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COMPLEX GINZBURG-LANDAU EQUATIONS WITH A DELAYED NONLOCAL PERTURBATION

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Abstract. We consider an initial boundary value problem of the complex Ginzburg-Landau equation with some delayed feedback terms proposed for the control of chemical turbulence in reaction diffusion systems. We consider the equation in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \le 3)$,

$$\frac{\partial u}{\partial t} - (1+i\epsilon)\Delta u + (1+i\beta)|u|^2 u - (1-i\omega)u = F(u(x,t-\tau))$$

for $t > 0$, with

$$F(u(x,t-\tau)) = e^{i\chi_0}\big\{\frac{\mu}{|\Omega|}\int_\Omega u(x,t-\tau)dx + \nu u(x,t-\tau)\big\},$$

where μ, $\nu \ge 0$, $\tau > 0$ but the rest of real parameters ϵ, β, ω and χ_0 do not have a prescribed sign. We prove the existence and uniqueness of weak solutions of problem for a range of initial data and parameters. When $\nu = 0$ and $\mu > 0$ we prove that only the initial history of the integral on Ω of the unknown on $(-\tau, 0)$ and a standard initial condition at $t = 0$ are required to determine univocally the existence of a solution. We prove several qualitative properties of solutions, such as the finite extinction time (or the zero exact controllability) and the finite speed of propagation, when the term $|u|^2u$ is replaced by $|u|^{m-1}u$, for some $m \in (0,1)$. We extend to the delayed case some previous results in the literature of complex equations without any delay.

1. Introduction

It is well-known that feedback delayed term can be introduced to control very complex phenomena (see for example the expositions in [4, 18, 29]). Our main interest in this paper concerns a model, of complex Ginzburg and Landau equations type, introduced for the control of turbulence in oscillatory reaction-diffusion systems made through a combination of global and local delayed feedback. We recall that, after the pioneering work of Ginzburg and Landau [19] in 1950 in superconductivity, Ginzburg-Landau equation has been systematically used to study different types of phenomena in superconductor theory. A rich variety of mathematical models of PDEs have also been inspired by the original model of Ginzburg and Landau to study a large number of physical phenomena, (see for instance Kuramoto [23], Levy [24], Temam [28] and references therein).

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In 1996, Battogtokh and Mikhailov [5], introduced a nonlocal delayed term in the generalized equation in order to control the system and suppress turbulence (see also Battogtokh, A. Preusser and Mikhailov [6]). The equation appears in the study of some chemical reactions and models the concentration of various reacting species. D. Battogtokh and A. Mikhailov analyze numerically this model and control the turbulence thanks to the delayed term. The idea is to adjust two real parameters: the feedback intensity μ and the delay time τ. The results were made rigorous later in a series of articles, which we indicate below. This work is a natural companion of those rigorous studies. For instance, a first rigorous approach was presented in Casal and Díaz [14], where the control of turbulence in oscillatory reaction-diffusion systems is made through a combination of global and local feedback by means of a pseudo-linearization technique (see also Casal and Díaz [13, 14], Casal, Díaz and Stich [15, 16] and Casal, Díaz, Stich and Vegas [17]). In this paper, we consider weaker assumptions on the initial data and parameters than in the above mentioned papers and others results in the literature (see, e.g. [2]).

Although our results can be stated under a great generality, here we consider only the framework motivated by the control problem goal. As a first model we will consider the case of a global delayed problem in which two real parameters play a fundamental role: the feedback intensity, μ, and the delay time, τ. The problem is reduced to find a complex valued field u in $Q := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain for $N \leq 3$ with regular boundary $\partial\Omega$ and $t > 0$.

$$\begin{gathered}
\frac{\partial u}{\partial t} - (1+i\epsilon)\Delta u + (1+i\beta)|u|^2 u - (1-i\omega)u = F(u(x,t-\tau)), \quad \text{in } Q, \\
\frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \partial\Omega \times (0,T), \\
u(x,0) = u_0(x) \quad \text{on } \Omega, \\
F(u(s)) = F_0(s) \quad s \in (-\tau, 0),
\end{gathered} \tag{1.1}$$

where the global delayed feedback term is

$$\begin{aligned}
F(u(x,t-\tau)) &= F_1(u(x,t-\tau)) + iF_2(u(x,t-\tau)) \\
&:= \mu e^{i\chi_0}\Big\{\frac{1}{|\Omega|}\int_\Omega u(x,t-\tau)dx\Big\},
\end{aligned} \tag{1.2}$$

here ω, β, ϵ, τ, μ and χ_0 are given real numbers without prescribed sign, $u_0(x)$ and $F_0(s)$ are given complex functions and $\vec{n}$ is the outward normal vector to $\partial\Omega$. We point out that, in contrast with most of the delayed problems, here the *initial past history* is composed of a pointwise information at $t = 0$ (the usual initial condition $u(x,0) = u_0(x)$ on Ω) and only a partial information on the function $u(s)$ when $s \in (-\tau, 0)$: only the integral of the unknown is prescribed $s \in (-\tau, 0)$. Under suitable conditions on $u_0(x)$ and $F_0(s)$ we prove (in Theorems 2.3 and 3.1) that there exists a unique solution of (1.1).

A second model concerns the case, already used in [5, 6, 14], in which the delayed feedback term involves the unknown

$$F(u(x,t-\tau)) = e^{i\chi_0}\Big\{\frac{\mu}{|\Omega|}\int_\Omega u(x,t-\tau)dx + \nu u(x,t-\tau)\Big\}. \tag{1.3}$$

In that case it is clear that the required *initial past history* must be more complete and so the new formulation is the usual one for delayed problems. As a matter of facts, as we mentioned later, the nonlinear perturbation can be easily treated under a more general growth condition of the type $(1+i\beta)|u|^{m-1}u$, for all $m>0$. In particular, when $m\in(0,1)$ we comment how to apply the techniques introduced in a series of works concerning the pure Schrödinger equation with a non-Lipschitz perturbation to our case (see [7, 8, 10, 11]). See also the study, for complex Ginzburg-Landau equations without any delayed term, made in [2]. Thus our second problem can be formulated as

$$\begin{gathered}
\frac{\partial u}{\partial t}-(1+i\epsilon)\Delta u+(1+i\beta)|u|^{m-1}u-(1-i\omega)u=F(u(x,t-\tau)),\quad \text{in } Q,\\
\frac{\partial u}{\partial \vec{n}}=0,\quad \text{on } \partial\Omega\times(0,T),\\
u(x,s)=U_0(x,s),\quad s\in[-\tau,0],\ x\in\Omega.
\end{gathered}\tag{1.4}$$

In the special case of $m\in(0,1)$ and F given by (1.3) with $\mu=0$ (i.e., with only local delayed feedback terms) we prove that several qualitative properties as the finite speed of propagation or the finite extinction time property obtained previously in the literature for complex formulations problems without delayed term (see [2, 7, 8, 10, 11]) can be easily extended to the mentioned delayed formulation.

This article is organized as follows: the existence of solutions for problems (1.1) and (1.4) is obtained in Section 2. The proof of the existence of solutions use an iterative argument as well as a Galerkin method when $t\in[0,\tau)$ jointly with suitable a priori estimates which allow to justify the passing to the limit. The uniqueness of solutions is given in Section 3 for $N\le 3$. Finally the study of some qualitative properties, for $m\in(0,1)$ and F given by (1.3) with $\mu=0$, will be collected in Section 4 where some energy methods will be applied.

Notation. $W^{s,p}(D)$ and $H^s(D)$ denotes the standard Sobolev spaces which consist of real scalar (or vector) valued functions defined on D (an open subset of $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$). Sobolev spaces of complex valued functions are denoted by $\mathcal{W}^{s,p}(D)$ and $\mathcal{H}^s(D)$ with calligraphic letters, as well, as continuous functions $\mathcal{C}(D)$ defined over a domain D. We use $\|\cdot\|$ and $(\cdot,\cdot)$ for the usual norm and the inner product of $L^2(D)$ (or $\mathcal{L}^2(D)$) respectively. Given a general Banach space B, $\|\cdot\|_B$ denotes the norm of Banach space B. Its topological dual space will be denoted by B'. By $\langle\cdot,\cdot\rangle_{B',B}$ we denote the duality product between B' and B.

2. Existence of solutions

We first introduce the notion of weak solution of problem (1.1).

Definition 2.1. Let $T\le\infty$, and assume $u_0\in\mathcal{L}^4(\Omega)\cap\mathcal{H}^1(\Omega)$ and $F_0\in\mathcal{L}^2(-\tau,0)$. A function $u:\Omega\times(-\tau,T)\to\mathbb{C}$ is called a *weak solution* of problem (1.1) if

$$\begin{gathered}
u\in C([0,T]:\mathcal{L}^2(\Omega))\cap\mathcal{L}^2(0,T:\mathcal{H}^1(\Omega))\cap\mathcal{L}^4(0,T:\mathcal{L}^4(\Omega))\cap\mathcal{L}^2(-\tau,0:\mathcal{L}^1(\Omega)),\\
u_t\in\mathcal{L}^2(0,T:(\mathcal{H}^1(\Omega))'),
\end{gathered}$$

for every $t \in (0,T)$

$$
\begin{aligned}
&\langle \frac{\partial}{\partial t} u, \varphi \rangle_{(\mathcal{H}^1(\Omega))' \times \mathcal{H}^1(\Omega)} \\
&= (1 - i\omega) \int_\Omega u\bar{\varphi} dx - (1 + i\beta) \int_\Omega |u|^2 u\bar{\varphi} dx - (1 + i\epsilon) \int_\Omega \nabla u \cdot \nabla \bar{\varphi} dx \\
&\quad + F(u(t-\tau)) \int_\Omega \bar{\varphi} dx, \quad \forall \varphi \in \mathcal{H}^1(\Omega), \\
&\qquad u(x,0) = u_0(x) \quad \text{in } \mathcal{L}^2(\Omega)
\end{aligned} \tag{2.1}
$$

and

$$
F(u(\cdot)) = F_0(\cdot) \text{ in } \mathcal{L}^2(-\tau, 0),
$$

where $F(u(t-\tau))$ is given by (1.2).

In the case of problem (1.4) a stronger notion of weak solution must be introduced.

Definition 2.2. Let $T \leq \infty$, and assume that $U_0 \in \mathcal{C}([-\tau, 0] : \mathcal{L}^2(\Omega))$, $U_0(\cdot, 0) \in \mathcal{L}^{m+1}(\Omega) \cap \mathcal{H}^1(\Omega)$ $(m > 0)$. A function $u : \Omega \times (-\tau, T) \to \mathbb{C}$ is called a *weak solution* of (1.4) if

$$
\begin{gathered}
u \in \mathcal{L}^2(0, T : \mathcal{H}^1(\Omega)) \cap \mathcal{L}^{m+1}(0, T : \mathcal{L}^{m+1}(\Omega)) \cap \mathcal{L}^2(-\tau, 0 : \mathcal{L}^2(\Omega)), \\
u_t \in \mathcal{L}^2(0, T : (\mathcal{H}^1(\Omega))'),
\end{gathered}
$$

for every $t \in (0,T)$

$$
\begin{aligned}
&\langle \frac{\partial}{\partial t} u, \varphi \rangle_{\mathcal{H}^{-1}(\Omega) \times \mathcal{H}^1(\Omega)} \\
&= (1 - i\omega) \int_\Omega u\bar{\varphi} dx - (1 + i\beta) \int_\Omega |u|^{m-1} u\bar{\varphi} dx - (1 + i\epsilon) \int_\Omega \nabla u \cdot \nabla \bar{\varphi} dx \\
&\quad + \int_\Omega F(u(x, t-\tau)) \bar{\varphi} dx, \quad \forall \varphi \in \mathcal{H}^1(\Omega) \\
&\qquad u = U_0 \text{ in } \mathcal{C}([-\tau, 0] : \mathcal{L}^2(\Omega)),
\end{aligned} \tag{2.2}
$$

where $F(u(x, t-\tau))$ is given by (1.3).

It is useful to rewrite the complex Gingzburg-Landau problem (1.1) in terms of the real components (u_1, u_2) of the solution u, i.e. $u = u_1 + iu_2$. The associated real system in Q is

$$
\begin{gathered}
\frac{\partial u_1}{\partial t} = \Delta u_1 - \epsilon \Delta u_2 + (u_1^2 + u_2^2)(-u_1 + \beta u_2) + u_1 + \omega u_2 + F_1(u(x, t-\tau)), \quad \text{in } Q, \\
\frac{\partial}{\partial t} u_2 = \epsilon \Delta u_1 + \Delta u_2 - (u_1^2 + u_2^2)(\beta u_1 + u_2) + u_2 - \omega u_1 + F_2(u(x, t-\tau)), \quad \text{in } Q, \\
u_1(x,t) = \mathrm{Re}(U_0(x,t)) \quad \text{and} \quad u_1(x,t) = \mathrm{Im}(U_0(x,t)), \quad \text{in } (-\tau, 0) \times \Omega, \\
\frac{\partial u_1}{\partial \vec{n}} = \frac{\partial u_2}{\partial \vec{n}} = 0, \quad \text{on } \partial\Omega \times (0,T).
\end{gathered}
$$

for F_1 and F_2 defined in (1.2) as the real and imaginary part of F respectively.

The main result of this section is stated as follows.

Theorem 2.3. *(i) Assume $F_0 \in \mathcal{L}^2(-\tau, 0)$ and let u_0 be such that $u_0 \in \mathcal{L}^4(\Omega) \cap \mathcal{H}^1(\Omega)$. Then there exists at least a weak solution to* (1.1) *in* $(0, \infty)$.

(ii) Assume $U_0 \in \mathcal{C}([-\tau,0]:\mathcal{L}^2(\Omega))$, $U_0(x,0) \in \mathcal{L}^{m+1}(\Omega)\cap\mathcal{H}^1(\Omega)$. Then, there exists at least a weak solution to (1.4) in $(0,\infty)$.

Remark 2.4. Although there are some works in the literature dealing with a partial information on the initial history (see, e.g. [1] and its references) we point out that the initial information required in problem (1.1) is weaker than in those series of works.

To prove the existence of weak solution of (1.1) we first obtain some a priori estimates in the following lemma.

Lemma 2.5. *Let $T<\infty$ and assume $F_0 \in \mathcal{L}^2(-\tau,0)$ and let u_0 be such that*

$$u_0 \in \mathcal{L}^4(\Omega)\cap\mathcal{H}^1(\Omega).$$

Let $u \in \mathcal{L}^2(0,T:\mathcal{L}^4(\Omega))$ be a weak solution of (1.1). Then

$$u \in \mathcal{L}^\infty(0,T:\mathcal{L}^2(\Omega)). \tag{2.3}$$

Moreover the norm of u in this space, as well as in the spaces $\mathcal{L}^2(0,T:\mathcal{H}^1(\Omega))$ and $\mathcal{L}^2(0,T:\mathcal{L}^4(\Omega))$ has a bound only depending of F_0, u_0, μ, τ, β and T.

Proof. Let u be a weak solution of (1.1). Let $\varphi = u$ in (2.1) and let $t \in (0,\tau)$. Taking the real part of the resultant equation,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx \\ &= \int_\Omega |u|^2dx - \int_\Omega |u|^4dx - \int_\Omega |\nabla u|^2dx + \operatorname{Re}\Big\{\Big(\int_\Omega \bar{u}dx\Big)F_0(t-\tau)\Big\}. \end{aligned} \tag{2.4}$$

Applying Hölder and Young inequalities we obtain

$$\operatorname{Re}\Big\{\Big(\int_\Omega \bar{u}dx\Big)F_0(t-\tau)\Big\} \le \frac{1}{2}\int_\Omega |u(t)|^2dx + \frac{|\Omega|}{2}|F_0(t-\tau)|^2. \tag{2.5}$$

Then, from the last inequalities and equation (2.4), it results that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx \le \frac{3}{2}\int_\Omega |u|^2dx - \int_\Omega |u|^4dx - \int_\Omega |\nabla u|^2dx + \frac{|\Omega|}{2}|F_0(t-\tau)|^2.$$

That is

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|^2_{\mathcal{L}^2(\Omega)} \\ &\le 3\|u(t)\|^2_{\mathcal{L}^2(\Omega)} - 2\|u(t)\|^4_{\mathcal{L}^4(\Omega)} - 2\|\nabla u(t)\|^2_{\mathcal{L}^2(\Omega)} + |\Omega||F_0(t-\tau)|^2. \end{aligned} \tag{2.6}$$

Step 1. We first prove that $u \in \mathcal{L}^\infty(0,\tau:\mathcal{L}^2(\Omega))$. Since $\mathcal{L}^4(\Omega)\hookrightarrow\mathcal{L}^2(\Omega)$, we have

$$\|u(t)\|^4_{\mathcal{L}^4(\Omega)} = \int_\Omega |u(t)|^4dx \ge \frac{1}{|\Omega|}(\int_\Omega |u(t)|^2dx)^2 = \frac{1}{|\Omega|}(\|u(t)\|^2_{\mathcal{L}^2(\Omega)})^2 \tag{2.7}$$

and thanks to (2.6), we obtain

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|^2_{\mathcal{L}^2(\Omega)} \\ &\le 3\|u(t)\|^2_{\mathcal{L}^2(\Omega)} - \frac{2}{|\Omega|}\|u(t)\|^4_{\mathcal{L}^2(\Omega)} - 2\|\nabla u(t)\|^2_{\mathcal{L}^2(\Omega)} + |\Omega||F_0(t-\tau)|^2. \end{aligned} \tag{2.8}$$

Denoting $f(t) := \|u(t)\|^2_{\mathcal{L}^2(\Omega)} (\geq 0)$, and dropping $-2\|\nabla u(t)\|^2_{\mathcal{L}^2(\Omega)}$ in the last equation, it results

$$\frac{d}{dt} f(t) \leq 3f(t) - \frac{2}{|\Omega|} f^2(t) + |\Omega| |F_0(t-\tau)|^2, \quad \forall t \in (0,\tau). \tag{2.9}$$

Let K_1 be a positive constant defined by

$$K_1 := e^{3\tau} \Big[\|u_0\|^2_{\mathcal{L}^2(\Omega)} + |\Omega| \int_0^\tau e^{-3s} |F_0(s-\tau)|^2 ds \Big]. \tag{2.10}$$

Then by Gronwall's lemma, we obtain that $f(t) \leq K_1$ for $t \in (0,\tau)$.

Step 2. In this step we prove (2.3), i.e. $u \in \mathcal{L}^\infty(0,T : \mathcal{L}^2(\Omega))$. If $t \in [\tau, 2\tau)$ we argue in a similar way but taking $\varphi(\cdot) = u(\cdot,t)$ in (2.1) we obtain now that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx &= \int_\Omega |u|^2 dx - \int_\Omega |u|^4 dx - \int_\Omega |\nabla u|^2 dx \\ &\quad + \operatorname{Re}\Big\{ \Big(\int_\Omega \bar{u} dx \Big) \Big(\frac{\mu e^{i\chi_0}}{|\Omega|} \int_\Omega u(x,t-\tau) dx \Big) \Big\}. \end{aligned} \tag{2.11}$$

Then

$$\Big|\frac{\mu e^{i\chi_0}}{|\Omega|} \int_\Omega u(x,t-\tau) dx\Big| \leq \frac{\mu}{|\Omega|} \Big| \int_\Omega u(x,t-\tau) dx\Big| \leq \frac{\mu}{|\Omega|^{1/2}} \|u(t-\tau)\|_{\mathcal{L}^2(\Omega)}$$

and therefore

$$\begin{aligned} &\operatorname{Re}\Big\{ \Big(\int_\Omega \bar{u} dx \Big) \Big(\frac{\mu e^{i\chi_0}}{|\Omega|} \int_\Omega u(x,t-\tau) dx \Big) \Big\} \\ &\leq \frac{\mu}{|\Omega|} \Big| \int_\Omega \bar{u} dx \Big| \, \Big| \int_\Omega u(x,t-\tau) dx \Big| \\ &\leq \frac{\mu}{|\Omega|} \Big[|\Omega|^{1/2} \Big(\int_\Omega |u(t)|^2 dx \Big)^{1/2} \Big] \Big[|\Omega|^{1/2} \Big(\int_\Omega |u(t-\tau)|^2 dx \Big)^{1/2} \Big] \\ &= \mu \Big(\int_\Omega |u(t)|^2 dx \Big)^{1/2} \Big(\int_\Omega |u(t-\tau)|^2 dx \Big)^{1/2}. \end{aligned}$$

Finally, by Young's inequality,

$$\begin{aligned} &\operatorname{Re}\Big\{ \Big(\int_\Omega \bar{u} dx \Big) \Big(\frac{\mu e^{i\chi_0}}{|\Omega|} \int_\Omega u(x,t-\tau) dx \Big) \Big\} \\ &\leq \frac{1}{2} \int_\Omega |u(t)|^2 dx + \frac{\mu^2}{2} \int_\Omega |u(t-\tau)|^2 dx. \end{aligned} \tag{2.12}$$

Then, from the last inequalities and equation (2.4), it results that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx \leq \frac{3}{2} \int_\Omega |u|^2 dx - \int_\Omega |u(t)|^4 dx - \int_\Omega |\nabla u|^2 dx + \frac{\mu^2}{2} \int_\Omega |u(t-\tau)|^2 dx.$$

That is

$$\begin{aligned} &\frac{d}{dt} \|u(t)\|^2_{\mathcal{L}^2(\Omega)} \\ &\leq 3\|u(t)\|^2_{\mathcal{L}^2(\Omega)} - 2\|u(t)\|^4_{\mathcal{L}^4(\Omega)} - 2\|\nabla u(t)\|^2_{\mathcal{L}^2(\Omega)} + \mu^2 \|u(t-\tau)\|^2_{\mathcal{L}^2(\Omega)}. \end{aligned} \tag{2.13}$$

Notice that if $t \in (\tau, 2\tau)$ then $t-\tau \in (0,\tau)$ we have

$$\|u(t-\tau)\|^2_{\mathcal{L}^2(\Omega)} \leq K_1 \,.$$

Therefore, using again Gronwall's inequality we arrive at

$$\|u(t)\|^2_{\mathcal{L}^2(\Omega)} \le K_2, \tag{2.14}$$

for some $K_2 > 0$ depending only on $F_0, u_0, \mu, \tau, \Omega$ and T. Iterating this argument we obtain that $u \in \mathcal{L}^\infty(0, T : \mathcal{L}^2(\Omega))$ and, in particular, $u \in\mathcal{L}^2(Q)$ (for all $T < \infty$).

Step 3. Here, we obtain $\nabla u \in\mathcal{L}^2(0, T : (\mathcal{L}^2(\Omega))^N)$. Again we start arguing on $(0, \tau)$. By integrating inequality (2.6) over $(0, t)$, for $t \in (0, \tau)$ we obtain

$$\begin{aligned}\int_0^t \frac{d}{dt}\|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds &\le 3\int_0^t \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds - 2\int_0^t \|u(s)\|^4_{\mathcal{L}^4(\Omega)} ds\\ &\quad - 2\int_0^t \|\nabla u(s)\|^2_{\mathcal{L}^2(\Omega)} ds + |\Omega|\int_0^t |F_0(s-\tau)|^2 ds\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^t \|\nabla u(s)\|^2_{\mathcal{L}^2(\Omega)} ds\\ &\le \frac{1}{2}\|u(0)\|^2_{\mathcal{L}^2(\Omega)} - \frac{1}{2}\|u(t)\|^2_{\mathcal{L}^2(\Omega)} + \frac{3}{2}\int_0^t \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds\\ &\quad - \int_0^t \|u(s)\|^4_{\mathcal{L}^4(\Omega)} ds + \frac{|\Omega|}{2}\int_0^t |F_0(s-\tau)|^2 ds.\end{aligned} \tag{2.15}$$

Since

$$|\Omega|\int_0^t \|u(s)\|^4_{\mathcal{L}^2(\Omega)} ds \ge \Big(\int_0^t \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds\Big)^2$$

we obtain that

$$\int_0^t \|\nabla u(s)\|^2_{\mathcal{L}^2(\Omega)} ds \le \frac{|\Omega|}{2}\|F_0\|^2_{\mathcal{L}^2((-\tau,0),\mathbb{C})} + \frac{3}{2}\|u\|^2_{\mathcal{L}^2(Q)} - \frac{2}{|\Omega|}\|u\|^4_{\mathcal{L}^2(Q)} ds.$$

Analogously, when $t \in (\tau, T)$, By integration over $(0, t)$, (2.13) becomes

$$\begin{aligned}\int_0^t \frac{d}{dt}\|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds &\le 3\int_0^t \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds - 2\int_0^t \|u(s)\|^4_{\mathcal{L}^4(\Omega)} ds\\ &\quad - 2\int_0^t \|\nabla u(s)\|^2_{\mathcal{L}^2(\Omega)} ds + \mu^2\int_0^t \|u(s-\tau)\|^2_{\mathcal{L}^2(\Omega)} ds,\end{aligned}$$

and since

$$\int_0^{t-\tau} \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds = \int_{-\tau}^0 \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds + \int_0^{t-\tau} \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds,$$

we obtain the desired estimate by using the previous step.

Step 4. From (2.15), we obtain

$$\begin{aligned}&\int_0^t \|u(s)\|^4_{\mathcal{L}^4(\Omega)} ds\\ &\le \frac{1}{2}\|u(0)\|^2_{\mathcal{L}^2(\Omega)} + \frac{3}{2}\int_0^t \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds + \frac{\mu^2}{2}\int_{-\tau}^{t-\tau} \|u(s)\|^2_{\mathcal{L}^2(\Omega)} ds,\end{aligned} \tag{2.16}$$

so that $u \in \mathcal{L}^4(0, T : \mathcal{L}^4(\Omega))$. Finally, from the above estimate and (2.14) we have the last assertion of the lemma. $\square$

Lemma 2.6. *Let $T < \infty$ and assume $u_0 \in \mathcal{L}^4(\Omega) \cap \mathcal{H}^1(\Omega)$, $F_0 \in \mathcal{L}^2(-\tau, 0)$. Let u be a "strong" solution of* (1.1)*. Then*

$$u \in \mathcal{H}^1(0, T : \mathcal{L}^2(\Omega)) \cap \mathcal{L}^\infty(0, T : \mathcal{H}^1(\Omega)) \cap \mathcal{L}^\infty(0, T : \mathcal{L}^4(\Omega)).$$

Moreover, there exists $K > 0$ such that

$$\begin{aligned} &\frac{1}{2}\int_0^t \int_\Omega |u_t|^2 dx + \frac{1}{4}\int_\Omega |u(t)|^4 dx + \frac{1}{4}\int_\Omega |\nabla u(t)|^2 dx \\ &\le K \int_{-\tau}^0 |F_0(s)|^2 ds + \frac{1}{4}\int_\Omega |u_0(x)|^4 dx + \frac{1}{4}\int_\Omega |\nabla u_0(x)|^2 dx \end{aligned} \tag{2.17}$$

for almost every $t \in (0, T)$.

Proof. The proof is similar to the proof of Lemma 2.5 (step 1). We assume that u is a "strong" solution of (1.1) (i.e., such that $u \in \mathcal{H}^1(0, T : \mathcal{L}^2(\Omega))$) and we take $\varphi = u_t$ in the identity (2.1). By taking the real part of the resultant equation, then, for all $t \in (0, \tau)$

$$\begin{aligned} \int_\Omega |u_t|^2 dx = \frac{d}{dt}\frac{1}{2}\int_\Omega |u|^2 dx - \frac{d}{dt}\frac{1}{4}\int_\Omega |u|^4 dx - \frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u|^2 dx \\ + \operatorname{Re}\Big\{\Big(\int_\Omega \bar{u}_t dx\Big) F_0(t-\tau)\Big\}. \end{aligned}$$

Since

$$\begin{aligned} \operatorname{Re}\Big\{\Big(\int_\Omega \bar{u}_t dx\Big) F_0(t-\tau)\Big\} &\le |\int_\Omega \bar{u}_t(t) dx||F_0(t-\tau)| \\ &\le |\Omega|^{1/2}\Big(\int_\Omega |\bar{u}_t(t)|^2 dx\Big)^{1/2} |F_0(t-\tau)|, \end{aligned}$$

by Young's inequality,

$$\operatorname{Re}\Big\{\Big(\int_\Omega \bar{u}_t dx\Big) F_0(t-\tau)\Big\} \le \frac{1}{2}\int_\Omega |\bar{u}_t(t)|^2 dx + \frac{|\Omega|^{1/2}}{2}|F_0(t-\tau)|^2.$$

Then, from the last inequalities it results

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u_t|^2 dx \\ &\le \frac{d}{dt}\frac{1}{2}\int_\Omega |u|^2 dx - \frac{d}{dt}\frac{1}{4}\int_\Omega |u(t)|^4 dx - \frac{d}{dt}\frac{1}{4}\int_\Omega |\nabla u|^2 dx + \frac{|\Omega|}{2}|F_0(t-\tau)|^2, \end{aligned}$$

after integration we obtain the estimate (2.17) result on $(0, \tau)$ with $K = |\Omega|$. By iterating, we obtain the desired estimate on $(0, T)$. $\square$

Proof of Theorem 2.3. To prove part (i) we use Galerkin's method. We consider the set of pairs $(\lambda_k, \varphi_k)_{k \ge 1}$ of eigenvalues and eigenfunctions of the $-\Delta$ operator with Neumann boundary conditions such that

$$\int_\Omega \varphi_i \varphi_j dx = \delta_{i,j}.$$

Let $\mathcal{V}_m$ be the complex vector space spanned by $\{\varphi_1, \dots, \varphi_m\}$. For all $v \in \mathcal{V}_m$, $v = \sum_{j=1}^m v^j \varphi_j$. The approximate problem on the interval $(0, \tau)$ is the following: to find

$$u_m \in \mathcal{L}^2(0, \tau : \mathcal{V}_m), \quad u_m(t) = \sum_{j=1}^m u_m^j(t)\varphi_j,$$

satisfying

$$\begin{aligned}\int_\Omega \bar{\varphi}\frac{\partial u_m}{\partial t}dx &= (1-i\omega)\int_\Omega u_m\bar{\varphi}dx - (1+i\beta)\int_\Omega |u_m|^2 u_m\bar{\varphi}dx \\ &\quad - (1+i\epsilon)\int_\Omega \nabla u_m\cdot\nabla\bar{\varphi}dx + F_0(t-\tau)\int_\Omega \bar{\varphi}dx,\end{aligned} \tag{2.18}$$

for all $\varphi \in L^2(0,\tau : \mathcal{V}_m)$, and

$$u_m(0) = u_{m0} := \sum_{j=1}^m u_{0m}^j \varphi_j \quad \text{with} \quad u_{0m}^j = \int_\Omega u_0(x)\varphi_j dx. \tag{2.19}$$

The approximate problem becomes a coupled system of m non homogeneous ODEs on the coefficients $u_m^j(t)$. The standard results on the existence and uniqueness of local solutions with a right hand side in $L^2(0,\tau)$ apply to (2.18)-(2.19). Moreover, the a priori estimates found in Lemmata 2.5 and 2.6 also holds for this special solutions and thus we know that there exist some positive constants K_i (only dependent on the norms of u_0 in $\mathcal{L}^4(\Omega)\cap\mathcal{H}^1(\Omega)$ and the norm of $F_0(t)$ in $\mathcal{L}^2(-\tau,0)$) such that

$$\begin{gathered}\|u_m(t)\|_{\mathcal{L}^2(\Omega)} \le K_1^{1/2}, \\ \|u_m\|_{\mathcal{L}^2((0,\tau)\times\Omega)} \le |\tau K_1|^{1/2}, \\ \|\nabla u_m\|_{\mathcal{L}^2((0,\tau)\times\Omega)} \le [K_2]^{1/2}, \\ \|u_m(t)\|_{\mathcal{L}^4(\Omega)} \le K_3, \\ \int_0^\tau \int_\Omega |\frac{\partial u_m}{\partial t}|^2 dx \le K_4.\end{gathered}$$

For $N \le 3$, we have that $\mathcal{H}^1(\Omega) \subset \mathcal{L}^k(\Omega)$ for $k = 2$ and 4 is a compact embedding. Then, thanks to Aubin-Lions Lemma we claim that there exists a subsequence $\{u_j\}_{j\in\mathbb{N}}$ of $\{u_m\}_{m\in\mathbb{N}}$, such that

$$\begin{gathered}u_j \to u^*, \quad \text{in } \mathcal{L}^2(0,\tau : \mathcal{L}^2(\Omega)); \\ u_j \to u^*, \quad \text{in } \mathcal{L}^4(0,\tau : \mathcal{L}^4(\Omega)); \\ \frac{\partial u_j}{\partial t} \rightharpoonup \frac{\partial u^*}{\partial t}, \quad \text{in } \mathcal{L}^2(0,\tau : \mathcal{L}^2(\Omega)); \\ u_j \rightharpoonup u^*, \quad \text{in } \mathcal{L}^2(0,\tau : \mathcal{H}^1(\Omega)).\end{gathered}$$

Then we take limits in the weak formulation (2.18) to obtain the existence of solutions to (1.1) in $(0,\tau)$. Moreover, as consequence of Aubin-Lions Lemma, we also obtain that $u \in C([0,\tau] : \mathcal{L}^2(\Omega))$. By an iterative argument on the intervals $(n\tau,(n+1)\tau)$ with $n \ge 1$ we obtain the existence of solution on the whole interval $(0,T)$, for all fixed $T > 0$.

The proof of part (ii) is entirely similar when $m \ge 1$ (it suffices to apply Hölder and Young inequalities in their general version with the corresponding exponents $p, p' \in (1,+\infty)$, $1/p + 1/p' = 1$). Notice that, in fact $u_j \to u^*$, in $\mathcal{L}^2(-\tau,\tau : \mathcal{L}^2(\Omega))$ and since $u \in \mathcal{C}([0,\tau] : \mathcal{L}^2(\Omega))$ and $U_0 \in \mathcal{C}([-\tau,0] : \mathcal{L}^2(\Omega))$ we conclude (as in of [29, Theorem 1.1]) that $u \in \mathcal{C}([-\tau,\tau] : \mathcal{L}^2(\Omega))$ and that $u = U_0$ in $\mathcal{C}([-\tau,0] : \mathcal{L}^2(\Omega))$. Finally, to treat the case $m \in (0,1)$ it is enough to start by considering the initial interval $(0,\tau)$ and to apply the existence results given in [2] for a right hand side $f(x,t)$ in $\mathcal{L}^2(0,\tau : \mathcal{L}^2(\Omega))$ and then to proceed by iteration on the rest of intervals

$(n\tau,(n+1)\tau)$ with $n\geq 1$. Notice that although the formulation of the equation considered in [2] was slightly different

$$\frac{\partial u}{\partial t}-e^{i\gamma}\Delta u+e^{i\gamma}|u|^{m-1}u=f(x,t)$$

their key assumption $\gamma\in(-\frac{\pi}{2},\frac{\pi}{2})$ allows to extend their main arguments to our framework in which the diffusion coefficient is $(1+i\epsilon)$ and the absorption coefficient is $(1+i\beta)$ (i.e. always with a positive real part). $\square$

3. Uniqueness of a solution

In this section we prove the uniqueness of a weak solution of (1.1) and (1.4). The proof follows a contradiction argument using the estimates obtained in the previous section. We recall that in all the paper we are assuming that $N\leq 3$.

Theorem 3.1. *Assume the conditions on F_0, u_0, and U_0 given in parts (i) and (ii) of Theorem 2.3. Then problems* (1.1) *and* (1.4) *have at most one weak solution for the following cases:*

- $m\in[1,\infty)$, *if* $N=1,2$,
- $m\in[1,5)$, *if* $N=3$,
- $m\in(0,1)$, *if* $N=1,2,3$ *provided* β *satisfies*

$$|\beta|\leq\frac{1-m}{2m^{1/2}}. \tag{3.1}$$

Proof. Let us start by considering problem (1.1). We assume there exists two solutions $u=u_1+iu_2$ and $v=v_1+iv_2$. We consider $U=U_1+iU_2$ defined by $U=u-v=u_1-u_2+i(u_2-v_2)$, then it satisfies

$$\begin{aligned}\frac{\partial U}{\partial t}&=(1-i\omega)U-(1+i\beta)(|u|^{m-1}u-|v|^{m-1}v)+(1+i\epsilon)\Delta U\\&\quad+\mu e^{i\chi_0}(L)^{-1}\int_\Omega U(x,t-\tau)dx+\nu U(t-\tau),\quad\text{in }\Omega\times(0,T),\end{aligned}\tag{3.2}$$

$$U(t)=0,\quad\text{on } t\in(-\tau,0), \tag{3.3}$$

$$\frac{\partial U}{\partial\vec n}=0,\quad\text{on }\partial\Omega\times(0,T). \tag{3.4}$$

Thus, on the initial interval $(0,\tau)$ the weak solution U of (3.2)–(3.4) satisfies

$$\begin{aligned}&\langle\frac{\partial}{\partial t}U,\varphi\rangle_{\mathcal{W}^{-1,2}(\Omega)\times\mathcal{W}_0^{1,2}(\Omega)}\\&=(1-i\omega)\int_\Omega U\bar\varphi dx-(1+i\beta)\int_\Omega(|u|^{m-1}u-|v|^{m-1}v)\bar\varphi dx\\&\quad-(1+i\epsilon)\int_\Omega\nabla U\cdot\nabla\bar\varphi dx,\quad\forall\varphi\in\mathcal{H}^1(\Omega).\end{aligned}$$

Thanks to Lemmas 2.5 and 2.6, we replace φ by U in the definition of weak solution and take the real part of the identity to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\frac12\int_\Omega|U|^2dx&=\int_\Omega|U|^2dx-\mathrm{Re}\Big\{(1+i\beta)\int_\Omega(|u|^{m-1}u-|v|^{m-1}v)\bar Udx\Big\}\\&\quad-(1+i\epsilon)\int_\Omega|\nabla U|^2dx,\end{aligned}\tag{3.5}$$

for every $t\in(0,\tau)$. We now consider two cases: $m\in(0,1)$ and $m\in[1,3]$.

Case 1: $m \in (0,1)$**.** Note that, for any two complex numbers x, y and $m > 0$ we have that (see [25, Lemma 2.2] and [2, 12])

$$\operatorname{Re}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] = |x|^{m+1} + |y|^{m+1} - (|x|^{m-1} + |y|^{m-1}) \operatorname{Re}(x\bar{y})$$

i.e.

$$\operatorname{Re}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] = |x|^{m+1} + |y|^{m+1} - (|x|^{m-1} + |y|^{m-1})|x||y| \cos\omega$$

where $\omega = arg(x\bar{y})$. In view of Young inequality we obtain

$$\operatorname{Re}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] \geq 0.$$

In the same way we have

$$\operatorname{Im}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] = (|x|^{m-1} - |y|^{m-1})|x||y| \sin\omega.$$

Thanks to [25], Lemma 2.2, we have that

$$\operatorname{Im}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] \leq \frac{1-m}{2m^{1/2}} \operatorname{Re}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})]$$

then, thanks to (3.1)we have $|\beta| \leq \frac{|m-1|}{2m^{1/2}}$ and then

$$\begin{aligned}
&\operatorname{Re}[(1+i\beta)(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] \\
&= \operatorname{Re}[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] - \beta Im[(|x|^{m-1}x - |y|^{m-1}y)(\bar{x} - \bar{y})] \geq 0.
\end{aligned}$$

Therefore

$$\operatorname{Re}\Big\{(1+i\beta)\int_\Omega (|u|^{m-1}u - |v|^{m-1}v)\bar{U}dx\Big\} \leq 0,$$

and (3.5) becomes

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |U|^2 dx \leq \int_\Omega |U|^2 dx - (1+i\epsilon)\int_\Omega |\nabla U|^2 dx, \tag{3.6}$$

Case 2: $m \geq 1$**.** In this case the real part of $(1+i\beta)\int_\Omega (|u|^{m-1}u - |v|^{m-1}v)\bar{U}dx$ satisfies

$$\operatorname{Re}\Big\{(1+i\beta)\int_\Omega (|u|^{m-1}u - |v|^{m-1}v)\bar{U}dx\Big\} \leq C(\beta,m)\int_\Omega (|u|^{m-1} + |v|^{m-1})|U|^2 dx.$$

Substituting in (3.5) it results,

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |U|^2 dx \leq C(\beta,m)\int_\Omega (1 + |u|^{m-1} + |v|^{m-1})|U|^2 dx - \int_\Omega |\nabla U|^2 dx.$$

Since u and v are weak solutions of the problem, and $u, v \in \mathcal{L}^\infty(0, T : \mathcal{H}^1(\Omega))$ and $\mathcal{H}^1(\Omega) \subset \mathcal{L}^{m+1}(\Omega)$, it results that

$$\begin{aligned}
&\int_\Omega (1 + |u|^{m-1} + |v|^{m-1})|U|^2 dx \\
&\leq (\|u\|_{\mathcal{L}^\infty(0,T:\mathcal{L}^{m+1}(\Omega))} + \|v\|_{\mathcal{L}^\infty(0,T:\mathcal{L}^{m+1}(\Omega))})^{\frac{m-1}{m+1}} | \int_\Omega |U|^{m+1} dx|^{\frac{2}{m+1}} \\
&\leq c\Big|\int_\Omega |U|^{m+1} dx\Big|^{\frac{2}{m+1}}.
\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\int_\Omega |U|^2 dx \leq c\Big|\int_\Omega |U|^{m+1} dx\Big|^{\frac{2}{m+1}} - \int_\Omega |\nabla U|^2 dx. \tag{3.7}$$

So, if $m=1$ we have

$$\frac{\partial}{\partial t}\int_\Omega |U|^2dx \le c\Big|\int_\Omega |U|^2dx\Big|. \tag{3.8}$$

And for $m>1$ and $N\le 3$, we apply Gagliardo Nirenberg inequality to obtain

$$\|U\|_{\mathcal{L}^{m+1}(\Omega)} \le c\|U\|^{\alpha}_{\mathcal{H}^1(\Omega)}\|U\|^{1-\alpha}_{\mathcal{L}^2(\Omega)} + c\|U\|_{\mathcal{L}^2(\Omega)}$$

for

$$\alpha = N\big(\frac{1}{2}-\frac{1}{m+1}\big).$$

Then, after some computations we obtain

$$\|U\|^2_{\mathcal{L}^{m+1}(\Omega)} \le \epsilon\|U\|^2_{\mathcal{H}^1(\Omega)} + c(\epsilon)\|U\|^2_{\mathcal{L}^2(\Omega)}.$$

We replace in (3.7) for ϵ small enough to obtain

$$\frac{\partial}{\partial t}\int_\Omega |U|^2dx \le c\int_\Omega |U|^2dx. \tag{3.9}$$

To complete the proof, we apply Gronwall's lemma to (3.6), (3.8) and (3.9) to obtain uniqueness of solutions on the interval $(0,\tau)$. By induction in intervals of the form $(\frac{n\tau}{2},\frac{(n+1)\tau}{2})$, we obtain again the uniqueness of solutions for $t\in(0,\infty)$. □

Remark 3.2. The results of this and the precedent section also holds, with minor changes, for other type of boundary conditions such as, Dirichlet boundary conditions or periodic boundary conditions (as considered in [5, 6, 14]). We also point out that the assumption $(1+i\epsilon)$ on the coefficient of the complex diffusion operator is absolutely crucial since when the real part of such a coefficient vanishes the equation becomes a nonlinear Schrödinger delayed equation and some additional conditions on the coefficient of the nonlinear part (in this paper assumed of the form $(1+i\beta)$) are required (see, e.g. the existence and uniqueness results for the case $m\in(0,1)$ given in [12]).

4. Properties of solutions of the delayed problem when $m\in(0,1)$

The main goal of this section is to explain how to adapt to the case of delayed problems the energy methods presented in the monograph [3] and, more concretely, their adaptation to complex Ginzburg-Landau equations with absorption made in [2]. Because of the presence of the "bad term" $-(1-i\omega)u$ in the equation in all this section we shall need a extra information on the solutions: we will always assume that the solution is bounded. This condition could be avoided in absence of such a term in the equation.

Our first result concerns the so called *finite extinction time.* This property is of interest in many different contexts. For instance in Control Theory it usually associated to the "zero exact controllability property".

Theorem 4.1. *Let $m\in(0,1)$ and $\alpha\in(0,1)$ such that*

$$\|u\|_{L^\infty(Q)} \le |1-\alpha|^{\frac{1}{1-m}}. \tag{4.1}$$

(i) Assume that

$$\|u_0\|^2_{\mathcal{L}^2(\Omega)} \text{ is small enough.} \tag{4.2}$$

Assume also that there exists $t^*\in(0,\tau)$ such that

$$|F_0(s)|^{\frac{m+1}{m}} \le c\,[(t^*-\tau)-s]_+^{\frac{\delta}{1-\delta}} \quad \text{for a.e. } s\in(-\tau,0), \tag{4.3}$$

for some $c > 0$ and some $\delta \in (0,1)$. Then any bounded solution of the nonlocal problem (1.4) (i.e. with $\nu = 0$) satisfies

$$\|u(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)} \le c\kappa[t^* - t]_+^{\frac{1}{1-\delta}} \quad \text{for all } t \in [0,\tau)$$

for some $c > 0$. In particular $u(\cdot,t) \equiv 0$ in Ω for all $t \in [t^*,\tau)$. In addition, $u(\cdot,t) \equiv 0$ in Ω for all $t \in [nt^*, n\tau)$ for all $n \in \mathbb{N}$.

(ii) Assume that

$$\|U_0(\cdot,s)\|_{\mathcal{L}^2(\Omega)}^{\frac{2(m+1)}{m}} \le \kappa\,[(t^* - \tau) - s]_+^{\frac{\delta}{1-\delta}} \ \text{ for a.e. } s \in (-\tau,0), \tag{4.4}$$

for some $\kappa > 0$ and some $\delta \in (0,1)$. Then any bounded solution of the (1.4), with $\nu > 0$, satisfies that

$$\|u(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)} \le c\kappa[t^* - t]_+^{\frac{1}{1-\delta}} \ \text{ for all } t \in [0,\tau)$$

for some $c > 0$. In particular $u(\cdot,t) \equiv 0$ in Ω for all $t \in [t^*,\tau)$. In addition $u(\cdot,t) \equiv 0$ in Ω for any $t \in [nt^*, n\tau)$, for all $n \in \mathbb{N}$.

Roughly speaking, for the proof of this results we follow the energy method presented in Section 6.2 of the monograph [3] (see the applications to complex equations made in [2] and [12]). In fact, we will use the following improvement of a suitable energy inequality.

Lemma 4.2 ([12]). *Let $y \in W^{1,1}_{\rm loc}\big([0,\infty);\mathbb{R}\big)$ with $y \ge 0$ over $[0,\infty)$, $\delta \in (0,1)$, $\alpha, T_0 > 0$, and*

$$y_\star = (\kappa\,\delta^\delta(1-\delta))^{\frac{1}{1-\delta}}, \tag{4.5}$$

$$x_\star = (\kappa\,\delta\,(1-\delta)\,T_0)^{\frac{1}{1-\delta}}. \tag{4.6}$$

If $y(0) \le x_\star$, and if for almost every $t > 0$,

$$y'(t) + \kappa\, y(t)^\delta \le y_\star(T_0 - t)_+^{\frac{\delta}{1-\delta}},$$

then there exists $k^* > 0$ such that

$$y(t) \le k^*(T_0 - t)_+^{\frac{1}{1-\delta}} \ \text{ for all } t > 0. \tag{4.7}$$

Proof of Theorem 4.1. As in the proof of Lemma 2.5, we take $\varphi = u$ as test function in the equation

$$\frac{\partial u}{\partial t} - (1+i\epsilon)\Delta u + (1+i\beta)|u|^{m-1}u - (1-i\omega)u = F(u(x,t-\tau))\,.$$

Integrating by parts, thanks to Hölder and Young inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2dx + \int_\Omega [|\nabla u(x,t)|^2 + (1-\frac{\alpha}{2})|u(x,t)|^{m+1}]dx \\
&\le \int_\Omega |u(x,t)|^2dx + c(\alpha)|F_0(t-\tau)|^{\frac{m+1}{m}}.
\end{aligned}$$

Since $\|u\|_{\mathcal{L}^\infty(Q)} \le |1-\alpha|^{\frac{1}{1-m}}$, we have

$$\int_\Omega |u(x,t)|^2dx \le |1-\alpha|\int_\Omega |u(x,t)|^{m+1}dx,$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2dx + \int_\Omega \Big[|\nabla u(x,t)|^2 + \frac{\alpha}{2}|u(x,t)|^{m+1}\Big]\,dx \le c_2|F_0(t-\tau)|^{\frac{m+1}{m}} \tag{4.8}$$

for some positive constants α and c_2. We then use the Gagliardo-Nirenberg inequality [2, Theorem 6.1] to arrive to the energy inequality

$$y'(t) + c_3 y(t)^\delta \le c_2 |F_0(t-\tau)|^{\frac{m+1}{m}}$$

for some $\delta \in (0,1)$ and some $c_3 > 0$, with

$$y(t) = \|u(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)}\,.$$

Since we arrive to the energy inequality

$$y'(t) + c_3 y(t)^\delta \le c_2 |F_0(t-\tau)|^{\frac{m+1}{m}}$$

for some $\delta \in (0,1)$ and some $c_3 > 0$, with

$$y(t) = \|u(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)}.$$

Then we apply Lemma 4.2 on the interval $t \in [0,\tau)$, with $\kappa = c_3$ and $T_0 = t^* \in (0,\tau)$, so if we assume that

$$\|u_0(\cdot)\|^2_{\mathcal{L}^2(\Omega)} \le (c_3\,\delta\,(1-\delta)t^*)^{\frac{1}{1-\delta}},$$
$$c_2|F_0(t-\tau)|^{\frac{m+1}{m}} \le y_\star(t^*-t)_+^{\frac{\delta}{1-\delta}}$$

with $y_\star = (\,c_3\delta^\delta(1-\delta))^{\frac{1}{1-\delta}}$. Then we conclude that

$$\|u(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)} \le k^*(t^*-t)_+^{\frac{1}{1-\delta}} \text{ for all } t \in [t^*,\tau), \tag{4.9}$$

which proves the first conclusion of part (i). Arguing in a similar way, now on the interval $t \in [\tau, 2\tau)$, we obtain

$$y'(t) + c_3 y(t)^\delta \le \widehat{c}_2 \Big| \int_\Omega u(x,t-\tau)dx \Big|^{\frac{m+1}{m}}$$

for the same $\delta \in (0,1)$ and $c_3 > 0$ and some $\widehat{c}_2 > 0$. Applying the Hölder inequality and conclusion (4.9) we have

$$\Big| \int_\Omega u(x,t-\tau)dx \Big|^{\frac{m+1}{m}} \le |\Omega|^{\frac{m+1}{2m}} k^{*\,\frac{2(m+1)}{m}} ((t^*+\tau)-t)_+^{\frac{2}{(1-\delta)}\,\frac{m+1}{m}}$$

for all $t \in [t^*+\tau, 2\tau)$. Then, since

$$\frac{2}{(1-\delta)}\frac{m+1}{m} > \frac{\delta}{1-\delta},$$

we van apply again Lemma 4.2 to conclude that

$$\|u(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)} \le k^*((t^*+\tau)-t)_+^{\frac{1}{1-\delta}} \quad \text{for all } t \in [t^*,\tau). \tag{4.10}$$

The proof of part (ii) is similar but now, on the initial interval $[0,\tau)$ we require the stronger assumption (4.4). □

Remark 4.3. If the initial history $F_0(s)$ and u_0 (respectively $U_0(\cdot,s)$) does not vanishes also for $s \in [-\tau, \underline{t}) \cup \{0\}$, for some $\underline{t} < 0$ then Theorem 4.1 proves that the solution of the nonlocal (1.4), i.e. with $\nu = 0$ (respectively the local case, with $\nu > 0$) is discontinuous at the time $t = n\tau$ for all $n \in \mathbb{N}$. The technique of proof in Theorem 4.1 could also be used to prove that the solutions may vanishes on intervals of the form $[n\tau, n\tau + \varepsilon]$ avowing the above mentioned discontinuity.

Remark 4.4. The detailed analysis made in [2] shows that, in fact,

$$\delta = \frac{m+1}{\theta(m+1)+2(1-\theta)} \quad \text{with} \quad \theta = \frac{N(1-m)}{N(1-m)+2(1+m)}.$$

Remark 4.5. The above assumptions are, in some sense, necessary. Indeed, if for instance $\|u_0\|^2_{\mathcal{L}^2(\Omega)}$ is big enough then it is possible to take the parameters such that any function $y(t)$ satisfying the ordinary differential inequality with zero in the right hand side satisfy that $y(\tau) > 0$ and thus the contribution of the global initial memory can not be of any help in the rest of values of time $t \in [\tau, +\infty)$. Analogously, the decay condition indicated in the assumption (4.3) is the optimal decay which is compatible with the decay of any function $y(t)$ satisfying the ordinary differential inequality with zero in the right hand side and with an exponent $\delta \in (0,1)$. On the other hand, it is well known that if $\delta \geq 1$ then $y(t) > 0$ for all $t \in [0, +\infty)$.

Remark 4.6. Assumption (4.1) is used to obtain the finite time extinction of the solution. If assumption (4.1) is not satisfied, for some initial data the solution achieves an homogeneous in space function in finite time, such function is, in fact, the average (in space) of the solution and its behavior is determined by an ordinary differential equation. See for instance [3], Chapter 2, section 7.2 and references therein where finite time convergence to the average of the solution is studied for a porous-media type equation.

The assumption $m \in (0,1)$ (jointly with the structure conditions on the coefficients $(1+i\beta)$ of the corresponding nonlinear term) makes possible the finite speed of propagation property and other qualitative properties related with the spatial support of the solutions $u(\cdot,t)$ for a fixed time $t > 0$. That was show in the nice paper [2] for the case without any delayed term and can be easily adapted to the problems considered in this work once that suitable conditions on the initial history are assumed. We shall follow the usual notations in this type of local methods (see [3]): B_ρ denotes the open ball of radius ρ of $\mathbb{R}^N$ contained in Ω (we shall not specify the dependence with respect the center of the ball x_0). Moreover, we shall use the notation $Q_{\rho,T_0} := (0,T_0) \times B_\rho$ and $\Sigma_{\rho,T_0} := (0,T_0) \times \partial B_\rho$. We introduce the local energies

$$\begin{gathered}
E(\rho,T_0) = \int_{Q_{\rho,T_0}} |\nabla u|^2 \,dx\,dt, \\
b(\rho,T_0) = \frac{1}{2} \operatorname{ess\,sup}_{0\leq t\leq T_0} \int_{B_\rho} |u(x,t)|^2 dx, \\
c(\rho,T_0) = \int_{Q_{\rho,T_0}} |u|^{m+1} \,dx\,dt.
\end{gathered}$$

Theorem 4.7. *Let $m \in (0,1)$ and $\alpha \in (0,1)$ be such that*

$$\|u\|_{L^\infty(Q)} \leq |1-\alpha|^{\frac{1}{1-m}}. \tag{4.11}$$

(i) Assume that there exists $\rho_0 > 0$ such that

$$u_0 = 0 \ \text{in } B_{\rho_0}. \tag{4.12}$$

Assume also that there exists $s_{F_0} \in (0,\tau)$ such that

$$F_0(s) = 0 \quad \text{for a.e. } s \in (-\tau, -s_{F_0}). \tag{4.13}$$

Let u be a bounded solution of the nonlocal (1.4) *(i.e. with $\nu = 0$). Then there exists $\rho_1 \in (0, \rho_0)$ and $t_1 \in (0, \tau)$ (both depending of the energies associated to u) such that $u = 0$ in Q_{ρ_1,t_1}.*

(ii) Assume that there exists $s_{F_0} \in (0, \tau)$ and $\rho_0 > 0$ such that

$$U_0(\cdot, s) = 0 \quad \text{on } B_{\rho_0} \text{ for } s = 0 \text{ and for a.e. } s \in (-\tau, -s_{F_0}). \tag{4.14}$$

Let u be a bounded solution of the local (1.4) *(i.e. with $\nu > 0$). Then there exists $\rho_1 \in (0, \rho_0)$ and $t_1 \in (0, \tau)$ (both depending of the energies associated to u) such that $u = 0$ in Q_{ρ_1,t_1}.*

Proof. (i) It is an easy modification of the adaptation, made in [2, Theorems 5.1 and 6.1], of the local energy method (presented in [3, Chapter 3]) to the case of complex Ginzburg-Landau equations without delay terms. By multiplying by $\overline{u}$ and integrating on B_ρ, for almost all $\rho \in (0, \rho_0)$, and using assumptions (4.12), (4.13) $s \in (-\tau, -s_{F_0})$, and the boundedness of u we obtain the "local integration by parts inequality" for $T_0 = s_{F_0}$

$$\begin{aligned} b(\rho, T_0) + E(\rho, T_0) + c(\rho, T_0) &\le \int_{\Sigma_{\rho,T_0}} |\nabla u||u|\, dx\, dt, \text{ a.e. } \rho \in (0, \rho_0) \\ &\le \|Du\|_{L^2(\Sigma_{\rho,T_0})} \|u\|_{L^2(\Sigma_{\rho,T_0})}. \end{aligned} \tag{4.15}$$

Here, again, we used the assumption (4.11) to obtain (4.8). The method continues, as usual, by applying some interpolation-trace inequalities and some estimates of the involved terms. Using that

$$\|Du\|^2_{L^2(\Sigma_{\rho,T_0})} = \frac{\partial E}{\partial \rho}(\rho, T_0), \tag{4.16}$$

we obtain the ordinary differential inequality

$$\rho^{2\vartheta} E(\rho, T_0)^\xi \le C K(\rho_0, T_0) \frac{\partial E}{\partial \rho}(\rho, T_0) \tag{4.17}$$

for some $\xi \in (0, 1)$, $\vartheta > 0$ and some positive constants C and $K(\rho_0, T_0)$, which implies the conclusion. Assumption (4.14) also allows to get the same "local integration by parts inequality" and thus the proof of (ii) follows the same arguments. $\square$

Remark 4.8. As in [2, Theorem 5.3], it is possible to show a "waiting time property" (showing that in fact $\rho_1 = \rho_0$ for all $t \in (0, t_0)$, for some $t_0 \in (0, \tau)$) for the solution u of the nonlocal (1.4) (i.e. with $\nu = 0$), if we make the decay stronger assumption

$$\int_{B_\rho} |u_0(x)|^2 dx \le \delta(\rho - \rho_0)_+^{\frac{1}{1-\xi}} \quad \text{for a.e. } \rho \in (0, \rho_0 + \varepsilon)$$

for some $\delta, \varepsilon > 0$ and with $\xi \in (0, 1)$ the exponent arising in (4.17). in the case of the local (1.4) (i.e. with $\nu > 0$) it must be required a similar decay stronger assumption now on $U_0(\cdot, s)$: to be more precise, we must assume that there exists $s_{F_0} \in (0, \tau]$, $\rho_0 > 0$ and $\delta, \varepsilon > 0$ such that

$$\int_{B_\rho} |U_0(x, 0)|^2 dx + \int_{-\tau}^{-s_{F_0}} \int_{B_\rho} |U_0(x, s)|^{\frac{m+1}{m}}\, dx\, ds \le \delta(\rho - \rho_0)_+^{\frac{1}{1-\xi}} \tag{4.18}$$

for almost every $\rho \in (0, \rho_0 + \varepsilon)$.

Remark 4.9. As in the case of the equation without delay terms, it remains an open question to know if the above finite speed of propagation also holds for the pure Schrödinger equation with the same absorption perturbation term. A partial answer was given in [11].

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