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CROSSING LIMIT CYCLES FOR A CLASS OF PIECEWISE LINEAR DIFFERENTIAL CENTERS SEPARATED BY A CONIC

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Abstract. In previous years the study of the version of Hilbert's 16th problem for piecewise linear differential systems in the plane has increased. There are many papers studying the maximum number of crossing limit cycles when the differential system is defined in two zones separated by a straight line. In particular in [11, 13] it was proved that piecewise linear differential centers separated by a straight line have no crossing limit cycles. However in [14, 15] it was shown that the maximum number of crossing limit cycles of piecewise linear differential centers can change depending of the shape of the discontinuity curve. In this work we study the maximum number of crossing limit cycles of piecewise linear differential centers separated by a conic.

1. Introduction and statement of the main results

The study of discontinuous piecewise linear differential systems in the plane started with Andronov, Vitt and Khaikin in [1]. After that these systems have been a topic of great interest in the mathematical community because of their applications in various areas. They are used for modeling real phenomena and different modern devices, see for instance the books [4, 24] and references therein.

In the qualitative theory of differential systems in the plane a *limit cycle* is a periodic orbit which is isolated in the set of all periodic orbits of the system. This concept was defined by Poincaré [20, 21]. In several papers as [3, 10, 25] it was shown that the limits cycles model many phenomena of the real world. After these works the non-existence, existence, the maximum number and other properties of the limit cycles have been extensively studied by mathematicians and physicists, and more recently, by biologists, economist and engineers, see for instance [4, 17, 18, 19, 26].

As for the general case of planar differential systems one of the main problems for the particular case of the piecewise linear differential centers is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit. In this paper we study the *crossing limit cycles* which are periodic orbits isolated in the set of all periodic orbits of the piecewise linear differential centers, which only have isolated points of intersection with the discontinuity curve.

To establish an upper bound for the number of crossing limit cycles for the family of piecewise linear differential systems in the plane separated by a straight line has been the subject of many recent papers, see for instance [2, 5, 7, 23]. In 1990 Lum

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and Chua [16] conjectured that the continuous piecewise linear systems in the plane separated by one straight line have at most one limit cycle, in 1998 this conjecture was proved by Freire et al [6]. Afterwards in 2010 Han and Zhang [8] conjectured that discontinuous piecewise linear differential systems in the plane separated by a straight line have at most two crossing limit cycles but in 2012 Huan and Yang [9] gave a negative answer to this conjecture through a numerical example with three crossing limit cycles, later on Llibre and Ponce in [12] proved the existence of these three limit cycles analytically, but it is still an open problem to know if 3 is the maximum number of crossing limit cycles that this class of systems can have.

In [11] the problem by Lum and Chua was extended to the class of discontinuous piecewise linear differential systems in the plane separated by a straight line. In particular it was proved that the class of planar discontinuous piecewise linear differential centers has no crossing limit cycles. However, recently in [14, 15] were studied planar discontinuous piecewise linear differential centers where the curve of discontinuity is not a straight line. It was shown that the number of crossing limit cycles in these systems is non-zero. For this reason it is interesting to study the role which plays the shape of the discontinuity curve in the number of crossing limit cycles that planar discontinuous piecewise linear differential centers can have.

In this paper we provide an upper bound for the maximum number of crossing limit cycles of the planar discontinuous piecewise linear differential centers separated by a conic Σ.

Using an affine change of coordinates, any conic can be written in one of following nine canonical forms:

- (p) $x^2 + y^2 = 0$ two complex straight lines intersecting at a real point;
- (CL) $x^2 + 1 = 0$ two complex parallel straight lines;
- (CE) $x^2 + y^2 + 1 = 0$ complex ellipse;
- (DL) $x^2 = 0$ one double real straight line;
- (PL) $x^2 - 1 = 0$ two real parallel straight lines;
- (LV) $xy = 0$ two real straight lines intersecting at a real point;
- (E) $x^2 + y^2 - 1 = 0$ ellipse;
- (H) $x^2 - y^2 - 1 = 0$, hyperbola;
- (P) $y - x^2 = 0$ parabola.

We do not consider conics of type (p), (CL) or (CE) because they do not separate the plane in connected regions.

We observe that we have two options for crossing limit cycles of discontinuous piecewise linear differential centers separated by a conic Σ. First we have the crossing limit cycles such that intersect the discontinuity curve in exactly two points and second we have the crossing limit cycles such that intersect the discontinuity curve Σ in four points; we study these two cases in the following sections.

1.1. **Crossing limit cycles intersecting the discontinuity curve Σ in two points.** The maximum number of crossing limit cycles of piecewise linear differential centers separated by a conic Σ such that intersect Σ in exactly two points is given in the following theorems.

Theorem 1.1. *Consider a planar discontinuous piecewise linear differential centers where Σ is a conic. If Σ is of the type*

(a) (LV), (PL) *or* (DL)*, then there are no crossing limit cycles.*

(b) (E), *then the maximum number of crossing limit cycles intersecting* Σ *in two points is two.*
(a) (P), *then the maximum number of crossing limit cycles intersecting* Σ *in two points is three.*

The above Theorem is proved in section 2. In the cases studied up to now, there is no a result determining the maximum number of crossing limit cycles for discontinuous piecewise linear differential centers when Σ is a hyperbola (H). We determine it in the following Theorem

Theorem 1.2. *Consider a family of planar discontinuous piecewise linear differential centers,* $\mathcal{F}_0$*, where* Σ *is a hyperbola* (H)*. Then the following statement hold:*
(a) *There are systems in* $\mathcal{F}_0$ *without crossing limit cycles.*
(b) *There are systems in* $\mathcal{F}_0$ *having exactly one crossing limit cycle that intersects* Σ *in two points, see Figure 2.*
(c) *There are systems in* $\mathcal{F}_0$ *having exactly two crossing limit cycles that intersect* Σ *in two points, see Figure 3.*
(d) *For this family of systems* $\mathcal{F}_0$ *we have that the maximum number of crossing limit cycles that intersect* Σ *in two points is two.*

The above Theorem is proved in section 3.

1.2. **Crossing limit cycles intersecting the discontinuity curve** Σ **in four points.** Here we do not consider the case where the discontinuity curve is the conic (DL), because first in [11, 13] it was proved that discontinuous piecewise linear differential systems separated by a straight line have no crossing limit cycles and second because the crossing limit cycles of these discontinuous piecewise linear centers cannot have four points on the discontinuity curve.

In the following theorems we analyze the maximum number of crossing limit cycles for planar discontinuous piecewise linear differential centers with four points on discontinuity curve, where the plane is divided by the curve of discontinuity Σ of the type (PL), (LV),(P),(E) or (H).

Theorem 1.3. *Let* $\mathcal{F}_1$ *be the family of planar discontinuous piecewise linear differential systems formed by three linear centers and with* Σ *of type* (PL)*. Then for this family the maximum number of crossing limit cycles that intersect* Σ *in four points is one. Moreover there are systems in this class having one crossing limit cycle.*

Theorem 1.3 for a particular linear center between the two parallel straight lines was done in [13], in section 4 we prove it for any linear center.

If the discontinuity curve Σ is of the type (LV), then we have the following 4 regions in the plane:

$$
\begin{aligned}
R_1 &= [(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0],\\
R_2 &= [(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } y > 0],\\
R_3 &= [(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } y < 0],\\
R_4 &= [(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y < 0].
\end{aligned}
$$

Moreover, $\Sigma = \Gamma_1^+ \cup \Gamma_1^- \cup \Gamma_2^+ \cup \Gamma_2^-$, where

$$
\Gamma_1^+ = [(x,y) \in \mathbb{R}^2 : x = 0, y \geq 0], \quad \Gamma_1^- = [(x,y) \in \mathbb{R}^2 : x = 0, y \leq 0],
$$

$$\Gamma_2^+ = [(x,y) \in \mathbb{R}^2 : y = 0, x \geq 0], \quad \Gamma_2^- = [(x,y) \in \mathbb{R}^2 : y = 0, x \leq 0].$$

In this case we have two types of crossing limit cycles, namely crossing limit cycles of type 1 which intersect only two branches of Σ in exactly two points in each branch, and crossing limit cycles of type 2 which intersect in a unique point each branch of the set Σ.

Theorem 1.4. *Let $\mathcal{F}_2$ be the family of planar discontinuous piecewise linear differential systems formed by four linear centers and with Σ of the type* (LV)*. The maximum number of crossing limit cycles type* 1 *is one. Moreover there are systems in this class having one crossing limit cycle.*

The above theorem is proved in Section 5.

Theorem 1.5. *Consider a family of planar discontinuous piecewise linear differential centers $\mathcal{F}_2$. Then the following statement hold.*

(a) *There are systems in $\mathcal{F}_2$ with exactly one crossing limit cycle of type* 2*, see Figure 7.*
(b) *There are systems in $\mathcal{F}_2$ with exactly two crossing limit cycles of type* 2*, see Figure 8.*
(c) *There are systems in $\mathcal{F}_2$ with exactly three crossing limit cycles of type* 2*, see Figure 9*

The above theorem is proved in Section 6. By the calculations made for this case and the illustrated examples in Theorem 1.5 we get the following conjecture

Conjecture 1.6. *For the family of systems $\mathcal{F}_2$, the maximum number of crossing limit cycles of type* 2 *is three.*

Theorem 1.7. *Let $\mathcal{F}_3$ be a family of planar discontinuous piecewise linear differential systems formed by two linear centers and with Σ of type* (P)*. Then for this family the maximum number of crossing limit cycles that intersect Σ in four points is one. Moreover there are systems in this class having one crossing limit cycle.*

The above theorem is proved in Section 7.

Theorem 1.8. *Let $\mathcal{F}_4$ be a family of planar discontinuous piecewise linear differential systems formed by two linear centers and with Σ of type* (E)*. Then for this family the maximum number of crossing limit cycles that intersect Σ in four points is one. Moreover there are systems in this class having one crossing limit cycle.*

The above Theorem is proved in Section 8.

Theorem 1.9. *Let $\mathcal{F}_5$ be a family of planar discontinuous piecewise linear differential systems formed by three linear centers and with Σ of type* (H)*. Then for this family the maximum number of crossing limit cycles that intersect Σ in four points is one. Moreover there are systems in this class having one crossing limit cycle.*

The above theorem is proved in Section 9.

1.3. **Crossing limit cycles with four and with two points on the discontinuity curve Σ simultaneously.** Here we study the maximum number of crossing limit cycles of planar discontinuous piecewise linear differential centers that intersect the discontinuity curve Σ in two and in four points simultaneously.

We do not consider planar discontinuous piecewise linear differential centers with discontinuity curve a conic of type (DL), (PL) and (LV) because as in the proof of

Theorem 1.1 they do not have crossing limit cycles that intersect the discontinuity curve in two points. Then we study the maximum number of crossing limit cycles with two and with four points in Σ simultaneously by the families $\mathcal{F}_3, \mathcal{F}_4$ and $\mathcal{F}_5$.

Theorem 1.10. *The following statements hold.*

(a) *The planar discontinuous piecewise linear differential centers that belong to the family $\mathcal{F}_3$, can have simultaneous one crossing limit cycle that intersects* (P) *in two points and one crossing limit cycle that intersects* (P) *in four points.*
(b) *The planar discontinuous piecewise linear differential centers that belong to the family $\mathcal{F}_4$, can have simultaneous one crossing limit cycle that intersects* (E) *in two points and one crossing limit cycle that intersects* (E) *in four points.*
(c) *The planar discontinuous piecewise linear differential centers that belong to the family $\mathcal{F}_5$, can have simultaneous one crossing limit cycle that intersects* (H) *in two points and one crossing limit cycle that intersects* (H) *in four points.*

The above Theorem is proved in Section 10. In Subsection 1.3 we do not consider the planar discontinuous piecewise linear differential centers in the family $\mathcal{F}_2$, because they do not have crossing limit cycles that intersect the discontinuity curve (LV) in two points. However in this family there are two types of crossing limit cycles like it was defined in Subsection 1.2.

1.4. **Crossing limit cycles of types 1 and 2 simultaneously for planar discontinuous piecewise linear differential centers in $\mathcal{F}_2$.** In this case we study the maximum number of crossing limit cycles of types 1 and 2 that planar discontinuous piecewise linear differential centers in the family $\mathcal{F}_2$ can have simultaneously.

Theorem 1.11. *There are planar discontinuous piecewise linear differential centers that belong to the family $\mathcal{F}_2$ such that have one crossing limit cycle of type* 1 *and three crossing limit cycles of type* 2 *simultaneously.*

The above theorem is proved in Section 11. By the illustrated examples in Theorem 1.11 we get the following conjecture

Conjecture 1.12. *The planar discontinuous piecewise linear differential centers that belong to the family $\mathcal{F}_2$ can have at most one crossing limit cycle of type* 1 *and three crossing limit cycles of type* 2 *simultaneously.*

2. Proof of Theorem 1.1

Analyzing the case of discontinuous piecewise linear differential centers with discontinuity curve a conic of the type (LV), (PL) or (DL) the maximum number of crossing limit cycles is equal to the maximum number of crossing limit cycles in discontinuous piecewise linear differential centers in the plane separated by a single straight line which was studied in [11, 13]. In these papers it was proved that the discontinuous piecewise linear differential centers separated by one straight line have no crossing limit cycles. This proves the statement (a) of Theorem 1.1.

In [15] the authors considered discontinuous piecewise linear differential centers separated by the parabola $y = x^2$ and proved that they have at most three crossing limit cycles that intersect Σ in two points, i.e. statement (b) of Theorem 1.1.

With regard to the discontinuous piecewise linear differential systems separated by an ellipse, in the paper [14] the authors shown that the class of planar discontinuous piecewise linear differential centers separated by the circle $\mathbb{S}^1$ has at most two crossing limit cycles. Moreover, there are discontinuous piecewise linear differential centers which reach the upper bound of 2 crossing limit cycles, see Example 2.1. Then we have the statement (c) of Theorem 1.1.

Example 2.1. We consider the discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by the ellipse (E) and both linear differential centers are defined as follows:

$$\dot{x} = -2x - 2y - \sqrt{2} - 1, \quad \dot{y} = 4x + 2y + \sqrt{2},$$

in the unbounded region limited by the ellipse (E), and in the bounded region with boundary the ellipse (E) we have the linear differential center

$$\dot{x} = -x + \frac{5}{4}y - \frac{1}{\sqrt{2}} - \frac{1}{8}, \quad \dot{y} = -x + y + \frac{1}{\sqrt{2}}.$$

This discontinuous piecewise differential system has exactly two crossing limit cycles, see Figure 1.

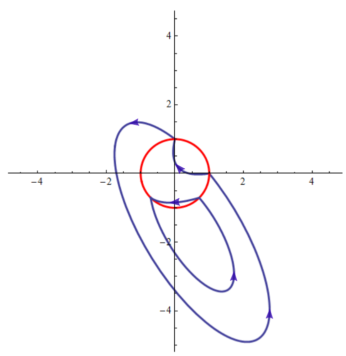


FIGURE 1. The two limit cycles of the discontinuous piecewise linear differential of Example 2.1.

3. Proof of Theorem 1.2

For the systems of the class $\mathcal{F}_0$ we have following regions in the plane:

$$R_1 = [(x, y) \in \mathbb{R}^2 : x^2 - y^2 > 1],$$

which is a region that consist of two connected components, and the region

$$R_2 = [(x, y) \in \mathbb{R}^2 : x^2 - y^2 < 1].$$

To have a crossing limit cycle, which intersects the hyperbola $x^2 - y^2 = 1$ in two different points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, these points must satisfy the *closing equations*

$$\begin{aligned} H_1(x_1, y_1) &= H_1(x_2, y_2), \\ H_2(x_2, y_2) &= H_2(x_1, y_1), \\ x_1^2 - y_1^2 &= 1, \\ x_2^2 - y_2^2 &= 1. \end{aligned} \tag{3.1}$$

Proof of statement (a) of Theorem 1.2. We consider a discontinuous piecewise linear differential system which has the linear center

$$\dot{x} = -y, \quad \dot{y} = x, \tag{3.2}$$

in the region R_2, the orbits of this center intersect the hyperbola in two or in four points, when it intersects the hyperbola in exactly two points these are $(\pm 1, 0)$, which are points of tangency between the hyperbola and the solution curves of the center (3.2), then it is impossible that there are crossing periodic orbits independent of the linear differential center that can be considered in the region R_1. So the orbits which can produce a crossing limit cycle intersect the hyperbola in four points and clearly these orbits cannot be crossing limit cycles with exactly two points on the discontinuity curve (H). □

Proof of statement (b) of Theorem 1.2. In the region R_1 we consider the linear differential center

$$\dot{x} = 27 - \sqrt{5} - 25y, \quad \dot{y} = -2 + x, \tag{3.3}$$

this system has the first integral $H_1(x, y) = 4(-4 + x)x + 4y(-54 + 2\sqrt{5} + 25y)$. In the region R_2 we have the linear differential center

$$\dot{x} = 2 - \frac{3\sqrt{5}}{4} - \frac{y}{4}, \quad \dot{y} = -\frac{3}{2} + x, \tag{3.4}$$

which has the first integral $H_2(x, y) = 4(-3 + x)x + y(-16 + 6\sqrt{5} + y)$.

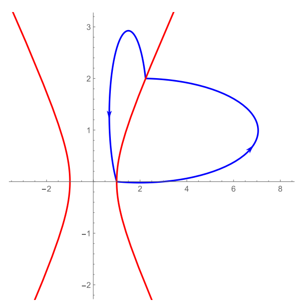


FIGURE 2. The crossing limit cycle of the discontinuous piecewise linear differential system formed by the centers (3.3) and (3.4).

This discontinuous piecewise differential system formed by the linear differential centers (3.3) and (3.4) has one crossing limit cycle, because the unique real solution (p, q) with $p \neq q$ of the *closing equations* given in (3.1), is $p = (1, 0)$ and $q = (\sqrt{5}, 2)$. See the crossing limit cycle of this system in Figure 2. □

Proof of statement (c) of Theorem 1.2. In the region R_1 we consider the linear differential center

$$\begin{aligned} \dot{x} &= \frac{289 - 48\sqrt{2} + 289\sqrt{3} - 305\sqrt{6}}{768} + \frac{x}{8\sqrt{3}} - \frac{49}{192}y, \\ \dot{y} &= \frac{\left(32\sqrt{3} - 289\sqrt{2}\right)\left(1 + \sqrt{2 + \sqrt{3}}\right)}{768} + x - \frac{y}{8\sqrt{3}}, \end{aligned} \tag{3.5}$$

which has the first integral

$$\begin{aligned} H_1(x,y) = \frac{1}{96}\Big(384x^2 + x\Big(\big(32\sqrt{3} - 289\sqrt{2}\big)\big(1+\sqrt{2+\sqrt{3}}\big) - 32\sqrt{3}y\Big) \\ + y\Big(98y - \sqrt{3(85057 - 9248\sqrt{6})} + 305\sqrt{6} - 289\Big)\Big). \end{aligned}$$

In the region R_2 we have the linear differential center

$$\dot{x} = \frac{1}{8}\big(-3 + 8\sqrt{2} + \sqrt{3} - \sqrt{6}\big) - \frac{x}{2} - \frac{y}{2}, \quad \dot{y} = \frac{1}{8}\big(-1 - 5\sqrt{2} - \sqrt{3}\big) + x + \frac{y}{2}, \tag{3.6}$$

this system has the first integral

$$H_2(x,y) = 4x^2 - x\big(1 + 5\sqrt{2} + \sqrt{3} - 4y\big) + y\big(3 - 8\sqrt{2} - \sqrt{3} + \sqrt{6} + 2y\big).$$

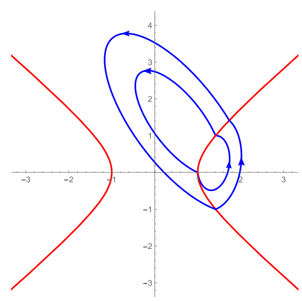


Figure 3. The two limit cycles of the discontinuous piecewise linear differential system formed by the centers (3.5) and (3.6).

This discontinuous piecewise differential system formed by the linear differential centers (3.5) and (3.6) has two crossing limit cycles, because the unique real solutions (p,q) of system (3.1) are $(1,0,\sqrt{2},1)$ and $(\sqrt{2},-1,\sqrt{3},\sqrt{2})$, therefore the intersection points of the two crossing limit cycles with the hyperbola are the pairs $(1,0)$, $(\sqrt{2},1)$ and $(\sqrt{2},-1)$, $(\sqrt{3},\sqrt{2})$. See these two crossing limit cycles in Figure 3. $\square$

We will use the following lemma which provides a normal form for an arbitrary linear differential center, for a proof see [13].

Lemma 3.1. *Through a linear change of variables and a rescaling of the independent variable every center in $\mathbb{R}^2$ can be written*

$$\dot{x} = -bx - \frac{4b^2 + \omega^2}{4a}y + d, \quad \dot{y} = ax + by + c, \tag{3.7}$$

with $a \neq 0$ and $\omega > 0$. This system has the first integral

$$H_1(x,y) = 4(ax+by)^2 + 8a(cx - dy) + y^2\omega^2. \tag{3.8}$$

Proof of statement (d) of Theorem 1.2. In the region R_1 we consider the arbitrary linear differential center (3.7) which has first integral (3.8). In the region R_2 we consider the arbitrary linear differential center

$$\dot{x} = -Bx - \frac{4B^2 + \Omega^2}{4A}y + D, \quad \dot{y} = Ax + By + C, \tag{3.9}$$

with $A \neq 0$ and $\Omega > 0$. Which has the first integral

$$H_2(x,y) = 4(Ax+By)^2 + 8A(Cx-Dy) + y^2\Omega^2.$$

It is possible to do a rescaling of time in the two above systems. Suppose $\tau = at$ in R_1 and $s = At$ in R_2. These two rescaling change the velocity in which the orbits of systems (3.7) and (3.9) travel, nevertheless they do not change the orbits, therefore they will not change the crossing limit cycles that the discontinuous piecewise linear differential system can have. After these rescalings of the time we can assume without loss of generality that $a = A = 1$, and the dot in system (3.7) (resp. (3.9)) denotes derivative with respect to the new time τ (resp. s).

We assume that the discontinuous piecewise linear differential system formed by the two linear differential centers (3.7) and (3.9) has three crossing periodic solutions. For this we must impose that the system of equations (3.1) has three pairs of points as solution, namely (p_i, q_i), $i = 1,2,3$, since these solutions provide crossing periodic solutions. We consider

$$p_i = (\cosh r_i, \sinh r_i), \quad q_i = (\cosh s_i, \sinh s_i), \quad \text{for } i = 1,2,3. \tag{3.10}$$

These points are the points where the three crossing periodic solutions intersect the hyperbola (H). Now we consider that the point (p_1, q_1) satisfies system (3.1) and with this condition we obtain the following expression

$$\begin{aligned} d = \frac{1}{8(\sinh r_1 - \sinh s_1)}\Big(&4\cosh^2 r_1 - 4\cosh^2 s_1 + 8\cosh r_1(c + b\sinh r_1) \\ &- 8\cosh s_1(c + b\sinh s_1) + (4b^2+\omega^2)(\sinh^2 r_1 - \sinh^2 s_1)\Big), \end{aligned}$$

and D has the same expression that d changing (b,c,ω) by (B,C,Ω).

We assume that the point (p_2, q_2) satisfies system (3.1) and we get the expression

$$\begin{aligned} c = &\frac{-1}{8(\sinh(r_1-r_2) + \sinh(r_2-s_1) - \sinh(r_1-s_2) + \sinh(s_1-s_2))} \\ &\times \Big((\sinh r_2 - \sinh s_2)\big(4\cosh^2 s_1 + 4b\sinh(2s_1) - 4\cosh^2 r_1 - 4b\sinh(2r_1)\big) \\ &+ (\sinh r_1 - \sinh s_1)\Big(4\cosh^2 r_2 - 4\cosh^2 s_2 + 8b\cosh r_2\sinh r_2 \\ &- 8b\cosh s_2\sinh s_2 + (4b^2+\omega^2)(\sinh r_2 - \sinh s_2)(-\sinh r_1 + \sinh r_2 \\ &- \sinh s_1 + \sinh s_2)\Big)\Big), \end{aligned}$$

and C has the same expression that c changing (b,ω) by (B,Ω).

Finally we impose that the point (p_3, q_3) satisfies system (3.1) and we get an expression for ω^2. In this case $\omega^2 = K/L$, where the expression for K is

$$\begin{aligned} &4\Big((1+b^2)\operatorname{csch}\big(\frac{r_1-r_2+s_1-s_2}{2}\big)\sinh\big(\frac{r_3-s_3}{2}\big) \\ &\times \Big(\cosh\big(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\big) - \cosh\big(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\big) \\ &+ \cosh\big(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\big) - \cosh\big(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\big) \\ &- \cosh\big(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\big) + \cosh\big(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\big) \\ &- \cosh\big(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\big) + \cosh\big(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&+ b^2\Big(\cosh\big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\big) - \cosh\big(\frac{r_1 - r_2 + 3r_3 + s_1 - s_2 + s_3}{2}\big)\Big) \\
&- 2b\Big(\sinh\big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\big) - \sinh\big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\big) \\
&- \sinh\big(\frac{2r_1 - r_2 + 2r_3 + s_1 - s_2 + s_3}{2}\big) \sinh\big(\frac{r_1 - r_3}{2}\big) \\
&+ 2\cosh\big(\frac{2r_1 + 2r_2 - r_3 + s_1 + s_2 - s_3}{2}\big) \sinh\big(\frac{r_1 - r_2}{2}\big) \\
&- 2\cosh\big(\frac{2r_1 - r_2 + 2r_3 + s_1 - s_2 + s_3}{2}\big) \sinh\big(\frac{r_1 - r_3}{2}\big) \\
&+ 2\sinh\big(\frac{r_2 - r_3}{2}\big) \cosh\big(\frac{r_1 - 2r_2 - 2r_3 + s_1 - s_2 - s_3}{2}\big) \\
&+ 2\cosh\big(\frac{r_1 + r_2 - r_3 + 2(s_1 + s_2) - s_3}{2}\big) \sinh\big(\frac{s_1 - s_2}{2}\big) \\
&- 2\cosh\big(\frac{r_1 - r_2 + r_3 + 2s_1 - s_2}{2} + s_3\big) \sinh\big(\frac{s_1 - s_3}{2}\big)\Big) \\
&+ 2(1 + b^2) \sinh\big(\frac{r_1 - r_2 + r_3 + 2s_1 - s_2}{2} + s_3\big) \sinh\big(\frac{s_1 - s_3}{2}\big)\Big)
\end{aligned}
$$

and the expression for L is

$$
\begin{aligned}
&\operatorname{csch}\big(\frac{r_1 - r_2 + s_1 - s_2}{2}\big) \sinh\big(\frac{r_3 - s_3}{2}\big)\Big(- \cosh\big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\big) \\
&+ \cosh\big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\big) - \cosh\big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\big) \\
&+ \cosh\big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\big) + \cosh\big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\big) \\
&- \cosh\big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\big) + \cosh\big(\frac{r_1 + r_2 - r_3 + 3s_1 + s_2 - s_3}{2}\big) \\
&- \cosh\big(\frac{r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3}{2}\big) - \cosh\big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\big) \\
&+ \cosh\big(\frac{r_1 - r_2 + 3r_3 + s_1 - s_2 + s_3}{2}\big) - \sinh\big(\frac{r_1 - r_2 + r_3 + 2s_1 - s_2}{2} + s_3\big) \\
&2\sinh\big(\frac{s_1 - s_3}{2}\big)\Big),
\end{aligned}
$$

and the expression for Ω^2 is the same than the expression for ω^2 changing b to B.

Now we replace d, c, ω^2 in the expression of the first integral $H_1(x,y)$ and we have

$$
H_1(x,y) = 4(x^2 - y^2) + h(x,y,r_1,r_2,r_3,s_1,s_2,s_3)b, \tag{3.11}
$$

and analogously we have

$$
H_2(x,y) = 4(x^2 - y^2) + h(x,y,r_1,r_2,r_3,s_1,s_2,s_3)B. \tag{3.12}
$$

Now we analyze if the discontinuous piecewise linear differential system formed by (3.7) and (3.9) has more crossing periodic solutions than the three supposed in (3.10). Taking into account (3.11) and (3.12) the *closing equations* (3.1) becomes

$$
\begin{gathered}
h(x_1,y_1,r_1,r_2,r_3,s_1,s_2,s_3) = h(x_2,y_2,r_1,r_2,r_3,s_1,s_2,s_3), \\
x_1^2 - y_1^2 = 1, \\
x_2^2 - y_2^2 = 1.
\end{gathered} \tag{3.13}
$$

This means, we must solve a system with three equations and four unknowns x_1, y_1, x_2, y_2, which we know that have at least the three solutions (3.10), so system (3.13) has a continuum of solutions which produce a continuum of crossing periodic solutions, so such systems cannot have crossing limit cycles. Since in statement (c), we have proved that there are systems in $\mathcal{F}_0$ with two crossing limit cycles, it follows that the maximum number of crossing limit cycles that intersect Σ in two points is two. This completes the proof of Theorem 1.2. $\square$

4. Proof of Theorem 1.3

When Σ is of the type (PL), we have following three regions in the plane:

$$\begin{aligned} R_1 &= [(x,y) \in \mathbb{R}^2 : x < -1], \\ R_2 &= [(x,y) \in \mathbb{R}^2 : -1 < x < 1], \\ R_3 &= [(x,y) \in \mathbb{R}^2 : x > 1]. \end{aligned}$$

We consider a planar discontinuous piecewise differential system separated by two parallel straight lines and formed by three arbitrary linear centers. By Lemma 3.1, we have that these linear centers can be as follows

$$\begin{gathered} \dot{x} = -bx - \frac{4b^2+\omega^2}{4a}y + d, \quad \dot{y} = ax + by + c, \quad \text{in } R_1, \\ \dot{x} = -Bx - \frac{4B^2+\Omega^2}{4A}y + D, \quad \dot{y} = Ax + By + C, \quad \text{in } R_2, \\ \dot{x} = -\beta x - \frac{4\beta^2+\lambda^2}{4\alpha}y + \delta, \quad \dot{y} = \alpha x + \beta y + \gamma, \quad \text{in } R_3. \end{gathered} \tag{4.1}$$

These linear centers have the first integrals

$$\begin{aligned} H_1(x,y) &= 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2, \\ H_2(x,y) &= 4(Ax+By)^2 + 8A(Cx-Dy) + y^2\Omega^2, \\ H_3(x,y) &= 4(\alpha x+\beta y)^2 + 8\alpha(\gamma x-\delta y) + y^2\lambda^2, \end{aligned}$$

respectively.

We are going to analyze if the discontinuous piecewise linear differential center (4.1) has crossing periodic solutions. Since the orbits in each region R_i, for $i = 1,2,3$, are ellipses or pieces of one ellipse, we have that if there is a crossing limit cycle this must intersect each straight line $x = \pm 1$ in exactly two points, namely $(1,y_1), (1,y_2)$ and $(-1,y_3), (-1,y_4)$, with $y_1 > y_2$ and $y_3 > y_4$. Therefore we must study the solutions of the system

$$\begin{aligned} H_3(1,y_2) &= H_3(1,y_1), \\ H_2(1,y_1) &= H_2(-1,y_3), \\ H_1(-1,y_3) &= H_1(-1,y_4), \\ H_2(-1,y_4) &= H_2(\ 1,y_2), \end{aligned}$$

or equivalently, we have the system

$$\begin{gathered}-(y_1-y_2)(8\beta-8\delta+(4\beta^2+\lambda^2)(y_1+y_2)=0,\\ 16C-8D(y_1-y_3)+8B(y_1+y_3)+(4B^2+\Omega^2)(y_1^2-y_3^2)=0,\\ (y_3-y_4)(-8b-8d+(4b^2+\omega^2)(y_3+y_4)=0,\\ -16C+8D(y_2-y_4)-8B(y_2+y_4)-(4B^2+\Omega^2)(y_2^2-y_4^2)=0.\end{gathered} \tag{4.2}$$

By hypothesis $y_1>y_2$ and $y_3>y_4$ and therefore system (4.2) is equivalently to the system

$$\begin{gathered}\gamma_3-\delta_3+l_3(y_1+y_2)=0,\\ \eta-\delta_2(y_1-y_3)+\gamma_2(y_1+y_3)+l_2(y_1^2-y_3^2)=0,\\ -\gamma_1-\delta_1+l_1(y_3+y_4)=0,\\ -\eta+\delta_2(y_2-y_4)-\gamma_2(y_2+y_4)-l_2(y_2^2-y_4^2)=0,\end{gathered} \tag{4.3}$$

where $\gamma_1=8b,\gamma_2=8B,\gamma_3=8\beta,\delta_1=8d,\delta_2=8D,\delta_3=8\delta,l_1=4b^2+\omega^2,l_2=4B^2+\Omega^2,l_3=4\beta^2+\lambda^2$ and $\eta=16C$. As $l_1\neq 0$ and $l_3\neq 0$, we can isolated y_1 and y_4 of the first and the third equations of system (4.3), respectively. Then, we obtain

$$y_1=\frac{-l_3y_2+\gamma_3-\delta_3}{l_3},\quad y_4=\frac{-l_1y_3+\gamma_1+\delta_1}{l_1}.$$

Now replacing these expressions for y_1 and y_4 in the second and fourth equations of (4.3), we have the system of two equations

$$\begin{aligned}E_1=\Big(&l_2(l_3(y_2-y_3)+\psi_3)(l_3(y_2+y_3)+\psi_3)\\ &+l_3(l_3(\eta+(y_3-y_2)\gamma_2+(y_2+y_3)\delta_2)-\psi_2\psi_3)\Big)/l_3^2,\\ E_2=\Big(&l_2\psi_1^2-l_1\psi_1(2l_2y_3+\gamma_2+\delta_2)\\ &-l_1^2(\eta+(y_2-y_3)(l_2(y_2+y_3)+\gamma_2)-(y_2+y_3)\delta_2)\Big)/l_1^2.\end{aligned}$$

Doing the Groebner basis of the two polynomials E_1 and E_2 with respect to the variables y_2 and y_3, we obtain the equations

$$m_0+m_1y_3+m_2y_3^2=0,\quad k_0+k_1y_3+k_2y_2=0, \tag{4.4}$$

with

$$\begin{aligned}m_0=\frac{1}{l_1^4l_3^2}\Big(&2l_1^3l_3^3\psi_2^2(l_3\psi_1(\gamma_2+\delta_2)+l_1(2l_3\eta-\psi_2\psi_3))\\ &-l_1^2l_2l_3^2\Big(l_3^2\psi_1^2(\gamma_2^2-6\gamma_2\delta_2+\delta_2^2)+4l_1l_3\psi_2(2l_1\eta+\psi_1(\gamma_2+\delta_2))\psi_3\\ &-5l_1^2\psi_2^2\psi_3^2\Big)+2l_1l_2^2l_3(-l_3^3\psi_1^3(\gamma_2+\delta_2)+2l_1l_3^2\psi_1^2\psi_2\psi_3-2l_1^3\psi_2\psi_3^3)\\ &+l_1^2l_3(2l_1\eta+\psi_1(\gamma_2+\delta_2))\psi_3^2+l_2^3(l_3\psi_1+l_1\psi_3)^2(l_3\psi_1-l_1\psi_3)^2\Big),\\ m_1=&\frac{4l_2\psi_1(-2l_1l_3\psi_1+l_1\psi_3))(2l_1l_3\delta_2-l_2(l_3\psi_1-l_1\psi_3))}{l_1^3},\\ m_2=&\frac{4l_2(l_2(l_3\psi_1+l_1\psi_3-2l_1l_3\gamma_2))(l_2(l_3\psi_1-l_1\psi_3-2l_1l_2\delta_2))}{l_1^2},\end{aligned}$$

$$k_0 = \frac{(l_1 l_3(l_3\psi_1(\gamma_2+\delta_2)\psi_1^2 + l_1^2\psi_3^2))}{l_1^2 l_3}, \quad k_1 = \frac{(2l_3(l_2\psi_1 - l_1(\gamma_2+\delta_2))}{l_1},$$
$$k_2 = 2(l_3\psi_2 - l_2\psi_3),$$

where $\psi_1 = \gamma_1 + \delta_1, \psi_2 = \gamma_2 - \delta_2$ and $\psi_3 = \gamma_3 - \delta_3$.

We recall that Bézout Theorem states that if a polynomial differential system of equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials which appear in the system, see [22]. Then by Bézout Theorem in this case, we have that system (4.4) has at most two solutions. Moreover, from these two solutions (y_2^1, y_3^1) and (y_2^2, y_3^2) of (4.4), we will have two solutions of (4.3) which are of the form $(y_1^1, y_2^1, y_3^1, y_4^1)$ and $(y_1^2, y_2^2, y_3^2, y_4^2)$, but analyzing system (4.3) we have that if $(y_1^1, y_2^1, y_3^1, y_4^1)$ is a solution, then $(y_2^1, y_1^1, y_4^1, y_3^1)$ is another solution. Finally due to the fact that $y_1 > y_2$ and $y_3 > y_4$, at most one of these two solutions will be satisfactory. Therefore we have proved that the planar discontinuous piecewise differential systems of the family $\mathcal{F}_1$, can have at most one crossing limit cycle.

Now we verify that this upper bound is reached, for this we present a discontinuous piecewise linear differential system that belongs to the family $\mathcal{F}_1$ and has exactly one crossing limit cycle.

We consider the discontinuous piecewise linear differential center

$$\begin{gathered}
\dot{x} = -\frac{3}{16} - \frac{x}{2} - \frac{5}{16}y, \quad \dot{y} = \frac{1}{16} + x + \frac{y}{2}, \quad \text{in } R_1, \\
\dot{x} = -\frac{67}{500} - \frac{x}{5} - \frac{29}{100}y, \quad \dot{y} = -\frac{43}{1000} + x + \frac{y}{5}, \quad \text{in } R_2, \\
\dot{x} = \frac{7}{60} - \frac{x}{3} - \frac{13}{36}y, \quad \dot{y} = \frac{1}{7} + x + \frac{y}{3}, \quad \text{in } R_3.
\end{gathered} \tag{4.5}$$

These systems have the first integrals

$$\begin{aligned}
H_1(x,y) &= 16x^2 + y(6+5y) + 2x(1+8y), \\
H_2(x,y) &= 4\big(x + \frac{y}{5}\big)^2 + y^2 + \frac{1}{125}(-43x + 134y), \\
H_3(x,y) &= \frac{8}{7}x - \frac{14}{15}y + y^2 + \frac{4}{9}(3x+y)^2,
\end{aligned}$$

respectively.

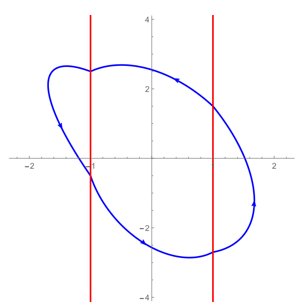


FIGURE 4. The crossing limit cycle of the discontinuous piecewise linear differential center (4.5) with three centers separated by the conic (PL).

Then the discontinuous piecewise differential system formed by the linear differential centers formed by the linear differential centers (4.5) has one crossing limit cycle that intersects (PL) in four points, because the unique real solution (y_1, y_2, y_3, y_4) with $y_1 > y_2$ and $y_3 > y_4$ of system (4.2) is the point $(y_1, y_2, y_3, y_4) = (3/2, -27/10, 5/2, -1/2)$. See the crossing limit cycle of this system in Figure 4. This completes the proof of Theorem 1.3.

5. Proof of Theorem 1.4

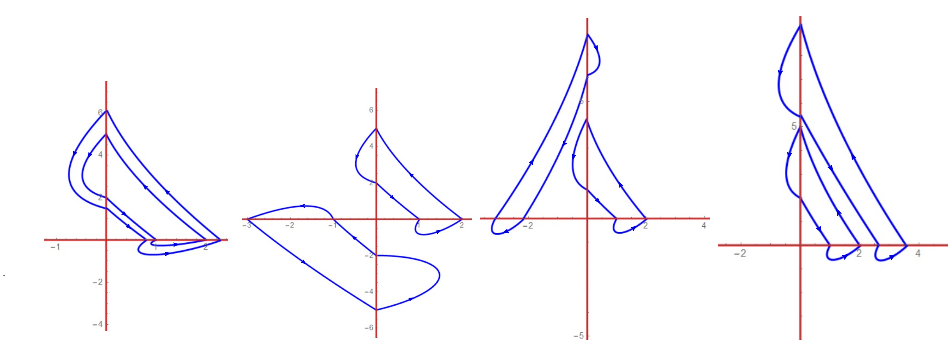
We consider a planar discontinuous piecewise linear differential system with four zones separated by (LV) and formed by four arbitrary linear centers. By Lemma 3.1 this piecewise linear differential system can be as follows

$$\begin{aligned}
\dot{x} &= -b_1 x - \frac{4b_1^2 + \omega_1^2}{4a_1} y + d_1, \quad \dot{y} = a_1 x + b_1 y + c_1, \quad \text{in } R_1,\\
\dot{x} &= -b_2 x - \frac{4b_2^2 + \omega_2^2}{4a_2} y + d_2, \quad \dot{y} = a_2 x + b_2 y + c_2, \quad \text{in } R_2,\\
\dot{x} &= -b_3 x - \frac{4b_3^2 + \omega_3^2}{4a_3} y + d_3, \quad \dot{y} = a_3 x + b_3 y + c_3, \quad \text{in } R_3,\\
\dot{x} &= -b_4 x - \frac{4b_4^2 + \omega_4^2}{4a_4} y + d_4, \quad \dot{y} = a_4 x + b_4 y + c_4, \quad \text{in } R_4,
\end{aligned} \tag{5.1}$$

with $a_i \neq 0$ and $\omega_i > 0$ for $i = 1, 2, 3, 4$. The regions R_i for $i = 1, 2, 3, 4$ are defined just before the statement of Theorem 1.4. These linear differential centers have the first integrals H_1, H_2, H_3 and H_4 respectively, where

$$H_i(x, y) = 4(a_i x + b_i y)^2 + 8a_i(c_i x - d_i y) + y^2\omega_i^2, quad \text{for } i = 1, 2, 3, 4. \tag{5.2}$$

If the discontinuous piecewise linear center (5.1) has two crossing limit cycles of type 1, these two crossing limit cycles should be some of Figure 5.



(a) (b) (c) (d)

Figure 5. Possible cases of two crossing limit cycles of type 1 of discontinuous piecewise linear center (5.1).

We observe that the cases of Figure 5 (b), (c), and (d) are not possible because in these cases the pieces of the ellipses of linear differential centers in the regions R_4, R_1 and R_2, respectively would not be nested which contradicts that the linear differential systems in each of these regions are linear centers. Therefore if the discontinuous piecewise linear center (5.1) has two crossing limit cycles of type 1 these could be as in Figure 5 (a).

Now we study the conditions in order that the piecewise linear differential center (5.1) has crossing limit cycles of type 1 and we will show that the maximum number of crossing limits cycles of type 1 is one. Without loss of generality we assume that the crossing limit cycles intersect the branches Γ_1^+ and Γ_2^+ in the points $(0, y_1), (0, y_2)$ and $(x_1, 0), (x_2, 0)$, respectively, where $0 < y_1 < y_2$ and $0 < x_1 < x_2$. Then taking into account the first integrals (5.2) for each linear center, these points must satisfy the following equations

$$\begin{aligned} H_1(x_2, 0) &= H_1(0, y_2), \\ H_2(0, y_2) &= H_2(0, y_1), \\ H_1(0, y_1) &= H_1(x_1, 0), \\ H_4(x_1, 0) &= H_4(x_2, 0), \end{aligned}$$

equivalently we have

$$\begin{aligned} &4a_1^2x_2^2 + 8a_1(c_1x_2 + d_1y_2) - y_2^2l_1 = 0, \\ &-(y_1 - y_2)(-8a_2d_2 + (y_1 + y_2)l_2) = 0, \\ &-4a_1^2x_1^2 - 8a_1(c_1x_1 + d_1y_1) + y_1^2l_1 = 0, \\ &4a_4(x_1 - x_2)(2c_4 + a_4(x_1 + x_2)) = 0, \end{aligned} \tag{5.3}$$

where $l_1 = 4b_1^2 + \omega_1^2, l_2 = 4b_2^2 + \omega_2^2$ and $\eta = (a_4c_1 - a_1c_4)$.

Moreover, by hypothesis $x_1 < x_2$ and $y_1 < y_2$, then from the second and the fourth equations of (5.3), we have

$$y_1 = \frac{8a_2d_2 - l_2y_2}{l_2}, \quad x_2 = -\frac{2c_4 + a_4x_1}{a_4}.$$

Substituting these expressions of y_1 and x_2 in the first and third equations of (5.3) we obtain the two equations

$$\begin{aligned} E_1 &= \frac{4a_1^2(2c_4 + a_4x_1)^2 - 8a_1a_4(2c_1c_4 + a_4c_1x_1 - a_4d_1y_2) - a_4^2y_2^2l_1}{a_4^2}, \\ E_2 &= 4a_1^2x_1^2 - \frac{l_1(y_2l_2 - 8a_2d_2)^2}{l_2^2} - 8a_1\Big(d_1y_2 - \frac{8a_2d_1d_2}{l_2} - c_1x_1\Big). \end{aligned}$$

Doing the Groebner basis of the two polynomials E_1 and E_2 with respect to the variables x_1 and y_2 we get the two equations

$$\alpha_0 + \alpha_1y_2 + \alpha_2y_2^2 = 0, \quad \beta_0 + \beta_1x_1 + \beta_2y_2 = 0, \tag{5.4}$$

where

$$\begin{aligned} \alpha_0 &= 4\Big(\frac{a_1c_4\eta^2(-2a_4c_1 + a_1c_4)}{a_4^2} + \frac{16a_2^3a_4^2d_2^3l_1(a_2d_2l_1 - 2a_1d_1l_1l_2)}{l_2^4} \\ &\quad + (8a_2d_2(a_1\eta^2d_1l_2 + a_2d_2(2a_1^2a_4^2d_1^2 - \eta^2l_1))\frac{1}{l_2^2}\Big), \\ \alpha_1 &= \frac{8a_2d_2}{l_2^3}\Big(- 32a_2^2a_4^2b_1^2d_2^2(2b_1^2 + \omega_1^2) - 4a_2^2a_4^2d_2^2\omega_1^4 + 8a_1a_2a_4^2d_1d_2l_1l_2 + a_4^2c_1^2l_2^2l_1^2 \\ &\quad - 2a_1a_4c_1c_4l_1l_2^2 + a_1^2(-4a_4^2d_1^2 + c_4^2l_1l_2\Big), \end{aligned}$$

$$
\begin{gathered}
\alpha_2 = 2a_1a_4c_1c_4l_1 - a_1^2c_4^2l_1 + a_4^2\Big(-c_1^2l_1 + 4a_1^2d_1^2 + \frac{4a_2^2d_2^2l_1^2}{l_2^2} - \frac{8a_1a_2d_1d_2l_1}{l_2}\Big),\\
\beta_0 = -\frac{a_1c_4\eta}{a_4} + \frac{4a_2^2a_4d_2^2l_1}{l_2^2} - \frac{4a_1a_2a_4d_1d_2}{l_2},\\
\beta_1 = -a_1\eta, \quad \beta_2 = a_1a_4d_1 - \frac{a_2a_4d_2l_1}{l_2}.
\end{gathered}
$$

The Bézout Theorem (see [22]) applied to system (5.4) says that this system has at most two isolated solutions. Therefore system (5.3) has two solutions which are of the form $(x_1^1, x_2^1, y_1^1, y_2^1)$ and $(x_1^2, x_2^2, y_1^2, y_2^2)$, but it is possible to prove that if (x_1, x_2, y_1, y_2) is a solution of system (5.3), then (x_2, x_1, y_2, y_1) is also a solution of this system. Since we must have that $x_1 < x_2$ and $y_1 < y_2$, then system (5.3) has a unique solution, and therefore the discontinuous piecewise linear differential center (5.1) that belongs to the family $\mathcal{F}_2$ can have at most one crossing limit cycle of type 1 intersecting Γ_1^+ and Γ_2^+.

Now we verify that this upper bound is reached. That is, that there are piecewise linear differential centers in the family $\mathcal{F}_2$ having one crossing limit cycle of type 1. We consider the following discontinuous piecewise linear differential center

$$
\begin{gathered}
\dot x = \frac{23177}{9000} - \frac{11}{10}x - \frac{557}{450}y, \quad \dot y = -\frac{1837}{1125} + x + \frac{11}{10}y, \quad \text{in } R_1,\\
\dot x = \frac{477}{64} - \frac{x}{2} - \frac{53}{16}y, \quad \dot y = 1 + x + \frac{y}{2}, \quad \text{in } R_2,\\
\dot x = -y - \beta, \quad \dot y = x + \alpha, \quad \text{in } R_3,\\
\dot x = 2 - \frac{x}{2} - \frac{17}{4}y, \quad \dot y = -2 + x + \frac{y}{2}, \quad \text{in } R_4.
\end{gathered} \tag{5.5}
$$

In the region R_3 we can consider any linear differential center, because the crossing limit cycle will be formed by parts of the orbits of the centers of the regions R_1, R_2 and R_4.

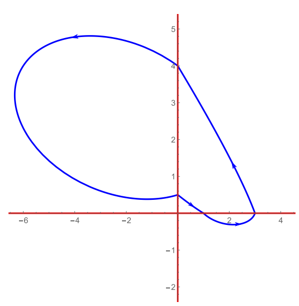


Figure 6. The crossing limit cycle of type 1 of discontinuous piecewise linear differential system (5.5) separated by the conic (LV).

The centers in (5.5) have the first integrals

$$
\begin{gathered}
H_1(x,y) = 4500x^2 + 44x(-334 + 225y) + y(-23177 + 5570y),\\
H_2(x,y) = 4x^2 + 4x(2+y) + \frac{53}{8}y(-9 + 2y),
\end{gathered}
$$

$$
\begin{aligned}
&H_3(x,y) = (x+\alpha)^2 + (y+\beta)^2,\\
&H_4(x,y) = 4(-4+x)x + 4(-4+x)y + 17y^2,
\end{aligned}
$$

in R_i, $i = 1,2,3,4$, respectively. Then for the discontinuous piecewise linear differential center (5.5) system (5.3) becomes

$$
\begin{gathered}
-14696x_2 + 4500x_2^2 + (23177 - 5570y_2)y_2 = 0,\\
(y_1 - y_2)(-9 + 2y_1 + 2y_2) = 0,\\
14696x_1 - 4500x_1^2 + y_1(-23177 + 5570y_1) = 0,\\
(x_1 - x_2)(-4 + x_1 + x_2) = 0.
\end{gathered}
\tag{5.6}
$$

Taking into account that the solutions (x_1, x_2, y_1, y_2) must satisfy $x_1 < x_2$ and $y_1 < y_2$, we have that the unique solution of system (5.6) is the point $(x_1, x_2, y_1, y_2) = (1, 3, 1/2, 4)$. See the crossing limit cycle of type 1 of discontinuous piecewise linear differential system (5.5) in Figure 6. This completes the proof of Theorem 1.4.

6. Proof of Theorem 1.5

Proof of statement (*a*) *of Theorem 1.5.* In the region R_1 we consider the linear differential center

$$
\dot{x} = -\frac{13}{4} - \frac{x}{2} - \frac{y}{2}, \quad \dot{y} = 1 + x + \frac{y}{2}, \tag{6.1}
$$

this system has the first integral $H_1(x,y) = 2(2x^2 + 2x(2+y) + y(13+y))$. In the region R_2 we have the linear differential center

$$
\dot{x} = -\frac{851}{3600} - \frac{x}{3} - \frac{181}{900}y, \quad \dot{y} = \frac{3}{2} + x + \frac{y}{3}, \tag{6.2}
$$

which has the first integral $H_2(x,y) = 4x^2 + 4x(9+2y)/3 + y(851 + 362y)/450$. In the region R_3 we have the linear differential center

$$
\dot{x} = -\frac{43}{32} + \frac{x}{4} - \frac{5}{16}y, \quad \dot{y} = -\frac{1}{2} + x - \frac{y}{4}, \tag{6.3}
$$

which has the first integral $H_3(x,y) = 4x^2 - 3x(2+y) + y(-43+5y)/4$. And in the region R_4 we have the linear differential center

$$
\dot{x} = \frac{137}{72} + \frac{x}{3} - \frac{25}{144}y, \quad \dot{y} = \frac{3}{2} + x - \frac{y}{3}, \tag{6.4}
$$

which has the first integral $H_4(x,y) = 4x(3+x) - (137+24x)y/9 + 25y^2/36$.

To have a crossing limit cycle of type 2, which intersects the discontinuity conic (LV) in four different points $p_1 = (x_1, 0)$, $q_1 = (0, y_1)$, $p_2 = (x_2, 0)$ and $q_2 = (0, y_2)$, with $x_1, y_1 > 0$ and $x_2, y_2 < 0$, these points must satisfy the *closing equations*

$$
\begin{aligned}
e_1 &= H_1(x_1,0) - H_1(0,y_1) = 0,\\
e_2 &= H_2(0,y_1) - H_2(x_2,0) = 0,\\
e_3 &= H_3(x_2,0) - H_3(0,y_2) = 0,\\
e_4 &= H_4(0,y_2) - H_4(x_1,0) = 0.
\end{aligned}
\tag{6.5}
$$

Considering the four above linear differential centers (6.1), (6.2), (6.3) and (6.4) and their respective first integrals $H_i(x,y)$, $i = 1,2,3,4$, we have the following equivalent system

$$
\begin{aligned}
4x_1(2+x_1) - 2y_1(13+y_1) &= 0,\\
-4x_2(3+x_2) + \frac{1}{450}y_1(851+361y_1) &= 0,\\
4(x_2-1)x_2 + \frac{1}{4}(43-5y_2)y_2 &= 0,\\
-4x_1(x_1+3) + \frac{1}{36}y_2(-548+25y_2) &= 0,
\end{aligned}
\tag{6.6}
$$

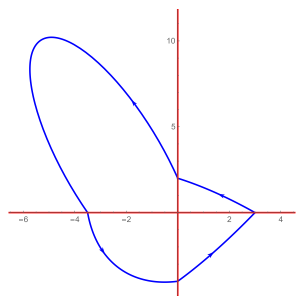


Figure 7. The crossing limit cycle of type 2 of the discontinuous piecewise linear differential system formed by the linear centers (6.1), (6.2), (6.3) and (6.4) separated by (LV).

The unique real solution (p_1, q_1, p_2, q_2) of (6.6) is $p_1 = (3, 0)$, $q_1 = (0, 2)$, $p_2 = (-7/2, 0)$ and $q_2 = (0, -4)$, therefore the piecewise differential system formed by the linear differential centers (6.1), (6.2), (6.3) and (6.4) has exactly one crossing limit cycle of type 2. See the crossing limit cycle of this system in Figure 7. □

Proof of statement (b) of Theorem 1.5. In the region R_1 we consider the linear differential center

$$\dot{x} = -\frac{25}{8} + \frac{x}{2} + \frac{y}{2}, \quad \dot{y} = \frac{11}{2} - x - \frac{y}{2}, \tag{6.7}$$

which has the first integral $H_1(x, y) = 4x^2 + 4x(-11 + y) + y(-25 + 2y)$. In the region R_2 we consider the linear differential center

$$\dot{x} = -\frac{251}{400} - x - \frac{109}{100}y, \quad \dot{y} = -\frac{293}{200} + x + y, \tag{6.8}$$

this system has the first integral $H_2(x, y) = 200x^2 + y\,(251 + 218y) + x(-586 + 400y)$. In the region R_3 we have the linear differential center

$$\dot{x} = \frac{5}{96} + \frac{x}{4} - \frac{5}{16}y, \quad \dot{y} = \frac{23}{24} + x - \frac{y}{4}, \tag{6.9}$$

this system has the first integral $H_3(x, y) = 4x^2 + x(23/3 - 2y) + 5y(-1 + 3y)/12$. In the region R_4 we have the linear differential center

$$\dot{x} = -\frac{73}{800} + \frac{x}{10} - \frac{29}{400}y, \quad \dot{y} = -\frac{31}{40} + x - \frac{y}{10}, \tag{6.10}$$

this system has the first integral $H_4(x, y) = 400x^2 - 20x(31 + 4y) + y(73 + 29y)$. This discontinuous piecewise linear differential center formed by the linear differential centers (6.7), (6.8), (6.9) and (6.10) has two crossing limit cycles of type 2, because the unique real solutions $(p_1^i, q_1^i, p_2^i, q_2^i)$, with $i = 1, 2$ of system (6.5) are $p_1^1 = (3/2, 0)$, $q_1^1 = (0, 3), p_2^1 = (-5/2, 0)$ and $q_2^1 = (0, -2)$ and $p_1^2 = (2, 0), q_1^2 = (0, 9/2), p_2^2 = (-4, 0)$ and $q_2^2 = (0, -5)$. See these two crossing limit cycles in Figure 8. □

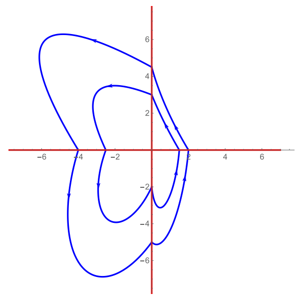


FIGURE 8. The two crossing limit cycle of type 2 of the discontinuous piecewise linear differential system formed by the linear centers (6.7), (6.8), (6.9) and (6.10) separated by (LV).

Proof of statement (c) of Theorem 1.5. In the region R_1 we consider the linear differential center

$$\dot{x} = \frac{813}{803} - \frac{x}{2} - \frac{300}{803}y, \quad \dot{y} = -\frac{1207}{730} + x + \frac{y}{2}, \tag{6.11}$$

which has the first integral $H_1(x, y) = 4x^2 + x(-4828/365 + 4y) + 24y(-271 + 50y)/803$. In the region R_2 we have the linear differential center

$$\dot{x} = \frac{210061}{55055} + \frac{11}{10}x - \frac{15760}{11011}y, \quad \dot{y} = \frac{63667}{20020} + x - \frac{11}{10}y, \tag{6.12}$$

this system has the first integral $H_2(x, y) = 110110x^2 + x(700337 - 242242y) + 4y(-210061 + 39400y)$. In the region R_3 we have the linear differential center

$$\dot{x} = -\frac{79831}{38904} - \frac{7}{10}x - \frac{3875}{4863}y, \quad \dot{y} = \frac{421379}{194520} + x + \frac{7}{10}y, \tag{6.13}$$

this system has the first integral $H_3(x, y) = 97260x^2 + 5y(79831 + 15500y) + 7x(60197 + 19452y)$. In the region R_4 we have the linear differential center

$$\dot{x} = -\frac{15513}{28057} + \frac{2}{5}x - \frac{5700}{28057}y, \quad \dot{y} = -\frac{330343}{280570} + x - \frac{2}{5}y, \tag{6.14}$$

this system has the first integral $H_4(x, y) = 140285x^2 + 30y(5171 + 950y) - x(330343 + 112228y)$. This discontinuous piecewise linear differential center formed by the linear differential centers (6.11), (6.12), (6.13) and (6.14) has three crossing limit cycles of type 2, because the unique real solutions $(p_1^i, q_1^i, p_2^i, q_2^i)$, with $i = 1, 2, 3$ of system (6.5) are $p_1^1 = (9/5, 0)$, $q_1^1 = (0, 3)$, $p_2^1 = (-7/2, 0)$ and $q_2^1 = (0, -43/10)$; $p_1^2 = (2, 0)$, $q_1^2 = (0, 33/10)$, $p_2^2 = (-39/10, 0)$ and $q_2^2 = (0, -47/10)$; and $p_1^3 = (17/10, 0)$, $q_1^3 = (0, 289/100)$, $p_2^3 = (-33/10, 0)$ and $q_2^3 = (0, -411/100)$. See these three crossing limit cycles of type 2 in Figure 9. □

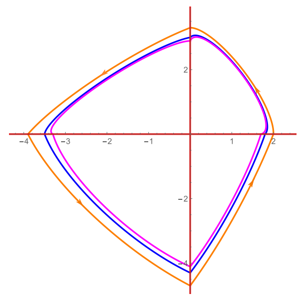


FIGURE 9. The three crossing limit cycle of type 2 of the discontinuous piecewise differential center formed by the centers (6.11), (6.12), (6.13) and (6.14) separated by (LV).

7. Proof of Theorem 1.7

If the discontinuity curve Σ is of the type (P), we have following two regions in the plane: $R_1 = [(x,y) \in \mathbb{R}^2 : x^2 < y]$, and $R_2 = [(x,y) \in \mathbb{R}^2 : x^2 > y]$.

We consider a planar discontinuous piecewise linear differential system formed by two linear arbitrary centers. By Lemma 3.1 these piecewise linear differential centers can be as follows

$$\begin{aligned} \dot{x} &= -bx - \frac{4b^2+\omega^2}{4a}y + d, \quad \dot{y} = ax + by + c, \quad \text{in } R_1, \\ \dot{x} &= -\beta x - \frac{4\beta^2+\omega^2}{4\alpha}y + \delta, \quad \dot{y} = \alpha x + \beta y + \gamma, \quad \text{in } R_2. \end{aligned} \tag{7.1}$$

These linear differential centers have the first integrals

$$\begin{aligned} H_1(x,y) &= 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2, \\ H_2(x,y) &= 4(\alpha x+\beta y)^2 + 8\alpha(\gamma x-\delta y) + y^2\Omega^2, \end{aligned} \tag{7.2}$$

respectively. After two rescalings of time as in the proof Theorem 1.2 we can assume without loss of generality that $a = \alpha = 1$.

In order that the piecewise linear differential centers (7.1) has crossing limit cycles with four point on (P). We must study the solutions of the system:

$$\begin{aligned} e_1 &: H_1(x_1,x_1^2) - H_1(x_2,x_2^2) = 0, \\ e_2 &: H_2(x_2,x_2^2) - H_2(x_3,x_3^2) = 0, \\ e_3 &: H_1(x_3,x_3^2) - H_1(x_4,x_4^2) = 0, \\ e_4 &: H_2(x_4,x_4^2) - H_2(x_1,x_1^2) = 0, \end{aligned} \tag{7.3}$$

or equivalently

$$
\begin{aligned}
&e_1: 4x_1^2(1+bx_1)^2-4x_2^2(1+bx_2)^2+\ 8x_1(c-dx_1)+8x_2(dx_2-c)\\
&\quad +(x_1^4-x_2^4)\omega^2=0,\\
&e_2: 4x_2^2(1+\beta x_2)^2+8x_2(\gamma-\delta x_2)^2-4x_3^2(1+\beta x_3)^2+8x_3(\delta x_3-\gamma)\\
&\quad +(x_2^4-x_3^4)\Omega=0,\\
&e_3: 4x_3^2(1+bx_3)^2-4x_4^2(1+bx_4)^2\ +8x_3(c-dx_3)+8x_4(dx_4-c)\\
&\quad +(x_3^4-x_4^4)\omega^2=0,\\
&e_4: 4x_4^2(1+\beta x_4)^2+8x_1(\delta x_1-\gamma)-4x_1^2(1+\beta x_1)^2+8x_4(\gamma-\delta x_4)\\
&\quad +(x_4^4-x_1^4)\Omega^2=0.
\end{aligned}
\tag{7.4}
$$

We assume that the discontinuous piecewise linear differential centers (7.1) has two crossing periodic solutions. For this we must have that system of equations (7.4) has two real solutions, namely (p_1,p_2,p_3,p_4) and (q_1,q_2,q_3,q_4), where $p_i=(k_i,k_i^2)$ and $q_i=(L_i,L_i^2)$, with $i=1,2,3,4$. These points are the points where the two crossing periodic solution intersect discontinuity curve (P).

If the point (p_1,p_2,p_3,p_4) satisfies system (7.4), by the equation e_1 of (7.4) we obtain the expression

$$
d=\frac{8c+4(k_1+k_2)(1+b(k_1+k_2))+4b(k_1^2+k_2^2)+(k_1+k_2)(k_1^2+k_2^2)l_1}{8(k_1+k_2)},
$$

by equation e_2 of (7.4) we get the expression

$$
\delta=\frac{8\gamma+4(k_2+k_3)(1+\beta(k_2+k_3))+4\beta(k_2^2+k_3^2)+(k_2+k_3)(k_2^2+k_3^2)l_2}{8(k_2+k_3)},
$$

by equation e_3 of (7.4) we obtain the expression

$$
\begin{aligned}
c=&\frac{k_1+k_2}{2(k_1+k_2-k_3-k_4)(k_3-k_4)}\Big((k_4^2-k_3^2)\Big(\frac{(k_1^2+k_2^2)l_1}{4}+1+(1+b(k_1+k_2))\\
&+b\frac{k_1^2+k_2^2}{k_1+k_2}\Big)+2b(k_4^3-k_3^3)+(k_4^4-k_3^4)l_1\Big)
\end{aligned}
$$

and by equation e_4 of (7.4) we obtain the expression

$$
\begin{aligned}
\gamma=&\frac{1}{8(k_1-k_2-k_3+k_4)}\Big(8\beta(k_1+k_4)k_4-l_2(k_2^2+k_3^2-k_4^2)(k_2+k_3)(k_1+k_4)\\
&-2(k_2^2+k_2k_3+k_3^2)+(k_2+k_3)(k_1^3l_2+k_1^2(8\beta+k_4l_2)\Big),
\end{aligned}
$$

here we consider $l_1=4b^2+\omega^2$ and $l_2=4\beta^2+\Omega^2$.

We assume that the point (q_1,q_2,q_3,q_4) satisfies system (7.4), then we can obtain the remaining parameters of discontinuous piecewise linear differential center (7.1).

By equation e_1 of (7.4) we obtain $\omega^2=S/T$, where

$$
\begin{aligned}
S=&\frac{-4b(L_1-L_2)}{k_1+k_2-k_3-k_4}\Big((bk_1+(bk_2+2))(k_3+k_4-L_1-L_2)k_1^2\\
&+k_1\Big(-bk_3^3+k_2(bk_2+2)(k_3+k_4-L_1-L_2)-(k_3^2+k_3k_4+k_4^2)(bk_4+2)\\
&+bL_2^3+(L_1^2+L_1L_2+L_2^2)(bL_1+2)\Big)+k_2(bk_2+2)(k_3+k_4-L_1-L_2)
\end{aligned}
$$

$$+ k_2\Big((bL_1+2)(L_1^2+L_1L_2+L_2^2)-(k_3^2+k_3k_4+k_4^2)(bk_4+2)-bk_3^3\Big)$$
$$+ bk_2L_2^3+(L_2+L_1)\Big(bk_3^3+(k_3+k_4)(k_4-L_1)(b(k_4+L_1)+2)+k_3^2$$
$$(bk_4+2)\Big)-L_2^2(k_3+k_4)((bL_1+2)-bL_2)\Big),$$

and

$$T = \frac{L_1-L_2}{k_1+k_2-k_3-k_4}\Big((k_1^3+k_1^2k_2+k_1k_2^2+k_2^3)(k_3+k_4-L_1-L_2)$$
$$+(k_1+k_2)\Big((L_1+L_2)(L_1^2+L_2^2)-(k_3+k_4)(k_3^2+k_4^2)\Big)$$
$$+(k_3+k_4)(L_1+L_2)\big(k_3^2+k_4^2-L_1^2-L_2^2\big)\Big),$$

by equation e_2 of (7.4) we obtain $\Omega^2 = V/W$, where

$$V = \frac{-4\beta(L_2-L_3)}{k_1-k_2-k_3+k_4}\Big(k_1(\beta k_1+(\beta k_4+2))(k_2+k_3-L_2-L_3)$$
$$-2k_1\Big(k_2^2+k_2(k_3-k_4)+(k_3-L_2)(k_3-k_4+L_2)+L_3(k_4-L_2)-L_3^2\Big)$$
$$+\beta k_1\Big(-k_2^3-k_2^2k_3+k_2(k_4^2-k_3^2)-k_3^3+k_3k_4^2+(L_2+L_3)(-k_4^2+L_2^2+L_3^2)\Big)$$
$$+\beta k_2^3(-k_4+L_2+L_3)-k_2^2(\beta k_3+2)(k_4-L_2-L_3)$$
$$-\beta k_2\Big(k_3^2(k_4-L_2-L_3)-k_4^3+(L_2+L_3)(L_2^2+L_3^2)\Big)$$
$$-2k_2\Big(L_2^2-k_4^2+k_3(k_4-L_2-L_3)+L_2L_3+L_3^2\Big)$$
$$-(k_3-k_4)\Big((\beta k_3^2+k_3(\beta k_4+2))(k_4-L_2-L_3)$$
$$+\beta(L_2+L_3)(-k_4^2+L_2^2+L_3^2)+2\Big(-k_4(L_2+L_3)+L_2^2+L_2L_3+L_3^2\Big)\Big)\Big),$$

and

$$W = \frac{L_2-L_3}{k_1-k_2-k_3+k_4}\Big(k_1^2(k_2+k_3-L_2-L_3)(k_1+k_4)$$
$$+k_1\Big(k_2(k_4^2-k_3^2)-k_2^3-k_2^2k_3-k_3^3+k_3k_4^2+(L_2+L_3)\big(-k_4^2+L_2^2+L_3^2\big)\Big)$$
$$+L_2\Big((k_2+k_3)\big(k_2^2+k_3^2\big)-k_4^3\Big)-k_4(k_2+k_3)(k_2^2+k_3^2-k_4^2)$$
$$+L_3\Big(k_2^3+k_2^2k_3+k_2k_3^2-L_2^2(k_2+k_3-k_4)+k_3^3-k_4^3\Big)$$
$$-(k_2+k_3-k_4)(L_2^3+L_2L_3^2+L_3^3)\Big),$$

by equations e_3 and e_4 we get that $b=\beta=0$. This implies that the linear differential systems in R_1 and in R_2 are of the form

$$\dot{x} = \frac{1}{2}, \quad \dot{y} = x,$$

which is a contradiction because by hypothesis each of the linear differential systems considered is a center. Therefore we have proved that the maximum number of crossing limit cycles of the discontinuous piecewise linear differential centers in $\mathcal{F}_3$ is one.

Now we verify that this upper bound is reached. That is, that there are piecewise linear differential centers in the family $\mathcal{F}_3$ having one crossing limit cycle. We consider the discontinuous piecewise linear differential system formed by the following linear differential centers

$$\dot{x} = \frac{831}{128} + x - \frac{17}{16}y, \quad \dot{y} = \frac{587}{128} + x - y, \quad \text{in } R_1, \tag{7.5}$$

$$\dot{x} = \frac{21145}{4176} + \frac{x}{6} - \frac{5}{18}y, \quad \dot{y} = \frac{127}{174} + x - \frac{y}{6}, \quad \text{in } R_2. \tag{7.6}$$

These linear differential centers have the first integrals

$$H_1(x,y) = 64x^2 + x(587 - 128y) + y(-831 + 68y),$$
$$H_2(x,y) = \frac{1}{522}(3048x - 21145y) + 4\left(x - \frac{y}{6}\right)^2 + y^2,$$

respectively.

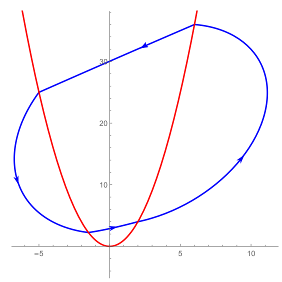


FIGURE 10. The crossing limit cycle of the discontinuous piecewise differential center formed by (7.5) and (7.6) separated by the conic (P).

The discontinuous piecewise differential center formed by the linear differential centers (7.5) and (7.6) has one crossing limit cycle, because the unique real solution (p_1, p_2, p_3, p_4) of system (7.4) is $p_1 = (6, 36)$, $p_2 = (-5, 25)$, $p_3 = (-3/2, 9/4)$, and $p_4 = (2, 4)$. See the crossing limit cycle of this discontinuous piecewise differential center in Figure 10.

8. Proof of Theorem 1.8

When the discontinuity curve Σ is of the type (E), we have following two regions in the plane: $R_1 = [(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1]$, and $R_2 = [(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1]$. By Lemma 3.1 a piecewise linear differential center of family $\mathcal{F}_4$ can be consider as (7.1) where the first integrals are given in (7.2).

Now we study the conditions in order that a piecewise linear differential center in the family $\mathcal{F}_4$ has crossing limit cycles intersecting the discontinuity curve (E) in exactly four points. Taking into account the first integrals (7.2) a piecewise linear differential center in $\mathcal{F}_4$ has crossing limit cycles if there are points (x_i, y_i) for $i = 1, 2, 3, 4$ satisfying the equations

$$e_1 = H_1(x_1, y_1) - H_1(x_2, y_2) = 0,$$
$$e_2 = H_2(x_2, y_2) - H_2(x_3, y_3) = 0,$$
$$e_3 = H_1(x_3, y_3) - H_1(x_4, y_4) = 0,$$

$$
\begin{aligned}
e_4 &= H_2(x_4,y_4) - H_2(x_1,y_1) = 0,\\
E_1 &= x_1^2 + y_1^2 - 1 = 0,\\
E_2 &= x_2^2 + y_2^2 - 1 = 0,\\
E_3 &= x_3^2 + y_3^2 - 1 = 0,\\
E_4 &= x_4^2 + y_4^2 - 1 = 0,
\end{aligned}
$$

considering $l_1 = 4b^2 + \omega^2$ and $l_2 = 4\beta^2 + \Omega^2$, we have the equivalent system

$$
\begin{aligned}
e_1 =& 4(x_1^2 - x_2^2) + 8(bx_1y_1 - bx_2y_2 + c(x_1 - x_2) - dy_1 + dy_2)\\
&+ l_1(y_1^2 - y_2^2) = 0,\\
e_2 =& 4(x_2^2 - x_3^2) + 8x_2(\gamma + \beta y_2) - 8x_3(\gamma + \beta y_3)\\
&+ (y_2 - y_3)(l_2(y_2 + y_3) - 8\delta) = 0,\\
e_3 =& 4(x_3^2 - x_4^2) + 8(bx_3y_3 - bx_4y_4 + c(x_3 - x_4) - dy_3 + dy_4)\\
&+ l_1(y_3^2 - y_4^2) = 0,\\
e_4 =& 8\delta(y_1 - y_4) + 8x_4(\gamma + \beta y_4) - 4(x_1^2 - x_4^2) - 8x_1(\gamma + \beta y_1)\\
&+ l_2(y_4^2 - y_1^2) = 0,\\
&E_1 = 0, \quad E_2 = 0, \quad E_3 = 0, \quad E_4 = 0.
\end{aligned} \tag{8.1}
$$

Where we consider without generality $a = \alpha = 1$ as in the proof Theorem 1.2.

We assume that this piecewise linear differential center has two crossing periodic solutions. For this we have that system (8.1) has two pairs of solutions, (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) with $p_i \neq p_j$, and $q_i \neq q_j$, for $i \neq j$ and $i, j = 1, 2, 3, 4$. Since these solution points are on the circle (E), then we can consider them in the following way

$$
\begin{aligned}
p_i &= (k_i, \lambda_i), \text{ with } k_i = \cos s_i, \quad \lambda_i = \sin s_i,\\
q_i &= (m_i, n_i), \text{ with } m_i = \cos t_i, \quad n_i = \sin t_i,
\end{aligned} \tag{8.2}
$$

where $s_i, t_i \in [0, 2\pi)$, for $i = 1, 2, 3, 4$.

Substituting the first solution (p_1, p_2, p_3, p_4) with p_i as in (8.2) in (8.1) we can determine the parameters d, δ, c, γ of the piecewise linear differential centers (7.1), and obtain

$$
\begin{aligned}
d =& \frac{(8c(k_1 - k_2) + 4(k_1 - k_2 + b\lambda_1 - b\lambda_2)(k_1 + k_2 + b(\lambda_1 + \lambda_2)) + (\lambda_1^2 - \lambda_2^2)\omega^2)}{8(\lambda_1 - \lambda_2)},\\
\delta =& \frac{4k_2^2 - 4k_3^2 + 8k_2(\lambda_2\beta + \gamma) - 8k_3(\lambda_3\beta + \gamma) + (\lambda_2^2 - \lambda_3^2)l_2}{8(\lambda_2 - \lambda_3)},\\
c =& \frac{1}{8((k_3 - k_4)(\lambda_1 - \lambda_2) - (k_1 - k_2)(\lambda_3 - \lambda_4))}\\
&\times \Big(4(\lambda_1 - \lambda_2)(k_4^2 - k_3^2 - 2b\lambda_3k_3 + 2bk_4\lambda_4) + (\lambda_3 - \lambda_4)(4(k_1^2 - k_2^2)\\
&+ 8b(k_1\lambda_1 - k_2\lambda_2) + (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3 - l_4)l_1)\Big),\\
\gamma =& \frac{1}{8(-(k_1 - k_4)(\lambda_2 - \lambda_3) + (k_2 - k_3)(\lambda_1 - \lambda_4))}\\
&\times \Big(4(\lambda_1 - \lambda_4)(k_3^2 - k_2^2 + 2k_3\lambda_3\beta - 2k_2\lambda_2) + (\lambda_2 - \lambda_3)(4k_1^2 - 4k_4^2 + 8k_1\lambda_1\beta
\end{aligned}
$$

$$- 8k_4\lambda_4\beta + (\lambda_1 - \lambda_4)(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)l_2)\Big).$$

Analogously, substituting the second solution (q_1, q_2, q_3, q_4) with q_i as in (8.2) in (8.1) we get remaining parameters ω, Ω, b, β. Substituting k_i, λ_i, m_i, n_i like (8.2) we obtain that $b = \beta = 0$, therefore we get that the piecewise linear differential centers is formed by linear differential center $\dot{x} = -y, \dot{y} = x$, in the regions R_1 and R_2. This is a contradiction because with this linear differential center is not possible to generate crossing limit cycles. Then we proved that the maximum number of crossing limits cycles for piecewise linear differential centers in $\mathcal{F}_4$ is one. Moreover this maximum number is reached, that is, there are piecewise linear differential centers in $\mathcal{F}_4$ such that have one crossing limit cycle with four points on (E), as we see below.

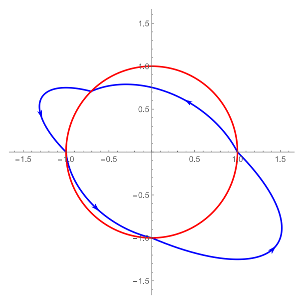


FIGURE 11. The crossing limit cycle of the discontinuous piecewise differential center formed by the centers (8.3) and (8.4) separated by the conic (E).

Consider the discontinuous piecewise differential center in the family $\mathcal{F}_4$ formed by the following two linear differential centers

$$\dot{x} = \frac{-107 - 89\sqrt{2}}{1024} - \frac{5}{16}x - \frac{345}{256}y, \quad \dot{y} = \frac{-71 + 89\sqrt{2}}{1024} + x + \frac{5}{16}y, \quad \text{in } R_1, \tag{8.3}$$

$$\dot{x} = -\frac{1}{4} - x - 2y, \quad \dot{y} = \frac{1}{4} + x + y, \quad \text{in } R_2. \tag{8.4}$$

These linear differential centers have the first integrals

$$H_1(x, y) = 512x^2 + x(-71 + 89\sqrt{2} + 320y) + y(107 + 89\sqrt{2} + 690y),$$
$$H_2(x, y) = x + 2x^2 + y + 4xy + 4y^2,$$

respectively. Then the discontinuous piecewise differential center formed by the linear differential centers (8.3) and (8.4) has one crossing limit cycle, because the unique real solution (p_1, p_2, p_3, p_4) of system (8.1) is $p_1 = (1, 0)$, $p_2 = (-\sqrt{2}/2, 1/\sqrt{2})$, $p_3 = (-1, 0)$, and $p_4 = (0, -1)$. See the crossing limit cycle of this system in Figure 11.

9. Proof of Theorem 1.9

If the discontinuity curve Σ is of the type (H) we have following three regions in the plane: $R_1 = [(x, y) \in \mathbb{R}^2 : x^2 - y^2 > 1, x > 0]$, $R_2 = [(x, y) \in \mathbb{R}^2 : x^2 - y^2 < 1]$ and $R_3 = [(x, y) \in \mathbb{R}^2 : x^2 - y^2 > 1, x < 0]$.

We consider a planar discontinuous piecewise linear differential system formed by three linear arbitrary centers. By Lemma 3.1 these linear differential centers can be as follows

$$\begin{aligned}
\dot{x} &= -b_1x - \frac{4b_1^2+\omega_1^2}{4a_1}y + d_1, \quad \dot{y} = a_1x + b_1y + c_1, \quad \text{in } R_1,\\
\dot{x} &= -b_2x - \frac{4b_2^2+\omega_2^2}{4a_2}y + d_2, \quad \dot{y} = a_2x + b_2y + c_2, \quad \text{in } R_2,\\
\dot{x} &= -b_3x - \frac{4b_3^2+\omega_3^2}{4a_3}y + d_3, \quad \dot{y} = a_3x + b_3y + c_3, \quad \text{in } R_3.
\end{aligned} \tag{9.1}$$

These linear differential centers have the first integrals

$$\begin{aligned}
H_1(x,y) &= 4(a_1x+b_1y)^2 + 8a_1(c_1x-d_1y) + y^2\omega_1^2,\\
H_2(x,y) &= 4(a_2x+b_2y)^2 + 8a_2(c_2x-d_2y) + y^2\omega_2^2,\\
H_3(x,y) &= 4(a_3x+b_3y)^2 + 8a_3(c_3x-d_3y) + y^2\omega_3^2,
\end{aligned} \tag{9.2}$$

respectively.

To have a crossing limit cycle, which intersects the discontinuity curve (H) in four different points $p_i = (x_i, y_i)$, $i = 1,2,3,4$, these points must satisfy the following equations

$$\begin{aligned}
e_1 &= H_1(x_1,y_1) - H_1(x_2,y_2) = 0,\\
e_2 &= H_2(x_2,y_2) - H_2(x_3,y_3) = 0,\\
e_3 &= H_3(x_3,y_3) - H_3(x_4,y_4) = 0,\\
e_4 &= H_2(x_4,y_4) - H_2(x_1,y_1) = 0,\\
E_1 &= x_1^2 - y_1^2 - 1 = 0,\\
E_2 &= x_2^2 - y_2^2 - 1 = 0,\\
E_3 &= x_3^2 - y_3^2 - 1 = 0,\\
E_4 &= x_4^2 - y_4^2 - 1 = 0,
\end{aligned} \tag{9.3}$$

equivalently, we have

$$\begin{aligned}
e_1 =& 4(x_1^2-x_2^2) + 8(c_1(x_1-x_2) - d_1y_1 + b_1x_1y_1 + d_1y_2 - b_1x_2y_2)\\
&+ (y_1^2-y_2^2)l_1 = 0,\\
e_2 =& 4(x_2^2-x_3^2) + 8(c_2(x_2-x_3) - d_2y_2 + b_2x_2y_2 + d_2y_3 - b_2x_3y_3)\\
&+ (y_2^2-y_3^2)l_2 = 0,\\
e_3 =& 4(x_3^2-x_4^2) + 8(c_3(x_3-x_4) - d_3y_3 + b_3x_3y_3 + d_3y_4 - b_3x_4y_4)\\
&+ (y_3^2-y_4^2)l_3 = 0,\\
e_4 =& 4(x_4^2-x_1^2) + 8(c_2(x_4-x_1) - d_2y_4 - b_2x_1y_1 + d_2y_1 + b_4x_4y_4)\\
&+ (y_4^2-y_1^2)l_2 = 0,\\
&E_1 = 0, \quad E_2 = 0, \quad E_3 = 0, \quad E_4 = 0,
\end{aligned} \tag{9.4}$$

where $l_i = 4b_i^2 + \omega_i^2$, for $i = 1,2,3$. Here we are taking without generality $a_1 = a_2 = a_3 = 1$ as in the proofs of the previous theorems.

We assume that the discontinuous piecewise linear differential system formed by the three linear differential centers in (9.1) has two crossing periodic solutions. For this we must impose that the system of equations (9.4) has two of real solutions, namely (p_1,p_2,p_3,p_4) and (q_1,q_2,q_3,q_4). Since these solutions provide crossing periodic solutions and these points are the points where the crossing periodic solutions intersect the hyperbola (H) we can consider

$$p_i=(k_i,\lambda_i)=(\cosh r_i,\sinh r_i)\quad\text{and}\quad q_i=(m_i,n_i)=(\cosh s_i,\sinh s_i), \tag{9.5}$$

with $r_i,s_i\in\mathbb{R}$ for $i=1,2,3,4$.

Now we assume that the point (p_1,p_2,p_3,p_4) with $p_i=(k_i,\lambda_i)$, $i=1,2,3,4$ satisfy (9.4), and then we obtain the following expressions

$$\begin{aligned}d_i=\Big(&8c_i(k_i-k_{i+1})+4(k_i-k_{i+1}+b_i(\lambda_i-\lambda_{i+1}))(k_i+k_{i+1}+b_i(\lambda_i+\lambda_{i+1}))\\&+(\lambda_i^2-\lambda_{i+1}^2)\omega_i^2\Big)/\big(8(\lambda_i-\lambda_{i+1})\big),\end{aligned}$$

for $i=1,2,3$, and

$$\begin{aligned}c_2=&\frac{1}{8((k_2-k_3)(\lambda_1-\lambda_4)-(k_1-k_4)(\lambda_2-\lambda_3))}\\&\times\Big(4(k_3^2+2b_2k_3\lambda_3-k_2^2-2b_2k_2\lambda_2)(\lambda_4-\lambda_1)+(\lambda_2-\lambda_3)(4(k_1^2-k_4^2)\\&+8b_2(k_1\lambda_1-k_4\lambda_4)+(\lambda_1-\lambda_4)(\lambda_1-\lambda_2-\lambda_3+\lambda_4)l_2)\Big).\end{aligned}$$

We assume that the point (q_1,q_2,q_3,q_4) with q_i as in (9.5) satisfies system (9.4). By the first equation in (9.4) we obtain that

$$\begin{aligned}c_1=&\frac{1}{8((\lambda_1-\lambda_2)(m_1-m_2)-(k_1-k_2)(n_1-n_2))}\\&\times\Big((\lambda_1-\lambda_2)(-4(m_1^2-m_2^2)-8b_1m_1n_1\\&+8b_1m_2n_2+(\lambda_1+\lambda_2)\omega_1^2(n_1-n_2)-(n_1^2-n_2^2)l_1)\\&+\frac{4(k_1-k_2+b_1(\lambda_1-\lambda_2))(k_1+k_2+b_1(\lambda_1+\lambda_2))(n_1-n_2)}{\lambda_1-\lambda_2}\Big).\end{aligned}$$

By the second equation of (9.4) we obtain that $\omega_2^2=K/S$ where

$$\begin{aligned}K=&\frac{4}{(k_1-k_4)(\lambda_2-\lambda_3)-(k_2-k_3)(\lambda_1-\lambda_4)}\\&\times\Big((k_3^2-k_2^2)\big((\lambda_1-\lambda_4)\psi_2-(k_1-k_4)\psi_1\big)+(\lambda_2-\lambda_3)\\&\times\big(k_1^2+b_2^2(\lambda_1-\lambda_4)(\lambda_1-\lambda_2-\lambda_3+\lambda_4)-k_4^2\big)\psi_2\\&+k_4\big(\psi_2-b_2^2(\lambda_2+\lambda_3-n_2-n_3)\psi_1+2b_2(-\lambda_4\psi_2+m_2n_2-m_3n_3)\big)\\&+k_1\big(-\psi_2+b_2^2(\lambda_2+\lambda_3-n_2-n_3)\psi_1+2b_2(\lambda_1\psi_2-m_2n_2+m_3n_3)\big)\Big)\\&+k_2\Big((\lambda_1-\lambda_4)(\psi_2-b_2^2(\lambda_1+\lambda_4-n_2-n_3)\psi_1)-(k_1^2-k_4^2)\psi_1\\&-2b_2\Big(\lambda_2(-\lambda_4\psi_2-(k_1-k_4)\psi_1)+\lambda_4((m_2-k_4)n_2+(k_4-m_3)n_3)\lambda_1(\lambda_2\psi_2\\&+(k_1-m_2)n_2+(m_3-k_1)n_3)\Big)\Big)+k_3((\lambda_4-\lambda_1)\psi_2+(k_1^2-k_4^2)\psi_1\\&+b_2^2(\lambda_1-\lambda_4)(\lambda_1+\lambda_4-n_2-n_3)\psi_1+2b_2\Big(\lambda_3(-\lambda_4\psi_2-(k_1-k_4)\psi_1)\\&+\lambda_4(m_2n_2-k_4\psi_1-m_3n_3)+\lambda_1(\lambda_3\psi_2+k_1\psi_1-m_2n_2+m_3n_3)\Big),\end{aligned}$$

and

$$
\begin{aligned}
S =& \frac{1}{(k_1-k_4)(\lambda_2-\lambda_3)-(k_2-k_3)(\lambda_1-\lambda_4)} \\
&\times \Big(\lambda_3^2\lambda_4\psi_2+\lambda_3\lambda_4^2\psi_2+(k_1-k_4)(\lambda_3^2n_2-\lambda_3n_2^2)+(k_3-k_2)(\lambda_4^2n_2-\lambda_4n_2^2) \\
&+\lambda_1^2((k_2-k_3)\psi_1-(\lambda_2-\lambda_3)\psi_2)+\lambda_2^2(\lambda_4\psi_2+(k_1-k_4)\psi_1) \\
&+((k_4-k_1)\lambda_3^2+(k_2-k_3)\lambda_4^2)n_3+(k_1\lambda_3-k_4\lambda_3-k_2\lambda_4+k_3\lambda_4)n_3^2 \\
&+\lambda_1((\lambda_2^2-\lambda_3^2)\psi_2-(k_2-k_3)\psi_1(n_2+n_3))+\lambda_2(\lambda_4^2\psi_2 \\
&+(k_1-k_4)\psi_1(n_2+n_3))\Big).
\end{aligned}
$$

By the third equation we obtain that

$$
\begin{aligned}
c_3 =& \frac{-1}{8(\lambda_4\psi_4-\lambda_3\psi_2+(k_3-k_4)\psi_1))}(((\lambda_3-\lambda_4)(4(m_4^2-m_3^2)+8b_3(m_4n_4-m_3n_3) \\
&+\frac{4(k_3-k_4+b_3(\lambda_3-\lambda_4))(k_3+k_4+b_3(\lambda_3+\lambda_4))\psi_1}{\lambda_3-\lambda_4} \\
&+(\lambda_3+\lambda_4)\omega_3^2\psi_1-(n_3^2-n_4^2)l_3)),
\end{aligned}
$$

where $\psi_1=n_2-n_3$, $\psi_2=m_2-m_3$, $\psi_3=n_3-n_4$ and $\psi_4=m_3-m_4$.

Finally by the fourth equation in (9.4) we have that $b_2=0$. With these expressions for d_2, c_2, ω_2 and b_2 we obtain that the linear differential system in the region R_2 is $\dot{x}=y$, $\dot{y}=x$, which is a linear differential system type saddle. This is a contradiction because we are working with centers in each region R_i for $i=1,2,3$. Therefore we have proved that the maximum number of crossing limit cycles for systems in $\mathcal{F}_5$ is one.

Moreover it is possible to show that there are piecewise linear differential centers in $\mathcal{F}_5$ such that have one crossing limit cycle. Indeed, consider the discontinuous piecewise linear differential system in the family $\mathcal{F}_5$ formed by the following linear differential centers

$$
\begin{gathered}
\dot{x}=\frac{-355+64\sqrt{10}+80\sqrt{21}}{64(6+\sqrt{21})}-\frac{x}{2}-\frac{29}{16}y, \quad \dot{y}=1+x+\frac{y}{2}, \quad \text{in } R_1, \\
\dot{x}=K_1-\frac{x}{10}-\frac{101}{100}y, \quad \dot{y}=K_2+x+\frac{y}{10}, \quad \text{in } R_2, \\
\dot{x}=\frac{3}{4}(-11+2\sqrt{3})+\frac{337(-3+2\sqrt{3})}{64\sqrt{5}}-\frac{3}{2}x-\frac{45}{16}y, \quad \dot{y}=-\frac{3}{2}x+\frac{3}{2}y, \quad \text{in } R_3.
\end{gathered} \tag{9.6}
$$

Where

$$
\begin{aligned}
K_1 =& \frac{1}{200\sqrt{81-12\sqrt{35}}(-20\sqrt{2}+26\sqrt{3}-14\sqrt{5}+7\sqrt{7}-13\sqrt{15}+2\sqrt{70})} \\
&\times \Big(27300-62790\sqrt{2}+8750\sqrt{3}-31356\sqrt{5}+15678\sqrt{7}+2500\sqrt{30} \\
&-2730\sqrt{35}-600\sqrt{42}+6279\sqrt{70}-420\sqrt{105}\Big) \\
K_2 =& \Big(3(200-800\sqrt{2}+5226\sqrt{3}-1846\sqrt{5}+403\sqrt{7}-2613\sqrt{15}+240\sqrt{35} \\
&+80\sqrt{70})\Big)/\Big(400(-20\sqrt{2}+26\sqrt{3}-14\sqrt{5}+7\sqrt{7}-13\sqrt{15}+2\sqrt{70})\Big).
\end{aligned}
$$

These linear differential centers have first integrals

$$
\begin{aligned}
H_1(x,y) &= (355-64\sqrt{10}-80\sqrt{21})y+2(6+\sqrt{21})(16x(2+x)+16xy+29y^2),\\
H_2(x,y) &= \frac{1}{200\sqrt{27-4\sqrt{35}}(-20\sqrt{2}+26\sqrt{3}-14\sqrt{5}+7\sqrt{7}-13\sqrt{15}+2\sqrt{70})}\\
&\quad \times \Big(y^2+\big(x+\frac{y}{10}\big)^2+(3\sqrt{27-4\sqrt{35}}(200-800\sqrt{2}+5226\sqrt{3}\\
&\quad -1846\sqrt{5}+403\sqrt{7}-2613\sqrt{15}+240\sqrt{35}+80\sqrt{70})x\\
&\quad -2(8750+9100\sqrt{3}-20930\sqrt{6}+2500\sqrt{10}-600\sqrt{14}-10452\sqrt{15}\\
&\quad +5226\sqrt{21}-420\sqrt{35}-910\sqrt{105}+2093\sqrt{210})y)\Big),\\
H_3(x,y) &= 4x^2+12x(-1+y)+\frac{1}{40}y(2640-480\sqrt{3}+1011\sqrt{5}\\
&\quad -674\sqrt{15}+450y),
\end{aligned}
$$

respectively.

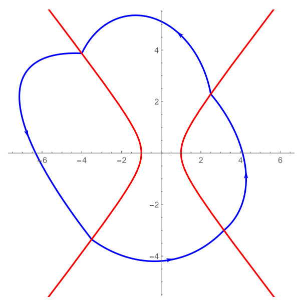


FIGURE 12. The crossing limit cycle of the discontinuous piecewise linear differential center (9.6) with discontinuity curve the conic (H).

The unique real solution (p_1,p_2,p_3,p_4) that satisfies (9.4) in this case is $p_1 = (\sqrt{10},-3)$, $p_2 = (-5/2,\sqrt{21/4})$, $p_3 = (-4,\sqrt{15})$, and $p_4 = (-7/2,-\sqrt{45/4})$. See the crossing limit cycle of this system in Figure 12.

10. Proof of Theorem 1.10

Proof of statement (a) of Theorem 1.10. In this case we use the notations given in the proof of Theorem 1.7, then we consider the planar discontinuous piecewise linear center (7.1) and the first integrals (7.2). In order that the discontinuous piecewise linear center (7.1) has crossing limit cycles with four points, namely $(x_1,x_1^2),(x_2,x_2^2),(x_3,x_3^2),(x_4,x_4^2)$ and one crossing limit cycle with two points, namely $(x_5,x_5^2),(x_6,x_6^2)$ on (P), we must study the solutions $(x_1,x_2,x_3,x_4,x_5,x_6)$ of system (7.3) and the equations

$$
\begin{aligned}
e_5 &= H_1(x_5,x_5^2)-H_1(x_6,x_6^2)=0,\\
e_6 &= H_2(x_6,x_6^2)-H_2(x_5,x_5^2)=0,
\end{aligned}
$$

or equivalently systems (7.4) and

$$\begin{aligned} e_5 =& 4x_5^2(1+bx_5)^2 + 8x_5(c-dx_5) - 4x_6^2(1+bx_6)^2 + 8x_6(dx_6-c) \\ & + (x_5^4 - x_6^4)\omega^2 = 0, \\ e_6 =& 4x_6^2(1+x_6\beta)^2 - 4x_5^2(1+x_5\beta)^2 + 8x_5(x_5\delta-\gamma) + 8x_6(\gamma - x_6\delta) \\ & + (x_6^4 - x_5^4)\Omega^2 = 0. \end{aligned} \tag{10.1}$$

We assume that systems (7.4) and (10.1) have two real solutions where each real solution provides one crossing limit cycle with four points on (P) and one crossing limit cycle whit two points on (P), but by Theorem 1.7 we have that discontinuous piecewise linear center (7.1) has at most 1 crossing limit cycle with four points on (P), therefore if we have two real solutions of systems (7.4) and (10.1) they are of the form $(x_1, x_2, x_3, x_4, x_5, x_6) = (k_1, k_2, k_3, k_4, k_5, k_6)$ and $(x_1, x_2, x_3, x_4, x_5, x_6) = (k_1, k_2, k_3, k_4, \lambda_5, \lambda_6)$, with $k_i, \lambda_5, \lambda_6 \in \mathbb{R}$ for $i = 1, 2, 3, 4, 5, 6$.

If the point $(k_1, k_2, k_3, k_4, k_5, k_6)$ satisfies systems (7.4) and (10.1), by the equations e_1, e_2, e_3 and e_4 of (7.4) we obtain expressions for the parameters d, δ, c and γ as in the proof of Theorem 1.7, by the equation e_5 of system (10.1) we obtain an expression for $\omega^2 = S/T$ with S and T as in the proof of Theorem 1.7 changing L_1 and L_2 by k_5 and k_6, respectively. By equation e_6 of system (10.1) we obtain $\Omega^2 = V/W$ where the expression for V and W are the same expressions that in the proof of Theorem 1.7 changing L_3 by k_5. We assume that the point $(k_1, k_2, k_3, k_4, \lambda_5, \lambda_6)$ satisfies systems (7.4) and (10.1), then we have $e_1 = e_2 = e_3 = e_4 = 0$ and by the equations e_5 and e_6 of system (10.1) we obtain $b = \beta = 0$. As in the proof of Theorem 1.7 we can conclude that the two linear centers in (7.1) became $\dot{x} = 1/2, \quad \dot{y} = x$, which is a contradiction. So systems (7.4) and (10.1) have at most one solution and therefore planar discontinuous piecewise linear centers in $\mathcal{F}_3$ have at most one crossing limit cycle with four point on (P) and one crossing limit cycle with two points on (P) simultaneously. Moreover this upper bound is reached, this is there are systems in $\mathcal{F}_3$ with one crossing limit cycle with four points on (P) and one crossing limit cycle with two points on (P) simultaneously.

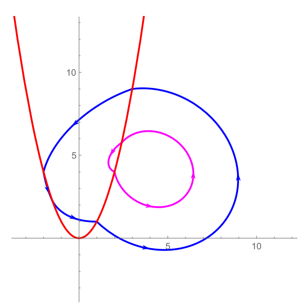


FIGURE 13. The two crossing limit cycles of the discontinuous piecewise linear differential system formed by the centers (10.2) and (10.3).

We consider the discontinuous piecewise linear differential system formed by the linear centers

$$\dot{x} = \frac{1225}{229} + \frac{x}{2} - \frac{310}{229}y, \quad \dot{y} = -\frac{103}{229} + x - \frac{y}{2}, \quad \text{in } R_1, \tag{10.2}$$

$$\dot{x} = \frac{6411}{1424} - \frac{x}{8} - \frac{85}{89}y, \quad \dot{y} = -\frac{3359}{712} + x + \frac{y}{8}, \quad \text{in } R_2. \tag{10.3}$$

These linear differential centers have the first integrals

$$\begin{aligned} H_1(x,y) &= 229x^2 + 10y(-245+31y) - x(206+229y),\\ H_2(x,y) &= 4x^2 + x\big(-\frac{3359}{89} + y\big) + \frac{y}{178}(-6411+680y), \end{aligned}$$

respectively.

The unique real solution of systems (7.4) and (10.1) is $(x_1,x_2,x_3,x_4,x_5,x_6) = (3,-2,-3/2,1,2,12/5)$, therefore we have one crossing limit cycle that intersects (P) in the points $(3,9)$, $(-2,4)$, $(-3/2,9/2)$ and $(1,1)$, and one crossing limit cycle that intersects (P) in the points $(2,4)$ and $(12/5,144/25)$. See these crossing limit cycles in Figure 13. □

Proof of statement (b) of Theorem 1.10. In this case we consider the notation of the proof of Theorem 1.8 and therefore we consider the planar discontinuous piecewise linear center (7.1) and the first integrals (7.2). In order that the discontinuous piecewise linear center (7.1) has crossing limit cycles with four points on (E), namely (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) and one crossing limit cycle with two points on (E), namely (x_5,y_5), (x_6,y_6), we must study the solutions $(p_1,p_2,p_3,p_4,p_5,p_6)$ of systems (8.1) and

$$\begin{aligned} e_5 =&4(x_5^2-x_6^2) + 8(c(x_5-x_6) - dy_5 + bx_5y_5 + dy_6 - bx_6y_6)\\ &+ (y_5^2-y_6^2)l_1 = 0,\\ e_6 =&4(x_6^2-x_5^2) + 8(\beta x_6y_6 - \beta x_5y_5 + y_5\delta - x_5\gamma + x_6\gamma - y_6\delta)\\ &+ (y_6^2-y_5^2)l_2 = 0,\\ &E_5 = x_5^2+y_5^2-1=0, \quad E_6 = x_6^2+y_6^2-1=0. \end{aligned} \tag{10.4}$$

We assume that systems (8.1) and (10.4) have two real solutions where each real solution provides one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E), like in Theorem 1.8 we proved that discontinuous piecewise linear center (7.1) has at most 1 crossing limit cycle with four points on (E), then we have that if there are two real solutions of systems (8.1) and (10.4) they are of the form $(p_1,p_2,p_3,p_4,p_5,p_6)$ and $(p_1,p_2,p_3,p_4,q_5,q_6)$, with p_i and q_j as (8.2) for $i=1,2,3,4,5,6$ and $j=5,6$.

Substituting the first solution $(p_1,p_2,p_3,p_4,p_5,p_6)$ in systems (8.1) and (10.4) we obtain from the equations e_1,e_2,e_3 and e_4 of (8.1) the same expressions than in the proof of Theorem 1.8 for d,δ,c,γ, and by the equations e_5 and e_6 of system (10.4) we obtain the same expressions than in the proof of Theorem 1.8 for ω and Ω changing (m_1,n_1) by (k_5,λ_5) and (m_2,n_2) by (k_6,λ_6), respectively. We assume that the point $(p_1,p_2,p_3,p_4,q_5,q_6)$ satisfies systems (8.1) and (10.4), then we have $e_1=e_2=e_3=e_4=0$ and by the equations e_5 and e_6 of system (10.4) we obtain $b=\beta=0$. As in the proof of Theorem 1.8 we obtain that both linear centers in (7.1) become $\dot{x}=-y, \quad \dot{y}=x$, in contradiction that they have limit cycles. So we can conclude that systems (8.1) and (10.4) have at most one solution and therefore planar discontinuous piecewise linear centers in $\mathcal{F}_4$ have at most one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E) simultaneously.

Now we verify that this upper bound is reached, that is there are systems in $\mathcal{F}_4$ with one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E) simultaneously. We consider the discontinuous piecewise linear differential system in $\mathcal{F}_4$ formed by the linear centers

$$\begin{gathered}\dot{x}=-\frac{(-6+3\sqrt{2}+\sqrt{6}+(6-4\sqrt{2}-6\sqrt{3})x+8(-1+\sqrt{2}+2\sqrt{3})y}{4(-3+2\sqrt{2}+3\sqrt{3})},\\ \dot{y}=-\frac{-4+3\sqrt{2}+2\sqrt{3}+\sqrt{6}}{2(-6+4\sqrt{2}+6\sqrt{3})}+x-\frac{y}{2},\quad \text{in } R_1,\end{gathered}\tag{10.5}$$

$$\begin{gathered}\dot{x}=-\Big(18-93\sqrt{2}+4\sqrt{3}+33\sqrt{6}-230(1+\sqrt{3})x+4(335-2\sqrt{2}\\ +261\sqrt{3}+20\sqrt{6})y\Big)/\big(920(1+\sqrt{3})\big),\\ \dot{y}=x+\frac{1}{920}\Big(9+34\sqrt{2}-67\sqrt{3}-41\sqrt{6}-230y\Big),\quad \text{in } R_1.\end{gathered}\tag{10.6}$$

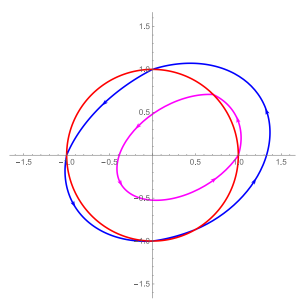


FIGURE 14. The two limit cycles of the discontinuous piecewise linear differential system formed by the centers (10.5) and (10.6).

The unique real solution of systems (8.1) and (10.4) in this case is $(p_1,p_2,p_3,p_4,p_5,p_6)$ with $p_1=(\cos(\pi/2),\sin(\pi/2))$, $p_2=(\cos(\pi),\sin(\pi))$, $p_3=(\cos(3\pi/2),\sin(3\pi/2))$, $p_4=(\cos(-\pi/3),\sin(-\pi/3))$, $p_5=(\cos(\pi/4),\sin(\pi/4))$ and $p_6=(\cos(0),\sin(0))$. See these crossing limit cycles in Figure 14. $\square$

Proof of statement (c) of Theorem 1.10. Here we consider the notation of the proof of Theorem 1.9 and therefore we consider the planar discontinuous piecewise linear center (9.1) and the first integrals (9.2). In order that the discontinuous piecewise linear center (9.1) has crossing limit cycles with four points on (H), namely $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ and one crossing limit cycle with two points on (H), namely $(x_5,y_5),(x_6,y_6)$, we must study the solutions $(p_1,p_2,p_3,p_4,p_5,p_6)$ of systems (9.4) and (10.7)

$$
\begin{aligned}
e_5 =& 4(x_5^2-x_6^2)+8(c_2(x_5-x_6)-d_2y_5+b_2x_5y_5+d_2y_6-b_2x_6y_6)\\
&+(y_5^2-y_6^2)l_1=0,\\
e_6 =& 4(x_6^2-x_5^2)+8(b_1x_6y_6-b_1x_5y_5+y_5d_1-x_5c_1+x_6c_1-y_6d_1)\\
&+(y_6^2-y_5^2)l_2=0,\\
&E_5=x_5^2-y_5^2-1=0,\quad E_6=x_6^2-y_6^2-1=0.
\end{aligned}
\tag{10.7}
$$

We assume that systems (9.4) and (10.7) have two real solutions where each real solution provides one crossing limit cycle with four points on (H) and one crossing limit cycle with two points on (H). By Theorem 1.9 the discontinuous piecewise linear center (9.1) has at most 1 crossing limit cycle with four points on (H), then we have that if there are two real solutions of systems (9.4) and (10.7) they are of the form $(p_1,p_2,p_3,p_4,p_5,p_6)$ and $(p_1,p_2,p_3,p_4,q_5,q_6)$, with p_i and q_j as (9.5) for $i=1,2,3,4,5,6$ and $j=5,6$.

Considering the first solution $(p_1,p_2,p_3,p_4,p_5,p_6)$ of systems (9.4) and (10.7) we obtain the same expressions that in the proof of Theorem 1.9 for $d_1,d_2,d_3,c_2,c_1,\omega_2$ changing (m_1,n_1) by (k_5,λ_5) and (m_2,n_2) by (k_6,λ_6), respectively.

Now we assume that the point $(p_1,p_2,p_3,p_4,q_5,q_6)$ satisfies systems (9.4) and (10.7), then we have $e_1=e_2=e_3=e_4=0$, and by the equation e_5 of system (10.7) we obtain $b_2=0$ and with this the linear system in the region R_2 becomes a saddle which is a contradiction, because we are working with linear centers in each regions R_i for $i=1,2,3$. Therefore the discontinuous piecewise linear center (9.1) has at most one crossing limit cycle with four points on (H) and one crossing limit cycle with two points on (H) simultaneously. Moreover this upper bound is reached, that is there are piecewise linear differential centers in $\mathcal{F}_5$ such that have one crossing limit cycle with four points on (H) and one crossing limit cycle with two points on (H) simultaneously. Indeed consider the piecewise linear differential system formed by the linear centers

$$
\begin{aligned}
\dot{x} &= \frac{-1215-576\sqrt{2}+256\sqrt{7}+112\sqrt{13}-384\sqrt{15}}{192(2\sqrt{7}+\sqrt{13}-\sqrt{\alpha})}-\frac{x}{2}-\frac{29}{16}y,\\
\dot{y} &= \frac{-1}{48\big(-1+2\sqrt{\frac{7}{\alpha}}+\sqrt{\frac{13}{\alpha}}\big)}\Big(-\frac{675}{4}-64\sqrt{7}-28\sqrt{13}\\
&\quad+288\sqrt{\frac{2}{7}(23-4\sqrt{30})}+945\sqrt{\frac{7}{\alpha}}+\frac{945}{2}\sqrt{\frac{13}{\alpha}}+144\sqrt{\frac{26}{\alpha}}\\
&\quad+192\sqrt{\frac{105}{\alpha}}+96\sqrt{\frac{195}{\alpha}}\Big)+x+\frac{y}{2}\quad \text{in } R_1,
\end{aligned}
\tag{10.8}
$$

$$
\begin{aligned}
\dot{x} =& \frac{1125+432\sqrt{14}+189\sqrt{26}+207\sqrt{30}+160\sqrt{105}+70\sqrt{195}}{6\xi}-\frac{x}{2}\\
&-\frac{54\sqrt{2}+336\sqrt{7}+174\sqrt{13}+24\sqrt{15}+68\sqrt{210}+35\sqrt{390}-\eta}{\eta}y,\\
\dot{y} =& -\frac{855\sqrt{2}+3516\sqrt{7}+1797\sqrt{13}+315\sqrt{15}+644\sqrt{210}+329\sqrt{390}}{6\xi}\\
&+x+\frac{y}{2},\quad \text{in } R_2,
\end{aligned}
\tag{10.9}
$$

$$\dot{x} = -\frac{9}{2} + \frac{73}{8\sqrt{2}} - \frac{3}{2}x - \frac{45}{16}y, \quad \dot{y} = -\frac{3}{2} + x + \frac{3}{2}y, \quad \text{in } R_3, \tag{10.10}$$

here $\alpha = 23+4\sqrt{30}$, $\eta = 42+48\sqrt{14}+24\sqrt{26}+9\sqrt{30}+24\sqrt{105}+12\sqrt{195}$ and $\xi = 87+108\sqrt{14}+54\sqrt{26}+16\sqrt{30}+40\sqrt{105}+20\sqrt{195}$. The unique real solution of systems (9.3) and (10.7) in this case is $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 = (3, -\sqrt{8})$, $p_2 = (4, \sqrt{15})$, $p_3 = (-3, \sqrt{8})$, $p_4 = (-1, 0)$, $p_5 = (7/6, -\sqrt{13}/6)$ and $p_6 = (4/3, \sqrt{7}/3)$. See these crossing limit cycles in Figure 15. $\square$

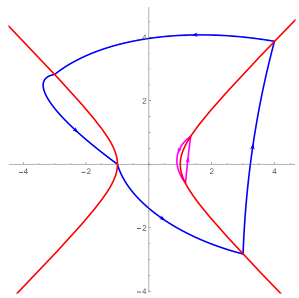


Figure 15. The two limit cycles of the discontinuous piecewise linear differential center formed by the centers (10.8), (10.9) and (10.10).

11. Proof of Theorem 1.11

Proof. To have a crossing limit cycle of type 1 and one crossing limit cycle of type 2, simultaneously, we must study the real solutions $(p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4)$, of systems (5.3) and (6.5) respectively, where $p_i = (x_i, 0)$ and $q_i = (0, y_i)$, with $x_1, x_2, x_3, y_1, y_2, y_3 > 0$ and $x_4, y_4 < 0$.

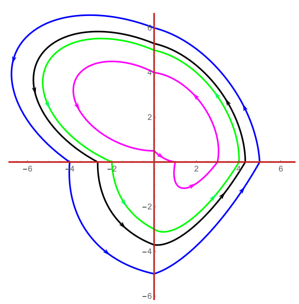


Figure 16. One crossing limit cycle of type 1 and three crossing limit cycles of type 2 of the discontinuous piecewise linear differential system formed by the linear centers (11.1), (11.2), (11.3) and (11.4) separated by (LV).

In the region R_1 we consider the linear differential center

$$\dot{x} = \frac{193}{134} - \frac{x}{3} - \frac{58}{67}y, \quad \dot{y} = -\frac{149}{134} + x + \frac{y}{3}, \tag{11.1}$$

this system has the first integral $H_1(x,y) = 201x^2 + x(134y - 447) + 3y(58y - 193))$. In the region R_2 we have the linear differential center

$$\dot{x} = \frac{9}{2} - \frac{x}{2} - 2y, \quad \dot{y} = -\frac{1}{4} + x + \frac{y}{2}, \tag{11.2}$$

which has the first integral $H_2(x,y) = 2x(2x-1) + 4y(x-9) + 8y^2$. In the region R_3 we have the linear differential center

$$\dot{x} = 1.068079\dots + \frac{\sqrt{3}}{4}x - 1.448022\dots y, \quad \dot{y} = -3.860171\dots + x - \frac{\sqrt{3}}{4}y, \tag{11.3}$$

which has the first integral $H_3(x,y) = x^2 + x(-7.720342\dots - 0.866025..y) + y(-2.136159\dots + 1.448022\dots y)$. And in the region R_4 we have the linear differential center

$$\dot{x} = \frac{51831 - 595\sqrt{16909}}{35912} + \frac{x}{2} + \frac{6775 - 119\sqrt{16909}}{17956}y, \quad \dot{y} = -2 + x - \frac{y}{2}, \tag{11.4}$$

which has the first integral $H_4(x,y) = 17956x^2 - 17956x(4+y) + y(-51831 + 595\sqrt{16909} + (-6775 + 119\sqrt{16909})y)$. The unique real solutions for systems (5.3) and (6.5) are $(p_1, q_1, p_2, q_2,\ p_3, q_3, p_4, q_4)$ with $p_1 = (1,0)$, $q_1 = (0,1/2)$, $p_2 = (3,0)$, $q_2 = (0,4)$, $p_3 = (5,0)$, $q_3 = (0,6)$, $p_4 = (-4,0)$ and $q_4 = (0,-5)$; $(p_1, q_1, p_2, q_2, l_3, m_3, l_4, m_4)$ with $l_3 = ((149 + 3\sqrt{16909})/134, 0)$, $m_3 = (0,5)$, $l_4 = (-2,0)$ and $m_4 = (0,-3)$; and $(p_1, q_1, p_2, q_2, \lambda_3, \eta_3, \lambda_4, \eta_4)$, where

$$\lambda_3 = (\ 4.319114\dots, 0), \quad \eta_3 = (\ 0, 53/10),$$
$$\lambda_4 = (-2.672755\dots, 0), \quad \eta_4 = (\ 0, -3.703965\dots).$$

See these crossing limit cycles of types 1 and 2 in Figure 16. □

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