Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 75, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

CYLINDRICAL HARDY INEQUALITIES ON HALF-SPACES

NGUYEN TUAN DUY, HUY BAC NGUYEN

Communicated by Jesus Ildefonso Diaz

Abstract. We study some versions of the cylindrical Hardy identities and inequalities in the style of Badiale-Tarantello [2]. We show that the best constants of the cylindrical Hardy inequalities can be improved when we consider functions on half-spaces.

1. Introduction

The main subject of this note is the celebrated Hardy inequality on $\mathbb{R}^N$, $N \geq 3$: for $u \in C_0^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \big(\frac{N-2}{2}\big)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \tag{1.1}$$

with optimal constant $(\frac{N-2}{2})^2$. Because of their important roles in many areas of mathematics, the Hardy type inequalities have been well-studied and there is a vast literature. See the monographs [3, 25, 28, 29, 40], for instance, that are typical references on the topic.

It is well-known that $(\frac{N-2}{2})^2$ in (1.1) is never achieved by nontrivial functions. Therefore, many efforts have been devoted to enhance the Hardy inequalities. One way to do so is to add extra nonnegative terms to the right-hand side of (1.1). The first result in this direction was established in [8] where Brezis and Vázquez proved that for $u \in W_0^{1,2}(\Omega)$. Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$, it holds

$$\int_\Omega |\nabla u|^2 \, dx \geq \big(\frac{N-2}{2}\big)^2 \int_\Omega \frac{|u|^2}{|x|^2} \, dx + z_0^2 \omega_N^{2/N} |\Omega|^{-2/N} \int_\Omega |u|^2 \, dx. \tag{1.2}$$

Here ω_N is the volume of the unit ball and $z_0 = 2.4048\dots$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2 \omega_N^{2/N} |\Omega|^{-2/N}$ is optimal when Ω is a ball. However, $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-2/N}$ is not attained in $W_0^{1,2}(\Omega)$. Hence, Brezis and Vázquez also conjectured that $z_0^2 \omega_N^{2/N} |\Omega|^{-2/N} \int_\Omega |u|^2 \, dx$ is just a first term of an infinite series of extra terms that can be added to the right-hand side of (1.2). This question was investigated by many authors. We refer the interested reader to [1, 4, 9, 10, 11, 12, 18, 21, 22, 23, 26, 37, 45, 46], to name just a few.

Ghoussoub and Moradifam [24, 25] proved the following result to improve, extend and unify several results about the Hardy type inequalities:

2010 *Mathematics Subject Classification.* 26D10, 35A23, 46E35.
Key words and phrases. Cylindrical Hardy inequality; Bessel pair.

Submitted May 24, 2020. Published July 16, 2020.

Theorem 1.1. *Let $0 < R \le \infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr = \infty$ and $\int_0^R r^{N-1}V(r)dr < \infty$. Then the following two statements are equivalent: (1) $(r^{N-1}V, r^{N-1}c_1W)$ is a Bessel pair on $(0,R)$ for some $c_1 > 0$. (2)*
$$\int_{B_R} V(|x|)|\nabla u|^2\,dx \ge c_2 \int_{B_R} W(|x|)|u|^2\,dx$$
for all $u \in C_0^\infty(B_R)$ and some $c_2 > 0$.

Here we say that a couple of C^1-functions (V,W) is a Bessel pair on $(0,R)$ if the ordinary differential equation
$$y''(r) + \frac{V_r(r)}{V(r)}y'(r) + \frac{W(r)}{V(r)}y(r) = 0$$
has a positive solution on the interval $(0,R)$. See the book [25] for more properties and examples about the Bessel pair.

Another line of research on the improvements of the Hardy type inequalities is to replace the usual ∇ by $\mathcal{R} := \frac{x}{|x|}\cdot\nabla$. It can be noted that $\mathcal{R}u$ is the radial gradient of u. Indeed, in the polar coordinate, $|\mathcal{R}u| = |\partial_r u(r\sigma)|$ while
$$|\nabla u| = \Big(|\partial_r u(r\sigma)|^2 + \frac{|\nabla_{\mathbb{S}^{N-1}} u(r\sigma)|^2}{r^2}\Big)^{1/2}.$$
Actually, the radial derivation plays an important part in the literature. The interested reader is referred to [42] for the roles of the radial derivation $\mathcal{R}$ in the functional and geometric inequalities on homogeneous groups. We also mention here that the Hardy type inequalities with radial gradient have been intensively studied recently. See [13, 14, 15, 16, 27, 30, 31, 39, 41, 42, 43], for example.

In an effort to unify many results about the Hardy type inequalities with radial derivation, and to compute the exact remainders of the Hardy type inequalities, the authors in [15] have proved the following result.

Theorem 1.2. *$0 < R \le \infty$, V and W be a positive C^1-functions on $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr = \infty$ and $\int_0^R r^{N-1}V(r)dr < \infty$. Assume that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0,R)$. Then for all $u \in C_0^\infty(B_R)$:*
$$\begin{aligned}&\int_{B_R} V(|x|)|\mathcal{R}u|^2\,dx - \int_{B_R} W(|x|)|u|^2\,dx\\ &= \int_{B_R} V(|x|)\Big|\mathcal{R}\Big(\frac{u}{\varphi_{r^{N-1}V,r^{N-1}W;R}}\Big)\Big|^2 \varphi^2_{r^{N-1}V,r^{N-1}W;R}\,dx\end{aligned}$$
and
$$\begin{aligned}&\int_{B_R} V(|x|)|\nabla u|^2\,dx - \int_{B_R} W(|x|)|u|^2\,dx\\ &= \int_{B_R} V(|x|)\Big|\nabla\Big(\frac{u}{\varphi_{r^{N-1}V,r^{N-1}W;R}}\Big)\Big|^2 \varphi^2_{r^{N-1}V,r^{N-1}W;R}\,dx\end{aligned}$$
where $\varphi_{r^{N-1}V,r^{N-1}W;R}$ is the positive solution of
$$y''(r) + \Big(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\Big)y'(r) + \frac{W(r)}{V(r)}y(r) = 0$$
on the interval $(0,R)$.

In [2], for investigating the existence and nonexistence of cylindrical solutions for a nonlinear elliptic equation that has been proposed as a model describing the dynamics of elliptic galaxies, Badiale and Tarantello established the following cylindrical Hardy type inequality,

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx \geq C_{N,k,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|y|^p} \, dx \tag{1.3}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. The optimal constant $C_{N,k,p} = (\frac{k-p}{p})^p$ was also conjectured in [2] and then verified in [44].

Recently, in [17, 31], the following result about the cylindrical Hardy type inequalities with Bessel pairs has been set up.

Theorem 1.3. *Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0, R)$. Assume that $(r^{k-1}V, r^{k-1}W)$ is a Bessel pair on $(0, R)$. Then for $u \in C_0^\infty(\{0 < |y| < R\})$:*

$$\begin{aligned}
&\int_{0<|y|<R} V(|y|)|\nabla u(y,z)|^2 dydz - \int_{0<|y|<R} W(|y|)|u(y,z)|^2 dydz \\
&= \int_{0<|y|<R} V(|y|)\varphi^2(|y|)|\nabla(\frac{u(y,z)}{\varphi(|y|)})|^2 dydz.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{0<|y|<R} V(|y|)|\frac{y}{|y|}\cdot\nabla_y u(y,z)|^2 dydz - \int_{0<|y|<R} W(|y|)|u(y,z)|^2 dydz \\
&= \int_{0<|y|<R} V(|y|)\varphi^2(|y|)|\frac{y}{|y|}\cdot\nabla_y(\frac{u(y,z)}{\varphi(|y|)})|^2 dydz.
\end{aligned}$$

Here φ is the positive solution of

$$(r^{k-1}V(r)y'(r))' + r^{k-1}W(r)y(r) = 0$$

on the interval $(0, R)$.

Because of their geometric meaning, Hardy's inequalities have been also studied extensively on the half-spaces $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_1 > 0\}$. For instance, Hardy's inequalities with distance to the boundary have been investigated in [6, 7, 19, 32, 33], to name just a few. Improved Hardy type inequalities on half-spaces have also been set up in, for instance, [5, 34, 35, 36].

It is interesting to note that when one restricts the domain to $\mathbb{R}^N_+$, the best constant of the Hardy inequality can be improved. Indeed, we have the Hardy inequality on half-space (see, e.g., [24, 38])

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 \, dx \geq \big(\frac{N}{2}\big)^2 \int_{\mathbb{R}^N_+} \frac{|u|^2}{|x|^2} \, dx \quad \text{for } u \in C_0^\infty(\mathbb{R}^N_+). \tag{1.4}$$

Here the constant $(N/2)^2$ is optimal. However, if we concern the Hardy inequality with radial derivation $\mathcal{R}$ on $\mathbb{R}^N_+$, then it is interesting to note that the best constant is still $((N-2)/2)^2$. Actually, in [32], the authors showed the following identities to provide a simple interpretation of the aforementioned phenomenon, a direct understanding of the Hardy inequality on half-spaces (1.4) as well as the "virtual" ground state in the sense of Frank and Seiringer [20]: for $u \in C_0^\infty(\mathbb{R}^N_+)$, it holds

$$\int_{\mathbb{R}^N_+} |\nabla u|^2\,dx - \Big(\frac{N}{2}\Big)^2 \int_{\mathbb{R}^N_+} \frac{|u|^2}{|x|^2}\,dx = \int_{\mathbb{R}^N_+} \Big|\nabla\Big(|x|^{N/2}\frac{u}{x_1}\Big)\Big|^2 |x|^{-N} x_1^2\,dx,$$

$$\int_{\mathbb{R}^N_+} |\mathcal{R}u|^2\,dx - \Big(\frac{N-2}{2}\Big)^2 \int_{\mathbb{R}^N_+} \frac{|u|^2}{|x|^2}\,dx = \int_{\mathbb{R}^N_+} \Big|\mathcal{R}\Big(|x|^{N/2}\frac{u}{x_1}\Big)\Big|^2 |x|^{-N} x_1^2\,dx.$$

More generally, the authors in [32] used the factorizations of suitable differential operators to study a version of Theorem 1.2 on $\mathbb{R}^N_+$. Let us denote $B_R^{(k)}$ the ball centered at 0 with radius R on $\mathbb{R}^k$. Then we have the following result in [32].

Theorem 1.4. *Let $0 < R \le \infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int_0^R \frac{1}{r^{N+1}V(r)}dr = \infty$ and $\int_0^R r^{N+1}V(r)dr < \infty$. If $(r^{N+1}V, r^{N+1}W)$ is a 1-dimensional Bessel pair on $(0,R)$, then for $u \in C_0^\infty(\mathbb{R}^N_+)$,*

$$\begin{aligned}&\int_{B_R^{(N)}\cap\mathbb{R}^N_+} V(|x|)|\nabla u|^2\,dx - \int_{B_R^{(N)}\cap\mathbb{R}^N_+} \big[W(|x|) - \frac{V'(|x|)}{|x|}\big]|u|^2\,dx\\ &= \int_{B_R^{(N)}\cap\mathbb{R}^N_+} V(|x|)\Big|\nabla\Big(\frac{u}{\varphi_{r^{N+1}V,r^{N+1}W;R}}\frac{1}{x_N}\Big)\Big|^2 \varphi^2_{r^{N+1}V,r^{N+1}W;R}x_N^2\,dx\end{aligned}$$

and

$$\begin{aligned}&\int_{B_R^{(N)}\cap\mathbb{R}^N_+} V(|x|)|\mathcal{R}u|^2\,dx - \int_{B_R^{(N)}\cap\mathbb{R}^N_+} \big[W(|x|) - \frac{V'(|x|)}{|x|} - (N-1)\frac{V(|x|)}{|x|^2}\big]|u|^2\,dx\\ &= \int_{B_R^{(N)}\cap\mathbb{R}^N_+} V(|x|)\Big|\mathcal{R}\big(\frac{1}{\varphi_{r^{N+1}V,r^{N+1}W;R}}\frac{u}{x_N}\big)\Big|^2 \varphi^2_{r^{N+1}V,r^{N+1}W;R}x_N^2\,dx.\end{aligned}$$

Here $\varphi_{r^{N+1}V,r^{N+1}W;R}$ is the positive solution of

$$y''(r) + \big(\frac{N+1}{r} + \frac{V_r(r)}{V(r)}\big)y'(r) + \frac{W(r)}{V(r)}y(r) = 0$$

on the interval $(0,R)$.

Motivated by the cylindrical Hardy type inequalities studied in [2, 17, 31], and the Hardy type inequalities on half-spaces in [32], our principal goal of this paper is to investigate the cylindrical Hardy type inequalities with Bessel pairs and with exact remainder terms on $\mathbb{R}^N_+$. More precisely, let $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $1 \le k \le N$ and $y = (x_1, w) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Our main result reads as follows.

Theorem 1.5. *Let $0 < R \le \infty$, V and W be positive C^1-functions on $(0,R)$. Assume that $(r^{k+1}V, r^{k+1}W)$ is a Bessel pair on $(0,R)$. Then for $u \in C_0^\infty(\{0 < |y| < R\} \cap \mathbb{R}^N_+)$,*

$$\begin{aligned}&\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)|\nabla u|^2\,dx\\ &- \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} \big[W(|y|) - \frac{V'(|y|)}{|y|}\big]|u|^2\,dx\\ &= \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\nabla\Big(\frac{u}{\varphi(|y|)}\frac{1}{x_1}\Big)\Big|^2 \varphi^2(|y|)x_1^2\,dx\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\frac{y}{|y|}\cdot\nabla_y u\Big|^2\,dx\\
&-\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\big[W(|y|)-\frac{V'(|y|)}{|y|}-(k-1)\frac{V(|y|)}{|y|^2}\big]|u|^2\,dx\\
&=\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\frac{y}{|y|}\cdot\nabla_y\Big(\frac{u}{\varphi(|y|)}\frac{1}{x_1}\Big)\Big|^2\varphi^2(|y|)x_1^2\,dx.
\end{aligned}$$

Here φ is the positive solution of

$$(r^{k+1}V(r)y'(r))'+r^{k+1}W(r)y(r)=0$$

on the interval $(0,R)$.

In Section 4, we will present some cylindrical Hardy type inequalities on half-spaces as consequences of our main result. Actually, we can obtain as many cylindrical Hardy type inequalities as we can construct Bessel pairs. For several examples of the Bessel pairs, the interested reader is referred to [25].

2. Some useful calculations

For $\widehat{x}=(\widehat{y},z)\in\mathbb{R}^{k+2}\times\mathbb{R}^{N-k}$, $1\le k\le N$ and $\widehat{y}=(t_1,t_2,t_3,w)\in\mathbb{R}^3\times\mathbb{R}^{k-1}$, we denote $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $1\le k\le N$, and $y=(x_1,w)\in\mathbb{R}\times\mathbb{R}^{k-1}$ where $x_1=\sqrt{t_1^2+t_2^2+t_3^2}$.

Let $u\in C_0^\infty(\mathbb{R}^N_+)$. Define a function $v:\mathbb{R}^{N+2}\to\mathbb{R}$ by $v(\widehat{x})=\frac{1}{x_1}u(x)$ where $x_1=\sqrt{t_1^2+t_2^2+t_3^2}$. Then $v\in C_0^\infty(\mathbb{R}^{N+2})$. Moreover,

$$\begin{aligned}
\nabla_{\mathbb{R}^{N+2}}v=\frac{1}{x_1}\Big(&\frac{\partial u}{\partial x_1}\frac{t_1}{x_1}-\frac{1}{x_1}u\frac{t_1}{x_1},\frac{\partial u}{\partial x_1}\frac{t_2}{x_1}-\frac{1}{x_1}u\frac{t_2}{x_1},\frac{\partial u}{\partial x_1}\frac{t_3}{x_1}-\frac{1}{x_1}u\frac{t_3}{x_1},\\
&\frac{\partial u}{\partial x_2},\dots,\frac{\partial u}{\partial x_N}\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
|\nabla_{\mathbb{R}^{N+2}}v|^2&=\sum_{i=2}^N\Big(\frac{1}{x_1}\Big)^2\Big(\frac{\partial u}{\partial x_i}\Big)^2+\sum_{j=1}^3\Big(\frac{1}{x_1}\frac{\partial u}{\partial x_1}\frac{t_j}{x_1}-\frac{1}{x_1^2}u\frac{t_j}{x_1}\Big)^2\\
&=\sum_{i=2}^N(\frac{1}{x_1})^2\Big(\frac{\partial u}{\partial x_i}\Big)^2+\Big(\frac{1}{x_1}\Big)^2\Big(\frac{\partial u}{\partial x_1}\Big)^2+\frac{|u|^2}{x_1^4}-2\frac{1}{x_1^3}u\frac{\partial u}{\partial x_1}\\
&=\Big(\frac{1}{x_1}\Big)^2\big[|\nabla_{\mathbb{R}^N}u|^2+\frac{|u|^2}{x_1^2}-2\frac{1}{x_1}u\frac{\partial u}{\partial x_1}\big].
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
\Delta_{\mathbb{R}^{N+2}}v&=\sum_{i=2}^N\frac{1}{x_1}\frac{\partial^2 u}{\partial x_i^2}+\frac{2}{x_1}\big[\frac{1}{x_1}\frac{\partial u}{\partial x_1}-\frac{1}{x_1^2}u\big]+\frac{1}{x_1}\frac{\partial^2 u}{\partial x_1^2}-\frac{2}{x_1^2}\frac{\partial u}{\partial x_1}+\frac{2}{x_1^3}u\\
&=\frac{1}{x_1}\Delta_{\mathbb{R}^N}u.
\end{aligned}\tag{2.2}$$

Also

$$\nabla_{\mathbb{R}^{k+2}}|\widehat{y}|=\Big(\frac{x_1}{|y|}\frac{t_1}{x_1},\frac{x_1}{|y|}\frac{t_2}{x_1},\frac{x_1}{|y|}\frac{t_3}{x_1},\frac{x_2}{|y|},\dots,\frac{x_k}{|y|}\Big).$$

Hence
$$
\begin{gathered}
|\nabla_{\mathbb{R}^{k+2}}|\widehat{y}||=|\nabla_{\mathbb{R}^k}|y||=1,\\
\Delta_{\mathbb{R}^{k+2}}|\widehat{y}|=\frac{k+1}{|y|}=\frac{k+1}{|\widehat{y}|}.
\end{gathered}
$$
We also have
$$
\begin{aligned}
\frac{\widehat{y}}{|\widehat{y}|}\cdot\nabla_{\widehat{y}}v&=\sum_{i=2}^k\frac{x_i}{|y|}\frac{1}{x_1}\frac{\partial u}{\partial x_i}+\sum_{j=1}^3\frac{t_j}{|y|}\Big(\frac{1}{x_1}\frac{\partial u}{\partial x_1}\frac{t_j}{x_1}-\frac{1}{x_1^2}u\frac{t_j}{x_1}\Big)\\
&=\sum_{i=2}^k\frac{x_i}{|y|}\frac{1}{x_1}\frac{\partial u}{\partial x_i}+\frac{x_1}{|y|}\Big(\frac{1}{x_1}\frac{\partial u}{\partial x_1}-\frac{1}{x_1^2}u\Big)\\
&=\frac{1}{x_1}\Big(\frac{y}{|y|}\cdot\nabla_y u-\frac{u}{|y|}\Big).
\end{aligned}\tag{2.3}
$$

3. Proof of main results

Proof of Theorem 1.5. For $\widehat{x}=(\widehat{y},z)\in\mathbb{R}^{k+2}\times\mathbb{R}^{N-k}$, $1\le k\le N$ and $\widehat{y}=(t_1,t_2,t_3,w)\in\mathbb{R}^3\times\mathbb{R}^{k-1}$, we denote $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $1\le k\le N$, and $y=(x_1,w)\in\mathbb{R}\times\mathbb{R}^{k-1}$ where $x_1=\sqrt{t_1^2+t_2^2+t_3^2}$. Let $u\in C_0^\infty(\mathbb{R}^N_+)$. Define a function $v:\mathbb{R}^{N+2}\to\mathbb{R}$ by $v(\widehat{x})=\frac{1}{x_1}u(x)$ where $x_1=\sqrt{t_1^2+t_2^2+t_3^2}$.

Using (2.1) and polar coordinates we have
$$
\begin{aligned}
&\int_{0<|\widehat{y}|<R}V(|\widehat{y}|)|\nabla v|^2d\widehat{x}\\
&=|\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}V(|y|)\big(\frac{1}{x_1}\big)^2\big[|\nabla_{\mathbb{R}^N}u|^2+\frac{|u|^2}{x_1^2}-2\frac{1}{x_1}u\frac{\partial u}{\partial x_1}\big]x_1^2\,dx_1\,dw\,dz\\
&=|\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}V(|y|)\big[|\nabla_{\mathbb{R}^N}u|^2+\frac{|u|^2}{x_1^2}-\frac{1}{x_1}\frac{\partial|u|^2}{\partial x_1}\big]\,dx_1\,dw\,dz.
\end{aligned}
$$
By integrations by parts, we obtain
$$
\begin{aligned}
&-\int_0^{\sqrt{R^2-w|^2}}V(|y|)\frac{1}{x_1}\frac{\partial|u|^2}{\partial x_1}\,dx_1\\
&=\int_0^{\sqrt{R^2-w|^2}}V(|y|)|u|^2\frac{\partial\frac{1}{x_1}}{\partial x_1}\,dx_1+\int_0^{\sqrt{R^2-w|^2}}\frac{|u|^2}{x_1}\frac{\partial V(|y|)}{\partial x_1}\,dx_1\\
&=-\int_0^{\sqrt{R^2-w|^2}}V(|y|)\frac{|u|^2}{x_1^2}\,dx_1+\int_0^{\sqrt{R^2-w|^2}}\frac{|u|^2}{x_1}V'(|y|)\frac{x_1}{|y|}\,dx_1.
\end{aligned}
$$
Hence
$$
\int_{0<|\widehat{y}|<R}V(|\widehat{y}|)|\nabla v|^2d\widehat{x}=|\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}V(|y|)\big[|\nabla u|^2+\frac{V'(|y|)}{|y|}|u|^2\big]\,dx.\tag{3.1}
$$
On the other hand, using polar coordinate agains, we have
$$
\begin{aligned}
\int_{0<|\widehat{y}|<R}W(|\widehat{y}|)|v|^2d\widehat{x}&=|\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}W(|y|)\big(\frac{1}{x_1}\big)^2|u|^2x_1^2\,dx_1\,dw\,dz\\
&=|\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}W(|y|)|u|^2\,dx
\end{aligned}\tag{3.2}
$$

and

$$\begin{aligned} &\int_{0<|\widehat{y}|<R} V(|\widehat{y}|)\Big|\nabla\Big(\frac{v}{\varphi(|\widehat{y}|)}\Big)\Big|^2\varphi^2(|\widehat{y}|)d\widehat{x} \\ &= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\nabla\Big(\frac{u}{\varphi(|y|)}\frac{1}{x_1}\Big)\Big|^2\varphi^2(|y|)x_1^2\,dx. \end{aligned} \tag{3.3}$$

Note that by applying Theorem 1.3 to the Bessel pair $(r^{k+1}V, r^{k+1}W)$, we obtain

$$\begin{aligned} \int_{0<|\widehat{y}|<R} V(|\widehat{y}|)|\nabla v|^2 d\widehat{x} &= \int_{0<|\widehat{y}|<R} W(|\widehat{y}|)|v|^2 d\widehat{x} \\ &\quad + \int_{0<|\widehat{y}|<R} V(|\widehat{y}|)\Big|\nabla\Big(\frac{v}{\varphi(|\widehat{y}|)}\Big)\Big|^2\varphi^2(|\widehat{y}|)d\widehat{x}. \end{aligned}$$

Hence, from (3.1), (3.2) and (3.3), we deduce that

$$\begin{aligned} &\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)|\nabla u|^2\,dx \\ &- \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} \big[W(|y|) - \frac{V'(|y|)}{|y|}\big]|u|^2\,dx \\ &= \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\nabla\Big(\frac{u}{\varphi(|y|)}\frac{1}{x_1}\Big)\Big|^2\varphi^2(|y|)x_1^2\,dx. \end{aligned}$$

Next, from (2.2), we obtain

$$\begin{aligned} &\int_{0<|\widehat{y}|<R} V(|\widehat{y}|)\Big|\frac{\widehat{y}}{|\widehat{y}|}\cdot\nabla_{\widehat{y}}v\Big|^2 d\widehat{x} \\ &= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\frac{1}{x_1}\Big(\frac{y}{|y|}\cdot\nabla_y u - \frac{u}{|y|}\Big)|^2 x_1^2\,dx \\ &= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\Big(\frac{y}{|y|}\cdot\nabla_y u - \frac{u}{|y|}\Big)\Big|^2\,dx \\ &= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)|\frac{y}{|y|}\cdot\nabla_y u|^2\,dx + |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\frac{|u|^2}{|y|^2}\,dx \\ &- 2|\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big(\frac{y}{|y|}\cdot\nabla_y u\Big)\frac{u}{|y|}\,dx. \end{aligned}$$

Noting that by polar coordinates and integration by parts, we obtain

$$\begin{aligned} &- 2\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)(\frac{y}{|y|}\cdot\nabla_y u)\frac{u}{|y|}dy \\ &= -2\int_0^R\int_{\mathbb{S}^{k-1}_+} V(r)u_r u r^{k-2}\,d\sigma\,dr \\ &= -\int_0^R\int_{\mathbb{S}^{k-1}_+} V(r)\frac{d|u|^2}{dr}r^{k-2}\,d\sigma\,dr \\ &= \int_0^R\int_{\mathbb{S}^{k-1}_+} |u|^2\frac{d}{dr}(V(r)r^{k-2})\,d\sigma\,dr \\ &= \int_0^R\int_{\mathbb{S}^{k-1}_+} |u|^2\big[\frac{V'(r)}{r} + (k-2)\frac{V(r)}{r^2}\big]r^{k-1}\,d\sigma\,dr \end{aligned}$$

$$= \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} \big[\frac{V'(|y|)}{|y|} + (k-2)\frac{V(|y|)}{|y|^2}\big]|u|^2\,dx.$$

Hence

$$\begin{aligned}
&\int_{0<|\widehat{y}|<R} V(|\widehat{y}|)\Big|\frac{\widehat{y}}{|\widehat{y}|}\cdot\nabla_{\widehat{y}}v\Big|^2 d\widehat{x}\\
&= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)|\frac{y}{|y|}\cdot\nabla_y u|^2\,dx\\
&+ |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} \big[\frac{V'(|y|)}{|y|} + (k-1)\frac{V(|y|)}{|y|^2}\big]|u|^2\,dx.
\end{aligned}\tag{3.4}$$

Similarly,

$$\begin{aligned}
\int_{0<|\widehat{y}|<R} W(|\widehat{y}|)|v|^2 d\widehat{x} &= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} W(|y|)(\frac{1}{x_1})^2|u|^2x_1^2\,dx_1\,dw\,dz\\
&= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} W(|y|)|u|^2\,dx
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&\int_{0<|\widehat{y}|<R} V(|\widehat{y}|)|\frac{\widehat{y}}{|\widehat{y}|}\cdot\nabla_{\widehat{y}}(\frac{v}{\varphi(|\widehat{y}|)})|^2\varphi^2(|\widehat{y}|)d\widehat{x}\\
&= |\mathbb{S}^2|\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)|\frac{y}{|y|}\cdot\nabla_y(\frac{u}{\varphi(|y|)}\frac{1}{x_1})|^2\varphi^2(|y|)x_1^2\,dx.
\end{aligned}\tag{3.6}$$

Applying Theorem 1.3 to the Bessel pair $(r^{k+1}V, r^{k+1}W)$ we obtain

$$\begin{aligned}
&\int_{0<|\widehat{y}|<R} V(|\widehat{y}|)|\frac{\widehat{y}}{|\widehat{y}|}\cdot\nabla_{\widehat{y}}v|^2 d\widehat{x} - \int_{0<|\widehat{y}|<R} W(|\widehat{y}|)|v|^2 d\widehat{x}\\
&= \int_{0<|\widehat{y}|<R} V(|\widehat{y}|)\Big|\frac{\widehat{y}}{|\widehat{y}|}\cdot\nabla_{\widehat{y}}(\frac{v}{\varphi(|\widehat{y}|)})\Big|^2\varphi^2(|\widehat{y}|)d\widehat{x}.
\end{aligned}$$

Then from (3.4), (3.5) and (3.6) we obtain

$$\begin{aligned}
&\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)|\frac{y}{|y|}\cdot\nabla_y u|^2\,dx\\
&- \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} \big[W(|y|) - \frac{V'(|y|)}{|y|} - (k-1)\frac{V(|y|)}{|y|^2}\big]|u|^2\,dx\\
&= \int_{\{0<|y|<R\}\cap\mathbb{R}^N_+} V(|y|)\Big|\frac{y}{|y|}\cdot\nabla_y\Big(\frac{u}{\varphi(|y|)}\frac{1}{x_1}\Big)\Big|^2\varphi^2(|y|)x_1^2\,dx.
\end{aligned}$$

□

4. Some consequences of our main result

Now we list a few applications of our results. First, since $(r^{k+1}, \frac{k^2}{4}r^{k-1})$ is a Bessel pair on $(0,\infty)$ with $\varphi = r^{-\frac{k}{2}}$, from Theorem 1.5 we deduce the following result.

Corollary 4.1. *For $u \in C_0^\infty(\mathbb{R}^N_+)$ we have*

$$\int_{\mathbb{R}^N_+} |\nabla u|^2\,dx - (\frac{k}{2})^2\int_{\mathbb{R}^N_+}\frac{|u|^2}{|y|^2}\,dx = \int_{\mathbb{R}^N_+}\frac{1}{|y|^k}\Big|\nabla\Big(|y|^{k/2}u\frac{1}{x_1}\Big)\Big|^2 x_1^2\,dx$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N_+}\Big|\frac{y}{|y|}\cdot\nabla_y u\Big|^2\,dx-\big(\frac{k-2}{2}\big)^2\int_{\mathbb{R}^N_+}\frac{|u|^2}{|y|^2}\,dx\\&=\int_{\mathbb{R}^N_+}\frac{1}{|y|^k}\Big|\frac{y}{|y|}\cdot\nabla_y(|y|^{k/2}u\frac{1}{x_1})\Big|^2x_1^2\,dx.\end{aligned}$$

We note that by Theorem 1.3,

$$\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\big(\frac{k-2}{2}\big)^2\int_{\mathbb{R}^N}\frac{|u|^2}{|y|^2}\,dx=\int_{\mathbb{R}^N}\frac{1}{|y|^{k-2}}|\nabla(|y|^{\frac{k-2}{2}}u)|^2\,dx.$$

Hence, when we restrict the domain to half-spaces, the optimal constant of the cylindrical Hardy inequality has been improved from $(\frac{k-2}{2})^2$ to $(\frac{k}{2})^2$.

More generally, since $(r^{k+1-\alpha},\frac{(k-\alpha)^2}{4}r^{k-1-\alpha})$ is a Bessel pair on $(0,\infty)$ with $\varphi=r^{-\frac{k-\alpha}{2}}$, we obtain the following result.

Corollary 4.2. *For $u\in C_0^\infty(\mathbb{R}^N_+)$,*

$$\int_{\mathbb{R}^N_+}\frac{|\nabla u|^2}{|y|^\alpha}\,dx-\big[\big(\frac{k-\alpha}{2}\big)^2+\alpha\big]\int_{\mathbb{R}^N_+}\frac{|u|^2}{|y|^{2+\alpha}}\,dx=\int_{\mathbb{R}^N}\frac{1}{|y|^k}|\nabla(|y|^{\frac{k-\alpha}{2}}u\frac{1}{x_1})|^2x_1^2\,dx$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N_+}\frac{|\frac{y}{|y|}\cdot\nabla_y u|^2}{|y|^\alpha}\,dx-\big[\big(\frac{k-\alpha}{2}\big)^2+\alpha-(k-1)\big]\int_{\mathbb{R}^N_+}\frac{|u|^2}{|y|^{2+\alpha}}\,dx\\&=\int_{\mathbb{R}^N}\frac{1}{|y|^k}\Big|\frac{y}{|y|}\cdot\nabla_y\Big(|y|^{\frac{k-\alpha}{2}}u\frac{1}{x_1}\Big)\Big|^2x_1^2\,dx.\end{aligned}$$

We note again that by Theorem 1.3,

$$\int_{\mathbb{R}^N}\frac{|\nabla_y u|^2}{|y|^\alpha}\,dx-\big(\frac{k-2-\alpha}{2}\big)^2\int_{\mathbb{R}^N}\frac{|u|^2}{|y|^{2+\alpha}}\,dx=\int_{\mathbb{R}^N}\frac{1}{|y|^{k-2}}|\nabla(|y|^{\frac{k-2-\alpha}{2}}u)|^2\,dx.$$

Hence, in this case, the sharp constant of the cylindrical Hardy type inequality has been improved from $(\frac{k-2-\alpha}{2})^2$ to $(\frac{k-\alpha}{2})^2+\alpha$ when we consider the functions on half-spaces. Now, since $(r^{k+1}\frac{1}{r^k},r^{k+1}\frac{1}{4r^{k+2}|\log\frac{r}{R}|^2})$ is a Bessel pair on $(0,R)$ with $\varphi=\sqrt{|\log(r/R)|}$. By Theorem 1.5, we obtain the cylindrical critical Hardy inequalities on half-space.

Corollary 4.3. *For $u\in C_0^\infty(\{0<|y|<R\}\cap\mathbb{R}^N_+)$:*

$$\begin{aligned}&\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\frac{|\nabla u(x)|^2}{|y|^k}\,dx-\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\Big[\frac{1}{4}\frac{1}{|\log\frac{R}{|y|}|^2}+k\Big]\frac{|u(x)|^2}{|y|^{k+2}}\,dx\\&=\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\frac{1}{|y|^k}\log\frac{R}{|y|}\Big|\nabla\Big(\frac{u(x)}{\sqrt{\log\frac{R}{|y|}}}\frac{1}{x_1}\Big)\Big|^2x_1^2\,dx\end{aligned}$$

and

$$\begin{aligned}&\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\frac{|\frac{y}{|y|}\cdot\nabla_y u(x)|^2}{|y|^k}\,dx-\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\Big[\frac{1}{4}\frac{1}{|\log\frac{R}{|y|}|^2}+1\Big]\frac{|u(x)|^2}{|y|^{k+2}}\,dx\\&=\int_{\{0<|y|<R\}\cap\mathbb{R}^N_+}\frac{1}{|y|^k}\log\frac{R}{|y|}\Big|\frac{y}{|y|}\cdot\nabla_y\big(\frac{u(x)}{\sqrt{\log\frac{R}{|y|}}}\frac{1}{x_1}\big)\Big|^2x_1^2\,dx.\end{aligned}$$

These versions of the cylindrical critical Hardy inequalities on half-spaces seem new in the literature.

References

[1] Adimurthi; Chaudhuri, N.; Ramaswamy, M.; *An improved Hardy-Sobolev inequality and its application*. Proc. Amer. Math. Soc. 130 (2002), no. 2, 489–505.

[2] Badiale, M.; Tarantello, G.; *A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics*. Arch. Ration. Mech. Anal. 163 (2002), no. 4, 259–293.

[3] Balinsky, A. A.; Evans, W. D.; Lewis, R. T.; *The analysis and geometry of Hardy's inequality* Universitext. Springer, Cham, 2015. xv+263 pp.

[4] Barbatis, G.; Filippas, S.; Tertikas, A.; *A unified approach to improved L^p Hardy inequalities with best constants*. Trans. Amer. Math. Soc. 356 (2004), no. 6, 2169–2196.

[5] Benguria, R. D.; Frank, R. L.; Loss, M.; *The sharp constant in the Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half-space*. Math. Res. Lett. 15 (2008), no. 4, 613–622.

[6] Brezis, H.; Marcus, M.; *Hardy's inequalities revisited*. Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 1-2, 217–237 (1998).

[7] Brezis, H.; Marcus, M.; Shafrir, I.; *Extremal functions for Hardy's inequality with weight*. J. Funct. Anal. 171 (2000), no. 1, 177–191.

[8] Brezis, H.; Vázquez, J. L.; *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid, 10, 1997, 443–469.

[9] Cazacu, C.; Zuazua, E.; *Improved multipolar Hardy inequalities*. Studies in phase space analysis with applications to PDEs, 35–52, Progr. Nonlinear Differential Equations Appl., 84, Birkhäuser/Springer, New York, 2013.

[10] D'Ambrosio, L.; Dipierro, S.; *Hardy inequalities on Riemannian manifolds and applications*. Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 3, 449–475.

[11] Dao, N. A.; Díaz, J. I.; Nguyen, Q.-H.; *Generalized Gagliardo-Nirenberg inequalities using Lorentz spaces, BMO, Hölder spaces and fractional Sobolev spaces*. Nonlinear Anal. 173 (2018), 146–153.

[12] Devyver, B.; Pinchover, Y.; Psaradakis, G.; *Optimal Hardy inequalities in cones*. Proc. Roy. Soc. Edinburgh Sect. A 147 (2017), no. 1, 89–124.

[13] Duy, N. T.; Lam, N.; Triet, N.; *Hardy and Rellich inequalities with exact missing terms on homogeneous groups*. J. Math. Soc. Japan 71 (2019), no. 4, 1243–1256.

[14] Duy, N. T.; Lam, N.; Triet, N.; *Hardy-Rellich identities with Bessel pairs*. Arch. Math. (Basel) 113 (2019), no. 1, 95–112.

[15] Duy, N. T.; Lam, N.; Triet, N.; *Improved Hardy and Hardy-Rellich type inequalities with Bessel pairs via factorizations*. To appear in J. Spectr. Theory.

[16] Duy, N. T.; Lam, N.; Triet, N.; Yin, W.; *Improved Hardy inequalities with exact remainder terms*. To appear in Math. Inequal. Appl.

[17] Duy, N. T.; Phi, L. L.; *Cylindrical Hardy type inequalities with Bessel pairs*. To appear in Oper. Matrices.

[18] Filippas, S.; Maz'ya, V.; Tertikas, A.; *Critical Hardy-Sobolev inequalities*. J. Math. Pures Appl. (9) 87 (2007), no. 1, 37–56.

[19] Filippas, S.; Psaradakis, G.; *The Hardy-Morrey & Hardy-John-Nirenberg inequalities involving distance to the boundary*. J. Differential Equations 261 (2016), no. 6, 3107–3136.

[20] Frank, R. L.; Seiringer, R.; *Non-linear ground state representations and sharp Hardy inequalities*. J. Funct. Anal. 255 (2008), no. 12, 3407–3430.

[21] Gazzola, F.; Grunau, H.-C.; Mitidieri, E.; *Hardy inequalities with optimal constants and remainder terms*. Trans. Amer. Math. Soc. 356 (2004), no. 6, 2149–2168.

[22] Gesztesy, F.; Littlejohn, L. L.; *Factorizations and Hardy-Rellich-type inequalities*. Non-linear partial differential equations, mathematical physics, and stochastic analysis, 207–226, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2018.

[23] Gesztesy, F.; Littlejohn, L.; Michael, I.; Pang, M.; *Radial and logarithmic refinements of Hardy's inequality*. Reprinted in St. Petersburg Math. J. 30 (2019), no. 3, 429–436. Algebra i Analiz 30 (2018), no. 3, 55–65.

[24] Ghoussoub, N.; Moradifam, A.; *Bessel pairs and optimal Hardy and Hardy-Rellich inequalities.* Math. Ann. 349 (2011), no. 1, 1–57.
[25] Ghoussoub, N.; Moradifam, A.; *Functional inequalities: new perspectives and new applications.* Mathematical Surveys and Monographs, 187. American Mathematical Society, Providence, RI, 2013. xxiv+299.
[26] Goldstein, J. A.; Kombe, I.; Yener, A.; *A unified approach to weighted Hardy type inequalities on Carnot groups.* Discrete Contin. Dyn. Syst. 37 (2017), no. 4, 2009–2021.
[27] Ioku, N.; Ishiwata, M.; Ozawa, T.; *Sharp remainder of a critical Hardy inequality.* Arch. Math. (Basel) 106 (2016), no. 1, 65–71.
[28] Kufner, A.; Maligranda, L.; Persson, L.-E.; *The Hardy Inequality. About its History and Some Related Results.* Vydavatelský Servis, Pilsen, 2007.
[29] Kufner, A.; Persson, L.-E.; Samko, N.; *Weighted inequalities of Hardy type* Second edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. xx+459 pp.
[30] Lam, N.; *A note on Hardy inequalities on homogeneous groups.* Potential Anal. 51 (2019), no. 3, 425–435.
[31] Lam, N.; *Hardy and Hardy-Rellich type inequalities with Bessel pairs.* Ann. Acad. Sci. Fenn. Math. 43 (2018), 211-223.
[32] Lam, N.; Lu, G.; Zhang, L.; *Factorizations and Hardy's type identities and inequalities on upper half spaces.* Calc. Var. Partial Differential Equations 58 (2019), no. 6, Paper No. 183, 31 pp.
[33] Lam, N.; Lu, G.; Zhang, L.; *Geometric Hardy's inequalities with general distance functions.* J. Funct. Anal. 279 (2020), no. 8, 108673.
[34] Lu, G.; Yang, Q.; *Paneitz operators on hyperbolic spaces and high order Hardy-Sobolev-Maz'ya inequalities on half spaces.* Amer. J. Math. 141 (2019), no. 6, 1777–1816.
[35] Lu, G.; Yang, Q.; *Green's functions of Paneitz and GJMS operators on hyperbolic spaces and sharp Hardy-Sobolev-Maz'ya inequalities on half spaces.* https: //arxiv.org/abs/1903.10365.
[36] Maz'ya, V.; *Sobolev spaces with applications to elliptic partial differential equations. Second, revised and augmented edition.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 342. Springer, Heidelberg, 2011. xxviii+866 pp.
[37] Mitidieri, È.; *A simple approach to Hardy inequalities.* (Russian) Mat. Zametki 67 (2000), no. 4, 563–572; translation in Math. Notes 67 (2000), no. 3-4, 479–486
[38] Nazarov, A. I.; *Hardy-Sobolev inequalities in a cone.* J. Math. Sci. 132(4), 419-427 (2006)
[39] Nguyen, V. H.; *New sharp Hardy and Rellich type inequalities on Cartan–Hadamard manifolds and their improvements.* Proc. Roy. Soc. Edinburgh Sect. A, 1-30. doi: 10.1017/prm.2019.37
[40] Opic, B.; Kufner, A.; *Hardy-type inequalities.* Pitman Research Notes in Mathematics Series, 219. Longman Scientific & Technical, Harlow, 1990. xii+333 pp.
[41] Ruzhansky, M.; Sabitbek, B.; Suragan, D.; *Subelliptic geometric Hardy type inequalities on half-spaces and convex domains.* https: //arxiv.org/abs/1806.06226, to appear in Ann. Funct. Anal.
[42] Ruzhansky, M.; Suragan, D.; *Hardy inequalities on homogeneous groups.* Progress in Math., Vol. 327, Birkhäuser, 2019. 588pp.
[43] Sano, M.; Takahashi, F.; *Scale invariance structures of the critical and the subcritical Hardy inequalities and their improvements.* Calc. Var. Partial Differential Equations 56 (2017), no. 3, Art. 69, 14 pp.
[44] Secchi, S.; Smets, D.; Willem, M.; *Remarks on a Hardy-Sobolev inequality.* C. R. Math. Acad. Sci. Paris 336 (2003), no. 10, 811–815.
[45] Vázquez, J. L.; Zuazua, E.; *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential.* J. Funct. Anal. 173 (2000), no. 1, 103–153.
[46] Wang, Z.-Q.; Zhu, M.; *Hardy inequalities with boundary terms.* Electron. J. Differential Equations, 2003 (2003), No. 43, 8 pp.

Nguyen Tuan Duy
Department of Fundamental Sciences, University of Finance-Marketing, 2/4 Tran Xuan Soan St., Tan Thuan Tay Ward, Dist. 7, Ho Chi Minh City, Vietnam
Email address: `nguyenduy@ufm.edu.vn`

Huy Bac Nguyen (Corresponding Author)
Faculty of Electrical Engineering & Computer Science, Technical University of Ostrava, Czech Republic
Email address: huy.bac.nguyen.st@vsb.cz