac{x}{2},x]}\|g^*(s)\|_X\Big(1 - e^{-\delta x/2}\Big) \to 0, \quad \text{as } x \to +\infty.
\end{aligned}$$

Now, it suffices to seek g^* in an appropriate space such that the following representation

$$\begin{aligned}
u(x) &= e^{xK_\lambda(x)}\varphi - \frac{1}{2}\int_0^{+\infty} e^{(x+s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\
&\quad + \frac{1}{2}\int_0^x e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\
&\quad + \frac{1}{2}\int_x^{+\infty} e^{(s-x)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\
&= u_0(x) + m_0(x,g^*) + w(x,g^*),
\end{aligned} \tag{2.8}$$

where $w(x,g^*)$ is defined by the last two integrals, gives a classical solution for Problem (1.1)-(1.2). By Lemma 2.1, all these integrals are absolutely convergent. Indeed, for the last term in (2.8), one can write (the other terms will be treated likewise)

$$\begin{aligned}
&\|\frac{1}{2}\int_x^{+\infty} e^{(s-x)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds\|_X \\
&\le C\int_x^{+\infty} e^{-(s-x)\delta}\|(K_\lambda(x))^{-1}g^*(s)\|_X ds \\
&\le C\|g^*\|_{C_\infty([0,\infty);X)} < +\infty.
\end{aligned}$$

3. Regularity of the solution

From the previous representation of the solution (see (2.8)), let us start this section by considering the operator defined by

$$Op(u)(x) = u''(x) + B(x)u'(x) + A_\lambda(x)u(x), \tag{3.1}$$

for all $x > 0$. Now, we shall analyze the behavior of (3.1) near 0.

3.1. **Regularity of operator** $Op(u_0)$**.** To study the regularity of the function $x \mapsto Op(u_0(x))$, where

$$u_0(x) = e^{xK_\lambda(x)}\varphi = -\frac{1}{2i\pi}\int_\Gamma e^{xz}(K_\lambda(x) - zI)^{-1}\varphi\,dz,$$

we first study the behavior of semigroups $e^{xK_\lambda(x)}\varphi$ and their derivatives $\frac{d}{dx}(e^{xK_\lambda(x)})\varphi$ and $\frac{d^2}{dx^2}(e^{xK_\lambda(x)})\varphi$ near 0. For this purpose, we need the use of Lemmas 2.1, 2.2 and the basic results proved in [5, Section 3, p. 7-15].

Proposition 3.1. *Let* $\varphi \in D(A(0))$*. Under Assumptions* (1.5)–(2.6)*, the function* $x \mapsto Op(u_0(\cdot))(x)$ *belongs to* $C^{\min(\alpha,\rho)}([0, +\infty[; X)$*.*

Proof. For $x > 0$, we have

$$A_\lambda(x)u_0(x) = -(K_\lambda(x))^2 e^{xK_\lambda(x)}\varphi = \frac{1}{2i\pi}\int_\Gamma z^2 e^{xz}(K_\lambda(x) - zI)^{-1}\varphi\,dz,$$

and

$$\begin{aligned} u_0'(x) &= \frac{d}{dx}e^{xK_\lambda(x)}\varphi \\ &= -\frac{1}{2i\pi}\int_\Gamma z e^{xz}(K_\lambda(x) - zI)^{-1}\varphi\,dz - \frac{1}{2i\pi}\int_\Gamma e^{xz}\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1}\varphi\,dz, \end{aligned}$$

which leads to

$$\begin{aligned} B(x)u_0'(x) &= B(x)\frac{d}{dx}e^{xK_\lambda(x)}\varphi \\ &= B(x)K_\lambda(x)e^{xK_\lambda(x)}\varphi - \frac{B(x)}{2i\pi}\int_\Gamma e^{xz}\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1}\varphi\,dz. \end{aligned}$$

By [5, Lemma 3.5 p. 10] we can prove that

$$x \mapsto B(x)u_0'(x) \in C^{\min(\alpha,\rho)}([0,\infty); X), \quad B(x)u_0'(x) \to B(0)K_\lambda(0)\varphi,$$

as $x \to 0$. Moreover,

$$\begin{aligned} u_0''(x) &= \frac{d^2}{dx^2}\big(e^{xK_\lambda(x)}\varphi\big) \\ &= -\frac{1}{2i\pi}\int_\Gamma z^2 e^{xz}(K_\lambda(x) - zI)^{-1}\varphi\,dz - \frac{1}{i\pi}\int_\Gamma z e^{xz}\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1}\varphi\,dz \\ &\quad - \frac{1}{2i\pi}\int_\Gamma e^{xz}\frac{\partial^2}{\partial x^2}(K_\lambda(x) - zI)^{-1}\varphi\,dz. \end{aligned}$$

By [5, Lemmas 3.6, 3.7 and 3.8], all these integrals are absolutely convergent. Therefore,

$$\begin{aligned} Op(u_0(x)) &= -\frac{1}{i\pi}\int_\Gamma z e^{xz}\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1}\varphi\,dz \\ &\quad - \frac{1}{2i\pi}\int_\Gamma e^{xz}\frac{\partial^2}{\partial x^2}(K_\lambda(x) - zI)^{-1}\varphi\,dz \\ &\quad + B(x)e^{xK_\lambda(x)}\varphi - \frac{B(x)}{2i\pi}\int_\Gamma e^{xz}\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1}\varphi\,dz \qquad (3.2) \\ &= G_\lambda(x)\varphi = \sum_{i=1}^{4} G_{\lambda i}(x)\varphi. \end{aligned}$$

Now, by [5, Lemma 3.7 p. 13]) we can show that

$$x \mapsto G_{\lambda 1}(x)\varphi \in C^{\min(\alpha,\rho)}([0,\infty);X), \quad G_{\lambda 1}(x)\varphi \to 0, \quad \text{as } x \to 0.$$

According to [5, Lemma 3.8 p. 14]) we obtain that, for all $x > 0$,

$$G_{\lambda,2}(x)\varphi = e^{xK_\lambda(0)}(-\frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0}(K_\lambda(0)\varphi)) + G(x)\varphi,$$

where the function $x \mapsto G(x)\varphi$ belongs to $C^{\min(\alpha,\rho)}([0,\infty);X)$. Moreover, from Lemma 2.2-(1) we obtain

$$G_{\lambda,2}(x)\varphi \to -\frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0}(K_\lambda(0))\varphi, \quad \text{as } x \to 0$$

if and only if

$$-\frac{d^2}{dx^2}\big(K_\lambda(x)\big)^{-1}\big|_{x=0}(K_\lambda(0))\varphi \in \overline{D(K_\lambda(0))} = \overline{D(A(0))}.$$

On the other hand, the function $x \mapsto G_{\lambda 3}(x)\varphi$ belongs to the space $C([0,\infty);X)$ because of $\varphi \in D(A(0))$, see Lemma 2.2-(1). Finally, as in [5, Lemma 3.5 p. 10]) it follows that $G_{\lambda 4}(x)\varphi \to 0$ as $x \to 0$. Consequently,

$$Op(u_0) \in C^{\min(\alpha,\rho)}([0,\infty);X).$$

□

3.2. **Regularity of operator $Op(m_0)$.** Observe that by using Dunford functional calculus, one can write for all $x > 0$,

$$\begin{aligned} m_0(x,g^*) &= -\frac{1}{2}\int_0^{+\infty} e^{(x+s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds \\ &= \frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty} e^{(x+s)z}(K_\lambda(x)-zI)^{-1}(K_\lambda(x))^{-1}g^*(s)dsdz. \end{aligned}$$

In the sequel, we will treat very carefully the convergence of all integrals obtained with respect to the variables z on the curve Γ and s at $+\infty$.

Proposition 3.2. *Assume* (1.5)–(2.6). *Then the function* $x \mapsto Op(m_0(\cdot,g^*))(x)$ *belongs to* $C_\infty^{\alpha+\rho-1}([0,\infty);X)$.

Proof. For $g^* \in C_\infty^\beta([0,\infty);X)$, we have

$$\begin{aligned} A_\lambda(x)m_0(x,g^*) &= \frac{1}{2}\int_0^{+\infty} K_\lambda(x)e^{(x+s)K_\lambda(x)}g^*(s)\,ds \\ &= -\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty} ze^{(x+s)z}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz. \end{aligned}$$

This term is well defined. Indeed, for $z \in \Gamma$ and $|z|$ sufficiently large, we have

$$|e^{(x+s)z}| \le e^{-(x+s)\operatorname{Re}(z)} = e^{-C_0(x+s)|z|}, \quad \text{where } C_0 = \sin\theta_1.$$

It follows that

$$\begin{aligned} \|A_\lambda(x)m_0(x,g^*)\|_X &\le C\int_\Gamma\Big(\int_0^{+\infty} e^{-C_0(x+s)|z|}ds\Big)\|g^*\|_{C_\infty([0,\infty);X)}d|z| \\ &\le C\Big(\int_\Gamma \frac{e^{-C_0x|z|}}{|z|}d|z|\Big)\|g^*\|_{C_\infty([0,\infty);X)} < +\infty. \end{aligned}$$

Now, since

$$(K_\lambda(x)-zI)^{-1}(K_\lambda(x))^{-1}=\frac{1}{z}[(K_\lambda(x)-zI)^{-1}-(K_\lambda(x))^{-1}],$$

we can write

$$m_0(x,g^*)=\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty}\frac{e^{(x+s)z}}{z}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz.$$

Therefore,

$$\begin{aligned}m_0'(x,g^*)&=\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty}e^{(x+s)z}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz,\\&\quad+\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty}\frac{e^{(x+s)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz,\end{aligned}$$

and

$$\begin{aligned}B(x)m_0'(x,g^*)&=\frac{B(x)}{4i\pi}\int_\Gamma\int_0^{+\infty}e^{(x+s)z}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz,\\&\quad+\frac{B(x)}{4i\pi}\int_\Gamma\int_0^{+\infty}\frac{e^{(x+s)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}m_0''(x,g^*)&=\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty}ze^{(x+s)z}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\&\quad+\frac{1}{2i\pi}\int_\Gamma\int_0^{+\infty}e^{(x+s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\&\quad+\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty}\frac{e^{(x+s)z}}{z}\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz.\end{aligned}$$

Hence

$$\begin{aligned}Op(m_0(x,g^*))&=\frac{1}{2i\pi}\int_\Gamma\int_0^{+\infty}e^{(x+s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\&\quad+\frac{1}{4i\pi}\int_\Gamma\int_0^{+\infty}\frac{e^{(x+s)z}}{z}\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\&\quad+\frac{B(x)}{4i\pi}\int_\Gamma\int_0^{+\infty}e^{(x+s)z}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\&\quad+\frac{B(x)}{4i\pi}\int_\Gamma\int_0^{+\infty}\frac{e^{(x+s)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\&=T_\lambda(g^*)(x)=\sum_{i=1}^{4}T_{\lambda i}(g^*)(x).\end{aligned}\tag{3.3}$$

Now, by Hypotheses (2.1)-(2.3), all these integrals are absolutely convergent. In fact, for the first and second terms (similarly we treat the other terms), by Hypothesis ((2.2), we have

$$\begin{aligned}&T_{\lambda 1}(g^*)(x)\\&=\frac{1}{2i\pi}\int_\Gamma\int_0^1 e^{(x+s)z}\Big(\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}-\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\big|_{x=0}\Big)\,ds\,dz\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{2i\pi}\int_\Gamma\int_0^1 e^{(x+s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\big|_{x=0}\,ds\,dz\\
&+ \frac{1}{2i\pi}\int_\Gamma\int_1^{+\infty} e^{(x+s)z}\Big(\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}-\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\big|_{x=0}\Big)\,ds\,dz\\
&+ \frac{1}{2i\pi}\int_\Gamma\int_1^{+\infty} e^{(x+s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\big|_{x=0},\,ds\,dz\\
&= (a_1)+(a_2)+(a_3)+(a_4).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(a_1)\|_X &\le C\int_0^1\int_\Gamma e^{-C_0(x+s)|z|}\frac{x^\alpha}{|z|^\rho}\|g^*\|_{C_\infty([0,\infty);X)}d|z|ds\\
&\le C\int_0^1\int_\Gamma \frac{e^{-C_0\sigma}}{\sigma^\rho}\frac{d\sigma}{x+s}(x+s)^\rho\|g^*\|_{C_\infty([0,\infty);X)}ds\\
&\le Cx^\alpha((x+1)^\rho-x^\rho)\|g^*\|_{C_\infty([0,\infty);X)}\\
&\le Cx^{\alpha+\rho-1}\|g^*\|_{C_\infty([0,\infty);X)}.
\end{aligned}$$

Similarly, it follows that

$$\begin{gathered}
\|(a_2)\|_X \le C((x+1)^\rho-x^\rho)\|g^*\|_{C_\infty([0,\infty);X)}<+\infty,\\
\|(a_3)\|_X \le Cx^\alpha\Big(\int_\Gamma\frac{e^{-C_0x|z|}}{|z|^{\rho+1}}d|z|\Big)\|g^*\|_{C_\infty([0,\infty);X)}<+\infty,\\
\|(a_4)\|_X \le C\Big(\int_\Gamma\frac{e^{-C_0x|z|}}{|z|^{\rho+1}}d|z|\Big)\|g^*\|_{C_\infty([0,\infty);X)}<+\infty\,.
\end{gathered}$$

From (2.3), it follows that

$$\|T_{\lambda 2}(g^*)(x)\|_X \le C\Big(\int_\Gamma\frac{e^{-C_0x|z|}}{|z|^{\rho+1}}d|z|\Big)\|g^*\|_{C_\infty([0,\infty);X)}<+\infty.$$

Finally, by Remark 2.3, it results that

$$Op(m_0)\in C_\infty^{\min(\alpha,\rho,\alpha+\rho-1)}([0,\infty);X)=C_\infty^{\alpha+\rho-1}([0,\infty);X).$$

□

3.3. Regularity of operator $Op(w)$.

Proposition 3.3. *Assume (1.5)–(2.6). Then the function $x\mapsto Op(w(\cdot,g^*))(x)$ belongs to $C_\infty^{\min(\beta,\alpha+\rho-1)}([0,\infty);X)$.*

Proof. Recall that for $x>0$, we have

$$\begin{aligned}
w(x,g^*) &= \frac12\int_0^x e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds\\
&\quad+\frac12\int_x^{+\infty} e^{(s-x)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)\,ds,
\end{aligned}$$

where $g^* \in C^\beta_\infty([0,\infty);X)$ (β will be specified, $0<\beta<1$). Thus

$$\begin{aligned} A_\lambda(x)w(x,g^*) &= -\frac{1}{2}\int_0^x K_\lambda(x)e^{(x-s)K_\lambda(x)}(g^*(s)-g^*(x))\,ds \\ &\quad -\frac{1}{2}\int_x^{+\infty} K_\lambda(x)e^{(s-x)K_\lambda(x)}(g^*(s)-g^*(x))\,ds \\ &\quad + g^*(x) - \frac{1}{2}e^{xK_\lambda(x)}g^*(x). \end{aligned} \tag{3.4}$$

From the properties of analytic semigroups and g^* being Hölder continuous, the first two integrals are absolutely convergent. For instance, for the second integral, we have

$$\begin{aligned} &\|-\frac{1}{4\pi i}\int_\Gamma\int_x^{+\infty} ze^{(s-x)z}(K_\lambda(x)-zI)^{-1}(g^*(s)-g^*(x))\,ds\,dz\|_X \\ &\le C\int_\Gamma\Big(\int_x^{+\infty} e^{-C_0(s-x)|z|}(s-x)^\beta ds\Big)\|g^*\|_{C^\beta_\infty([0,\infty);X)}d|z|. \end{aligned}$$

Now, by using the Hölder inequality, we obtain

$$\begin{aligned} &\int_x^{+\infty} e^{-C_0(s-x)|z|}(s-x)^\beta ds \\ &\le \Big(\int_x^{+\infty} e^{-C_0(s-x)|z|}ds\Big)^{1-\beta}\Big(\int_x^{+\infty} e^{-C_0(s-x)|z|}(s-x)\,ds\Big)^\beta \\ &\le \Big(\frac{C_1}{|z|}\Big)^{1-\beta}\Big(\frac{C_2}{|z|^2}\Big)^\beta \\ &\le \frac{C}{|z|^{1+\beta}}. \end{aligned}$$

Hence

$$\begin{aligned} &\|-\frac{1}{4\pi i}\int_\Gamma\int_x^{+\infty} ze^{(s-x)z}(K_\lambda(x)-zI)^{-1}(g^*(s)-g^*(x))\,ds\,dz\|_X \\ &\le \Big(C\int_\Gamma\frac{d|z|}{|z|^{1+\beta}}\Big)\|g^*\|_{C^\beta_\infty([0,\infty);X)} < +\infty. \end{aligned}$$

Moreover,

$$\begin{aligned} w'(x,g^*) &= \frac{1}{2}\int_0^x e^{(x-s)K_\lambda(x)}g^*(s)\,ds - \frac{1}{2}\int_x^{+\infty} e^{(s-x)K_\lambda(x)}g^*(s)\,ds \\ &\quad + \frac{1}{2}\int_0^x e^{(x-s)K_\lambda(x)}\frac{d}{dx}(K_\lambda(x))^{-1}g^*(s)\,ds \\ &\quad + \frac{1}{2}\int_x^{+\infty} e^{(s-x)K_\lambda(x)}\frac{d}{dx}(K_\lambda(x))^{-1}g^*(s)\,ds \\ &\quad - \frac{1}{4i\pi}\int_\Gamma\int_0^x e^{(x-s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}(K_\lambda(x))^{-1}g^*(s)\,ds\,dz \\ &\quad - \frac{1}{4i\pi}\int_\Gamma\int_x^{+\infty} e^{(s-x)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}(K_\lambda(x))^{-1}g^*(s)\,ds\,dz \\ &= \sum_{i=1}^6 w_i(x,g^*). \end{aligned}$$

All these integrals are absolutely convergent (see Lemma 2.1-(2) and Assumption (2.1)). On the other hand, it is very important to note here, that the calculation of the term $w''(x,g^*)$ is not easy to justify. For this purpose, we need to simplify the terms $(w_3(x,g^*)+w_5(x,g^*))$ and $(w_4(x,g^*)+w_6(x,g^*))$ in order to justify their derivatives. Indeed, by using Dunford's calculus and similar calculus as in [5, p. 23, 24] we obtain

$$\begin{aligned}
&(w_3(x,g^*)+w_5(x,g^*))+(w_4(x,g^*)+w_6(x,g^*))\\
&=-\frac{1}{4i\pi}\int_\Gamma\int_0^x \frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad-\frac{1}{4i\pi}\int_\Gamma\int_x^{+\infty} \frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&=S_\lambda(g^*)(x).
\end{aligned}\tag{3.5}$$

Therefore,

$$\begin{aligned}
w'(x,g^*)&=w_1(x,g^*)+w_2(x,g^*)\\
&\quad-\frac{1}{4i\pi}\int_\Gamma\int_0^x \frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad-\frac{1}{4i\pi}\int_\Gamma\int_x^{+\infty} \frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz,
\end{aligned}$$

and

$$\begin{aligned}
B(x)w'(x,g^*)&=B(x)w_1(x,g^*)+B(x)w_2(x,g^*)\\
&\quad-\frac{B(x)}{4i\pi}\int_\Gamma\int_0^x \frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad-\frac{B(x)}{4i\pi}\int_\Gamma\int_x^{+\infty} \frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&=N_\lambda(g^*)(x).
\end{aligned}\tag{3.6}$$

All these terms are well defined because of Lemma 2.1-(2) and Assumptions (1.5) and (2.1). Now, to differentiate $w'(x,g^*)$, we also need to justify the term $w_1'(\cdot,g^*)+w_2'(\cdot,g^*)$. For this end, we use the method presented, for instance, in [19, Theorem 3.3.4, p. 70]. For $0<\varepsilon\le x$, let us introduce the function

$$w_\varepsilon'(x,g^*)=w_1^\varepsilon(x,g^*)+w_2^\varepsilon(x,g^*)+S_\lambda(g^*)(x),$$

where

$$\begin{aligned}
w_1^\varepsilon(x,g^*)+w_2^\varepsilon(x,g^*)&=\frac{1}{2}\int_0^{x-\varepsilon} e^{(x-s)K_\lambda(x)}g^*(s)\,ds\\
&\quad-\frac{1}{2}\int_{x+\varepsilon}^{+\infty} e^{(s-x)K_\lambda(x)}g^*(s)\,ds.
\end{aligned}$$

It is a simple matter to see that these integrals are absolutely convergent and

$$\lim_{\varepsilon\to 0} w_\varepsilon'(x,g^*)=w'(x,g^*).$$

The calculus of the term $(w_1^\varepsilon)'(x,g^*)+(w_2^\varepsilon)'(x,g^*)$ gives

$$\begin{aligned}
&(w_1^\varepsilon)'(x,g^*)+(w_2^\varepsilon)'(x,g^*)\\
&=\frac{1}{2}e^{\varepsilon K_\lambda(x)}(g^*(x-\varepsilon)+g^*(x+\varepsilon))-e^{\varepsilon K_\lambda(x)}g^*(x)+\frac{1}{2}e^{xK_\lambda(x)}g^*(x)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^{x-\varepsilon} K_\lambda(x)e^{(x-s)K_\lambda(x)}(g^*(s)-g^*(x))\,ds\\
&+\frac{1}{2}\int_{x+\varepsilon}^{+\infty} K_\lambda(x)e^{(s-x)K_\lambda(x)}(g^*(s)-g^*(x))\,ds\\
&-\frac{1}{4i\pi}\int_\Gamma\int_0^{x-\varepsilon} e^{(x-s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&+\frac{1}{4i\pi}\int_\Gamma\int_{x+\varepsilon}^{+\infty} e^{(s-x)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}[(w_1^\varepsilon)'(x,g^*)+(w_2^\varepsilon)'(x,g^*)]\\
&=\frac{1}{2}e^{xK_\lambda(x)}g^*(x)+\frac{1}{2}\int_0^{x} K_\lambda(x)e^{(x-s)K_\lambda(x)}(g^*(s)-g^*(x))\,ds\\
&\quad+\frac{1}{2}\int_{x}^{+\infty} K_\lambda(x)e^{(s-x)K_\lambda(x)}(g^*(s)-g^*(x))\,ds\\
&\quad-\frac{1}{4i\pi}\int_\Gamma\int_0^{x} e^{(x-s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad+\frac{1}{4i\pi}\int_\Gamma\int_{x}^{+\infty} e^{(s-x)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&=w_1'(x,g^*)+w_2'(x,g^*).
\end{aligned}\tag{3.7}
$$

On the other hand, from (3.5) we obtain

$$
\begin{aligned}
(S_\lambda(g^*))'(x)&=-\frac{1}{4i\pi}\int_\Gamma\int_0^{x} e^{(x-s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad-\frac{1}{4i\pi}\int_\Gamma\int_0^{x} \frac{e^{(x-s)z}}{z}\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad+\frac{1}{4i\pi}\int_\Gamma\int_{x}^{+\infty} e^{(s-x)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&\quad-\frac{1}{4i\pi}\int_\Gamma\int_{x}^{+\infty} \frac{e^{(s-x)z}}{z}\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz.
\end{aligned}\tag{3.8}
$$

Now, since

$$
w_\varepsilon''(x,g^*)=(w_1^\varepsilon)'(x,g^*)+(w_2^\varepsilon)'(x,g^*)+(S_\lambda(g^*))'(x),
$$

by summing (3.7) and (3.8), we obtain

$$
\lim_{\varepsilon\to 0} w_\varepsilon''(x,g^*)=w''(x,g^*).\tag{3.9}
$$

Finally, by summing (3.4), (3.6) and (3.9), it follows that

$$
Op(w(x,g^*))=g^*(x)+V_\lambda(g^*)(x)+N_\lambda(g^*)(x),\tag{3.10}
$$

where

$$
\begin{aligned}
V_\lambda(g^*)(x) = &-\frac{1}{2i\pi}\int_\Gamma\int_0^x e^{(x-s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&+\frac{1}{2i\pi}\int_\Gamma\int_x^{+\infty} e^{(s-x)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&-\frac{1}{4i\pi}\int_\Gamma\int_0^x \frac{e^{(x-s)z}}{z}\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
&-\frac{1}{4i\pi}\int_\Gamma\int_x^{+\infty} \frac{e^{(s-x)z}}{z}\frac{\partial^2}{\partial x^2}(K_\lambda(x)-zI)^{-1}g^*(s)\,ds\,dz\\
= &\sum_{i=1}^4 V_{\lambda i}(g^*)(x).
\end{aligned}\tag{3.11}
$$

These integrals are absolutely convergent because of (2.1), (2.2) and (2.3). To see this, let us for instance treat the convergence of $V_{\lambda 1}(g^*)(x)$.

$$
\begin{aligned}
&V_{\lambda 1}(g^*)(x)\\
&= -\frac{1}{2i\pi}\int_0^x\int_\Gamma e^{(x-s)z}\Big(\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}-\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\big|_{x=0}\Big)g^*(s)\,dz\,ds\\
&\quad-\frac{1}{2i\pi}\int_0^x\int_\Gamma e^{(x-s)z}\frac{\partial}{\partial x}(K_\lambda(x)-zI)^{-1}\big|_{x=0}g^*(s)\,dz\,ds\\
&= \phi_1(x)+\phi_2(x).
\end{aligned}
$$

From (2.2), we obtain

$$
\begin{aligned}
\|(\phi_1(x))\|_X &\le C\|g^*\|_{C_\infty([0,\infty);X)}\int_0^x\int_\Gamma e^{-C_0(x-s)|z|}\frac{x^\alpha}{|z|^\rho}d|z|ds\\
&\le C\|g^*\|_{C_\infty([0,\infty);X)}\int_0^x\int_\Gamma e^{-C_0\sigma}x^\alpha\frac{(x-s)^\rho}{\sigma^\rho}\frac{d\sigma}{(x-s)}ds\\
&\le Cx^{\alpha+\rho}\|g^*\|_{C_\infty([0,\infty);X)}.
\end{aligned}
$$

Assumption (2.1) leads to

$$
\|\phi_2(x)\|_X \le Cx^\rho\|g^*\|_{C_\infty([0,\infty);X)}.
$$

By Remark 2.3, it follows that $V_\lambda(g^*)+N_\lambda(g^*)\in C_\infty^{\alpha+\rho-1}([0,\infty);X)$. Finally, the function $x\mapsto Op(w(x,g^*))$ belongs to $C_\infty^{\min(\beta,\alpha+\rho-1)}([0,\infty);X)$, because $g^*\in C_\infty^\beta([0,\infty);X)$. □

Remark 3.4. Note that all the Hölderianities studied above were done near 0 which allow us to deduce the Hölderianities in $[0,\infty)$.

4. Equation satisfied by the solution and its resolution

The previous calculus prove that the representation given in (2.8) satisfies the abstract equation

$$
\begin{aligned}
&u''(x)+B(x)u'(x)+Q(x)u(x)\\
&=Op(u(x,g^*))\\
&=g^*(x)+G_\lambda(x)\varphi+T_\lambda(g^*)(x)+V_\lambda(g^*)(x)+N_\lambda(g^*)(x)\\
&=f(x),\quad \text{for } x\in[0,\infty).
\end{aligned}\tag{4.1}
$$

To determine the unknown function g^*, we need to prove the following result.

Proposition 4.1. *Let $\varphi \in D(A(0))$ and $f \in C_\infty^\theta([0,\infty);X)$, where $0 < \theta < 1$. Under Hypotheses* (1.5)–((2.6)*, suppose that u given in* (2.8) *is a classical solution of Problem* (1.1)-(1.2)*. Then in the space $C_\infty([0,\infty);X)$, the function g^* satisfies*

$$(I + P_\lambda)(g^*)(x) = f(x) - G_\lambda(x)\varphi, \tag{4.2}$$

where

$$P_\lambda(g^*)(\cdot) = T_\lambda(g^*)(x) + V_\lambda(g^*)(x) + N_\lambda(g^*)(x). \tag{4.3}$$

Moreover, there exists $\lambda^* > 0$ such that for every $\lambda \geq \lambda^*$, operator $I + P_\lambda$ is invertible in the space $C_\infty^{\min(\alpha,\rho)}([0,\infty);X)$ and

$$g^*(\cdot) = (I + P_\lambda)^{-1} f(\cdot). \tag{4.4}$$

Proof. To solve (4.2) in $C_\infty([0,\infty);X)$, we should estimate $\|P_\lambda\|_{L(C_\infty([0,\infty);X))}$ (see (4.3)), for a large $\lambda > 0$. Let us, for instance, estimate some terms contained in $V_\lambda(g^*)(x)$ (see formula (3.11)). By applying the same arguments used in [5, p. 27, 28], it follows that

$$\|V_{\lambda 1}(g^*)(x)\|_X \leq \frac{C}{\lambda^{(1-\alpha)/2}} \|g^*\|_{C_\infty([0,\infty);X)},$$
$$\|V_{\lambda 3}(g^*)(x)\|_X \leq C\Big(\frac{1}{\lambda^{1/4}} + \frac{1}{\lambda^{(\rho+\alpha-1)/2}}\Big)\|g^*\|_{C_\infty([0,\infty);X)}.$$

A similar analysis for the remaining terms proves the existence of some λ^* such that for all $\lambda \geq \lambda^*$, we have $\|P_\lambda\|_{L(C([0,\infty);X))} \leq \frac{1}{2}$. Therefore operator $I + P_\lambda$ is invertible for $\lambda \geq \lambda^*$ in $C_\infty([0,\infty);X)$ and thus (4.4) follows. $\square$

In the sequel, we need the following result concerning $g^*(0)$.

Proposition 4.2. *Let $\varphi \in D(A(0))$ and $f \in C_\infty^\theta([0,\infty);X)$, where $0 < \theta < 1$. Under Hypotheses* (1.5)–(2.6)*, suppose that u given in* (2.8) *is a classical solution of Problem* (1.1)-(1.2)*. Then*

$$g^*(0) = f(0) + \frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0} K_\lambda(0)\varphi + \Psi_0^*(\varphi) + s_0(g^*),$$

where $\Psi_0^*(\varphi)$, $s_0(g^*) \in \overline{D(K_\lambda(0))} = \overline{D(A(0))}$. Moreover $g^* \in C_\infty^\beta([0,\infty);X)$, where $\beta = \min(\alpha + \rho - 1, \theta)$.

Proof. We have

$$g^*(0) = f(0) - [G_\lambda(0)\varphi + T_\lambda(g^*)(0) + V_\lambda(g^*)(0) + N_\lambda(g^*)(0)],$$

and the term

$$s_0(g^*) = T_\lambda(g^*)(0) + V_\lambda(g^*)(0) + N_\lambda(g^*)(0)$$

is in $\overline{D(K_\lambda(0))} = \overline{D(A(0))}$. From (3.2), it follows that

$$G_\lambda(0)\varphi = \Psi_0^*(\varphi) - \frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0} K_\lambda(0)\varphi,$$

where $\Psi_0^*(\varphi) \in \overline{D(K_\lambda(0))}$ (see (2.5)). Therefore

$$g^*(0) = f(0) + \frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0} K_\lambda(0)\varphi + \Psi_0^*(\varphi) + s_0(g^*),$$

where $\Psi_0^*(\varphi)$, $s_0(g^*) \in \overline{D(K_\lambda(0))}$. On the other hand, the function

$$x \mapsto G_\lambda(x)\varphi + T_\lambda(g^*)(x) + V_\lambda(g^*)(x) + N_\lambda(g^*)x)$$

belongs to $C_\infty^{\alpha+\rho-1}([0,\infty);X)$, and thus we deduce that if g^* exists, then it belongs necessarily to $C_\infty^{\min(\theta,\alpha+\rho-1)}([0,\infty);X)$. Therefore $\beta=\min(\theta,\alpha+\rho-1)$. □

5. Main results

According to the above study on the regularity of the solution, we are ready to state our first main result on the existence and the uniqueness of the classical solution of Problem (1.1)-(1.2).

Theorem 5.1. *Let $\varphi \in D(A(0))$ and $f \in C_\infty^\theta([0,\infty);X)$, where $0<\theta<1$. Then, under Hypotheses* (1.5)–(2.6)*, there exists $\lambda^*>0$ such that for all $\lambda \geq \lambda^*$, the function u given in the representation* (2.7) *is the unique classical solution of Problem* (1.1)-(1.2) *if and only if*

$$(-A(0))\varphi + f(0) + \frac{d^2}{dx^2}(\lambda I - A(x))^{-1/2}\big|_{x=0}(\lambda I - A(0))^{1/2}\varphi \in \overline{D(A(0))}. \tag{5.1}$$

Proof. It suffices to prove that

$$x \mapsto A_\lambda(x)u(x) = -(K_\lambda(x))^2u(x) \in C_\infty([0,\infty),X).$$

We have

$$\begin{aligned}
(K_\lambda(x))^2u(x) &= (K_\lambda(x))^2[u_0(x)\varphi + m_0(x,g^*) + w(x,g^*)]\\
&= (K_\lambda(x))^2 e^{xK(x)}\varphi - g^*(x) + e^{xK(x)}g^*(x)\\
&\quad - \frac{1}{2}\int_0^{+\infty} K_\lambda(x)e^{(x+s)K_\lambda(x)}(g^*(s)-g^*(0))\,ds\\
&\quad + \frac{1}{2}\int_0^{x} K_\lambda(x)e^{(x-s)K_\lambda(x)}(g^*(s)-g^*(x))\,ds\\
&\quad + \frac{1}{2}\int_x^{+\infty} K_\lambda(x)e^{(s-x)K_\lambda(x)}(g^*(s)-g^*(x))\,ds.
\end{aligned}$$

We treat only the regularity of the following term (the other terms are regular near 0),

$$\begin{aligned}
R(x) &= (K_\lambda(x))^2e^{xK_\lambda(x)}\varphi + e^{xK_\lambda(x)}g^*(0)\\
&= [(K_\lambda(x))^2-(K_\lambda(0))^2]e^{xK_\lambda(x)}\varphi + (K_\lambda(0))^2[e^{xK_\lambda(x)}-e^{xK_\lambda(0)}]\varphi\\
&\quad + [e^{xK_\lambda(x)}-e^{xK_\lambda(0)}]g^*(0) + (K_\lambda(0))^2e^{xK_\lambda(0)}\varphi + e^{xK_\lambda(0)}g^*(0)\\
&= \sum_{i=1}^5 R_i(x).
\end{aligned}$$

Hence $R_1(x)$ and $R_2(x)$ tend to 0, as $x\to 0$. Moreover, (2.1) leads to

$$\begin{aligned}
\|R_3(x)\|_X &= \Big\|-\frac{1}{2i\pi}\int_\Gamma e^{xz}\Big(\int_0^x \frac{\partial}{\partial r}(K_\lambda(r)-zI)^{-1}dr\Big)g^*(0)dz\Big\|_X\\
&\leq Cx^\rho\|g^*\|_{C_\infty([0,\infty);X)} \to 0, \quad \text{as } x\to 0.
\end{aligned}$$

Now, by Proposition 4.2 and Lemma 2.2-(1), the term

$$\begin{aligned}
R_4(x)+R_5(x) &= e^{xK_\lambda(0)}[(K_\lambda(0))^2\varphi + f(0) + \frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0}K_\lambda(0)\varphi]\\
&\quad + e^{xK_\lambda(0)}[\Phi_0^*(\varphi) + s_0(g^*)]
\end{aligned}$$

is in $C([0,\infty);X)$ if and only if

$$(K_\lambda(0))^2\varphi + f(0) + \frac{d^2}{dx^2}(K_\lambda(x))^{-1}\big|_{x=0}K_\lambda(0)\varphi \in \overline{D(K_\lambda(0))} = \overline{D(A(0))}$$

and $\Phi_0^*(\varphi) + s_0(g^*) \in \overline{D(K_\lambda(0))} = \overline{D(A(0))}$. This completes the proof. $\square$

Before giving our second main result on the maximal regularity of the classical solution of problem (1.1)-(1.2), we recall the interpolation spaces that play a crucial role in our proof. For this reason, consider the interpolation spaces

$$D_{(-A(0))}(\theta/2,+\infty) = D_{K_\lambda(0)}(\theta,+\infty) \subset \overline{D(K_\lambda(0))} = \overline{D(A(0))}.$$

We replace Assumptions (2.5)-(2.6) by the following Hypotheses:

$$B(0)(X) \subset D_{K_\lambda(0)}(\theta,+\infty), \tag{5.2}$$

$$\frac{d}{dx}(K_\lambda(x))^{-1}\big|_{x=0}(D(K_\lambda(0))) \subset D_{K_\lambda(0)}(\theta,+\infty). \tag{5.3}$$

By using Lemma 2.2-(2) and similar arguments to those applied in the proof of Theorem 5.1, we can prove the following result.

Theorem 5.2. *Let $\varphi \in D(A(0))$ and $f \in C_\infty^\theta([0,\infty);X)$, where $0<\theta<1$. Then, under Hypotheses (1.5)–(2.4), (5.2) and (5.3), there exists $\lambda^*>0$ such that for all $\lambda \ge \lambda^*$, the function u given in the representation (2.7) is the unique classical solution of Problem (1.1)-(1.2) satisfying*

$$u''(\cdot), B(\cdot)u'(\cdot), A_\lambda(\cdot)u(\cdot) \in C_\infty^\beta([0,\infty);X), \quad \beta \in \min(\theta,\alpha+\rho-1)$$

if and only if

$$(-A(0))\varphi + f(0) + \frac{d^2}{dx^2}(\lambda I - A(x))^{-1/2}\big|_{x=0}(\lambda I - A(0))^{1/2}\varphi \in D_{A(0)}(\theta/2,+\infty). \tag{5.4}$$

6. Applications

Consider the Banach space $X = C([0,1])$ with its usual supremum norm and define the family of closed linear operators $(-A(x))^{1/2}$ for all $x \ge 0$ by

$$D(-(-A(x))^{1/2}) = \{\varphi \in C^2([0,1]) : \varphi(0) - a(x)\varphi'(0) = 0; \varphi(1) = 0\}$$
$$(-(-A(x))^{1/2}\varphi)(y) = \varphi''(y), y \in [0,1].$$

Then

$$\begin{gathered} D(A(x)) = \big\{\varphi \in C^4([0,1]) : \varphi(0) - a(x)\varphi'(0) = 0, \varphi(1) = 0, \\ \varphi''(0) - a(x)\varphi'''(0) = 0, \varphi''(1) = 0\big\} \\ (A(x)\varphi)(y) = -\varphi^{(iv)}(y),\ y \in [0,1]. \end{gathered}$$

We assume that $a \in C^{2,\kappa}[0,\infty)$, $a(x) \ge 0$ and $\min_{x\ge 0} a(x) > 0$. The family of bounded linear operators $(B(x))_{x\ge 0}$ is defined by

$$D(B(x)) = X, \quad (B(x)\varphi)(y) = \omega(x)\varphi(y).$$

Then, all our results apply to the following model on a concrete quasi-elliptic boundary-value problem, for a large positive λ,

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \omega(x)\frac{\partial u}{\partial x}(x,y) - \frac{\partial^4 u}{\partial y^4}(x,y) - \lambda u(x,y) = f(x,y), \quad x>0,\ y \in (0,1),$$

$$
\begin{gathered}
u(x,0)-a(x)\frac{\partial u}{\partial y}(x,0)=0, \quad x\geq 0,\\
\frac{\partial^2 u}{\partial y^2}(x,0)-a(x)\frac{\partial^3 u}{\partial y^3}(x,0)=0, \quad x\geq 0,\\
u(x,1)=0=\frac{\partial^2 u}{\partial y^2}(x,1), \quad x\geq 0,\\
u(0,y)=\varphi(y), \quad u(+\infty,y)=0, \quad y\in[0,1], \quad f\in C_\infty^\theta([0,\infty);X).
\end{gathered}
$$

Conclusions. In this article, we obtained interesting results, namely Theorems 5.1 and 5.2, on the existence, the uniqueness and the maximal regularity of the classical solution of Problem (1.1)-(1.2) by using semigroups theory, the fractional powers of linear operators, the Dunford's functional calculus and the interpolation spaces. Moreover, we established necessary and sufficient conditions of compatibility (see (5.4)) to obtain the solution. For future work, we think it is interesting to generalize the study of this problem to more complicated situations, where the variable operators $(A(x))_{x\geq 0}$ and $(B(x))_{x\geq 0}$ are closed (unbounded), in the two functional frames Hölderian and L^p.

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Fatiha Boutaous
Department of Mathematics, Faculty of Sciences, Saâd Dahlab University, 09000, Blida, Algeria.
Lab. NLPDE and Hist. of Maths, Ecole Normale Supérieure, 16050, Kouba, Algeria
Email address: boutaous.fatiha2009@hotmail.com