Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 46, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GLOBAL STABILITY OF TRAVELING WAVES FOR DELAY REACTION-DIFFUSION SYSTEMS WITHOUT QUASI-MONOTONICITY

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ABSTRACT. This article concerns the global stability of traveling waves of a reaction-diffusion system with delay and without quasi-monotonicity. We prove that the traveling waves (monotone or non-monotone) are exponentially stable in $L^\infty(\mathbb{R})$ with the exponential convergence rate $t^{-1/2}e^{-\mu t}$ for some constant $\mu > 0$. We use the Fourier transform and the weighted energy method with a suitably weight function.

1. INTRODUCTION

This article is devoted to studying the delay reaction-diffusion system

$$\begin{aligned}\frac{\partial}{\partial t}u_1(x,t) &= d_1\frac{\partial^2}{\partial x^2}u_1(x,t) - \alpha u_1(x,t) + h(u_2(x,t-\tau_1)),\\ \frac{\partial}{\partial t}u_2(x,t) &= d_2\frac{\partial^2}{\partial x^2}u_2(x,t) - \beta u_2(x,t) + g(u_1(x,t-\tau_2)).\end{aligned} \tag{1.1}$$

Here $u_1(x,t)$ and $u_2(x,t)$ stand for the spatial density of the bacterial population and the infective human population at point $x \in \mathbb{R}$ and time $t \geq 0$, respectively. Both bacteria and humans are assumed to diffuse, d_1 and d_2 are diffusion coefficients; the term αu_1 is the natural death rate of the bacterial population and the nonlinearity $h(u_2)$ is the contribution of the infective humans to the growth rate of the bacterial; βu_2 is the natural diminishing rate of the infective population due to the finite mean duration of the infectious population and the nonlinearity $g(u_1)$ is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic, and τ_1, τ_2 are time delays.

Wu and Hsu [23] have already established the existence and qualitative features of solutions of (1.1). For the particular case $\tau_i = 0$, $i = 1, 2$, system (1.1) becomes the non-delay reaction-diffusion system

$$\begin{aligned}\frac{\partial}{\partial t}u_1(x,t) &= d_1\frac{\partial^2}{\partial x^2}u_1(x,t) - \alpha u_1(x,t) + h(u_2(x,t)),\\ \frac{\partial}{\partial t}u_2(x,t) &= d_2\frac{\partial^2}{\partial x^2}u_2(x,t) - \beta u_2(x,t) + g(u_1(x,t)).\end{aligned} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35C07, 35B35, 92D30.
Key words and phrases. Delay reaction-diffusion system; traveling waves; global stability; Fourier transform; weighted energy method.

Submitted December 8, 2019. Published May 19, 2020.

Hsu and Yang [6] studied the existence, uniqueness, monotonicity and asymptotic behavior of monostable traveling wave solutions of (1.2). For $\tau_1 = 0$ and $h(u_2) = \gamma u_2$ in (1.1), Freedman and Zhao [3] presented a threshold result for the global dynamics of the epidemic system

$$\begin{aligned} \frac{\partial}{\partial t}u_1(x,t) &= d_1\frac{\partial^2}{\partial x^2}u_1(x,t) - \alpha u_1(x,t) + \gamma u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) &= d_2\frac{\partial^2}{\partial x^2}u_2(x,t) - \beta u_2(x,t) + g(u_1(x,t-\tau)). \end{aligned} \tag{1.3}$$

The epidemic model (1.3) with $\tau = 0$ was first proposed and analyzed by Capasso and Maddalena [1]. When $d_2 = 0$, system (1.3) becomes

$$\begin{aligned} \frac{\partial}{\partial t}u_1(x,t) &= d_1\frac{\partial^2}{\partial x^2}u_1(x,t) - \alpha u_1(x,t) + \gamma u_2(x,t), \\ \frac{\partial}{\partial t}u_2(x,t) &= -\beta u_2(x,t) + g(u_1(x,t-\tau)). \end{aligned} \tag{1.4}$$

Thieme and Zhao [20] investigated the existence of spreading speed and minimal wave speed of (1.4) in the quasi-monotone case. The results in [20] were then extended by Wu and Liu [22] to the non-quasi-monotone case by constructing two auxiliary monotone integral equations. Yang, Li and Wu [25, 26] studied the stability of traveling wave solutions of (1.4) in both the quasi-monotone case and the non-quasi-monotone case by using the weighted energy method. When $\tau = 0$ in (1.4), Xu and Zhao [24] proved the existence, uniqueness (up to translation) and globally exponential stability of bistable traveling wave fronts of (1.4), and Zhao and Wang [31] proved the existence and non-existence of monostable traveling wave fronts of (1.4).

More recently, Hsu, Yang and Yu [7] studied the existence and exponential stability of traveling wave solutions for general delay reaction-diffusion systems

$$\begin{aligned} \frac{\partial}{\partial t}u_1(x,t) &= d_1\frac{\partial^2}{\partial x^2}u_1(x,t) + h(u_1(x,t), u_1(x,t-\hat{\tau}_1), u_2(x,t-\tau_2)), \\ \frac{\partial}{\partial t}u_2(x,t) &= d_2\frac{\partial^2}{\partial x^2}u_2(x,t) + g(u_2(x,t), u_1(x,t-\tau_1), u_2(x,t-\hat{\tau}_2)). \end{aligned} \tag{1.5}$$

When system (1.5) is monotone, by applying the techniques of weighted energy method and the comparison principle, they showed that the traveling wave solutions of (1.5) are exponentially stable provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev space. To the best of our knowledge, global stability for traveling wave solutions of (1.1)-(1.5) without monotonicity have not been considered. The purpose of this article is to establish the global stability of traveling waves of (1.1) with $\tau_1 = \tau_2$, without quasi-monotonicity.

The stability of traveling waves for various evolution equations has been extensively studied. We refer the readers to [4, 5, 9, 10, 11, 13, 14, 15, 18, 19, 21, 26] for reaction-diffusion equations and to [8, 12, 17, 27, 28, 29, 30] for nonlocal dispersal equations. Note that when the evolution equations are non-monotone, the comparison principle is not applicable. Thus, the frequently used methods for the stability of traveling waves, such as the squeezing technique, the method of combination of the comparison principle and the weighted energy method are not applicable. Recently, the weighted energy method without the comparison principle was used to prove the stability of traveling waves of nonmonotone equations, see Chern et al.

[2], Huang et al. [8], Li et al. [9], Wu et al. [21], Yang et al. [26], Zhang and Ma [28] and Zhang et al. [30]. In particular, Yang et al. [26] studied the stability of traveling waves of (1.4) without quasi-monotonicity. Zhang, Li and Feng [30] further investigated the stability of traveling waves of (1.4) by replacing $d\frac{\partial^2}{\partial x^2}u_1$ with $d(J*u_1-u_1)$. However, local stability of traveling waves has been obtained only for perturbations around the traveling wave with properly small weighted norm. Recently, Mei et al. [16] established the global stability for the oscillatory traveling waves of local Nicholson's blowflies equations by using the anti-weighted energy method together with the Fourier transform. Zhang [29] applied this method to a nonlocal dispersal equation with time delay and obtained the global stability of traveling waves. Motivated by [12, 16, 29], we shall extend this method to the study of global stability of traveling waves of reaction-diffusion system (1.1) without quasi-monotonicity.

The rest of this article is organized as follows. In Section 2, we present some preliminaries and summarize our main results. Section 3 is dedicated to the global stability of traveling waves of (1.1) by the Fourier transform and the weighted energy method, when $h(u)$ and $g(u)$ are not monotone.

2. Preliminaries and statement main results

Throughout this article, we assume that $\tau_1 = \tau_2 = \tau$ in (1.1), that the initial data satisfies

$$u_i(x,s) = u_{i0}(x,s),\ x\in\mathbb{R},\ s\in[-\tau,0],\ i=1,2. \tag{2.1}$$

Now we state some basic assumptions on the nonlinearities g and h.

- (H1) $g\in C^2([0,K_1],\mathbb{R})$, $g(0)=h(0)=0$, $K_2=g(K_1)/\beta>0$, $h\in C^2([0,K_2],\mathbb{R})$, $h(g(K_1)/\beta)=\alpha K_1$, $h(g(u)/\beta)>\alpha u$ for $u\in(0,K_1)$, where K_1 is a positive constant.
- (H2) $|g'(u)|\le g'(0)$ and $|h'(v)|\le h'(0)$ for $u,v\in[0,+\infty)$.

From (H1), we see that the spatially homogeneous system of (1.1) admits two constant equilibria

$$(u_{1-},u_{2-}) = (0,0) =: \mathbf{0} \quad\text{and}\quad (u_{1+},u_{2+}) = (K_1,K_2) =: \mathbf{K}.$$

A *traveling wave solution* (in short, traveling wave) of (1.1) is a special translation invariant solution of the form $(u_1(x,t),u_2(x,t)) = (\phi_1(x+ct),\phi_2(x+ct))$, where $c>0$ is the wave speed. If ϕ_1 and ϕ_2 are monotone, then (ϕ_1,ϕ_2) is called a traveling wavefront. Substituting $(\phi_1(x+ct),\phi_2(x+ct))$ into (1.1) and letting $\xi=x+ct$, we obtain the following wave profile system with boundary conditions

$$\begin{gathered}
c\phi_1'(\xi) = d_1\phi_1''(\xi) - \alpha\phi_1(\xi) + h(\phi_2(\xi-c\tau)),\\
c\phi_2'(\xi) = d_2\phi_2''(\xi) - \beta\phi_2(\xi) + g(\phi_1(\xi-c\tau)),\\
(\phi_1,\phi_2)(-\infty) = (u_{1-},u_{2-}),\quad (\phi_1,\phi_2)(+\infty) = (u_{1+},u_{2+}).
\end{gathered} \tag{2.2}$$

It is clear that the characteristic function for (2.2) with respect to the trivial equilibrium $\mathbf{0}$ can be represented by

$$\Delta_1(\lambda,c) := f_1(c,\lambda) - f_2(c,\lambda)$$

for $c\ge 0$ and $\lambda\in\mathbb{C}$, where

$$\begin{gathered}
f_1(c,\lambda) := (d_1\lambda^2 - c\lambda - \alpha)(d_2\lambda^2 - c\lambda - \beta),\\
f_2(c,\lambda) := h'(0)g'(0)e^{-2c\lambda\tau}.
\end{gathered}$$

For convenience, we denote

$$\lambda_1^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_1\alpha}}{2d_1}, \quad \lambda_2^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_2\beta}}{2d_2}, \quad \lambda_m^c = \min\{\lambda_1^+, \lambda_2^+\}.$$

It is clear that $f_1(c, \lambda_1^{\pm}) = f_1(c, \lambda_2^{\pm})$. According to [23, Lemma 2.1], we have the following result.

Lemma 2.1. *There exist a positive number c_* such that the following items hold.*

- (i) *If $c \geq c_*$, then the equation $\Delta_1(\lambda, c) = 0$ has two positive real roots $\lambda_1 := \lambda_1(c)$ and $\lambda_2 := \lambda_2(c)$ with $0 < \lambda_1(c) \leq \lambda_2(c) < \lambda_m^c$.*
- (ii) *If $c = c_*$, then $\lambda_* = \lambda_1(c_*) = \lambda_2(c_*)$ and if $c > c_*$, then $\lambda_1(c) < \lambda_2(c)$ and $\Delta_1(\cdot, c) > 0$ in $(\lambda_1(c), \lambda_2(c))$.*

When $g'(u) \geq 0$ for $u \in [0, K_1]$ and $h'(v) \geq 0$ for $v \in [0, K_2]$, system (1.1) is a quasi-monotone system. The existence of traveling wave fronts has been obtained by Wu and Hsu, see [23, Theorem 2.3]. When the condition $g'(u) \geq 0$ for $u \in [0, K_1]$ or $h'(v) \geq 0$ for $v \in [0, K_2]$ does not hold, system (1.1) is a non-quasi-monotone system. The existence of traveling waves can also be obtained by using auxiliary equations and Schauder's fixed point theorem [22, 26], if we assume the following conditions:

- (H3) There exist $\mathbf{K}^{\pm} = (K_1^{\pm}, K_2^{\pm}) \gg 0$ with $\mathbf{K}^- < \mathbf{K} < \mathbf{K}^+$ and four continuous and twice piecewise continuous differentiable functions $g^{\pm} : [0, K_1^+] \to \mathbb{R}$ and $h^{\pm} : [0, K_2^+] \to \mathbb{R}$ such that
  - (i) $K_2^{\pm} = g^{\pm}(K_1^{\pm})/\beta$, $h^{\pm}(\frac{1}{\beta} g^{\pm}(K_1^{\pm})) = \alpha K_1^{\pm}$, and $h^{\pm}(\frac{1}{\beta} g^{\pm}(u)) > \alpha u$ for $u \in (0, K_1^{\pm})$;
  - (ii) $g^{\pm}(u)$ and $h^{\pm}(v)$ are non-decreasing on $[0, K_1^+]$ and $[0, K_2^+]$, respectively;
  - (iii) $(g^{\pm})'(0) = g'(0)$, $(h^{\pm})'(0) = h'(0)$ and
$$\begin{gathered} 0 < g^-(u) \leq g(u) \leq g^+(u) \leq g'(0)u \text{ for } u \in [0, K_1^+], \\ 0 < h^-(v) \leq h(v) \leq h^+(v) \leq h'(0)v \text{ for } v \in [0, K_2^+]. \end{gathered}$$

Proposition 2.2. *Assume that* (H1) *and* (H3) *hold, $\tau \geq 0$, and let c_* be defined as in Lemma 2.1. Then for every $c > c_*$, system* (1.1) *has a traveling wave $(\phi_1(\xi), \phi_2(\xi))$ satisfying $(\phi_1(-\infty), \phi_2(-\infty)) = (0, 0)$ and*

$$\begin{gathered} K_1^- \leq \liminf_{\xi \to +\infty} \phi_1(\xi) \leq \limsup_{\xi \to +\infty} \phi_1(\xi) \leq K_1^+, \\ 0 \leq \liminf_{\xi \to +\infty} \phi_2(\xi) \leq \limsup_{\xi \to +\infty} \phi_2(\xi) \leq K_2^+. \end{gathered}$$

Notation. $C > 0$ denotes a generic constant, while C_i $(i = 1, 2, \dots)$ represents a specific constant. Let $\|\cdot\|$ and $\|\cdot\|_\infty$ denote 1-norm and ∞-norm of the matrix (or vector), respectively. Let I be an interval, typically $I = \mathbb{R}$. Denote by $L^1(I)$ the space of integrable functions defined on I, and $W^{k,1}(I)(k \geq 0)$ the Sobolev space of the L^1-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^n}{dx^n} f(n = 1, \dots, k)$ also belong to $L^1(I)$. Let $L_w^1(I)$ be the weighted L^1-space with a weight function $w(x) > 0$ and norm

$$\|f\|_{L_w^1(I)} = \int_I w(x)|f(x)|dx\,.$$

Let $W_w^{k,1}(I)$ be the weighted Sobolev space with norm

$$\|f\|_{W_w^{k,1}(I)} = \sum_{i=0}^{k} \int_I w(x) \Big| \frac{d^i f(x)}{dx^i} \Big| dx.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$, and by $L^1([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued L^1-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly. For any function $f(x)$, its Fourier transform is

$$\mathcal{F}[f](\eta) = \widehat{f}(\eta) = \int_{\mathbb{R}} e^{-\mathbf{i}x\eta} f(x)dx$$

and the inverse Fourier transform is

$$\mathcal{F}^{-1}[\widehat{f}](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathbf{i}x\eta} \widehat{f}(\eta) d\eta,$$

where $\mathbf{i}$ is the imaginary number, $\mathbf{i}^2 = -1$.

To obtain stability of traveling waves of (1.1), we need the following assumptions.

- (H4) $d_1 > d_2$, $\alpha > \beta$ and $\max\{h'(0), g'(0)\} > \beta$.
- (H5) The initial data $(u_{10}(x,s), u_{20}(x,s)) \geq (0,0)$ satisfies

$$\lim_{x\to\pm\infty} (u_{10}(x,s), u_{20}(x,s)) = (u_{1\pm}, u_{2\pm}) \text{ uniformly in } \ s \in [-\tau, 0].$$

We consider the function

$$\Delta_2(\lambda, c) = d_2\lambda^2 - c\lambda - \beta + \max\{h'(0), g'(0)\} e^{-\lambda c\tau}.$$

It is easy to see that there exist $\lambda^* > 0$ and $c^* > 0$, such that $\Delta_2(\lambda^*, c^*) = 0$ and $\frac{\partial \Delta_2(\lambda,c)}{\partial\lambda}|_{(\lambda^*,c^*)} = 0$. When $c > c^*$, the equation $\Delta_2(\lambda, c) = 0$ has two positive real roots $\lambda_1^\natural(c)$ and $\lambda_2^\natural(c)$ with $0 < \lambda_1^\natural(c) < \lambda^* < \lambda_2^\natural(c)$. When $\lambda \in (\lambda_1^\natural(c), \lambda_2^\natural(c))$, $\Delta_2(\lambda, c) < 0$. Moreover, $(\lambda_1^\natural)'(c) < 0$ and $(\lambda_2^\natural)'(c) > 0$.

Since $(\lambda_1^\natural)'(c) < 0$, there exists a positive number $c^\natural$ such that when $c > c^\natural > c^*$, $\lambda_1^\natural(c) < \sqrt{\frac{\alpha-\beta}{d_1-d_2}}$. Define the weight function $w(\xi) > 0$ as

$$w(\xi) = e^{-2\lambda\xi},$$

where $\lambda > 0$ satisfies $\lambda_1^\natural(c) < \lambda < \min\big\{\sqrt{\frac{\alpha-\beta}{d_1-d_2}}, \lambda_2^\natural(c)\big\}$. Now we present the main result on global stability of traveling waves.

Theorem 2.3. *Assume that* (H1)–(H5) *hold. For any given traveling wave* $(\phi_1(x+ct), \phi_2(x+ct))$ *of* (1.1) *with speed* $c \geq \max\{c_*, c^\natural\}$ *connecting* $(0,0)$ *and* (K_1, K_2), *whether it is monotone or non-monotone, if the initial data satisfy*

$$u_{i0}(x,s) - \phi_i(x+cs) \in C_{\rm unif}[-\tau, 0] \cap C([-\tau,0]; W_w^{2,1}(\mathbb{R})), \quad i = 1,2,$$
$$\partial_s(u_{i0} - \phi_i) \in L^1([-\tau,0]; L_w^1(\mathbb{R})), \quad i = 1,2,$$

then there exists $\tau_0 > 0$ *such that for any* $\tau \leq \tau_0$, *the solution* $(u_1(x,t), u_2(x,t))$ *of* (1.1)-(2.1) *converges to the traveling wave* $(\phi_1(x+ct), \phi_2(x+ct))$ *with*

$$\sup_{x\in\mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \leq Ct^{-1/2} e^{-\mu t}, \quad t > 0,$$

where C and μ are two positive constants, and $C_{\rm unif}[r,T]$ is the space of uniformly continuous functions,

$$\begin{aligned} C_{\rm unif}[r,T] := \Big\{ & u \in C^2([r,T]\times\mathbb{R}) : \lim_{x\to+\infty} u(x,t) \text{ exists uniformly in } t\in[r,T], \\ & \lim_{x\to+\infty} u_x(x,t) = \lim_{x\to+\infty} u_{xx}(x,t) = 0 \text{ uniformly for } t\in[r,T]\Big\}. \end{aligned}$$

3. Global stability of traveling waves

In this section we prove Theorem 2.3. Let $(\phi_1(x+ct),\phi_2(x+ct)) = (\phi_1(\xi),\phi_2(\xi))$ be a given traveling wave with speed $c \ge c_*$ and define

$$\begin{gathered} U_i(\xi,t) := u_i(x,t) - \phi_i(x+ct) = u_i(\xi-ct,t) - \phi_i(\xi),\ i=1,2, \\ U_{i0}(\xi,s) := u_{i0}(x,s) - \phi_i(x+cs) = u_{i0}(\xi-cs,s) - \phi(\xi),\ i=1,2. \end{gathered}$$

Then from (1.1) and (2.2), $U_i(\xi,t)$ satisfies

$$\begin{gathered} U_{1t} + cU_{1\xi} - d_1U_{1\xi\xi} + \alpha U_1 = P_1(U_2(\xi-c\tau,t-\tau)), \\ U_{2t} + cU_{2\xi} - d_2U_{2\xi\xi} + \beta U_2 = P_2(U_1(\xi-c\tau,t-\tau)), \\ U_i(\xi,s) = U_{i0}(\xi,s), \quad (\xi,s)\in\mathbb{R}\times[-\tau,0],\ i=1,2. \end{gathered} \tag{3.1}$$

The nonlinear terms are

$$\begin{gathered} P_1(U_2) := h(\phi_2+U_2) - h(\phi_2) = h'(\tilde\phi_2)U_2, \\ P_2(U_1) := g(\phi_1+U_1) - g(\phi_1) = g'(\tilde\phi_1)U_1, \end{gathered} \tag{3.2}$$

for some $\tilde\phi_i$ between ϕ_i and ϕ_i+U_i, with $\phi_i = \phi_i(\xi-c\tau_i)$ and $U_i = U_i(\xi-c\tau_i,t-\tau_i)$.

We first prove the existence and uniqueness of solution $(U_1(\xi,t),U_2(\xi,t))$ to the initial value problem (3.1) in the space $C_{\rm unif}[-\tau,+\infty)\times C_{\rm unif}[-\tau,+\infty)$.

Proposition 3.1. *Assume that* (H1) *and* (H2) *hold. If the initial perturbation satisfies*

$$(U_{10}(\xi,s),U_{20}(\xi,s)) \in C_{\rm unif}[-\tau,0]\times C_{\rm unif}[-\tau,0]$$

for $c\ge c_*$, then a solution (U_1,U_2) of the perturbed equation (3.1) *is unique, exists globally in time, and belongs to $C_{\rm unif}[-\tau,+\infty)\times C_{\rm unif}[-\tau,+\infty)$.*

Proof. When $t\in[0,\tau]$, we have $t-\tau\in[-\tau,0]$ and $U_i(\xi-c\tau,t-\tau) = U_{i0}(\xi-c\tau,t-\tau)$, $i=1,2$, which imply that (3.1) is linear. Thus, the solution of (3.1) can be explicitly and uniquely solved:

$$\begin{aligned} U_1(\xi,t) &= e^{-\alpha t}\int_{-\infty}^{\infty} G_1(\eta,t)U_{10}(\xi-\eta,0)d\eta \\ &\quad + \int_0^t e^{-\alpha(t-s)}\int_{-\infty}^{\infty} G_1(\eta,t-s)P_1(U_{20}(\xi-\eta-c\tau,s-\tau))\,d\eta\,ds, \\ U_2(\xi,t) &= e^{-\beta t}\int_{-\infty}^{\infty} G_2(\eta,t)U_{20}(\xi-\eta,0)d\eta \\ &\quad + \int_0^t e^{-\beta(t-s)}\int_{-\infty}^{\infty} G_2(\eta,t-s)P_2(U_{10}(\xi-\eta-c\tau,s-\tau))\,d\eta\,ds \end{aligned} \tag{3.3}$$

for $t\in[0,\tau]$, where $G_i(\eta,t)$ is the heat kernel

$$G_i(\eta,t) = \frac{1}{\sqrt{4\pi d_i t}}\exp\big(-\frac{(\eta+ct)^2}{4d_it}\big), \quad i=1,2.$$

Since $U_{i0}(\xi,s)\in C_{\rm unif}[-\tau,0]$, $i=1,2$, namely, $\lim_{\xi\to+\infty}U_{i0}(\xi,s)=U_{i0}(\infty,s)$ and $\lim_{\xi\to+\infty}U_{i0,\xi}(\xi,s)=\lim_{\xi\to+\infty}U_{i0,\xi\xi}(\xi,s)=0$ uniformly in $s\in[-\tau,0]$, we immediately prove the following uniform convergence

$$
\begin{aligned}
&\lim_{\xi\to+\infty}U_1(\xi,t)\\
&=e^{-\alpha t}\int_{-\infty}^{\infty}G_1(\eta,t)\lim_{\xi\to+\infty}U_{10}(\xi-\eta,0)d\eta\\
&\quad+\int_0^t e^{-\alpha(t-s)}\int_{-\infty}^{\infty}G_1(\eta,t-s)\lim_{\xi\to+\infty}P_1(U_{20}(\xi-\eta-c\tau,s-\tau))\,d\eta\,ds\\
&=e^{-\alpha t}U_{10}(\infty,0)\int_{-\infty}^{\infty}G_1(\eta,t)d\eta\\
&\quad+\int_0^t e^{-\alpha(t-s)}P_1(U_{20}(\infty,s-\tau))\int_{-\infty}^{\infty}G_1(\eta,t-s)\,d\eta\,ds\\
&=e^{-\alpha t}U_{10}(\infty,0)+\int_0^t e^{-\alpha(t-s)}P_1(U_{20}(\infty,s-\tau))ds\\
&=:g_1(t),\quad\text{uniformly for } t\in[0,\tau],
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\xi\to+\infty}U_2(\xi,t)\\
&=e^{-\beta t}\int_{-\infty}^{\infty}G_2(\eta,t)\lim_{\xi\to+\infty}U_{20}(\xi-\eta,0)d\eta\\
&\quad+\int_0^t e^{-\beta(t-s)}\int_{-\infty}^{\infty}G_2(\eta,t-s)\lim_{\xi\to+\infty}P_2(U_{10}(\xi-\eta-c\tau,s-\tau))\,d\eta\,ds\\
&=e^{-\beta t}U_{20}(\infty,0)\int_{-\infty}^{\infty}G_2(\eta,t)d\eta\\
&\quad+\int_0^t e^{-\beta(t-s)}P_2(U_{10}(\infty,s-\tau))\int_{-\infty}^{\infty}G_2(\eta,t-s)\,d\eta\,ds\\
&=e^{-\beta t}U_{20}(\infty,0)+\int_0^t e^{-\beta(t-s)}P_2(U_{10}(\infty,s-\tau))ds\\
&=:g_2(t),\quad\text{uniformly for } t\in[0,\tau],
\end{aligned}
$$

where we used that $\int_{-\infty}^{\infty}G_i(\eta,t-s)d\eta=1$ for $i=1,2$. Furthermore, we obtain

$$
\begin{aligned}
&\lim_{\xi\to+\infty}\partial_\xi^k U_1(\xi,t)\\
&=e^{-\alpha t}\int_{-\infty}^{\infty}\partial_\eta^k G_1(\eta,t)\lim_{\xi\to+\infty}U_{10}(\xi-\eta,0)d\eta\\
&\quad+\int_0^t e^{-\alpha(t-s)}\int_{-\infty}^{\infty}\partial_\eta^k G_1(\eta,t-s)\lim_{\xi\to+\infty}P_1(U_{20}(\xi-\eta-c\tau,s-\tau))\,d\eta\,ds\\
&=e^{-\alpha t}U_{10}(\infty,0)\int_{-\infty}^{\infty}\partial_\eta^k G_1(\eta,t)d\eta\\
&\quad+\int_0^t e^{-\alpha(t-s)}P_1(U_{20}(\infty,s-\tau))\int_{-\infty}^{\infty}\partial_\eta^k G_1(\eta,t-s)\,d\eta\,ds\\
&=0,\quad\text{uniformly for } t\in[0,\tau],\ k=1,2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\xi\to+\infty}\partial_\xi^k U_2(\xi,t)\\
&= e^{-\beta t}\int_{-\infty}^{\infty}\partial_\eta^k G_2(\eta,t)\lim_{\xi\to+\infty}U_{20}(\xi-\eta,0)d\eta\\
&\quad+\int_0^t e^{-\beta(t-s)}\int_{-\infty}^{\infty}\partial_\eta^k G_2(\eta,t-s)\lim_{\xi\to+\infty}P_2(U_{10}(\xi-\eta-c\tau,s-\tau))\,d\eta\,ds\\
&= e^{-\beta t}U_{20}(\infty,0)\int_{-\infty}^{\infty}\partial_\eta^k G_2(\eta,t)d\eta\\
&\quad+\int_0^t e^{-\beta(t-s)}P_2(U_{10}(\infty,s-\tau))\int_{-\infty}^{\infty}\partial_\eta^k G_2(\eta,t-s)\,d\eta\,ds\\
&= 0,\quad \text{uniformly for } t\in[0,\tau],\ k=1,2.
\end{aligned}
$$

Here we used that

$$
G_i(\pm\infty,t-s)=0,\quad \partial_\eta G_i(\eta,t-s)\big|_{\eta=\pm\infty}=0,\ i=1,2.
$$

Thus, we have proved that $(U_1,U_2)\in C_{\rm unif}[-\tau,\tau]\times C_{\rm unif}[-\tau,\tau]$.

Now we consider (3.1) for $t\in[\tau,2\tau]$. Since $t-\tau\in[0,\tau]$ and $U_i(\xi,t-\tau)$ is solved already in (3.3), $P_1(U_2(\xi-c\tau,t-\tau))$ and $P_2(U_1(\xi-c\tau,t-\tau))$ are known for (3.1) with $t\in[0,2\tau]$, namely, the equation (3.1) is linear for $t\in[0,2\tau]$. As showed before, we can similarly prove the existence and uniqueness of the solution $(U_1(\xi,t),U_2(\xi,t))$ to (3.1) for $t\in[0,2\tau]$, and particularly $(U_1,U_2)\in C_{\rm unif}[-\tau,2\tau]\times C_{\rm unif}[-\tau,2\tau]$.

By repeating this process for $t\in[n\tau,(n+1)\tau]$ with $n\in\mathbb{Z}_+$, we prove that there exists a unique solution $(U_1,U_2)\in C_{\rm unif}[-\tau,(n+1)\tau]\times C_{\rm unif}[-\tau,(n+1)\tau]$ for (3.1), and step by step, we finally prove the uniqueness and existence global in time of the solution $(U_1,U_2)\in C_{\rm unif}[-\tau,\infty)\times C_{\rm unif}[-\tau,\infty)$ for (3.1). $\square$

Now we state a stability result for the perturbed equation (3.1), which automatically implies Theorem 2.3.

Proposition 3.2 (Stability of traveling waves)**.** *Assume that* (H1), (H2), (H4) *and* (H5) *hold. If*

$$
U_{i0}\in C_{\rm unif}[-\tau,0]\cap C([-\tau,0];W_w^{2,1}(\mathbb{R})),\ i=1,2,
$$

and $\partial_s U_{i0}\in L^1([-\tau,0];L_w^1(\mathbb{R}))$ *for* $i=1,2$, *then there exists* $\tau_0>0$ *such that for any* $\tau\le\tau_0$, *when* $c\ge\min\{c_*,c^\natural\}$, *it holds*

$$
\sup_{\xi\in\mathbb{R}}|U_i(\xi,t)|\le Ct^{-1/2}e^{-\mu t},\quad t>0,\ i=1,2, \tag{3.4}
$$

for some $\mu>0$ *and* $C>0$.

To prove Proposition 3.2, we first investigate the time-exponential decay estimate of $U_i(\xi,t)$ at $\xi=+\infty$, $i=1,2$.

Lemma 3.3. *There exist* $\tau_0>0$ *and a large number* $x_0\gg1$ *such that when* $\tau\le\tau_0$, *the solution* $U_i(\xi,t)$ *of* (3.1) *satisfies*

$$
\sup_{\xi\in[x_0,+\infty)}|U_i(\xi,t)|\le Ce^{-\mu_1 t},\quad t>0,\ i=1,2,
$$

for some $\mu_1>0$ *and* $C>0$.

Proof. Denote

$$z_i^+(t) := U_i(\infty, t), \quad z_{i0}^+(s) := U_{i0}(\infty, s), \quad s \in [-\tau, 0],\ i = 1, 2.$$

Since $(U_1, U_2) \in C_{\rm unif}[-\tau, +\infty) \times C_{\rm unif}[-\tau, +\infty)$, we have

$$\lim_{\xi\to+\infty} U_i(\xi, t) = z_i^+(t)$$

exists uniformly for t, and

$$\lim_{\xi\to+\infty} U_{i\xi}(\xi, t) = \lim_{\xi\to+\infty} U_{i\xi\xi}(\xi, t) = 0$$

uniformly for t. Let us take the limits in (3.1) as $\xi \to +\infty$. Then we have

$$\begin{aligned}
&\frac{dz_1^+}{dt} + \alpha z_1^+ - h'(u_{2+}) z_2^+(t-\tau) = Q_1(z_2^+(t-\tau)),\\
&\frac{dz_2^+}{dt} + \beta z_2^+ - g'(u_{1+}) z_1^+(t-\tau) = Q_2(z_1^+(t-\tau)),\\
&z_i^+(s) = z_{i0}^+(s), \quad s \in [-\tau, 0],\ i = 1, 2,
\end{aligned}$$

where

$$\begin{aligned}
Q_1(z_2^+) &= h(u_{2+} + z_2^+) - h(u_{2+}) - h'(u_{2+}) z_2^+,\\
Q_2(z_1^+) &= g(u_{1+} + z_1^+) - g(u_{1+}) - g'(u_{1+}) z_1^+.
\end{aligned}$$

Then by [9, Lemma 3.8], there exist positive constants τ_0, μ_1 and C such that when $\tau \le \tau_0$,

$$|U_i(\infty, t)| = |z_i^+(t)| \le Ce^{-\mu_1 t}, \quad t > 0,\ i = 1, 2,$$

provided that $|z_{i0}^+| \ll 1$, $i = 1, 2$.

Furthermore, by the continuity and the uniform convergence of $U_i(\xi, t)$ as $\xi \to +\infty$, there exists a large $x_0 \gg 1$ such that

$$\sup_{\xi\in[x_0,+\infty)} |U_i(\xi, t)| \le Ce^{-\mu_1 t}, \quad t > 0,\ i = 1, 2,$$

provided that $\sup_{\xi\in[x_0,+\infty)} |U_{i0}(\xi, s)| \ll 1$ for $s \in [-\tau, 0]$. Such a smallness for the initial perturbation (U_{10}, U_{20}) near $\xi \to +\infty$ can be easily verified, since

$$\lim_{x\to+\infty} (u_{10}(x, s), u_{20}(x, s)) = (K_1, K_2) \quad \text{uniformly in } s \in [-\tau, 0],$$

which implies

$$\lim_{\xi\to+\infty} U_{i0}(\xi, s) = \lim_{\xi\to+\infty} [u_{i0}(\xi, s) - \phi_i(\xi)] = K_i - K_i = 0$$

uniformly for $s \in [-\tau, 0]$, $i = 1, 2$. The proof is complete. □

Next we establish the a priori decay estimate of $\sup_{\xi\in(-\infty,x_0]} |U_i(\xi, t)|$. We shall use the anti-weighted technique [2, 8] together with Fourier transform to treat this problem. First of all, we shift $U_i(\xi, t)$ to $U_i(\xi + x_0, t)$ by the constant x_0 given in Lemma 3.3, and then introduce the transformation

$$V_i(\xi, t) = \sqrt{w(\xi)}\, U_i(\xi + x_0, t) = e^{-\lambda\xi} U_i(\xi + x_0, t), \quad i = 1, 2.$$

Substituting $U = w^{-1/2}V$ in (3.1) yields

$$\begin{gathered} V_{1t} + \rho_1(c)V_{1\xi} - d_1 V_{1\xi\xi} + \rho_2(c)V_1 = \tilde{P}_1(V_2(\xi - c\tau, t - \tau)), \\ V_{2t} + \rho_3(c)V_{2\xi} - d_2 V_{2\xi\xi} + \rho_4(c)V_2 = \tilde{P}_2(V_1(\xi - c\tau, t - \tau)), \\ (\xi, t) \in \mathbb{R} \times [0, +\infty), \\ V_i(\xi, s) = \sqrt{w(\xi)}U_{i0}(\xi + x_0, s) =: V_{i0}(\xi, s), \quad \xi \in \mathbb{R},\ s \in [-\tau, 0],\ i = 1, 2, \end{gathered} \tag{3.5}$$

where

$$\begin{gathered} \rho_1(c) := c - 2d_1\lambda, \quad \rho_2(c) := c\lambda - d_1\lambda^2 + \alpha, \\ \rho_3(c) := c - 2d_2\lambda, \quad \rho_4(c) := c\lambda - d_2\lambda^2 + \beta, \\ \tilde{P}_1(V_2) = e^{-\lambda\xi}P_1(U_2), \quad \tilde{P}_2(V_1) = e^{-\lambda\xi}P_2(U_1). \end{gathered}$$

By (3.2), $\tilde{P}_1(V_2)$ satisfies

$$\begin{aligned} \tilde{P}_1(V_2(\xi - c\tau, t - \tau)) &= e^{-\lambda\xi}P_1(U_2(\xi - c\tau + x_0, t - \tau)) \\ &= e^{-\lambda\xi}h'(\tilde{\phi}_2)U_2(\xi - c\tau + x_0, t - \tau) \\ &= e^{-\lambda c\tau}h'(\tilde{\phi}_2)V_2(\xi - c\tau, t - \tau) \end{aligned} \tag{3.6}$$

and $\tilde{P}_2(V_1)$ satisfies

$$\tilde{P}_2(V_1(\xi - c\tau, t - \tau)) = e^{-\lambda c\tau}g'(\tilde{\phi}_1)V_1(\xi - c\tau, t - \tau). \tag{3.7}$$

Furthermore, by (H2), we have

$$\begin{gathered} |\tilde{P}_1(V_2(\xi - c\tau, t - \tau))| \le h'(0)e^{-\lambda c\tau}|V_2(\xi - c\tau, t - \tau)|, \\ |\tilde{P}_2(V_1(\xi - c\tau, t - \tau))| \le g'(0)e^{-\lambda c\tau}|V_1(\xi - c\tau, t - \tau)|. \end{gathered}$$

Taking (3.6) and (3.7) into (3.5), we see that the coefficient $h'(\tilde{\phi}_2)$ and $g'(\tilde{\phi}_1)$ on the right side of (3.5) is variable and can be negative. Thus, the classical methods, such as the monotone technique and the Fourier transform cannot be applied directly to establish the decay estimate for (V_1, V_2). A new method should be introduced. The main ideas of this method can be described as follows.

(i) We replace $h'(\tilde{\phi}_2)$ in the first equation of (3.5) with a constant $h'(0)$, and $g'(\tilde{\phi}_1)$ in the second equation of (3.5) with a constant $g'(0)$, and then consider the following linear delayed reaction-diffusion system

$$\begin{gathered} V_{1t}^+ + \rho_1(c)V_{1\xi}^+ - d_1 V_{1\xi\xi}^+ + \rho_2(c)V_1^+ = h'(0)e^{-\lambda c\tau}V_2^+(\xi - c\tau, t - \tau), \\ V_{2t}^+ + \rho_3(c)V_{2\xi}^+ - d_2 V_{2\xi\xi}^+ + \rho_4(c)V_2^+ = g'(0)e^{-\lambda c\tau}V_1^+(\xi - c\tau, t - \tau), \\ V_i^+(\xi, s) = \sqrt{w(\xi)}U_{i0}(\xi + x_0, s) =: V_{i0}^+(\xi, s), \quad i = 1, 2, \end{gathered} \tag{3.8}$$

where $\xi \in \mathbb{R}$, $t \in [0, +\infty)$ and $s \in [-\tau, 0]$. Then we investigate the decay estimate of (V_1^+, V_2^+) by applying the Fourier transform to (3.8);

(ii) We prove that the solution (V_1, V_2) of (3.5) can be bounded by the solution (V_1^+, V_2^+) of (3.8).

Lemma 3.4 (Positiveness)**.** *When $(V_{10}^+(\xi, s), V_{20}^+(\xi, s)) \ge (0, 0)$ for $(\xi, s) \in \mathbb{R} \times [-\tau, 0]$, then $(V_1^+(\xi, t), V_2^+(\xi, t)) \ge (0, 0)$ for $(\xi, t) \in \mathbb{R} \times [0, +\infty)$.*

Proof. When $t \in [0, \tau]$, we have $t - \tau \in [-\tau, 0]$ and

$$h'(0)e^{-\lambda c\tau}V_2^+(\xi - c\tau, t - \tau) = h'(0)e^{-\lambda c\tau}V_{20}^+(\xi - c\tau, t - \tau)dy \ge 0. \tag{3.9}$$

Applying (3.9) to the first equation of (3.8), we obtain

$$\begin{gathered} V_{1t}^{+}+\rho_1(c)V_{1\xi}^{+}-d_1V_{1\xi\xi}^{+}+\rho_2(c)V_1^{+}\geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],\\ V_{10}^{+}(\xi,s)\geq 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{gathered}$$

By the comparison principle, we have

$$V_1^{+}(\xi,t)\geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{3.10}$$

Similarly, we obtain

$$\begin{gathered} V_{2t}^{+}+\rho_3(c)V_{2\xi}^{+}-d_2V_{2\xi\xi}^{+}+\rho_4(c)V_2^{+}\geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],\\ V_{20}^{+}(\xi,s)\geq 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{gathered}$$

Using the comparison principle again, we obtain

$$V_2^{+}(\xi,t)\geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{3.11}$$

When $t\in[n\tau,(n+1)\tau]$, $n=1,2,\dots$, repeating the above procedure step by step, we can similarly prove

$$(V_1^{+}(\xi,t),V_2^{+}(\xi,t))\geq(0,0), \quad (\xi,t)\in\mathbb{R}\times[n\tau,(n+1)\tau]. \tag{3.12}$$

Combining (3.10), (3.11) and (3.12), we obtain $(V_1^{+}(\xi,t),V_2^{+}(\xi,t))\geq(0,0)$ for $(\xi,t)\in\mathbb{R}\times[0,+\infty)$. The proof is complete. □

Now we establish the following crucial boundedness estimate for (V_1,V_2).

Lemma 3.5. *Let $(V_1(\xi,t),V_2(\xi,t))$ and $(V_1^{+}(\xi,t),V_2^{+}(\xi,t))$ be the solutions of (3.5) and (3.8), respectively. If*

$$|V_{i0}(\xi,s)|\leq V_{i0}^{+}(\xi,s) \quad \text{for } (\xi,s)\in\mathbb{R}\times[-\tau,0],\ i=1,2, \tag{3.13}$$

then

$$|V_i(\xi,t)|\leq V_i^{+}(\xi,t) \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,+\infty),\ i=1,2.$$

Proof. First of all, we prove $|V_i(\xi,t)|\leq V_i^{+}(\xi,t)$ for $t\in[0,\tau]$, $i=1,2$. In fact, when $t\in[0,\tau]$, namely, $t-\tau\in[-\tau,0]$, it follows from (3.13) that

$$\begin{aligned} |V_i(\xi-c\tau,t-\tau)| &= |V_{i0}(\xi-c\tau,t-\tau)|\\ &\leq V_{i0}^{+}(\xi-c\tau,t-\tau)\\ &= V_i^{+}(\xi-c\tau,t-\tau) \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau]. \end{aligned} \tag{3.14}$$

Then by $|h'(\tilde\phi_2)|<h'(0)$ and $|g'(\tilde\phi_1)|<g'(0)$ and (3.14), we obtain

$$\begin{aligned} &h'(0)e^{-\lambda c\tau}V_2^{+}(\xi-c\tau,t-\tau)\pm h'(\tilde\phi_2)e^{-\lambda c\tau}V_2(\xi-c\tau,t-\tau)\\ &\geq h'(0)e^{-\lambda c\tau}V_2^{+}(\xi-c\tau,t-\tau)-|h'(\tilde\phi_2)|e^{-\lambda c\tau}|V_2(\xi-c\tau,t-\tau)|\\ &\geq 0 \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &g'(0)e^{-\lambda c\tau}V_1^{+}(\xi-c\tau,t-\tau)\pm g'(\tilde\phi_1)e^{-\lambda c\tau}V_1(\xi-c\tau,t-\tau)\\ &\geq 0 \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau]. \end{aligned}$$

Let

$$v_i^{-}(\xi,t):=V_i^{+}(\xi,t)-V_i(\xi,t), \quad v_i^{+}(\xi,t):=V_i^{+}(\xi,t)+V_i(\xi,t), \quad i=1,2.$$

We are going to estimate $v_i^{\pm}(\xi,t)$. From (3.5), (3.6), (3.8) and (3.15), we see that $v_1^-(\xi,t)$ satisfies

$$\begin{gathered} v_{1t}^- + \rho_1(c)v_{1\xi}^- - d_1 v_{1\xi\xi}^- + \rho_2(c)v_1^- \geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau], \\ v_{10}^-(\xi,s) = V_{10}^+(\xi,s) - V_{10}(\xi,s) \geq 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{gathered}$$

By the comparison principle, we obtain

$$v_1^-(\xi,t)\geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

namely,

$$V_1(\xi,t)\leq V_1^+(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{3.16}$$

Similarly, one has

$$\begin{gathered} v_{2t}^- + \rho_3(c)v_{2\xi}^- - d_2 v_{2\xi\xi}^- + \rho_4(c)v_2^- \geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau], \\ v_{20}^-(\xi,s) = V_{20}^+(\xi,s) - V_{20}(\xi,s) \geq 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{gathered}$$

Applying the comparison principle again, we have

$$v_2^-(\xi,t)\geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

i.e.,

$$V_2(\xi,t)\leq V_2^+(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{3.17}$$

On the other hand, $v_1^+(\xi,t)$ satisfies

$$\begin{gathered} v_{1t}^+ + \rho_1(c)v_{1\xi}^+ - d_1 v_{1\xi\xi}^+ + \rho_2(c)v_1^+ \geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau], \\ v_{10}^+(\xi,s) = V_{10}^+(\xi,s) + V_{10}(\xi,s) \geq 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{gathered}$$

Then the comparison principle implies that

$$v_1^+(\xi,t) = V_1^+(\xi,t) + V_1(\xi,t) \geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau];$$

that is,

$$-V_1^+(\xi,t)\leq V_1(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{3.18}$$

Similarly, $v_2^+(\xi,t)$ satisfies

$$\begin{gathered} v_{2t}^+ + \rho_3(c)v_{2\xi}^+ - d_2 v_{2\xi\xi}^+ + \rho_4(c)v_2^+ \geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau], \\ v_{20}^+(\xi,s) = V_{20}^+(\xi,s) + V_{20}(\xi,s) \geq 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{gathered}$$

Therefore, we can prove that

$$v_2^+(\xi,t) = V_2^+(\xi,t) + V_2(\xi,t) \geq 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

namely

$$-V_2^+(\xi,t)\leq V_2(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{3.19}$$

Combining (3.16) and (3.18), we obtain

$$|V_1(\xi,t)|\leq V_1^+(\xi,t) \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau], \tag{3.20}$$

and combining (3.17) and (3.19), we prove

$$|V_2(\xi,t)|\leq V_2^+(\xi,t) \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau], \tag{3.21}$$

Next, when $t\in[\tau,2\tau]$, namely, $t-\tau\in[0,\tau]$, based on (3.20) and (3.21) we can similarly prove

$$|V_i(\xi,t)|\leq V_i^+(\xi,t) \quad \text{for } (\xi,t)\in\mathbb{R}\times[\tau,2\tau], \quad i=1,2.$$

Repeating this procedure, we then further prove

$$|V_i(\xi,t)|\leq V_i^+(\xi,t),\ (\xi,t)\in\mathbb{R}\times[n\tau,(n+1)\tau], \quad n=1,2,\dots,$$

which implies

$$|V_i(\xi,t)| \le V_i^+(\xi,t) \quad \text{for } (\xi,t) \in \mathbb{R}\times[0,\infty),\ i=1,2.$$

The proof is complete. □

In the following, we derive the stability of traveling waves for the linear system (3.8) by using the weighted method and by carrying out the crucial boundedness estimate on the fundamental solutions. Now let us recall the properties of the solutions to the delayed ODE system.

Lemma 3.6 ([12, Lemma 3.1]). *Let $z(t)$ be the solution to the scalar differential equation with delay*

$$\begin{gathered} \frac{d}{dt}z(t) = Az(t) + Bz(t-\tau), \quad t\ge 0,\ \tau>0, \\ z(s) = z_0(s), \quad s\in[-\tau,0]. \end{gathered} \tag{3.22}$$

where $A, B \in \mathbb{C}^{N\times N}$, $N\ge 2$, and $z_0(s)\in C^1([-\tau,0],\mathbb{C}^N)$. Then

$$z(t) = e^{A(t+\tau)}e_\tau^{B_1 t}z_0(-\tau) + \int_{-\tau}^0 e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}[z_0'(s) - Az_0(s)]ds,$$

where $B_1 = Be^{-A\tau}$ and $e_\tau^{B_1 t}$ is the so-called delayed exponential function in the form

$$e_\tau^{B_1 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ I, & -\tau \le t < 0, \\ I + B_1\frac{t}{1!}, & 0\le t<\tau, \\ I + B_1\frac{t}{1!} + B_1^2\frac{(t-\tau)^2}{2!}, & \tau \le t < 2\tau, \\ \dots & \dots \\ I + B_1\frac{t}{1!} + B_1^2\frac{(t-\tau)^2}{2!} + \dots + B_1^m\frac{[t-(m-1)\tau]^m}{m!}, & (m-1)\tau \le t < m\tau, \\ \dots & \dots \end{cases}$$

where $0, I\in\mathbb{C}^{N\times N}$, and 0 is zero matrix and I is the identity matrix.

Lemma 3.7 ([12, Theome 3.1]). *Suppose $\mu(A) := \frac{\mu_1(A)+\mu_\infty(A)}{2} < 0$, where $\mu_1(A)$ and $\mu_\infty(A)$ denote the matrix measure of A induced by the matrix 1-norm $\|\cdot\|_1$ and ∞-norm $\|\cdot\|_\infty$, respectively. If $\nu(B) := \frac{\|B\|+\|B\|_\infty}{2} \le -\mu(A)$, then there exists a decreasing function $\varepsilon_\tau = \varepsilon(\tau)\in(0,1)$ for $\tau>0$ such that any solution of system (3.22) satisfies*

$$\|z(t)\| \le C_0 e^{-\varepsilon_\tau\sigma t}, \quad t>0,$$

where C_0 is a positive constant depending on initial data $z_0(s)$, $s\in[-\tau,0]$ and $\sigma = |\mu(A)| - \nu(B)$. In particular,

$$\|e^{At}e_\tau^{B_1 t}\| \le C_0 e^{-\varepsilon_\tau\sigma t}, \quad t>0,$$

where $e_\tau^{B_1 t}$ is defined in Lemma 3.6.

It can be seen from the proof of [12, Theome 3.1] that

$$\mu_1(A) = \lim_{\theta\to 0^+}\frac{\|I+\theta A\|-1}{\theta} = \max_{1\le j\le N}\Big[\operatorname{Re}(a_{jj}) + \sum_{j\ne i}^N |a_{ij}|\Big],$$

$$\mu_\infty(A) = \lim_{\theta\to 0^+} \frac{\|I+\theta A\|_\infty - 1}{\theta} = \max_{1\le i\le N}\Big[\mathrm{Re}(a_{ii}) + \sum_{i\neq j}^N |a_{ij}|\Big].$$

Next, we shall estimate the decay rate for the solution $V^+(\xi,t)$.

Lemma 3.8. *Let the initial data $V_{i0}^+(\xi,s)$, $i=1,2$, be such that*

$$V_{i0}^+ \in C([-\tau,0];W^{2,1}(\mathbb{R})), \quad \partial_s V_{i0}^+ \in L^1([-\tau,0];L^1(\mathbb{R})), \quad i=1,2.$$

Then

$$\|V_i^+(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2}e^{-\mu_2 t} \quad \text{for } c\ge \max\{c_*, c^\natural\},\ i=1,2,$$

where $\mu>0$ and $C>0$.

Proof. Taking Fourier transform in (3.8) and denoting the transform of $V^+(\xi,t)$ by $\hat V^+(\eta,t)$, we obtain

$$\begin{gathered}
\hat V_{1t}^+(\eta,t) = -(d_1|\eta|^2+\rho_2(c)+\mathbf{i}\rho_1(c)\eta)\hat V_1^+(\eta,t) + h'(0)e^{-c\tau(\lambda+\mathbf{i}\eta)}\hat V_2^+(\eta,t-\tau),\\
\hat V_{2t}^+(\eta,t) = -(d_2|\eta|^2+\rho_4(c)+\mathbf{i}\rho_3(c)\eta)\hat V_2^+(\eta,t) + g'(0)e^{-c\tau(\lambda+\mathbf{i}\eta)}\hat V_1^+(\eta,t-\tau),\\
\hat V_i^+(\eta,s) = \hat V_{i0}^+(\eta,s), \quad \eta\in\mathbb{R},\ s\in[-\tau,0],\ i=1,2.
\end{gathered} \tag{3.23}$$

Let

$$A(\eta) = \begin{pmatrix} -(d_1|\eta|^2+\rho_2(c)+\mathbf{i}\rho_1(c)\eta) & 0\\ 0 & -(d_2|\eta|^2+\rho_4(c)+\mathbf{i}\rho_3(c)\eta)\end{pmatrix},$$

$$B(\eta) = \begin{pmatrix} 0 & h'(0)e^{-c\tau(\lambda+\mathbf{i}\eta)}\\ g'(0)e^{-c\tau(\lambda+\mathbf{i}\eta)} & 0\end{pmatrix}.$$

Then system (3.23) can be rewritten as

$$\hat V_t^+(\eta,t) = A(\eta)\hat V^+(\eta,t) + B(\eta)\hat V^+(\eta,t-\tau), \tag{3.24}$$

where $\hat V^+(\eta,t) = (\hat V_1^+(\eta,t), \hat V_2^+(\eta,t))^T$.

By Lemma 3.6, the linear delayed system (3.24) has solution

$$\begin{aligned}
\hat V^+(\eta,t) =& e^{A(\eta)(t+\tau)}e_\tau^{B_1(\eta)t}\hat V_0^+(\eta,-\tau)\\
&+\int_{-\tau}^0 e^{A(\eta)(t-s)}e_\tau^{B_1(\eta)(t-s-\tau)}\big[\partial_s\hat V_0^+(\eta,s) - A(\eta)\hat V_0^+(\eta,s)\big]ds\\
:=&\mathcal{I}_1(\eta,t) + \int_{-\tau}^0 \mathcal{I}_2(\eta,t-s)ds,
\end{aligned} \tag{3.25}$$

where $B_1(\eta) = B(\eta)e^{A(\eta)\tau}$. Let $V^+(\xi,t) := (V_1^+(\xi,t), V_2^+(\xi,t))^T$. Then by taking the inverse Fourier transform in (3.25), one has

$$\begin{aligned}
V^+(\xi,t) &= \mathcal{F}^{-1}[\mathcal{I}_1](\xi,t) + \int_{-\tau}^0 \mathcal{F}^{-1}[\mathcal{I}_2](\xi,t-s)ds\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty e^{\mathbf{i}\xi\eta}e^{A(\eta)(t+\tau)}e_\tau^{B_1(\eta)t}\hat V_0^+(\eta,-\tau)d\eta\\
&\quad+\frac{1}{2\pi}\int_{-\tau}^0\int_{-\infty}^\infty e^{\mathbf{i}\xi\eta}e^{A(\eta)(t-s)}e_\tau^{B_1(\eta)(t-s-\tau)}\\
&\quad\times\big[\partial_s\hat V_0^+(\eta,s) - A(\eta)\hat V_0^+(\eta,s)\big]\,d\eta\,ds.
\end{aligned} \tag{3.26}$$

From the definition of $\mu(\cdot)$ and $\nu(\cdot)$, we have

$$\begin{aligned}\mu(A(\eta)) =&\frac{\mu_1(A(\eta))+\mu_\infty(A(\eta))}{2}\\ =&\max\left\{-d_1\eta^2-\rho_2(c),-d_2\eta^2-\rho_4(c)\right\}\\ =&-d_2\eta^2-c\lambda+d_2\lambda^2-\beta,\end{aligned}$$

since $d_1>d_2$, $\alpha>\beta$ and $\lambda^2<\frac{\alpha-\beta}{d_1-d_2}$, and

$$\nu(B(\eta))=\max\{h'(0),g'(0)\}e^{-\lambda c\tau}.$$

By considering $\lambda\in(\lambda_1^\natural(c),\lambda_2^\natural(c))$, we obtain $\mu(A(\eta))<0$ and

$$\mu(A(\eta))+\nu(B(\eta))=-d_2\eta^2-c\lambda+d_2\lambda^2-\beta+\max\{h'(0),g'(0)\}e^{-\lambda c\tau}<0.$$

Furthermore, we obtain

$$\begin{aligned}|\mu(A(\eta))|-\nu(B(\eta)) =&d_2\eta^2+c\lambda-d_2\lambda^2+\beta-\max\{h'(0),g'(0)\}e^{-\lambda c\tau}\\ =&-\Delta_2(\lambda,c)+d_2\eta^2,\end{aligned}$$

where $\Delta_2(\lambda,c)=d\lambda^2-c\lambda-\beta+\max\{h'(0),g'(0)\}e^{-\lambda c\tau}<0$ for $c\ge\max\{c_*,c^\natural\}$. It then follows from Lemma 3.7 that there exists a decreasing function $\varepsilon_\tau=\varepsilon(\tau)\in(0,1)$ such that

$$\|e^{A(\eta)(t+\tau)}e^{B_1(\eta)t}\|\le C_1e^{-\varepsilon_\tau(|\mu(A(\eta))|-\nu(B(\eta)))t}\le C_1e^{-\varepsilon_\tau\mu_0t}e^{-\varepsilon_\tau d\eta^2t},$$

where C_1 is a positive constant and $\mu_0:=-\Delta_2(\lambda,c)>0$ with $c>c^\natural$. By the definition of Fourier transform, we have

$$\sup_{\eta\in\mathbb{R}}\|\hat V_0^+(\eta,-\tau)\|\le\int_{\mathbb{R}}\|V_0^+(\xi,-\tau)\|d\xi=\sum_{i=1}^2\|V_{i0}^+(\cdot,-\tau)\|_{L^1(\mathbb{R})}.$$

Therefore,

$$\begin{aligned}\sup_{\xi\in\mathbb{R}}\|\mathcal{F}^{-1}[\mathcal{I}_1](\xi,t)\| =&\sup_{\xi\in\mathbb{R}}\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathbf{i}\xi\eta}e^{A(\eta)(t+\tau)}e^{B_1(\eta)t}\hat V_0^+(\eta,-\tau)d\eta\Big\|\\ \le&C\int_{-\infty}^{\infty}e^{-\varepsilon_\tau d\eta^2t}e^{-\varepsilon_\tau\mu_0t}\|\hat V_0^+(\eta,-\tau)\|d\eta\\ \le&Ce^{-\varepsilon_\tau\mu_0t}\sup_{\eta\in\mathbb{R}}\|\hat V_0^+(\eta,-\tau)\|\int_{-\infty}^{\infty}e^{-\varepsilon_\tau d\eta^2t}d\eta\\ \le&Ce^{-\mu_2t}t^{-1/2}\sum_{i=1}^2\|V_{i0}^+(\cdot,-\tau)\|_{L^1(\mathbb{R})},\end{aligned}\tag{3.27}$$

with $\mu_2:=\varepsilon_\tau\mu_0$.

By using the property of Fourier transform, we obtain

$$\begin{aligned}\sup_{\eta\in\mathbb{R}}|d_i\eta^2\hat V_i^+(\eta,t)|&=\sup_{\eta\in\mathbb{R}}\left|d_i\mathcal{F}[V_{i\xi\xi}^+](\eta,t)\right|\\ &=d_i\|\partial_{\xi\xi}V_i^+(\cdot,t)\|_{L^1(\mathbb{R})}\\ &\le d_i\|V_i^+(\cdot,t)\|_{W^{2,1}(\mathbb{R})}\end{aligned}$$

and

$$\sup_{\eta\in\mathbb{R}}|(\mathbf{i}\eta)\hat V_i^+(\eta,t)|=\sup_{\eta\in\mathbb{R}}|\mathcal{F}[\partial_\xi V_i^+](\eta,t)|$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}} |\partial_\xi V_i^+(\xi,t)| d\xi \\
&= \|\partial_\xi V_i^+(\cdot,t)\|_{L^1(\mathbb{R})},
\end{aligned}
$$

for $i = 1, 2$. Thus,

$$
\sup_{\eta\in\mathbb{R}} \|A(\eta)\hat{V}_0^+(\eta,s)\| \le C\sum_{i=1}^{2} \|V_{i0}^+(\cdot,s)\|_{W^{2,1}(\mathbb{R})}.
$$

Similarly, we can derive that

$$
\begin{aligned}
&\sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[\mathcal{I}_2](\xi,t-s)\| \\
&= \sup_{\xi\in\mathbb{R}} \Big\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathbf{i}\xi\eta} e^{A(\eta)(t-s)} e^{B_1(\eta)(t-s-\tau)} [\partial_s \hat{V}_0^+(\eta,s) - A(\eta)\hat{V}_0^+(\eta,s)] d\eta \Big\| \\
&\le C \int_{-\infty}^{\infty} e^{-\varepsilon_\tau d\eta^2(t-s)} e^{-\varepsilon_\tau\mu_0(t-s)} \big\|\partial_s \hat{V}_0^+(\eta,s) - A(\eta)\hat{V}_0^+(\eta,s)\big\| d\eta \\
&\le Ce^{-\varepsilon_\tau\mu_0 t} e^{\varepsilon_\tau\mu_0 s} \sup_{\eta\in\mathbb{R}} \|\partial_s \hat{V}_0^+(\eta,s) - A(\eta)\hat{V}_0^+(\eta,s)\| \int_{-\infty}^{\infty} e^{-\varepsilon_\tau d\eta^2(t-s)} d\eta \\
&\le Ce^{-\varepsilon_\tau\mu_0 t}(t-s)^{-1/2}\mathcal{E}(s),
\end{aligned}
\tag{3.28}
$$

where

$$
\mathcal{E}(s) = \|\partial_s V_0^+(\cdot,s)\|_{L^1(\mathbb{R})} + \|V_0^+(\cdot,s)\|_{W^{2,1}(\mathbb{R})}.
$$

Furthermore, one has

$$
\begin{aligned}
&\int_{-\tau}^{0} (t-s)^{-1/2}\mathcal{E}(s) ds \\
&\le (1+t)^{-1/2} \int_{-\tau}^{0} \frac{(1+t)^{1/2}}{(t-s)^{1/2}} \mathcal{E}(s) ds \\
&\le Ct^{-1/2}\big(\|\partial_s V_0^+(s)\|_{L^1([-\tau,0];L^1(\mathbb{R}))} + \|V_0^+(s)\|_{L^1([-\tau,0];W^{2,1}(\mathbb{R}))}\big).
\end{aligned}
\tag{3.29}
$$

Substituting (3.27), (3.28) and (3.29) in (3.26), we obtain the the decay rate

$$
\sum_{i=1}^{2} \|V_i^+(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2}e^{-\mu_2 t}.
$$

This proof is complete. $\square$

Let us choose that $V_{i0}^+(\xi,s)$ such that

$$
\begin{gathered}
V_{i0}^+ \in C([-\tau,0]; W^{2,1}(\mathbb{R})), \quad \partial_s V_{i0}^+ \in L^1([-\tau,0]; L^1(\mathbb{R})), \\
V_{i0}^+(\xi,s) \ge |V_{i0}(\xi,s)|, \quad (\xi,s) \in \mathbb{R}\times[-\tau,0],\ i = 1,2.
\end{gathered}
$$

Combining Lemmas 3.5 and 3.8, we obtain the convergence rates for $V(\xi,t)$.

Lemma 3.9. *If $V_{i0} \in C([-\tau,0]; W^{2,1}(\mathbb{R}))$ and $\partial_s V_{i0} \in L^1([-\tau,0]; L^1(\mathbb{R}))$, then*

$$
\|V_i(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2}e^{-\mu_2 t},
$$

for some $\mu_2 > 0$, $i = 1, 2$.

Since $V_i(\xi,t)=\sqrt{w(\xi)}U_i(\xi+x_0,t)=e^{-\lambda\xi}U_i(\xi+x_0,t)$ and $\sqrt{w(\xi)}=e^{-\lambda\xi}\geq 1$ for $\xi\in(-\infty,0]$, it follows that

$$\sup_{\xi\in(-\infty,0]}|U_i(\xi+x_0,t)|\leq\|V_i(t)\|_{L^\infty(\mathbb{R})}\leq Ct^{-1/2}e^{-\mu_2 t}.$$

Thus, we obtain the following estimate for the unshifted $U(\xi,t)$.

Lemma 3.10. *It holds that*

$$\sup_{\xi\in(-\infty,x_0]}|U_i(\xi,t)|\leq Ct^{-1/2}e^{-\mu_2 t},\ i=1,2,$$

for some $\mu_2>0$.

Proof of Proposition 3.2. By Lemmas 3.3 and 3.10, we immediately obtain (3.4) for $0<\mu<\min\{\mu_1,\mu_2\}$. □

Acknowledgements. G.-B. Zhang was supported by the NSF of Gansu Province (18JR3RA093), and by the NSF of China (11861056). The authors are grateful to the anonymous referees for their careful reading of this paper and for their valuable suggestions.

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