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EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL LAPLACIAN EQUATIONS: THEORY AND NUMERICAL EXPERIMENTS

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Abstract. We consider a class of nonlinear fractional Laplacian problems satisfying the homogeneous Dirichlet condition on the exterior of a bounded domain. We prove the existence of positive weak solution for classes of sublinear nonlinearities including logistic type. A method of sub- and supersolution, without monotone iteration, is established to prove our existence results. We also provide numerical bifurcation diagrams and the profile of positive solutions, corresponding to the theoretical results using the finite element method in one dimension.

1. Introduction

We investigate the existence of positive solutions for a class of nonlocal problems of the form

$$\begin{gathered} (-\Delta)^s u = \lambda f(x,u) \quad \text{in } \Omega; \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,, \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary $\partial\Omega$ for $N \geq 2$ (or bounded open interval if $N = 1$). For a fixed $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x) := C_{N,s}\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,,$$

where $C_{N,s} := s2^{2s}\pi^{-\frac{N}{2}}\Gamma(\frac{N+2s}{2})/\Gamma(1-s)$ is a positive normalizing constant with Γ as the usual gamma function, and P.V. stands for the Cauchy principal value of the singular integral. The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, $f(\cdot,t)$ is measurable for each $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous for a.e. $x \in \Omega$).

We seek to establish the existence of a *weak solution* (to be defined) of the nonlocal problem (1.1) under suitable conditions on the nonlinearity f using the method of sub- and supersolution. In [1] and [16], sub- and supersolution methods for fractional Laplacian equations were established for L^1-very weak solutions, which required a rather complicated structure of the space of test functions. Therefore, we first present a sub- and supersolution result in a framework that is analogous

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to the weak solution framework for the Laplacian case. An advantage of our approach is in the possibility of employing principal eigenfunction corresponding to the variational principal eigenvalue of $(-\Delta)^s$ in the construction of positive sub- and supersolutions. See also [6], where a sub- and supersolution method was proved for the fractional $p(x)$-Laplacian equation for weak solutions, but with the additional requirement that the nonlinearity is monotone with respect to the solution variable.

Next, using the method of sub- and supersolutions we establish the existence of positive weak solutions to (1.1) for classes of nonlinearities: sublinear at infinity, weighted logistic problems, and logistic problems with constant yield harvesting.

We also discuss the existence and properties of a principal eigenvalue and the corresponding eigenfunction for a weighted fractional eigenvalue problem. This eigenfunction plays a crucial role in the construction of some positive sub- and supersolutions.

Finally, using a finite element method, we present numerical bifurcation diagrams and illustrate the typical profile of positive solutions for examples of nonlinearity f satisfying the hypotheses of our theoretical results. Using these numerical experiments, we formulate some relevant conjectures.

The fractional Laplacian operator $(-\Delta)^s$, is associated with superdiffusion driven by Lévy flights and appears to be of interest from the application point of view, see [10, 23, 24, 27, 31] and references therein. More specifically, such operators are associated with the efficient foraging strategy of living organisms, see e.g. [3, 21] and references therein.

In Section 2, we discuss function spaces, define terminologies, and state our main results. In particular, we first state a general sub- and supersolution result, Theorem 2.3. Then, we state existence results for classes of nonlinearity f that are sublinear at infinity. In particular, Theorem 2.5 deals with the positone case $f(0) > 0$ and Theorem 2.6 deals with the case $f(0) = 0$. Next, we state an existence result for a weighted logistic problem in Theorem 2.8, and a logistic problem with constant yield harvesting (semipositone, that is, $f(0) < 0$) in Theorem 2.9. In Section 3, we discuss a weighted fractional eigenvalue problem and a fractional linear problem necessary in the construction of sub- and supersolutions in later sections. In Section 4, we prove Theorem 2.3 using the Schauder fixed point theorem. In Section 5, we prove Theorem 2.5 and Theorem 2.6. In Section 6, we prove Theorem 2.8 and Theorem 2.9. In Section 7, we present numerical bifurcation diagrams and profiles of positive solutions obtained using the finite element method in one dimension. In the Appendix, we show that the norms generated by inner products (2.1) and (2.2) are equivalent in the fractional Sobolev space $H_0^s(\Omega)$ (defined below) in one dimension as well.

2. Preliminaries and statement of results

We first discuss some properties of fractional Sobolev spaces, see [15, 24] for more details. Let

$$H^s(\mathbb{R}^N) := \left\{ w \in L^2(\mathbb{R}^N) : \|w\|_{H^s(\mathbb{R}^N)} < +\infty \right\},$$

where $\|w\|_{H^s(\mathbb{R}^N)} := \left(\|w\|^2_{L^2(\mathbb{R}^N)} + [w]^2_{H^s(\mathbb{R}^N)} \right)^{1/2}$ and

$$[w]_{H^s(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y \Big)^{1/2}$$

is the *Gagliardo seminorm* of w. Then, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product

$$\langle v, w\rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} vw \,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)][w(x)-w(y)]}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\,. \tag{2.1}$$

Further, the fractional Sobolev space

$$H_0^s(\Omega) := \big\{w \in H^s(\mathbb{R}^N) : w \equiv 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega\big\}$$

is also a Hilbert space with respect to the inner product

$$\langle v, w\rangle_{H_0^s(\Omega)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)][w(x)-w(y)]}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\,. \tag{2.2}$$

The norms generated by (2.1) and (2.2) are equivalent in $H_0^s(\Omega)$. This fact follows from [24, Lemmas 1.28 and 1.29] for $N \geq 2$ and Lemma 8.1 for $N = 1$ with $\Omega = (0,1) \subset \mathbb{R}$. We utilize this important fact in our analysis.

To simplify notation, for $\psi, \phi \in H^s(\mathbb{R}^N)$, we define

$$\mathcal{E}(\psi, \phi) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[\psi(x)-\psi(y)][\phi(x)-\phi(y)]}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\,. \tag{2.3}$$

Definition 2.1. We say that a function $u \in H_0^s(\Omega)$ is a *weak solution* of (1.1) if for all $\phi \in H_0^s(\Omega)$, it holds

$$\mathcal{E}(u, \phi) = \lambda \int_\Omega f(x, u)\phi(x)\,\mathrm{d}x\,. \tag{2.4}$$

Definition 2.2. A function $\overline{u} \in H^s(\mathbb{R}^N)$ is called a *weak supersolution* of (1.1) if, for all $\phi \in H_0^s(\Omega)$ such that $\phi \geq 0$ a.e. Ω, the following inequality holds

$$\mathcal{E}(\overline{u}, \phi) \geq \lambda \int_\Omega f(x, \overline{u}(x))\phi(x)\,\mathrm{d}x\,, \tag{2.5}$$

$$\overline{u} \geq 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\,. \tag{2.6}$$

A function $\underline{u} \in H^s(\mathbb{R}^N)$ is called a *weak subsolution* of (1.1) if the inequalities are reversed in (2.5) and (2.6).

Now we state a general sub- and supersolution result without monotonicity assumption for the problem

$$\begin{gathered} (-\Delta)^s u = g(x, u) \quad \text{in } \Omega;\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{2.7}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following assumptions:

- (H1) for all $r > 0$, there is $a_r \in L^\infty(\Omega)$ such that $|g(x,t)| \leq a_r(x)$ for all $|t| \leq r$ a.e. $x \in \Omega$;
- (H2) for all $r > 0$, there is a continuous nondecreasing function b_r with $b_r(0) = 0$ such that $|g(x,t_1) - g(x,t_2)| \leq b_r(|t_1 - t_2|)$ for all $|t_1|, |t_2| \leq r$ a.e. $x \in \Omega$.

Theorem 2.3. *Assume* (H1) *and* (H2) *hold. Let* $\underline{u}$ *and* $\overline{u} \in H^s(\mathbb{R}^N) \cap L^\infty(\Omega)$ *be weak subsolution and weak supersolution, respectively of* (2.7) *satisfying* $\underline{u} \leq \overline{u}$ *a.e. in* Ω. *Then, there exists a weak solution* u *to* (2.7) *satisfying* $\underline{u} \leq u \leq \overline{u}$ *a.e. in* Ω.

Remark 2.4. The hypotheses of Theorem 2.3 are satisfied by a function of the form $g(x,t) = k(x)\tilde{g}(t)$, where $k \in L^\infty(\Omega)$ and $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is Hölder continuous. Indeed, clearly $g(x,t)$ is a Carathéodory function. For any $r > 0$ and for all $|t| \le r$, we have $|g(x,t)| \le \|k\|_{L^\infty(\Omega)} \max_{|t|\le r} |\tilde{g}(t)|$ and hence (H1) is satisfied. By the Hölder continuity of $\tilde{g}$, $|g(x,t_1)-g(x,t_2)| \le A\|k\|_{L^\infty(\Omega)}|t_1-t_2|^\eta$ for all $|t_1|, |t_2| \le r$ for some $\eta \in (0,1)$ and $A > 0$. Then, (H2) is satisfied with $b_r(|t_1-t_2|) := A\|k\|_{L^\infty(\Omega)}|t_1-t_2|^\eta$.

We employ Theorem 2.3 to discuss the existence of positive weak solutions of (1.1) for classes of nonlinearities f. First, we consider the case $f(x,t) = f(t)$ satisfying sublinear condition at infinity

$$\lim_{t\to+\infty} \frac{f(t)}{t} = 0\,. \tag{2.8}$$

Our first result deals with the case when $f > 0$.

Theorem 2.5. *Suppose $f : [0,\infty) \to (0,+\infty)$ is a Hölder continuous function and (2.8) is satisfied. Then, (1.1) has a positive weak solution for each $\lambda > 0$.*

To state our second result, let λ_1 be the principal eigenvalue of the eigenvalue problem (3.3), with $q \equiv 1$, corresponding to the fractional Laplacian operator $(-\Delta)^s$. Then, we prove the following existence result for the case $f(0) = 0$.

Theorem 2.6. *Suppose $f : [0,+\infty) \to [0,+\infty)$ is a C^1 function such that $f(0) = 0$, $f'(0) > 0$ with $f(t) > 0$ for all $t > 0$, and (2.8) is satisfied. Then, (1.1) has a positive weak solution for any $\lambda > \frac{\lambda_1}{f'(0)}$.*

Remark 2.7. An example satisfying the hypotheses of Theorem 2.5 is the reaction term $f(t) = e^{\frac{\kappa t}{\kappa + t}}$ for $t \ge 0$ with $\kappa > 0$, referred in the literature as *perturbed Gelfand problem* when considered with Laplacian operator, see [7, Chap. 2]. In Section 7, Figures 1-3 give numerical bifurcation diagrams corresponding to this nonlinearity f depending on the value of κ, illustrating the result in Theorem 2.5. A simple example satisfying the hypotheses of Theorem 2.6 is $f(t) = 3(1+t)^{1/3} - 3$ for $t \ge 0$. In Section 7, Figure 4 gives the numerical bifurcation diagrams illustrating the result in Theorem 2.6.

Finally, we discuss existence of positive weak solution for two logistic type problems. First, let $q \in L^\infty(\Omega)$ be such that $0 \le q \le 1$ a.e. in Ω and $q(x) > \frac{1}{2}$ on a set of positive measure. By $\lambda_{1,q}$, we denote the principal eigenvalue of the weighted eigenvalue problem (3.3) with weight q. Then, we prove the following result.

Theorem 2.8. *The fractional logistic problem*

$$\begin{gathered} (-\Delta)^s u = \lambda u(q(x) - u) \quad \text{in } \Omega; \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{2.9}$$

has a positive weak solution for any $\lambda > \lambda_{1,q}$.

Next, let λ_1 denote the first eigenvalue of (3.3) with $q \equiv 1$. Then, we prove the following existence result for the logistic problem with constant yield harvesting.

Theorem 2.9. *For any $\lambda > \lambda_1$, there exists $a^* = a^*(\lambda) > 0$ such that the logistic problem with constant yield harvesting*

$$\begin{gathered} (-\Delta)^s u = \lambda[u(1-u) - a] \quad \text{in } \Omega; \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{2.10}$$

has a positive weak solution for $a \in (0, a^*)$.

Remark 2.10. For derivation of the time dependent fractional logistic model $u_t + (-\Delta)^s u = \lambda u(1-u)$ with $u = u(x,t)$ and $(x,t) \in \mathbb{R}^2$, for a simple two particle reaction scheme, see [10]. Theorem 2.8 complements the existence result obtained in [12], where existence of nonnegative solution for logistic growth problem was established using energy minimization. Theorem 2.9 is analogous to the existence result obtained in [26] for the Laplacian case.

In Section 7, the numerical bifurcation diagrams given in Figures 7-9, illustrate the result of Theorem 2.8 and demonstrate the influence of $q(x)$ and $s \in (0,1)$ on the shape of positive solution. In Figure 6, we give the numerical bifurcation diagrams illustrating the result of Theorem 2.9 for $a = 0.05$. In this case, we observe numerically that the bifurcation diagrams contain positive as well as sign-changing solutions.

In the proof of Theorem 2.8, the difficulty is due to the presence of the weight function $q(x)$. In the proof Theorem 2.9, the harvesting term "a" poses a challenge in the investigation of positive solutions. Combining the methods used in the proofs of Theorem 2.8 and Theorem 2.9, one can prove an existence result if $f(x,u) = \lambda[u(q(x)-u)-a]$ with q as in Theorem 2.8.

3. Auxiliary problems

To prove Theorems 2.5-2.9, we utilize the positive weak solutions of the following auxiliary problems in the construction of weak sub- and supersolutions.

First, consider the linear problem

$$\begin{aligned} &(-\Delta)^s e = 1 \quad \text{in } \Omega; \\ &e = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,. \end{aligned} \tag{3.1}$$

Then, there exists a unique weak solution $e \in H_0^s(\Omega) \subset H^s(\mathbb{R}^N)$ of (3.1) such that $e > 0$ a.e. in Ω, see [20, Thm. 12] for $N \geq 2 > 2s$, and for $N = 1$ the explicit formula of the solution is given in [29, eqn. (1.4)]. Moreover, it follows from [28, Lem. 7.3] and [29, Thm. 1.2] that there exist $c_1, c_2 > 0$ such that

$$c_1\delta^s(x) \leq e(x) \leq c_2\delta^s(x) \quad \text{a.e. in } \Omega\,, \tag{3.2}$$

where $\delta(x)$ is the distance function to the boundary $\partial\Omega$. The function e plays a crucial role in the construction of both sub- and supersolutions in Section 5.

Second, we consider the following weighted fractional eigenvalue problem

$$\begin{aligned} &(-\Delta)^s \varphi = \lambda q(x)\varphi \quad \text{in } \Omega; \\ &\varphi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,, \end{aligned} \tag{3.3}$$

where $q \in L^\infty(\Omega)$ is such that $q \geq 0$ a.e. in Ω and positive on a set of positive measure. The following results hold true by using arguments similar to the case $q \equiv 1$, cf. [24, Prop 3.1 & Cor. 4.8]. We outline the proof for completeness.

Proposition 3.1. *Let $s \in (0,1)$ be fixed and $\Omega \subset \mathbb{R}^N$ be an open, bounded set. Then, the following holds.*

(a) *There exists a principle eigenvalue $\lambda_{1,q} > 0$ of (3.3) that can be characterized as*

$$\lambda_{1,q} = \inf_{\phi \in H_0^s(\Omega)\setminus\{0\}} \frac{\mathcal{E}(\varphi_{1,q}, \varphi_{1,q})}{\int_\Omega q(x)|\phi(x)|^2\,\mathrm{d}x}\,. \tag{3.4}$$

(b) *There exists a nonnegative eigenfunction* $\varphi_{1,q} \in H_0^s(\Omega)$ *corresponding to* $\lambda_{1,q}$*, attaining the minimum in* (3.4)*, that is,*

$$\lambda_{1,q} = \frac{\mathcal{E}(\varphi_{1,q}, \varphi_{1,q})}{\int_\Omega q(x)|\varphi_{1,q}(x)|^2 \,\mathrm{d}x} . \tag{3.5}$$

Moreover, $\varphi_{1,q}$ *satisfies*

$$\mathcal{E}(\varphi_{1,q}, \phi) = \lambda_{1,q} \int_\Omega q(x)\varphi_{1,q}(x)\phi(x) \,\mathrm{d}x \tag{3.6}$$

for every $\phi \in H_0^s(\Omega)$.

(c) $\lambda_{1,q}$ *is simple, that is, if* $\psi \in H_0^s(\Omega)$ *is a solution of the equation*

$$\mathcal{E}(\psi, \phi) = \lambda_{1,q} \int_\Omega q(x)\psi(x)\phi(x) \,\mathrm{d}x$$

for every $\phi \in H_0^s(\Omega)$*, then* $\psi = k\varphi_{1,q}$ *for some* $k \in \mathbb{R}$.

(d) *If* Ω *is* $C^{1,1}$ *for* $N \geq 2$ *(or bounded open interval if* $N = 1$*), then there exist positive constants* $\tilde{c}_1(q), \tilde{c}_2(q)$ *such that*

$$0 < \tilde{c}_1(q)\delta^s(x) \leq \varphi_{1,q}(x) \leq \tilde{c}_2(q)\delta^s(x) \quad \text{a.e. in } \Omega . \tag{3.7}$$

(e) *If* Ω *is* $C^{1,1}$ *for* $N \geq 2$ *(or bounded open interval if* $N = 1$*), then*

$$\lambda_{1,q} = \inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \geq \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi, \phi)}{\int_\Omega q(x)|\phi(x)|^2 \,\mathrm{d}x} . \tag{3.8}$$

Proof. For $N \geq 2$, parts (a)–(c) can be obtained by repeating the argument of [24, Prop 3.1] with $L^2(\Omega)$ norm replaced with weighted L^2 norm $\int_\Omega q(x)|\phi(x)|^2 \,\mathrm{d}x$ in constructing the principal eigenvalue $\lambda_{1,q}$ as Rayleigh quotient given by (3.4). For $N = 1$, these follow from the fact that our definition of $H_0^s(\Omega)$, via $H^s(\mathbb{R}^N)$, allows us to prove the compact embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ without considering an *extension domain* (cf. proof of [15, Thm. 7.1]). Then, $q \in L^\infty(\Omega)$ gives continuous embedding $L^2(\Omega) \hookrightarrow L^2((\Omega); q)$, and hence the principal eigenvalue can be constructed as the Rayleigh quotient given in (3.4).

For part (d), the $C^{1,1}$ smoothness assumption on $\partial\Omega$ is used to establish the inequalities of (3.7). In particular, the arguments used in establishing the left inequality in [28, Lem 7.3] and the right inequality in [29, Thm. 1.2] apply in this case as well, which are independent of dimension N.

For part (e), clearly, $\leq$ holds in (3.8). Using the definition of the infimum and using the fact that $\varphi_{1,q} \geq \delta^s$ a.e. in Ω (after suitable scaling of $\varphi_{1,q}$) due to (3.7), we find

$$\inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \geq \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi, \phi)}{\int_\Omega q(x)|\phi(x)|^2 \,\mathrm{d}x} \leq \frac{\mathcal{E}(\varphi_{1,q}, \varphi_{1,q})}{\int_\Omega q(x)|\varphi_{1,q}(x)|^2 \,\mathrm{d}x} = \lambda_{1,q} ,$$

which establishes $\geq$ in (3.8), completing part (e). This completes the proof of Proposition 3.1. $\square$

Next, for $k = 2, 3, \dots$, we consider the weighted fractional eigenvalue problem

$$\begin{gathered} (-\Delta)^s\varphi = \lambda\gamma_k(x)\varphi \quad \text{in } \Omega; \\ \varphi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega , \end{gathered} \tag{3.9}$$

where

$$\gamma_k(x) := \begin{cases} 0 & \text{if } 0 \le q(x) < 1/k \\ q(x) & \text{if } q(x) \ge 1/k\,, \end{cases} \tag{3.10}$$

for $q \in L^\infty(\Omega)$ with $0 \le q \le 1$ a.e. in Ω and $q(x) > 1/2$ on a set of positive measure. Then, for each $k = 2, 3, \dots$, the weighted fractional eigenvalue problem (3.9) has a principal eigenvalue λ_{1,γ_k} and a corresponding eigenfunction φ_{1,γ_k} satisfying (a)-(d) of Proposition 3.1.

Finally, we establish the following useful relationship between $\lambda_{1,q}$ and λ_{1,γ_k}.

Proposition 3.2. *Let $q \in L^\infty(\Omega)$ with $0 \le q \le 1$ a.e. in Ω and $q(x) > 1/2$ on a set of positive measure, and γ_k be given by* (3.10)*. Then $\lambda_{1,\gamma_k} \searrow \lambda_{1,q}$ as $k \to +\infty$.*

Proof. The properties of q and γ_k imply that the inequalities

$$\int_\Omega q(x)|\phi(x)|^2\,\mathrm{d}x \ge \int_\Omega \gamma_{k+1}(x)|\phi(x)|^2\,\mathrm{d}x \ge \int_\Omega \gamma_k(x)|\phi(x)|^2\,\mathrm{d}x > 0 \tag{3.11}$$

hold for every $k \ge 2$ for all $\phi \in H_0^s(\Omega)$ with $\phi \ge \delta^s$ a.e. in Ω. First, we show $\lambda_{1,q} \le \lambda_{1,\gamma_{k+1}} \le \lambda_{1,\gamma_k}$ for each $k \ge 2$. Indeed, it follows from (3.11) that the inequalities

$$\frac{\mathcal{E}(\phi,\phi)}{\int_\Omega q(x)|\phi(x)|^2\,\mathrm{d}x} \le \frac{\mathcal{E}(\phi,\phi)}{\int_\Omega \gamma_{k+1}(x)|\phi(x)|^2\,\mathrm{d}x} \le \frac{\mathcal{E}(\phi,\phi)}{\int_\Omega \gamma_k(x)|\phi(x)|^2\,\mathrm{d}x} \tag{3.12}$$

hold for all $\phi \in H_0^s(\Omega)$ with $\phi \ge \delta^s$ a.e. in Ω. By taking the infimum over all such ϕ, inequalities (3.12) imply $\lambda_{1,q} \le \lambda_{1,\gamma_{k+1}} \le \lambda_{1,\gamma_k}$, using (3.8), as desired.

Now we show $\lambda_{1,\gamma_k} \to \lambda_{1,q}$ as $k \to +\infty$. By (3.8) with $k \ge 2$, we see

$$\lambda_{1,\gamma_k} = \inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \ge \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi,\phi)}{\int_\Omega \gamma_k(x)|\phi(x)|^2\,\mathrm{d}x}\,.$$

Let $\varphi_{1,q}$ be the principal eigenfunction scaled such that $\varphi_{1,q} \ge \delta^s$ a.e. in Ω. Then, using the same argument as in the proof of Proposition 3.1 (e), we obtain

$$\lambda_{1,q} = \inf_{\phi \in H_0^s(\Omega)} \Big\{ \frac{\mathcal{E}(\phi,\phi)}{\int_\Omega q(x)|\phi(x)|^2\,\mathrm{d}x} : \phi \ge \delta^s \text{ a.e. in } \Omega\,, \|\phi\|_{H_0^s(\Omega)} \le \|\varphi_{1,q}\|_{H_0^s(\Omega)} \Big\}\,.$$

By the definition of the infimum, for each $k \in \mathbb{N}$, we can find $\phi_k \in H_0^s(\Omega)$, $\phi_k \ge \delta^s$ a.e. in Ω and $\|\phi_k\|_{H_0^s(\Omega)} \le \|\varphi_{1,q}\|_{H_0^s(\Omega)}$ such that

$$\lambda_{1,q} \ge \frac{\mathcal{E}(\phi_k,\phi_k)}{\int_\Omega q(x)|\phi_k(x)|^2\,\mathrm{d}x} - 2^{-k}\,.$$

Thus, for $k \ge 2$, we have

$$\begin{aligned}
\lambda_{1,q} &\ge \frac{\mathcal{E}(\phi_k,\phi_k)}{\int_\Omega q(x)|\phi_k(x)|^2\,\mathrm{d}x} - 2^{-k} \\
&= \frac{\mathcal{E}(\phi_k,\phi_k)}{\int_\Omega \gamma_k(x)|\phi_k(x)|^2\,\mathrm{d}x} \cdot \frac{\int_\Omega \gamma_k(x)|\phi_k(x)|^2\,\mathrm{d}x}{\int_\Omega q(x)|\phi_k(x)|^2\,\mathrm{d}x} - 2^{-k} \\
&\ge \Big(\inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \ge \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi,\phi)}{\int_\Omega \gamma_k(x)|\phi(x)|^2\,\mathrm{d}x} \Big) \frac{\int_\Omega \gamma_k(x)|\phi_k(x)|^2\,\mathrm{d}x}{\int_\Omega q(x)|\phi_k(x)|^2\,\mathrm{d}x} - 2^{-k} \\
&\ge \lambda_{1,\gamma_k} \frac{\int_\Omega \gamma_k(x)|\phi_k(x)|^2\,\mathrm{d}x}{\int_\Omega q(x)|\phi_k(x)|^2\,\mathrm{d}x} - 2^{-k}\,.
\end{aligned}$$

This yields

$$\lambda_{1,q} \le \lambda_{1,\gamma_k} \le (\lambda_{1,q} + 2^{-k}) \frac{\int_\Omega q(x)|\phi_k(x)|^2 \,\mathrm{d}x}{\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \,\mathrm{d}x} \,. \tag{3.13}$$

By the compact embedding of $H_0^s(\Omega)$ into $L^2(\Omega)$ and $\|\phi_k\|_{H_0^s(\Omega)} \le \|\varphi_{1,q}\|_{H_0^s(\Omega)}$, we can find a subsequence $\phi_{k_j} \to \psi$ in $L^2(\Omega)$, where ψ is some element of $L^2(\Omega)$. Then, $\phi_{k_j}^2 \to \psi^2$ in $L^1(\Omega)$. Since $q \in L^\infty(\Omega)$, $\int_\Omega q(x)|\phi_{k_j}(x)|^2 \,\mathrm{d}x \to \int_\Omega q(x)|\psi(x)|^2 \,\mathrm{d}x$. Now we show that

$$\int_\Omega \gamma_{k_j}(x)|\phi_{k_j}(x)|^2 \,\mathrm{d}x \to \int_\Omega q(x)|\psi(x)|^2 \,\mathrm{d}x$$

as well. Indeed,

$$\int_\Omega \gamma_{k_j}(x)|\phi_{k_j}(x)|^2 \,\mathrm{d}x = \int_\Omega (\gamma_{k_j}(x) - q(x))|\phi_{k_j}(x)|^2 \,\mathrm{d}x + \int_\Omega q(x)|\phi_{k_j}(x)|^2 \,\mathrm{d}x \,.$$

By (3.10), $q(x) - \gamma_{k_j}(x) \le 1/k_j$, thus

$$\Big| \int_\Omega (\gamma_{k_j}(x) - q(x))|\phi_{k_j}(x)|^2 \,\mathrm{d}x \Big| \le 1/k_j \int_\Omega |\phi_{k_j}|^2 \,\mathrm{d}x \le \frac{C}{k_j} \|\varphi_{1,q}\|^2_{H_0^s(\Omega)} \to 0$$

as $k_j \to +\infty$, where C is the constant of the embedding of $H_0^s(\Omega)$ into $L^2(\Omega)$. Observe that $\psi \ge \delta^s > 0$ a.e. in Ω since $\phi_{k_j} \ge \delta^s$ a.e. in Ω, and hence

$$\frac{\int_\Omega q(x)|\phi_k(x)|^2 \,\mathrm{d}x}{\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \,\mathrm{d}x} \ \to\ \frac{\int_\Omega q(x)|\psi(x)|^2 \,\mathrm{d}x}{\int_\Omega q(x)|\psi(x)|^2 \,\mathrm{d}x} = 1$$

as $k_j \to +\infty$. Thus, (3.13) establishes that $\lambda_{1,\gamma_{k_j}} \to \lambda_{1,q}$. Since λ_{1,γ_k} is monotone sequence, it must hold for entire sequence $\lambda_{1,\gamma_k} \searrow \lambda_{1,q}$. □

4. Proof of Theorem 2.3

We follow the idea of proof from Clement-Sweers [14], where this result was proven for the Laplacian case ($s = 1$) using the Schauder fixed point theorem.

Proof of Theorem 2.3. Consider the modified function $g^*\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined as

$$g^*(x,t) := \begin{cases} g(x, \underline{u}(x)) & \text{if } t < \underline{u}(x)\,, \\ g(x,t) & \text{if } \underline{u}(x) \le t \le \overline{u}(x)\,, \\ g(x, \overline{u}(x)) & \text{if } t > \overline{u}(x) \end{cases}$$

and note that g^* is clearly a Carathéodory function. We observe that any weak solution u of (2.7) satisfying $\underline{u} \le u \le \overline{u}$ a.e. in Ω is also a weak solution of the modified problem

$$\begin{gathered} (-\Delta)^s u = g^*(x,u) \quad \text{in } \Omega; \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,. \end{gathered} \tag{4.1}$$

Moreover, using the definition of g^*, it follows from the claim below that any weak solution of (4.1) is a weak solution of (2.7).

Claim: If u is a weak solution of (4.1), then $\underline{u} \le u \le \overline{u}$ a.e. in Ω.
First, we establish $u \le \overline{u}$ a.e. in Ω by showing that $\mathrm{meas}(A) = 0$, where $A := \{x \in \mathbb{R}^N \colon \overline{u}(x) < u(x)\}$. Clearly A is measurable (in the sense of Lebesgue) since $u \in H_0^s(\Omega)$ and $\overline{u} \in H^s(\mathbb{R}^N)$. Assume to the contrary that $\mathrm{meas}(A) > 0$. We note that $\mathrm{meas}\big(A \cap (\mathbb{R}^N \setminus \Omega)\big) = 0$ since $\overline{u} \ge 0 = u$ a.e. in $\mathbb{R}^N \setminus \Omega$. Hence, $\mathrm{meas}(A \cap \Omega) > 0$. Setting $z^+ := \max\{0, z\} \ge 0$, we see that $[u - \overline{u}]^+ \in H_0^s(\Omega)$ since $[u - \overline{u}]^+ \in H^s(\mathbb{R}^N)$

and it vanishes almost everywhere outside $A \subset \Omega$. Taking $\phi := [u - \overline{u}]^+$ as a test function in (2.4) and (2.5), and using the definitions of g^* and A, we obtain

$$\begin{aligned} \mathcal{E}(u, [u-\overline{u}]^+) &= \int_\Omega g^*(x, u(x))[u-\overline{u}]^+(x)\,\mathrm{d}x \\ &= \int_A g^*(x, u(x))[u-\overline{u}]^+(x)\,\mathrm{d}x \\ &= \int_A g(x, \overline{u}(x))[u-\overline{u}]^+(x)\,\mathrm{d}x \\ &= \int_\Omega g(x, \overline{u}(x))[u-\overline{u}]^+(x)\,\mathrm{d}x \\ &\le \mathcal{E}(\overline{u}, [u-\overline{u}]^+)\,. \end{aligned} \tag{4.2}$$

On one hand, subtracting the right-hand side from the left-hand side in (4.2) and rearranging the terms yields the inequality

$$\mathcal{E}((u-\overline{u}), [u-\overline{u}]^+) \le 0\,. \tag{4.3}$$

On the other hand, by taking $v = u - \overline{u}$, it follows from [24, Lem. 3.3] that

$$[v(x)-v(y)][v^+(x)-v^+(y)] \ge [v^+(x)-v^+(y)]^2 \text{ for a.e. } x, y \in \mathbb{R}^N\,. \tag{4.4}$$

Using (2.2) and (2.3), the pointwise estimate (4.4) yields

$$\mathcal{E}(v, v^+) \ge \frac{C_{N,s}}{2}\|v^+\|^2_{H_0^s(\Omega)} > 0\,,$$

since $\operatorname{meas}(A) > 0$, $v > 0$ in A, and $\|\cdot\|_{H_0^s(\Omega)}$ is a norm on $H_0^s(\Omega)$. This is a contradiction to (4.3). Hence, $\operatorname{meas}(A) = 0$, that is, $u(x) \le \overline{u}(x)$ for a.e. $x \in \Omega$. Similarly, by letting $\phi := [\underline{u} - u]^+$ as a test function, and repeating the argument above we can show that $\operatorname{meas}(B) = 0$, where $B := \{x \in \mathbb{R}^N : \underline{u}(x) > u(x)\}$. Hence, $u(x) \ge \underline{u}(x)$ a.e. $x \in \Omega$. This proves the claim.

Therefore, it suffices to show the existence of solution of (4.1) using the Schauder fixed point theorem. To construct a compact operator, consider the following linear problem

$$\begin{gathered} (-\Delta)^s w = \theta(x) \quad \text{in } \Omega; \\ w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\,. \end{gathered} \tag{4.5}$$

For each $\theta \in H^{-s}(\Omega)$ (the dual of $H_0^s(\Omega)$), there exists a unique weak solution $w \in H_0^s(\Omega)$ of (4.5), see [20, Thm. 12] for $N \ge 2$ and [9, Prop. 2.1] for $N = 1$. Moreover, if $\theta \in L^\infty(\Omega)$, then there exists $C > 0$ such that

$$\|w\|_{C^{0,s}(\overline{\Omega})} \le C\|\theta\|_{L^\infty(\Omega)}\,, \tag{4.6}$$

see [28, Prop. 7.2]. Then, the solution operator $K \colon L^\infty(\Omega) \to L^\infty(\Omega)$ given by $\theta \mapsto w$ is well defined, continuous, and compact since the following holds for some $s' \in (0, s)$

$$L^\infty(\Omega) \xrightarrow{K} C^{0,s}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{0,s'}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)\,.$$

The Nemytskii operator $H \colon L^\infty(\Omega) \to L^\infty(\Omega)$ defined by $u \mapsto g^*(x, u(x))$ is continuous (see [4, Thm. 3.17, p. 110]) since g^* satisfies (H1) and (H2). Then, $KH \colon L^\infty(\Omega) \to L^\infty(\Omega)$ is continuous and compact, and fixed points of KH are solutions of (4.1).

Next, we find a nonempty, closed, and convex subset of $L^\infty(\Omega)$ to apply the Schauder fixed point theorem. Without loss of generality, assume that $\underline{u} \not\equiv \overline{u}$ in Ω. Then, using $\underline{u}, \overline{u} \in L^\infty(\Omega)$, we define

$$r^* := \max\{\|\underline{u}\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\Omega)}\} > 0\,.$$

Then, it follows from (H1), applied to g^*, that there exists $a_{r^*} \in L^\infty(\Omega)$ such that $|g^*(x,t)| \le a_{r^*}(x)$ for all $|t| \le r^*$. Therefore, for any $u \in L^\infty(\Omega)$, we have

$$\|KH(u)\|_{L^\infty(\Omega)} \le \|K\|\|H(u)\|_{L^\infty(\Omega)} \le \|K\|\|a_{r^*}\|_{L^\infty(\Omega)}\,,$$

and hence the operator KH maps $\overline{B}_R(0)$ to itself where $R := \|K\|\|a_{r^*}\|_{L^\infty(\Omega)}$ and $\|\cdot\|$ is the operator norm. Hence, by the Schauder fixed point theorem, KH has a fixed point $u \in \overline{B}_R(0) \subset L^\infty(\Omega)$. This implies that the modified problem (4.1) and hence the original problem (2.7) has a weak solution $u \in L^\infty(\Omega)$ such that $\underline{u} \le u \le \overline{u}$. By the definition of K it follows that $u \in H_0^s(\Omega)$ as well. Hence, the proof of Theorem 2.3 is complete. □

5. Proofs of Theorems 2.5 and 2.6

To prove Theorems 2.5 and 2.6, we employ Theorem 2.3. For each case, we construct an ordered pair of weak sub- and supersolutions in $H_0^s(\Omega) \subset H^s(\mathbb{R}^N)$ of (1.1) where $f(x,t) = f(t)$.

Proof of Theorem 2.5. Since $f(0) > 0$, it follows that $\underline{u} \equiv 0 \in H_0^s(\Omega)$ is a weak subsolution of (1.1). Now let $\lambda > 0$ be fixed and $e \in H_0^s(\Omega)$ be the positive weak solution of (3.1). We show that there exists $M_\lambda > 0$ such that $\overline{u} := Me$ is a weak supersolution of (1.1) for all $M \ge M_\lambda$. We observe that while f is not assumed to be nondecreasing, $\overline{f}(t) := \max_{\sigma\in[0,t]} f(\sigma)$ is nondecreasing. Moreover, $f(t) \le \overline{f}(t)$ for all $t \ge 0$, and $\overline{f}$ satisfies the sublinear condition at infinity

$$\lim_{t\to+\infty} \frac{\overline{f}(t)}{t} = 0\,.$$

Therefore, there exists $M_\lambda > 0$ such that for all $M \ge M_\lambda$,

$$\frac{\overline{f}(M\|e\|_{L^\infty(\Omega)})}{M\|e\|_{L^\infty(\Omega)}} \le \frac{1}{\lambda\|e\|_{L^\infty(\Omega)}} \text{ or equivalently } \lambda\overline{f}(M\|e\|_{L^\infty(\Omega)}) \le M\,.$$

Then $\overline{u} = Me \in H_0^s(\Omega)$ satisfies

$$\begin{aligned} M\mathcal{E}(e,\phi) &= M\int_\Omega \phi(x)\,\mathrm{d}x \\ &\ge \lambda\int_\Omega \overline{f}(M\|e\|_{L^\infty(\Omega)})\phi(x)\,\mathrm{d}x \\ &\ge \lambda\int_\Omega \overline{f}(Me(x))\phi(x)\,\mathrm{d}x \\ &\ge \lambda\int_\Omega f(Me(x))\phi(x)\,\mathrm{d}x = \lambda\int_\Omega f(\overline{u})\phi(x)\,\mathrm{d}x \end{aligned}$$

for all $\phi \in H_0^s(\Omega)$ with $\phi \ge 0$ a.e. in Ω. Therefore, $\overline{u}$ is a weak supersolution of (1.1) for each $\lambda > 0$. Clearly $\overline{u} = Me \ge 0 = \underline{u}$ a.e. in Ω. Hence, by Theorem 2.3 and [28, Lem. 7.3], there exists a weak solution u of (1.1) such that $0 < u \le Me$ a.e. in Ω for any $\lambda > 0$. This completes the proof of Theorem 2.5. □

Proof of Theorem 2.6. Let $\lambda > \frac{\lambda_1}{f'(0)}$ be fixed, where λ_1 is the principal eigenvalue of (3.3) with $q \equiv 1$ and $\varphi_1 > 0$ is the corresponding principal eigenfunction. Note that since $f(0) = 0$, $u \equiv 0$ is a solution and hence a subsolution of (1.1). Hence, to complete the proof, we must construct a positive weak subsolution. We show that an appropriate constant multiple of φ_1 is a weak subsolution of (1.1). We find this constant by analyzing the function $\Theta(t) := \lambda_1 t - \lambda f(t)$ for $t \geq 0$. Clearly $\Theta(0) = 0$ and $\Theta'(t) = \lambda_1 - \lambda f'(t)$. Therefore, $\Theta'(0) < 0$ since $\lambda > \frac{\lambda_1}{f'(0)}$ and hence there exists $\theta(\lambda) > 0$ such that $\Theta(t) < 0$ for any $t \in (0, \theta(\lambda))$.

Now we show that $\underline{u} := m\varphi_1 \in H_0^s(\Omega)$ is a positive weak subsolution of (1.1) for any $m \in (0, m_\lambda)$, where $m_\lambda := \frac{\theta(\lambda)}{\|\varphi_1\|_{L^\infty(\Omega)}}$. Indeed, by (3.6) and the discussion above, $\underline{u}$ satisfies

$$
\begin{aligned}
m\,\mathcal{E}(\varphi_1, \phi) &= \lambda_1 \int_\Omega m\varphi_1(x)\phi(x)\,\mathrm{d}x \\
&\leq \lambda \int_\Omega f(m\varphi_1)\phi(x)\,\mathrm{d}x = \lambda \int_\Omega f(\underline{u})\phi(x)\,\mathrm{d}x
\end{aligned}
$$

for all $\phi \in H_0^s(\Omega)$ with $\phi \geq 0$ a.e. in Ω. Hence, for any $\lambda > \frac{\lambda_1}{f'(0)}$ and any $m \in (0, m_\lambda)$, $\underline{u} = m\varphi_1$ is a weak subsolution of (1.1).

As in the proof of Theorem 2.5, for any $\lambda > \frac{\lambda_1}{f'(0)}$ there exists $M_\lambda > 0$ such that for $M \geq M_\lambda$, the function $\overline{u} = Me \in H_0^s(\Omega)$ is a weak supersolution of (1.1). Using the left estimate of e in (3.2) and the right estimate of φ_1 in (3.7), and by choosing M sufficiently large and/or choosing m sufficiently small, we obtain $\underline{u} \leq \overline{u}$ a.e. in Ω. Hence, by Theorem 2.3, (1.1) has a positive weak solution u satisfying $m\varphi_1 \leq u \leq Me$ a.e. in Ω for any $\lambda > \frac{\lambda_1}{f'(0)}$. This completes the proof of Theorem 2.6. □

6. Proofs of Theorems 2.8 and 2.9

Here also we construct an ordered pair of weak sub- and supersolution of (1.1) in $H_0^s(\Omega) \subset H^s(\mathbb{R}^N)$.

Proof of Theorem 2.8. Let $\lambda > \lambda_{1,q}$ be fixed. First, we construct a positive weak subsolution of (2.9). Since $\lambda > \lambda_{1,q}$ and $\lambda_{1,\gamma_k} \searrow \lambda_{1,q}$ as $k \to +\infty$ (by Proposition 3.2), there exists $l \in \mathbb{N}$ such that $\lambda_{1,q} \leq \lambda_{1,\gamma_l} < \lambda$, where λ_{1,γ_l} is the principal eigenvalue of (3.9) with γ_l defined by (3.9). Let $\varphi_{1,\gamma_l} \in H_0^s(\Omega)$ be a positive eigenfunction corresponding to λ_{1,γ_l}, and $e \in H_0^s(\Omega)$ be a positive weak solution of (3.1)

We show that there exist $m_\lambda > 0$ and $\varepsilon > 0$ such that for all $m \in (0, m_\lambda)$, $\underline{u} := m(\varphi_{1,\gamma_l} - \varepsilon e) \in H_0^s(\Omega)$ is a positive weak subsolution of (2.9). Set $\alpha := \sqrt{\frac{\lambda_{1,\gamma_l}}{\lambda}} \in (0,1)$. Then, using (3.2), and (3.7) with $q = \gamma_l$, there exists $\varepsilon > 0$ such that

$$
\varphi_{1,\gamma_l} - \varepsilon e > \alpha\varphi_{1,\gamma_l} > 0 \quad \text{a.e. in } \Omega\,. \tag{6.1}
$$

Define

$$
m_\lambda := \min\Big\{\frac{\varepsilon}{\lambda\alpha\|\varphi_{1,\gamma_l}(\varphi_{1,\gamma_l} - \varepsilon e)\|_{L^\infty(\Omega)}}\,,\ \frac{1-\alpha}{l\|\varphi_{1,\gamma_l} - \varepsilon e\|_{L^\infty(\Omega)}}\Big\}
$$

and let $m \in (0, m_\lambda)$. Using the weak formulation of e and φ_{1,γ_l}, we see that $\underline{u} = m(\varphi_{1,\gamma_l} - \varepsilon e) \in H_0^s(\Omega)$ satisfies

$$\mathcal{E}(\underline{u}, \phi) = m \int_\Omega [\lambda_{1,\gamma_l}\, \gamma_l(x)\, \varphi_{1,\gamma_l}(x) - \varepsilon]\phi(x)\,\mathrm{d}x$$

for all $\phi \in H_0^s(\Omega)$. Therefore, $\underline{u}$ is a weak subsolution of (2.9) if

$$\begin{aligned} &m \int_\Omega [\lambda_{1,\gamma_l}\, \gamma_l(x)\varphi_{1,\gamma_l}(x) - \varepsilon]\phi(x)\,\mathrm{d}x \\ &\le \lambda\, m \int_\Omega [\varphi_{1,\gamma_l}(x) - \varepsilon e(x)]\big[q(x) - m(\varphi_{1,\gamma_l}(x) - \varepsilon e(x))\big]\phi(x)\,\mathrm{d}x \end{aligned} \tag{6.2}$$

for all $\phi \in H_0^s(\Omega)$ with $\phi \ge 0$ a.e. in Ω. Using the definition of γ_l and (6.1), we obtain

$$\lambda m \alpha \varphi_{1,\gamma_l}\big[\gamma_l - m(\varphi_{1,\gamma_l} - \varepsilon e)\big] \le \lambda m(\varphi_{1,\gamma_l} - \varepsilon e)\big[q - m(\varphi_{1,\gamma_l} - \varepsilon e)\big] \quad \text{a.e. in } \Omega\,.$$

Therefore, (6.2) holds if

$$\lambda_{1,\gamma_l}\, \gamma_l \varphi_{1,\gamma_l} - \varepsilon \le \lambda\, \alpha\, \varphi_{1,\gamma_l}\big[\gamma_l - m(\varphi_{1,\gamma_l} - \varepsilon e)\big] \quad \text{a.e. in } \Omega\,. \tag{6.3}$$

Let $\Omega_l := \{x \in \Omega : q(x) < 1/l\}$. If $x \in \Omega_l$, then $\gamma_l(x) = 0$. In this case, the inequality in (6.3) holds a.e. in Ω_l since

$$m < m_\lambda \le \frac{\varepsilon}{\lambda\alpha \|\varphi_{1,\gamma_l}(\varphi_{1,\gamma_l} - \varepsilon e)\|_{L^\infty(\Omega)}}\,.$$

If $x \in \Omega \setminus \Omega_l$, then $\gamma_l(x) = q(x) \ge 1/l$. In this case, the inequality in (6.3) holds a.e. in $\Omega \setminus \Omega_l$ since the inequality

$$\lambda_{1,\gamma_l}\, \gamma_l \varphi_{1,\gamma_l} \le \lambda\, \alpha\, \varphi_{1,\gamma_l}\big[\gamma_l - m(\varphi_{1,\gamma_l} - \varepsilon e)\big]$$

holds by choosing

$$m < m_\lambda \le \frac{1-\alpha}{l\|\varphi_{1,\gamma_l} - \varepsilon e\|_{L^\infty(\Omega)}}\,.$$

Hence, $\underline{u} = m(\varphi_{1,\gamma_l} - \varepsilon e)$ is a positive weak subsolution of (2.9) for any $m \in (0, m_\lambda)$.

Now we construct a positive weak supersolution. Since $q \in L^\infty(\Omega)$ with $0 \le q \le 1$ a.e. in Ω and $\max_{y\in\mathbb{R}} y(1-y) = 1/4$, the inequality

$$1 \ge M\, e(q - Me)$$

holds for all $M \ge M_\lambda = \lambda$ a.e. in Ω. Then $\overline{u} = M\, e \in H_0^s(\Omega)$ satisfies

$$\begin{aligned} \mathcal{E}(\overline{u}, \phi) &\ge \lambda \int_\Omega \phi(x)\,\mathrm{d}x \\ &\ge \lambda \int_\Omega Me(x)\,\big(q(x) - Me(x)\big)\phi(x)\,\mathrm{d}x \\ &= \lambda \int_\Omega \overline{u}\,\big(q(x) - \overline{u}\big)\phi(x)\,\mathrm{d}x \end{aligned}$$

for all $\phi \in H_0^s(\Omega)$ with $\phi \ge 0$ a.e. in Ω. Therefore, $\overline{u} = Me$ is a weak supersolution of (2.9) for any $M \ge \lambda$.

Finally, using the left estimate of (3.2), and the right estimate of (3.7) combined with (6.1), we can choose $M \ge \lambda$ sufficiently large and/or $0 < m < m_\lambda$ sufficiently small, so that $\underline{u} \le \overline{u}$ a.e. in Ω. Hence, by Theorem 2.3, (2.9) has a positive weak solution u satisfying $m(\varphi_{1,\gamma_l} - \varepsilon e) \le u \le Me$ a.e. in Ω for any $\lambda > \lambda_{1,q}$. This completes the proof of Theorem 2.8 $\square$

Proof of Theorem 2.9. First, we construct a positive weak subsolution for (2.10). Let $\lambda > \lambda_1$ be fixed and define $\beta := \sqrt{\frac{\lambda_1}{\lambda}} \in (0,1)$. Then, as in the proof of Theorem 2.8 there exists $\varepsilon > 0$ such that $\varphi_1 - \varepsilon e > \beta\varphi_1 > 0$ a.e. in Ω. Define $\underline{u} := m^*(\varphi_1 - \varepsilon e) \in H_0^s(\Omega)$ with fixed $m^* := \frac{1-\beta}{2\|\varphi_1 - \varepsilon e\|_{L^\infty(\Omega)}}$. We show that $\underline{u}$ is a positive weak subsolution of (2.10) for any $0 < a < a^* := \frac{\varepsilon m^*}{\lambda}$. Then $\underline{u}$ satisfies

$$\mathcal{E}(\underline{e}, \phi) = m^* \int_\Omega (\lambda_1 \varphi_1(x) - \varepsilon)\phi(x)\, \mathrm{d}x\,,$$

for all $\phi \in H_0^s(\Omega)$. Hence, $\underline{u}$ is a weak subsolution of (2.10) if

$$m^*(\lambda_1 \varphi_1 - \varepsilon) \le \lambda\, m^*(\varphi_1 - \varepsilon e)\big[1 - m^*(\varphi_1 - \varepsilon e)\big] - \lambda a \quad \text{a.e. in } \Omega\,. \tag{6.4}$$

Since $\varphi_1 - \varepsilon e > \beta\varphi_1 > 0$, (6.4) is satisfied if

$$\lambda_1\varphi_1 \le \lambda\beta\varphi_1\big[1 - m^*(\varphi_1 - \varepsilon e)\big] + \varepsilon - \frac{\lambda a}{m^*} \quad \text{a.e. in } \Omega\,. \tag{6.5}$$

But, $\varepsilon m^* - \lambda a \ge 0$ since $\varepsilon m^* - \lambda a^* = 0$ by our choice of a^*, and $a < a^*$. Then, using $\varphi_1 > \varphi_1 - \varepsilon e > 0$ a.e. in Ω and $m^* = \frac{1-\beta}{2\|\varphi_1 - \varepsilon\, e\|_{L^\infty(\Omega)}}$, (6.5) follows from the inequality

$$\lambda_1\varphi_1 \le \lambda\beta\varphi_1\big[1 - m^*\|\varphi_1 - \varepsilon e\|_{L^\infty(\Omega)}\big] \quad \text{a.e. in } \Omega\,.$$

Hence, $\underline{u} = m^*(\varphi_1 - \varepsilon e)$ is a subsolution of (2.10) for $a < a^*$.

As in the proof of Theorem 2.8, $\overline{u} = M\, e \in H_0^s(\Omega)$ is a supersolution of (2.10) for any $M \ge M_\lambda = \lambda$ since $1 \ge Me\big(1 - M\, e\big) - a$ a.e. in Ω.

Again, using the estimates (3.2) and (3.7), we can further refine the choice of $M \ge \lambda$ to be sufficiently large such that $\underline{u} \le \overline{u}$ in Ω. Therefore, by Theorem 2.3, for any $\lambda > \lambda_1$, (2.10) has a positive weak solution u satisfying $m^*(\varphi_1 - \varepsilon e) \le u \le Me$ a.e. in Ω for $0 < a < a^*$. This completes the proof of Theorem 2.9. $\square$

7. Finite element method for the fractional Laplacian in one dimension

The finite element approximation of the linear one dimensional problem

$$\begin{gathered} (-\Delta)^s u = z(x) \quad \text{in } (0,1) \\ u = 0 \quad \text{in } \mathbb{R} \setminus (0,1)\,, \end{gathered} \tag{7.1}$$

for $s \in (0,1)$ was investigated, including convergence results with z in appropriate function spaces, see [2, 8]. This motivated the investigation of numerical positive weak solutions for the nonlinear fractional Laplacian problems

$$\begin{gathered} (-\Delta)^s u = \lambda f(x,u) \quad \text{in } (0,1) \\ u = 0 \quad \text{in } \mathbb{R} \setminus (0,1)\,, \end{gathered} \tag{7.2}$$

where $\lambda > 0$ and $f : (0,1) \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function. For numerical experiments, we further assume that $f_t(x,t) := \frac{\partial f}{\partial t}(x,t)$ is continuous a.e. in $(0,1)$, and f satisfies certain Hölder type conditions with respect to $x \in (0,1)$, specified below. We consider examples of nonlinearity f satisfying the respective hypotheses of Theorems 2.5 - 2.9 with these additional assumptions.

We use the one dimensional finite element method (FEM) developed for the linear fractional Laplacian problems of the form (7.1) in [8] to construct a numerical solution u (often positive) with $\lambda > 0$ of the nonlinear problem (7.2). Moreover, using the branch following technique of [25], we construct bifurcation diagrams $\|u\|_\infty$

vs. λ, where $\|u\|_\infty = \|u\|_{L^\infty(0,1)}$. Additionally, we formulate relevant conjectures based on the qualitative features of the bifurcation diagrams from our numerical experiments.

As in [8], we use the weak formulation of (7.2) to seek solutions $u \in H_0^s(0,1)$ such that for all $\phi \in H_0^s(0,1)$

$$\frac{C_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y = \lambda\int_0^1 f(x,u(x))\phi(x)dx\,.$$

Remark 7.1. It is known that (see [9, 15, 30]) for u from a suitable class of functions,

$$\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s\to 0^+}(-\Delta)^s u = I\,, \tag{7.3}$$

where I is the identity operator. Furthermore, it was shown in [9] that the weak solution of the Poisson equation for $(-\Delta)^s$ with homogeneous Dirichlet condition on $\mathbb{R}^N\setminus\Omega$ approaches to the weak solution of Poisson equation for $-\Delta$ with homogeneous Dirichlet condition on $\partial\Omega$ as $s\to 1^-$. We utilize the limiting behavior information (7.3) as a hint for the correctness of our numerical scheme. In particular, throughout this section, we use the finite difference or the quadrature method to generate the bifurcation diagram for the Laplacian case ($s=1$) and then compare to the fractional Laplacian case ($s=0.99$) using the finite element method, before proceeding with other values of $s\in(0,1)$.

We describe our method below.

7.1. **Discretization.** We introduce a uniform partition $0 = x_0 < x_1 < x_2 \dots < x_{n+1} = 1$, of $[0,1]$, with step size $h = x_i - x_{i-1}$ for $i = 1,\dots,n+1$. Let V_h be an n-dimensional subspace of $H_0^s(0,1)$ spanned by $\{\phi_1,\dots,\phi_n\}$, where

$$\phi_i(x) := \begin{cases} 1-|x-x_i|/h & \text{if } x\in[x_{i-1},x_{i+1}]\,,\\ 0 & \text{if } x\in\mathbb{R}\setminus[x_{i-1},x_{i+1}]\end{cases} \tag{7.4}$$

for $i=1,\dots,n$. The finite element approximation $u_h\in V_h$ of weak solution $u\in H_0^s(0,1)$ of (7.2) is expressed as

$$u_h(x) := \sum_{i=1}^n u_i\phi_i(x)\,,$$

where $u_i\in\mathbb{R}$ are unknowns and u_h satisfies the system of n equations

$$\frac{C_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{[u_h(x)-u_h(y)][\phi_j(x)-\phi_j(y)]}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y = \lambda\int_0^1 f\left(x,u_h(x)\right)\phi_j(x)\,\mathrm{d}x \tag{7.5}$$

for $j=1,\dots,n$. To implement the finite element scheme, we express (7.5) in matrix notation. For a column vector $\mathbf{u} := [u_1,\dots,u_n]^T$, the left hand side of (7.5) can be expressed as $\mathcal{A}\mathbf{u}$, where $\mathcal{A}$ is the $n\times n$ stiffness matrix corresponding to the left hand side of (7.5) derived in [8].

To numerically compute the integral on the right hand side of (7.5), we assume that there exists $L>0$ such that for any $y_1,y_2\in(0,1)$ and any $t_1,t_2\ge 0$

$$|f(y_2,t_2)-f(y_1,t_1)| \le L(|y_2-y_1|^s + |t_2-t_1|)\,. \tag{7.6}$$

Then, the expectation that $\|u_h\|_{C^{0,s}([0,1])}\le K'$ holds (independent of h), yields

$$|f(x,u_h(x)) - f(x_j,u_h(x_j))| \le L(|x-x_j|^s + |u_h(x)-u_h(x_j)|) \le L(1+K')h^s\,.$$

Therefore, for all $j = 1, \dots, n$, using the definition (7.4) of ϕ_j one has

$$
\begin{aligned}
&\int_0^1 f\left(x, u_h(x)\right) \phi_j(x)\,\mathrm{d}x \\
&= \int_{x_{j-1}}^{x_{j+1}} f\left(x, u_h(x)\right) \phi_j(x)\,\mathrm{d}x \\
&= \int_{x_{j-1}}^{x_{j+1}} \left[f\left(x_j, u_h(x_j)\right) \phi_j(x) + f\left(x, u_h(x)\right) \phi_j(x) - f\left(x_j, u_h(x_j)\right) \phi_j(x) \right] \mathrm{d}x \\
&= f(x_j, u_j) \int_{x_{j-1}}^{x_{j+1}} \phi_j(x)\,\mathrm{d}x + \int_{x_{j-1}}^{x_{j+1}} \left[f\left(x, u_h(x)\right) \phi_j(x) - f\left(x_j, u_h(x_j)\right) \phi_j(x) \right] \mathrm{d}x \\
&= h f(x_j, u_j) + O(h^{1+s}) \,,
\end{aligned}
$$

where (more precisely)

$$
0 \le |O(h^{1+s})| \le L(1 + K')h^{1+s} \,.
$$

Then, defining the column vector $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$
\mathbf{F}(\mathbf{u}) := h[f(x_1, u_1)\,, f(x_2, u_2), \dots, f(x_n, u_n)]^T \,,
$$

we rewrite (7.5) as a matrix equation

$$
\mathcal{A}\mathbf{u} = \lambda \mathbf{F}(\mathbf{u}) \,. \tag{7.7}
$$

We solve this system (7.7) for a given nonlinearity f and $\lambda > 0$ with Newton's method provided we have a suitable initial guess for iteration. A multiple of the solution of the linear problem $(-\Delta)^s e = 1$ in $(0, 1)$ with $u = 0$ in $\mathbb{R} \setminus (0, 1)$ served as a good candidate for an initial guess in many cases.

Now we describe the pseudo-code for constructing numerical solutions and numerical bifurcation diagrams, where $|\cdot|_\infty$ will denote the maximum norm in $\mathbb{R}^n$.

Input:

$s \in (0, 1)$	(real parameter in $(-\Delta)^s$)
$0 \le \lambda_{min} < \lambda_{max}$	(range of values of λ in the bifurcation diagram)
$m \in \mathbb{N}$	(number of partitions of the interval $[\lambda_{min}, \lambda_{max}]$)
$n \in \mathbb{N}$	(number of interior nodes in partition of interval $[0, 1]$)
$6 < r < 15$	(10^{-r} is the tolerance in the Newton iteration)

Output:

$\mathcal{S}$	(list of points of the form $(\lambda, |\mathbf{u}|_\infty)$)

Begin

% Initialization

Create interior nodes of the uniform partition $\mathcal{P}$ of $[0,1]$
by setting $x_j \leftarrow j/(n+1), \quad j = 1, \dots, n$

$C_{1,s} \leftarrow \frac{2^{2s}\, s\, \Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1-s)}$

Assemble the $n \times n$ stiffness matrix $\mathcal{A}$ for the partition $\mathcal{P}$
and parameter s using the algorithm described in [8, page 12]

Create a uniform partition Λ of $[\lambda_{\min}, \lambda_{\max}]$
by setting $\mu_i \leftarrow \lambda_{\min} + \frac{\lambda_{\max} - \lambda_{\min}}{m} i, \quad i = 0, \dots, m$

$\mathbf{u}_{\text{init}} \leftarrow$ Solution of $\mathcal{A}\,\mathbf{e} = \mathbf{1}$

% Here $\mathbf{1}$ stands for $n \times 1$ column vector of 1s.

$\mathcal{S} \leftarrow$ Empty list

% End of Initialization

% Main Loop

For $i := 0 : m$ do

% Apply Newton iterations to: $\mathcal{A}\mathbf{u} = \mu_i \mathbf{F}(\mathbf{u})$

$\mathbf{u} \leftarrow \mathbf{u}_{\text{init}}$

% Compute $\mathbf{F}(\mathbf{u})$ componentwise (represented by column vector $\mathbf{b}$)

$[\mathbf{b}]_j \leftarrow h f(x_j, u_j)$ for $j = 1, \dots, n$

$\mathbf{res} \leftarrow \mathcal{A}\mathbf{u} - \mu_i \mathbf{b}$

% Newton loop

While $|\mathbf{res}|_\infty > 10^{-r}$ do

% Compute $\mathbf{J_F}$, the Jacobian matrix of $\mathbf{F}(\mathbf{u})$ componentwise

$[\mathbf{J_F}]_{j,j} \leftarrow h f_t(x_j, u_j)$ for $j = 1, \dots, n$, and
$[\mathbf{J_F}]_{i,j} \leftarrow 0$ for $i \neq j$, $i, j = 1, \dots n$

% Compute $\mathbf{J}$, the Jacobian matrix of the system (7.7)

$\mathbf{J} \leftarrow \mathcal{A} - \mu_i \mathbf{J_F}$

% Newton's update of $\mathbf{u}$

$\mathbf{u} \leftarrow \mathbf{u} - \mathbf{J}^{-1}\mathbf{res}$

% Update of $\mathbf{F}(\mathbf{u})$ componentwise

$[\mathbf{b}]_j \leftarrow h f(x_j, u_j)$ for $j = 1, \dots, n$

% Update of $\mathbf{res}$

$\mathbf{res} \leftarrow \mathcal{A}\mathbf{u} - \mu_i \mathbf{b}$

EndWhile

$\mathcal{S} \leftarrow \text{Append}(\mathcal{S}, (\mu_i, |\mathbf{u}|_\infty))$

EndFor

% End of Main Loop

Return $\mathcal{S}$

End

7.2. **Construction of bifurcation diagrams.** Let $\mathcal{S}$ be a closed connected set of $(\lambda, u) \in \mathbb{R} \times L^{\infty}(0,1)$ such that u is a weak solution of (7.2) corresponding to $\lambda > 0$. In each example below, we will discuss the shape of $\mathcal{S}$ via the bifurcation diagram obtained numerically in the $\|u\|_\infty$ vs. λ plane. These bifurcation diagrams are in agreement with the results obtained in Section 2 and furthermore, help formulate conjectures.

For each of the bifurcation diagrams, we also give a numerical positive solution for a specific value of λ for which existence is guaranteed by the results in Section 2.

We observe from the proofs of Theorems 2.5-2.9 that these positive weak solutions u satisfy $c_1\delta^s(x) \le u(x) \le c_2\delta^s(x)$ for $x \in [0,1]$. Hence, the influence of s on the behavior of positive solutions near the boundary ($x = 0$ and $x = 1$) becomes more pronounced for $s \in (0,1)$ small.

Example 7.2. Let $f(t) = e^{\frac{t}{1+t}}$ for $t \ge 0$ (satisfying the hypotheses of Theorem 2.5).

For existence result and the bifurcation diagram for the Laplacian case ($s = 1$), see [22, Sec. 2.2]. For the fractional Laplacian case, (7.2) has a positive weak solution, for each $\lambda > 0$, by Theorem 2.5. Clearly $u \equiv 0$ is a solution of (7.2) for $\lambda = 0$. Figure 1 shows the bifurcation diagrams for (A) $s = 1$ (B) $s = 0.99$ (C) $s = 0.9$ (D) $s = 0.7$ (E) $s = 0.5$ and (F) $s = 0.3$. We observe that the bifurcation diagrams are qualitatively similar for all the s values considered. The inset in each bifurcation diagram shows the typical profile of a numerical positive solution corresponding to $\lambda = 55$ and the influence of s on the behavior of positive solutions, in particular near the boundary points $x = 0$ and $x = 1$.

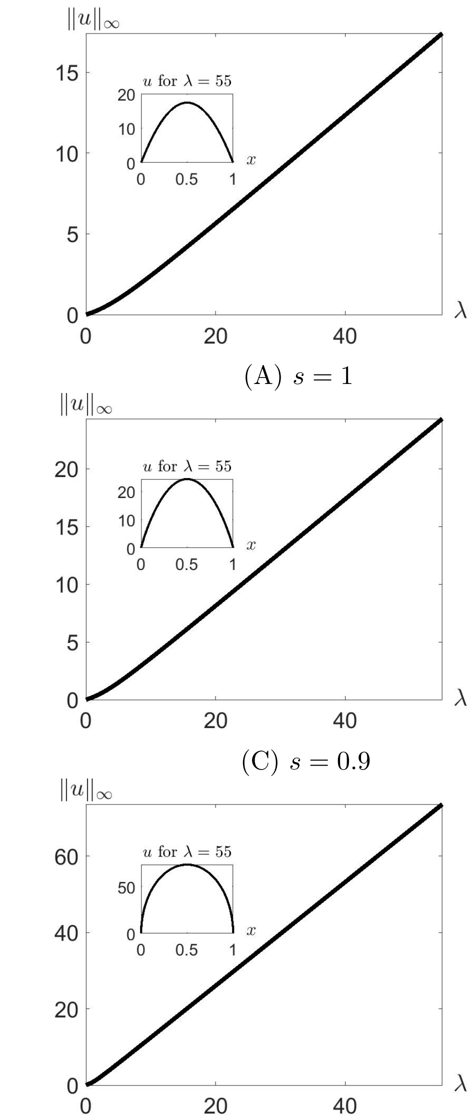
We see from the bifurcation diagrams in Figure 1 that the solution set $\mathcal{S}$ emanates from the origin and increases with respect to λ (hence there is a unique positive solution for each $\lambda > 0$). Moreover, $\|u\|_\infty \to 0$ as $\lambda \to 0^+$ and $\|u\|_\infty \to +\infty$ as $\lambda \to +\infty$. In [20], the authors prove uniqueness of positive solutions if $\frac{f(t)}{t}$ is decreasing in t. Note that this condition is satisfied by $f(t) = e^{\frac{t}{1+t}}$ and our bifurcation diagram confirms the uniqueness result in [20, Thm. 20].

Example 7.3. $f(t) = e^{\frac{5t}{5+t}}$ for $t \ge 0$ (satisfying the hypotheses of Theorem 2.5).

Investigation of the bifurcation diagrams for the perturbed Gelfand problem $f(t) = e^{\frac{\kappa t}{\kappa+t}}$ for $\kappa > 0$ and $t \ge 0$ has been of interest since the paper of Keller and Cohen [18]. It was shown in [11] that a sufficient condition for the bifurcation diagram to be S-shaped is satisfied if $\kappa \ge 4.07$ for the Laplacian case ($s = 1$). Indeed, we see in Figure 2 that the numerical bifurcation diagram is S-shaped for both $s = 1$ (obtained using quadrature method) and $s = 0.99$. As in Example 7.2, $\|u\|_\infty \to 0$ as $\lambda \to 0^+$ and $\|u\|_\infty \to +\infty$ as $\lambda \to +\infty$. However, the solution set $\mathcal{S}$ is not monotone with respect to λ. Additionally, there is a range of λ for which we see three numerical positive solutions.

In Figure 3, the bifurcation diagrams are given in (A) $s = 0.9$, (C) $s = 0.7$, (E) $s = 0.5$, $s = 0.3$ and corresponding profiles of three positive solutions are given in (B) $\lambda = 2.75$, (D) $\lambda = 1.5$, (F) $\lambda = 1.5$ and (H) $\lambda = 0.6$, respectively. We also observe that the interval of λ for which three solutions exist shifts to the left as s decreases. With these observations, we state the following conjecture.

Conjecture 7.4. Let $f(t) > 0$ and $t/f(t)$ be strictly increasing for small $t \ge 0$ as well as for large $t > 0$, and decreasing somewhere, then the bifurcation diagram is S-shaped.



(A) $s = 1$

(B) $s = 0.99$

(C) $s = 0.9$

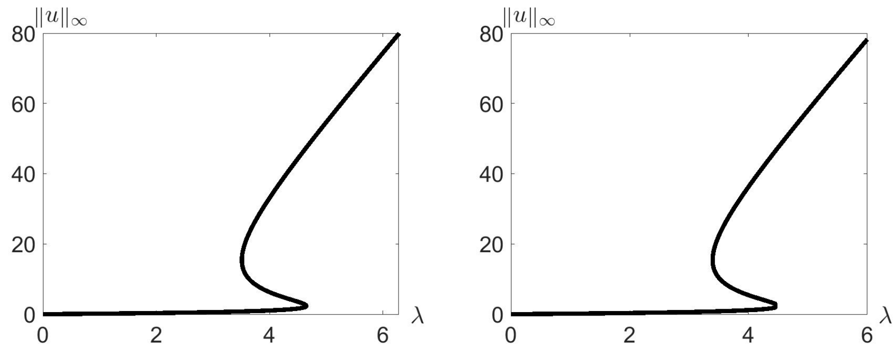
(D) $s = 0.7$

(E) $s = 0.5$

(F) $s = 0.3$

FIGURE 1. Bifurcations diagrams for $f(t) = e^{\frac{t}{1+t}}$ and numerical positive solutions with $\lambda = 55$

For nonlinearities like $f(t) = e^{\frac{5t}{5+t}}/t^{\eta},\ \eta \in (0,1)$, it was shown in [17] that there is a range of λ for which there exist three positive solutions and unique positive solution for λ large. Their result suggests the existence of the S-shaped bifurcation diagram. For existence and multiplicity results using critical point theory, see [5]. However, the connectedness of such solution set remains an important open question.



(A) $s = 1$ (B) $s = 0.99$

FIGURE 2. Bifurcation diagrams for $f(t) = e^{\frac{5t}{5+t}}$ for (A) $s = 1$ and (B) $s = 0.99$

Example 7.5. $f(t) = 3(1+t)^{1/3} - 3$ for $t \geq 0$ (satisfying the hypotheses of Theorem 2.6).

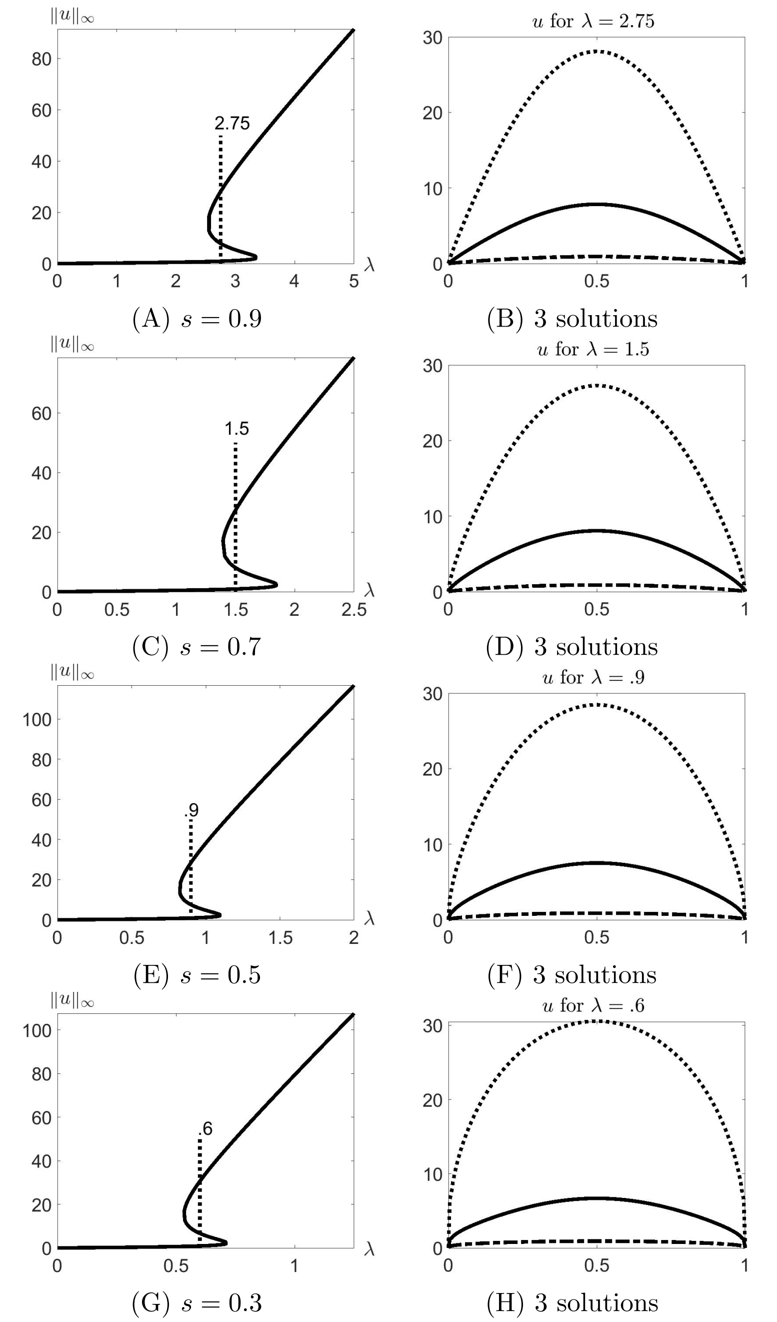
The bifurcation diagram for the Laplacian case ($s = 1$) was discussed in detail in [22, Sec. 1.2]. In particular, if $f(0) = 0$ and $f'(0) > 0$, then the positive solution bifurcates from the line of trivial solutions at $\lambda = \frac{\lambda_1}{f'(0)}$. Here $f'(0) = 1$, so the bifurcation from the trivial branch of solutions occurs at $\lambda = \lambda_1 = \pi^2$ for $s = 1$, see Figure 4 (A). The inset of Figure 4 (A) shows a numerical positive solution for $\lambda = 55$.

In Figure 4, (B)-(F) show the bifurcation diagrams and the insets give the numerical positive solution corresponding to $\lambda = 55$ for (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$ and (E) $s = 0.5$ and (F) $s = 0.3$, respectively. We observe again that the bifurcation diagrams for any $s \in (0,1)$ are qualitatively similar to those for $s = 1$. For $s = 0.99$, the bifurcation of positive solutions from the line of trivial solutions occurs near $\pi^2 \approx 9.8696$, see Figure 4 (B). The influence of $s \in (0,1)$ is noticeable in the location of the point of bifurcation from the line of trivial solutions. This can be justified by the estimate of the principal eigenvalue of $(-\Delta)^s$ on $(0,1)$, see [19]. Also, the profile of the numerical positive solutions corresponding to $\lambda = 55$ for values of $s \in (0,1]$ exhibit the boundary behavior similar to δ^s.

Conjecture 7.6. If f satisfies the hypotheses of Theorem 2.6, then there exists a continuum of positive weak solutions that bifurcates from the branch of trivial solutions at $\lambda = \frac{\lambda_1}{f'(0)}$ and from infinity at infinity in the positive λ direction.

Example 7.7. Logistic reaction term $f(t) = t(1-t)$ for $t \geq 0$ (corresponding to Theorem 2.8 with $q(x) \equiv 1$).

The logistic reaction term considered here is essentially a sublinear nonlinearity at infinity with $f(0) = 0$ and $f'(0) = 1$. Hence, the bifurcation diagrams in Figure 5 resemble those obtained for Example 7.5. The L^∞ norm of the solutions $\|u\|_\infty$ are bounded above by 1 for any $s \in (0,1]$. Therefore, to understand the influence of $s \in (0,1)$ on solutions, we compute the L^1 norm $\|u\|_{L^1(0,1)} = \|u\|_1$ of the numerical positive solution u for $\lambda = 55$. We observe that $\|u\|_1$ increases as s decreases. It appears, numerically, that $\|u\|_1 \nearrow 1$ as $s \to 0^+$.



(A) $s = 0.9$ (B) 3 solutions

(C) $s = 0.7$ (D) 3 solutions

(E) $s = 0.5$ (F) 3 solutions

(G) $s = 0.3$ (H) 3 solutions

FIGURE 3. Bifurcation diagrams for $f(t) = e^{\frac{5t}{5+t}}$ and three numerical positive solutions for the λ specified

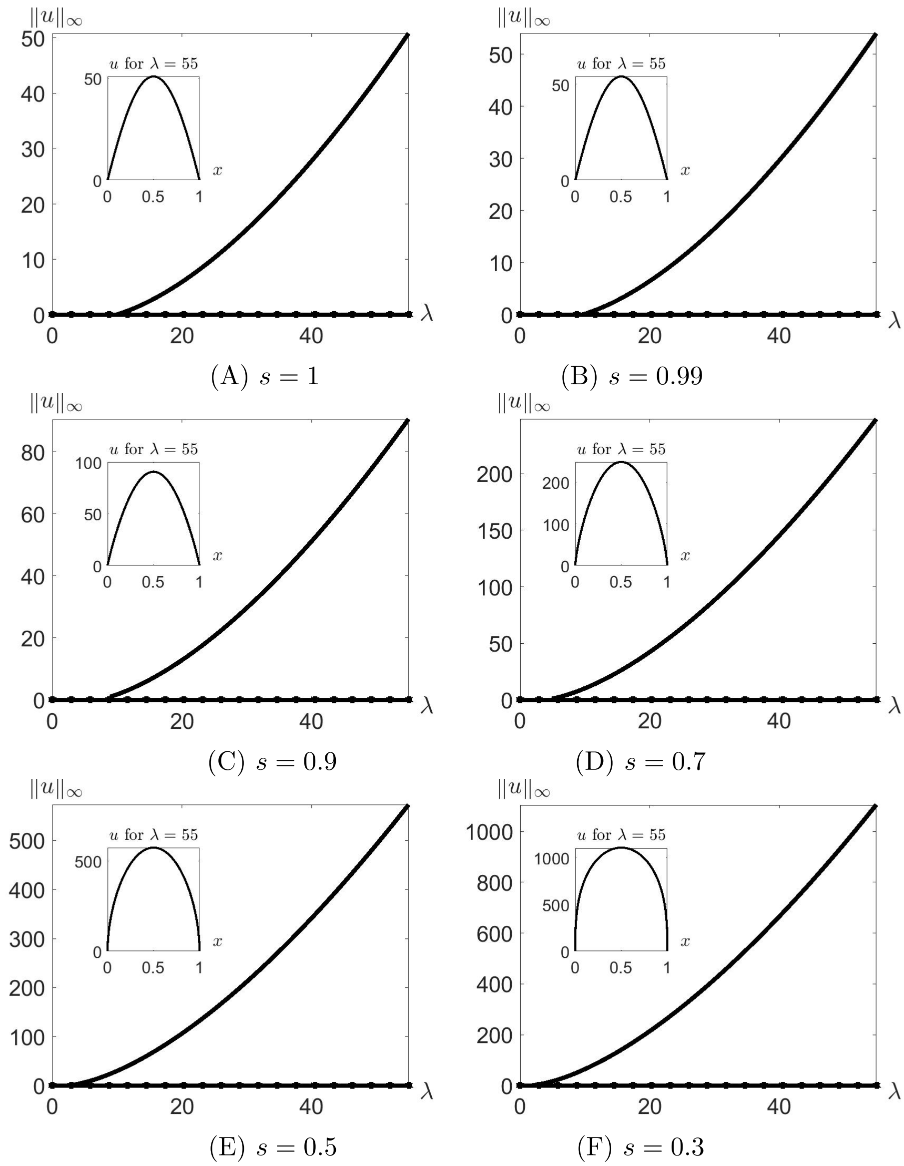
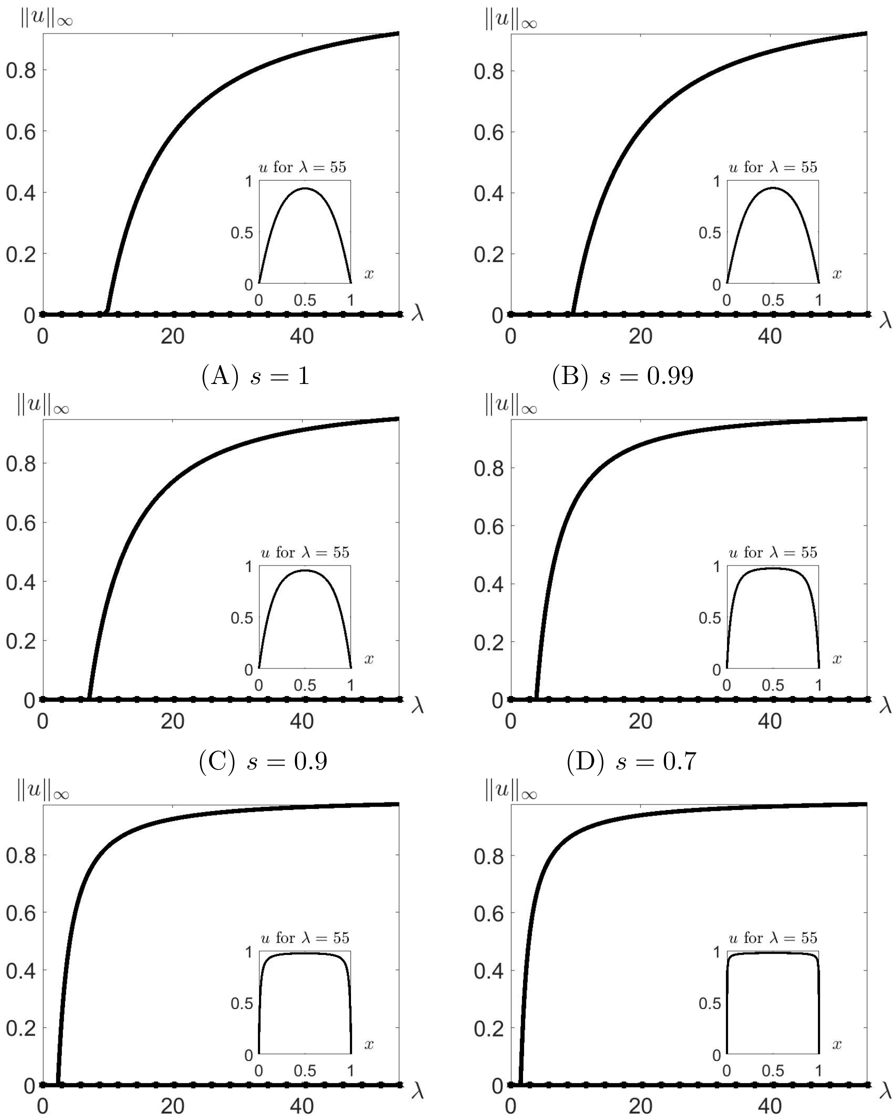


FIGURE 4. Bifurcation diagrams for $f(t) = 3(1+t)^{\frac{1}{3}} - 3$ and numerical positive solutions with $\lambda = 55$



(A) $s = 1$ (B) $s = 0.99$

(C) $s = 0.9$ (D) $s = 0.7$

(E) $s = 0.5$ (F) $s = 0.3$

s	1	0.99	0.9	0.7	0.5	0.3
$\|u\|_1$	0.652287	0.660978	0.725085	0.842018	0.921431	0.960814

FIGURE 5. Bifurcation diagrams for $f(t) = t(1-t)$, numerical positive solutions, and L^1 norms of solutions with $\lambda = 55$

Example 7.8. Logistic reaction with constant yield harvesting $f(t) = t(1-t) - .05$ for $t \geq 0$ (corresponding to Theorem 2.9).

For the Laplacian case ($s = 1$), the bifurcation diagram for the sublinear, semipositone problem was obtained in [13, Thm. 1.1 (B)] using the quadrature method. For the fractional Laplacian, the bifurcation diagrams are given in the first column of Figure 6 for (A) $s = 0.99$, (C) $s = 0.7$, (E) $s = 0.5$, and (G) $s = 0.3$ which retain the qualitative behavior observed for $s = 1$. The solid part of the solution set $\mathcal{S}$ contains positive solutions and the dashed part contains sign changing solutions. On the solution set $\mathcal{S}$, the markers $\Delta, \bigcirc$, and $*$ indicate the locations of a positive solution, the last positive solution in the positive λ direction on the lower branch of $\mathcal{S}$, and a sign changing solution in $(0,1)$, respectively. The Locations of Δ and $*$ are chosen so that the L^∞ norms of the solutions corresponding to these locations are approximately same but always greater than the one for the solution corresponding to $\bigcirc$.

The second column in Figure 6 shows three numerical solutions corresponding to the location of Δ, $\bigcirc$ and $*$ on $\mathcal{S}$ for $s \in (0,1)$ corresponding to the bifurcation diagrams in the first column. The solution corresponding to Δ is given by a solid line, the solution corresponding to $\bigcirc$ is given by a long dashed line, and the solution corresponding to $*$ is given by a short dashed lines.

Based on the numerical experiments, we expect the following multiplicity and uniqueness results for a general sublinear semipositone problem:

Conjecture 7.9. If $f : [0, +\infty) \to \mathbb{R}$ is such that $f(0) < 0$, eventually positive, and satisfies (2.8), then there exist $0 < \lambda_* < \lambda^*$ such that (1.1) has two positive solutions for $\lambda \in (\lambda_*, \lambda^*)$ and a unique positive solution for $\lambda > \lambda^*$.

We remark that it is straightforward to show that there is no nonnegative weak solution for λ sufficiently small. Indeed, it follows from the assumption of Conjecture 7.9 that there exists $a > 0$ such that $f(t) \leq a\,t$ for all $t \geq 0$. Let u be a nonnegative solution of (1.1). Taking $\varphi_1 > 0$ as test function in the definition of weak solution, we obtain

$$\lambda_1 \int_\Omega u\varphi_1 \,\mathrm{d}x = \mathcal{E}(u, \varphi_1) = \lambda \int_\Omega f(u)\varphi_1 \,\mathrm{d}x \leq \lambda\, a \int_\Omega u\varphi_1 \,\mathrm{d}x\,,$$

a contradiction if $\lambda < \frac{\lambda_1}{a}$.

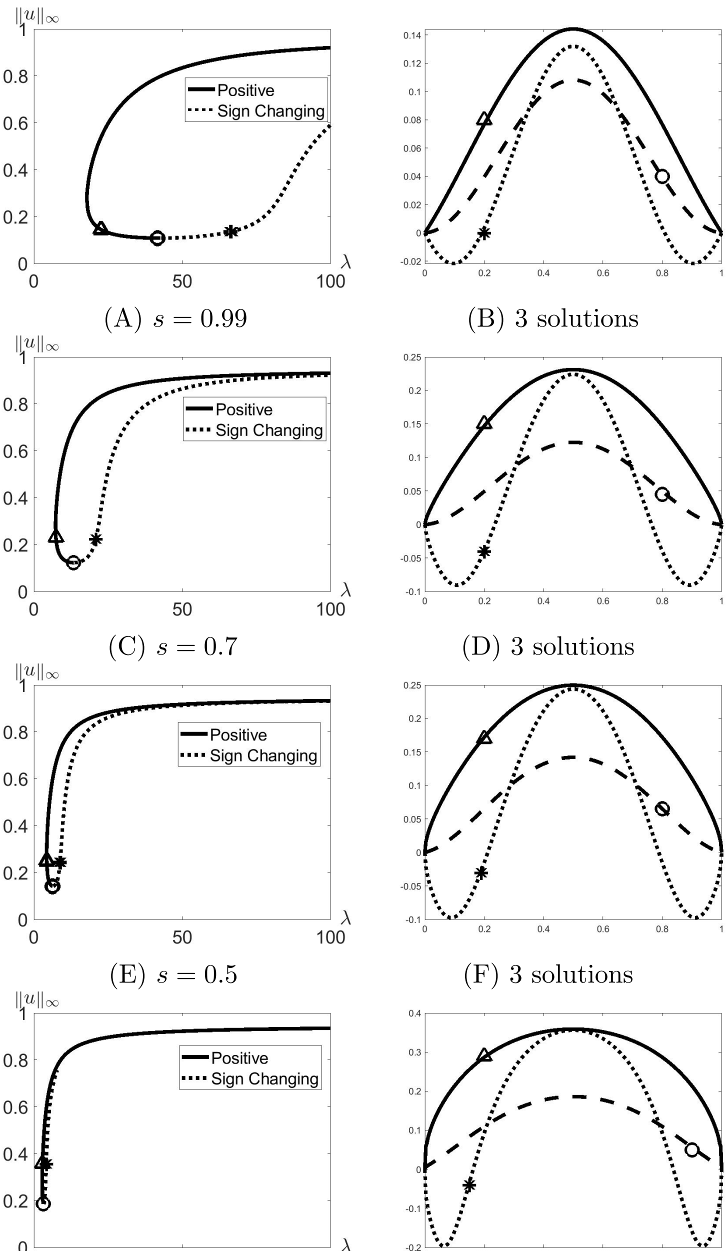
Example 7.10. Weighted logistic problem $f(x,t) = t(q(x) - t)$ for $t \geq 0$ and $x \in (0,1)$ (q satisfying the hypothesis of Theorem 2.8).

For numerical experiments, we consider three specific cases of q. Namely, let $q_1, q_2, q_3 : [0,1] \to \{0,1\}$ be given by the following:

(1) $q_1(x) = 1$ for $x \in [6/10, 7/10)$ and $q_1(x) = 0$ otherwise,
(2) $q_2(x) = 1$ for $x \in [0, 1/5) \cup [2/5, 4/5)$ and $q_2(x) = 0$ otherwise,
(3) $q_3(x) = 1$ for $x \in [1/5, 2/5) \cup [3/5, 4/5)$ and $q_3(x) = 0$ otherwise.

If we consider nonlinearities $f(x,t)$ that have discontinuities at finitely many points on $[0,1]$, with $q_1\,, q_2\,, q_3$ above, we partition the interval $[0,1]$ in such a way that the possible points of discontinuity occur at x_j with $j \in \{1, \dots, n\}$. In order to compute the integral on the right hand side of (7.5), we modify the Hölder type assumption (7.6) for any $t_1, t_2 \geq 0$ to local type as follows. For any $y_1, y_2 \in (x_{j-1}, x_j)$,

$$|f(y_2, t_2) - f(y_1, t_1)| \leq L(|y_2 - y_1|^s + |t_2 - t_1|) \tag{7.8}$$



(A) $s = 0.99$ (B) 3 solutions

(C) $s = 0.7$ (D) 3 solutions

(E) $s = 0.5$ (F) 3 solutions

(G) $s = 0.3$ (H) 3 solutions

FIGURE 6. Bifurcation diagrams for $f(t) = t(1-t) - .05$ and three numerical solutions with λ corresponding to Δ, $\bigcirc$, and $*$

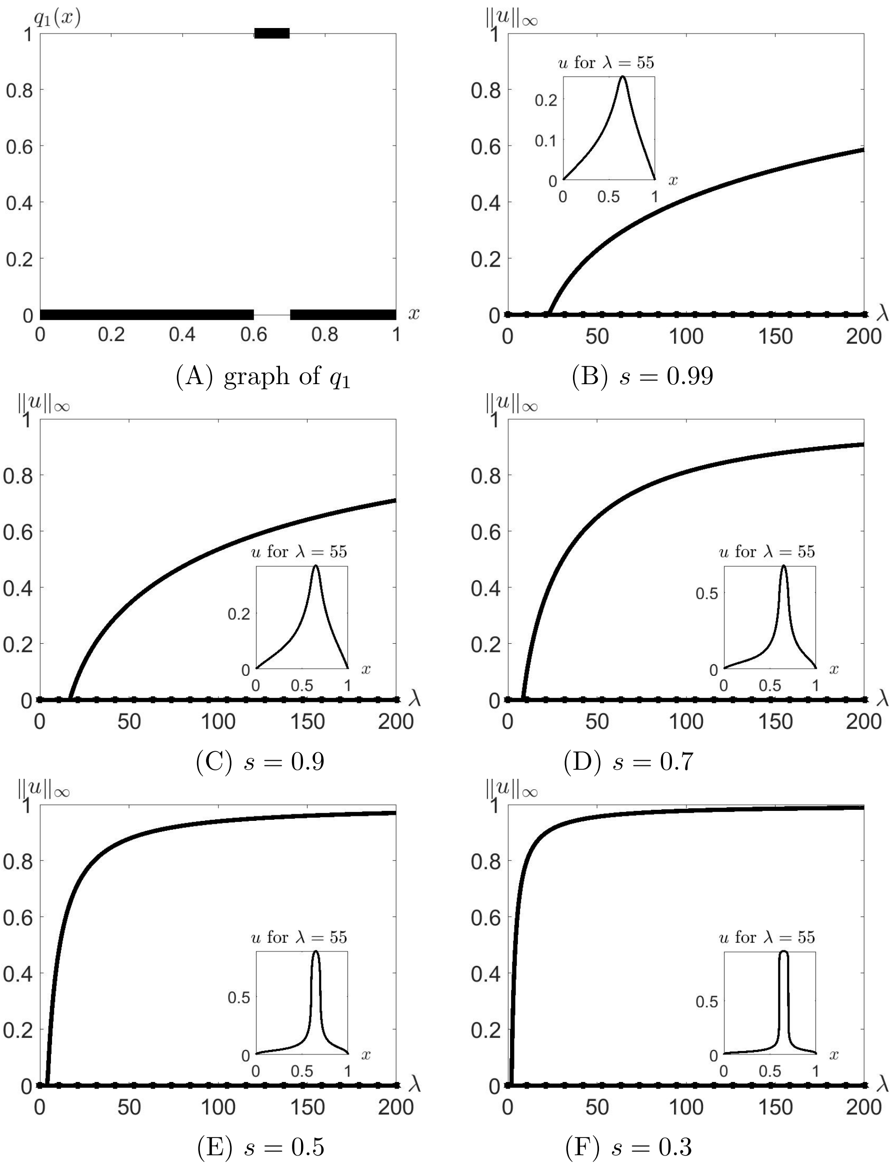
for all $j = 1, \dots, n$. We compute the integral on the right hand side of (7.5) as

$$\int_0^1 f\left(x, u_h(x)\right) \phi_j(x)\, \mathrm{d}x$$

$$
\begin{aligned}
&= \Big(\int_{x_{j-1}}^{x_j} + \int_{x_j}^{x_{j+1}}\Big) f\left(x, u_h(x)\right) \phi_j(x)\,\mathrm{d}x \\
&= \frac{h}{2}[f\big(\frac{x_{j-1}+x_j}{2}, u_j\big) + f\big(\frac{x_j+x_{j+1}}{2}, u_j\big)] \\
&\quad + \int_{x_{j-1}}^{x_j} [f(x, u_h(x)) - f(\frac{x_{j-1}+x_j}{2}, u_h(x_j))]\phi_j(x)\,\mathrm{d}x \\
&\quad + \int_{x_j}^{x_{j+1}} [f\left(x, u_h(x)\right) - f\left(\frac{x_j+x_{j+1}}{2}, u_h(x_j)\right)]\phi_j(x)\,\mathrm{d}x \\
&= \frac{h}{2}[f\big(\frac{x_{j-1}+x_j}{2}, u_j\big) + f\big(\frac{x_j+x_{j+1}}{2}, u_j\big)] + O(h^{1+s})
\end{aligned}
$$

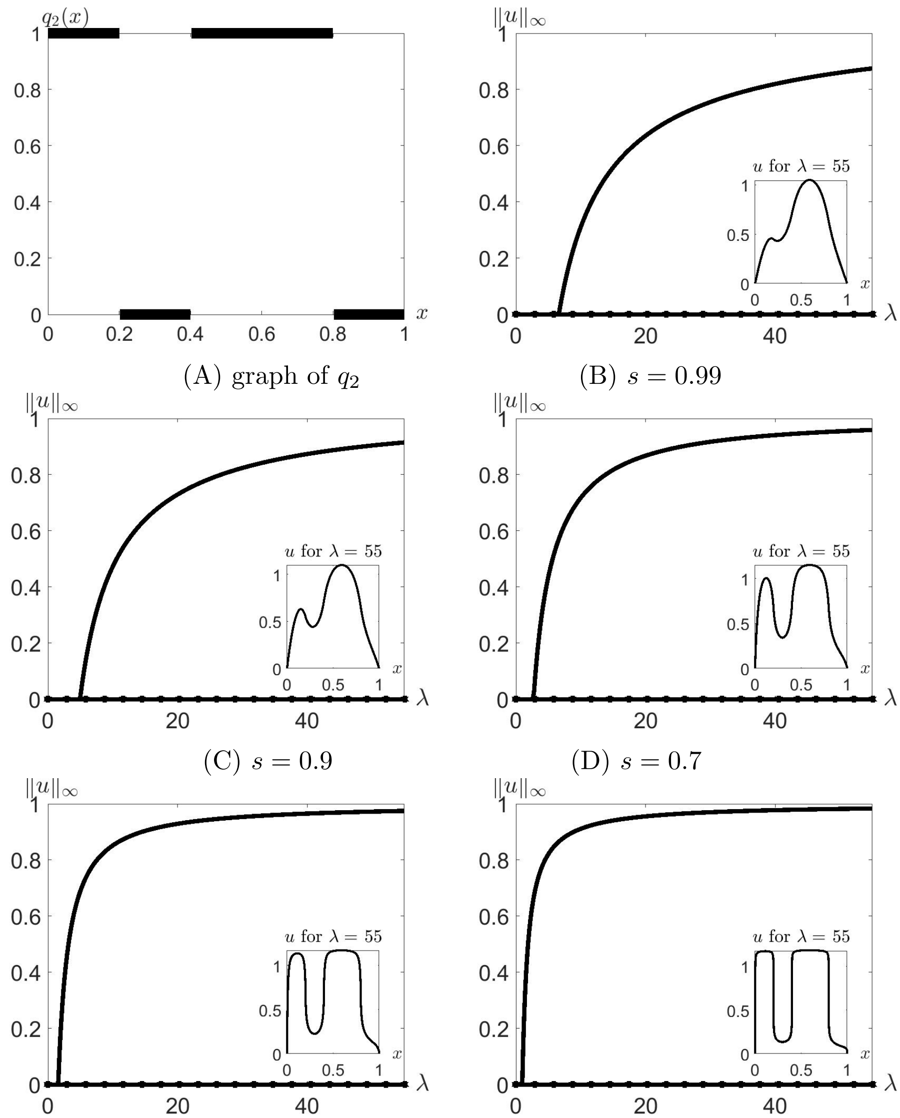
and proceed with the finite element scheme.

In Figures 7-9, (A) gives graph of $q_i (i = 1\,, 2\,, 3)$ and (B)-(F) show the bifurcation diagrams for $s = 0.99$, 0.9, 0.7, $0.5\,, 0.3$ and the insets give numerical positive solutions corresponding to the specified λ. The table at the end of each figure provides comparison of the L^1 norms of the positive solutions for $s = 0.99$, 0.9, 0.7, 0.5 and 0.3. We observe that on one hand, for the choices of the weight function q_1 and $\lambda = 55$, the L^1 norms in Figure 7 attain maximal value for s between 0.7 and 0.3. On the other hand, for the weight functions q_2 and q_3, the L^1 norms in Figures 8-9 appear to be monotone for $\lambda = 55$ and $\lambda = 25$, respectively for s between 0.99 and 0.3.



s	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.106618	0.131814	0.1610100	0.156378	0.137046

FIGURE 7. Graph of q_1, bifurcation diagrams for $f(x,t) = t(q_1(x) - t)$, numerical positive solutions, and the L^1 norms of the solutions with $\lambda = 55$



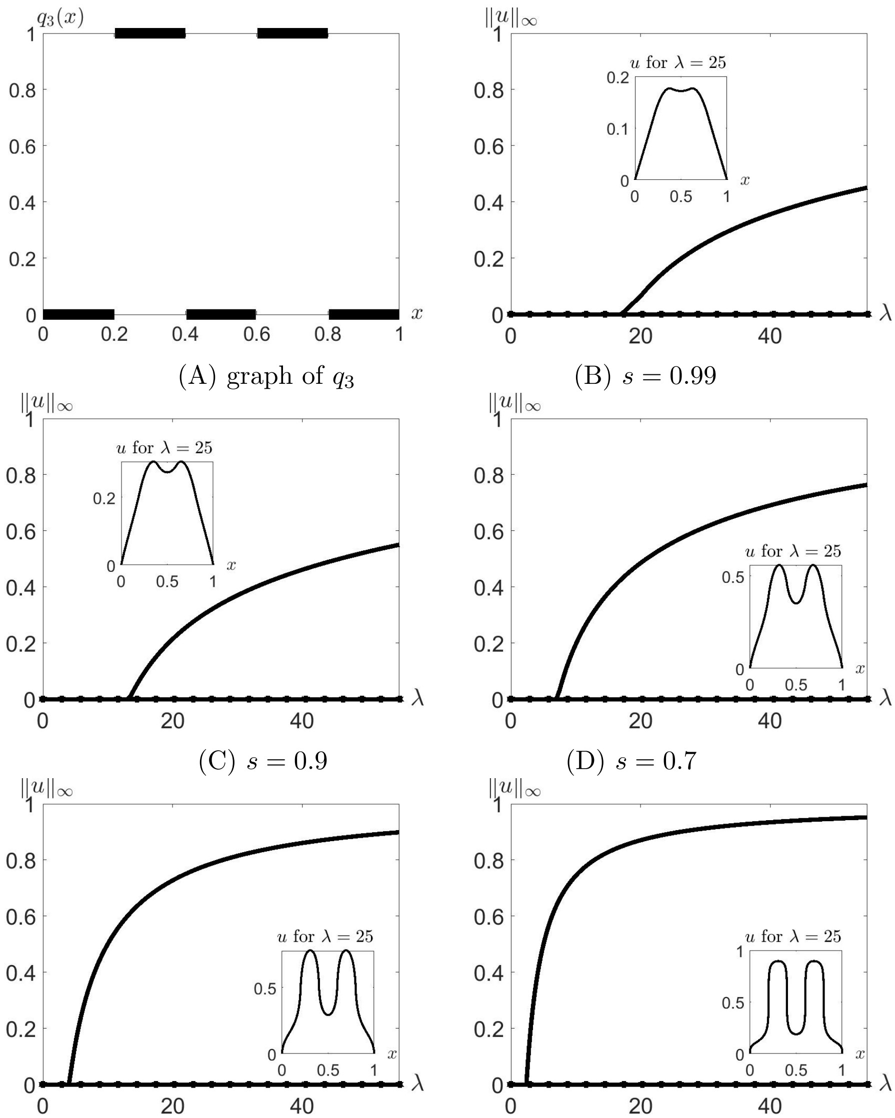
(A) graph of q_2 (B) $s = 0.99$

(C) $s = 0.9$ (D) $s = 0.7$

(E) $s = 0.5$ (F) $s = 0.3$

s	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.416029	0.463028	0.562381	0.617061	0.627710

FIGURE 8. Graph of q_2, bifurcation diagrams for $f(x,t) = t(q_2(x) - t)$, numerical positive solutions, and the L^1 norms of the solutions with $\lambda = 55$



(A) graph of q_3 (B) $s = 0.99$

(C) $s = 0.9$ (D) $s = 0.7$

(E) $s = 0.5$ (F) $s = 0.3$

s	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.122209	0.211938	0.357183	0.433505	0.449811

FIGURE 9. Graph of q_3, bifurcation diagrams for $f(x,t) = t(q_3(x) - t)$, numerical positive solutions, and the L^1 norms of the solutions with $\lambda = 25$

8. Appendix

We show that the norms generated by (2.1) and (2.2) are equivalent in dimension $N = 1$ as well. Let $\Omega = (0,1) \subset \mathbb{R}$.

Lemma 8.1. *Norms generated by $\langle\cdot,\cdot\rangle_{H_0^s(0,1)}$ and $\langle\cdot,\cdot\rangle_{H^s(\mathbb{R})}$ are equivalent in $H_0^s(0,1)$.*

Proof. Let $s \in (0,1)$ and $v \in H_0^s(0,1)$. Then,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y = \|v\|^2_{H_0^s(0,1)} \le \|v\|^2_{H^s(\mathbb{R})}\\ = \int_0^1 |v(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

To establish the reverse inequality, we compute the integral below using $v = 0$ in $\mathbb{R}\setminus(0,1)$,

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}}\int_0^1 \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}}\int_{\mathbb{R}\setminus(0,1)} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \Big[\int_0^1\int_0^1 + \int_{\mathbb{R}\setminus(0,1)}\int_0^1 + \int_{\mathbb{R}\setminus(0,1)}\int_{\mathbb{R}\setminus(0,1)} + \int_0^1\int_{\mathbb{R}\setminus(0,1)}\Big] \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\int_0^1 \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_0^1 |v(y)|^2\,\frac{y^{-2s}+(1-y)^{-2s}}{2s}\,\mathrm{d}y\\ &\ge \int_0^1 |v(y)|^2\omega(y)\,\mathrm{d}y\end{aligned}$$

where $\omega(y) := \frac{y^{-2s}+(1-y)^{-2s}}{s}$. Letting $B := \big(\min_{y\in(0,1)}\omega(y)\big)^{-1} > 0$, we obtain

$$\int_0^1 |v(y)|^2\,\mathrm{d}y \le B\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Then,

$$\|v\|_{H^s(\mathbb{R})} \le (1+B)^{1/2}\|v\|_{H_0^s(0,1)}$$

as desired. Hence, the two norms are equivalent in $H_0^s(0,1)$. □

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