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HARDY AND CAFFARELLI-KOHN-NIRENBERG INEQUALITIES WITH NONRADIAL WEIGHTS

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Abstract. We study the Hardy type inequalities and Caffarelli-Kohn-Nirenberg type inequalities with nonradial weights of the form $|x_1|^{A_1} \cdots |x_N|^{A_N}/|x|^{\lambda}$.

1. Introduction

Cabré and Ros-Oton [5] studied the regularity for stable solutions to reaction-diffusion problems of double revolution. Their motivation is an open question raised by Haïm Brezis [3, 4]. We note that one important tool in their proofs in [5] is a version of the Sobolev inequality with monomial weight. After that, the authors in [6] also set up the Sobolev, Morrey, Trudinger and isoperimetric inequalities with monomial weight x^A. Here

$$\begin{gathered} x^A = |x_1|^{A_1} \cdots |x_N|^{A_N} \\ A_1 \geq 0, \dots, A_N \geq 0 \\ A = (A_1, \dots, A_N). \end{gathered}$$

Also, the best constants of the Trudinger-Moser inequalities with monomial weights were computed explicitly in [32].

Bakry, Gentil and Ledoux [1] combined the stereographic projection and the Curvature-Dimension condition to set up the following Sobolev inequality with monomial weight: for $a \geq 0$, $N + a > 2$, there exists $S(N, a) > 0$ such that for all smooth, compactly supported function u on $\mathbb{R}^{N-1} \times \mathbb{R}_+$:

$$\Big[\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}_+} |u(x)|^{\frac{2(N+a)}{N+a-2}} x_N^a dx \Big]^{\frac{N+a-2}{2(N+a)}} \leq S(N,a) \Big[\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}_+} |\nabla u(x)|^2 x_N^a dx \Big]^{1/2}.$$

The best constant $S(N, a)$ was also exhibited in [1]. In [40], mass transport approach was used to study the sharp constants and optimizers for the Gagliardo-Nirenberg inequalities and logarithmic Sobolev inequalities with arbitrary norm and with monomial weights. We also mention that in [8], the author provided a

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simple proof for the Hardy-Sobolev-type inequalities with monomial weights. However, the best constant and the extremals for the inequalities were not studied there.

Our main motivation of this note is the results in [30] where Lam established general Caffarelli-Kohn-Nirenberg inequalities with nonradial weights of the form $\frac{x^A}{|x|^\lambda}$. It is worthy to note that because of the presence of the weights $\frac{x^A}{|x|^\lambda}$, the classical rearrangement arguments are not applicable. Nevertheless, the approach in [30] relied on a suitable quasiconformal mapping.

The Caffarelli-Kohn-Nirenberg inequalities were first introduced in 1984 by Caffarelli, Kohn and Nirenberg in their celebrated work [7]:

Theorem 1.1. *There exists a positive constant $C = C(N, r, p, q, \gamma, \alpha, \beta)$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$,*

$$\| \, |x|^\gamma u\|_r \le C\| \, |x|^\alpha |\nabla u| \, \|_p^a \| \, |x|^\beta u\|_q^{1-a}, \tag{1.1}$$

where $p, q \ge 1$, $r > 0$, $0 \le a \le 1$,

$$\frac{1}{p} + \frac{\alpha}{N}, \ \frac{1}{q} + \frac{\beta}{N}, \ \frac{1}{r} + \frac{\gamma}{N} > 0$$

where $\gamma = a\sigma + (1-a)\beta$,

$$\frac{1}{r} + \frac{\gamma}{N} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{N}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{N}\Big),$$

and $0 \le \alpha - \sigma$ if $a > 0$; and $\alpha - \sigma \le 1$ if $a > 0$ and

$$\frac{1}{p} + \frac{\alpha - 1}{N} = \frac{1}{r} + \frac{\gamma}{N}.$$

Because of their important roles in many areas of modern mathematics such as geometric analysis, partial differential equations, spectral theory, etc, the Caffarelli-Kohn-Nirenberg inequalities have been intensively investigated in many settings in the literature. See [10, 12, 13, 14, 15, 17, 21, 26, 33, 34, 39, 40, 42, 45, 47]. It is also worth mentioning that Caffarelli-Kohn-Nirenberg inequality is one of the most interesting inequalities in partial differential equations. It generalizes many well-known and important inequalities in analysis such as Gagliardo-Nirenberg inequalities, Sobolev inequalities, Hardy-Sobolev inequalities, Nash's inequalities, etc.

In the special case $a = 1$, $p = r = 2$, $\alpha = 0$, (1.1) reduces to the well-known L^2-Hardy inequality: for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge \Big(\frac{N-2}{2}\Big)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \tag{1.2}$$

The L^2-Hardy inequality is one of the most used inequalities in analysis and has been well-studied in the literature. Especially, since the constant $(\frac{N-2}{2})^2$ is optimal but cannot be achieved by nontrivial functions, the problem of finding improved versions of (1.2) has attracted great attention in the literature. Pioneering by Brezis and Vázquez in [4], this question has been tackled by many authors, by adding nonnegative terms to the left-hand side of (1.2), by replacing the usual ∇ by other operators, etc. The interested reader is referred to the monographs [2, 24, 27, 28, 38, 41, 44], that are standard references on the subject.

The first main purpose of this note is to study the L^2-Hardy type inequalities with the weight $\frac{x^A}{|x|^\lambda}$. More precisely, motivated by the functional inequalities with non-radial weight of the form $\frac{x^A}{|x|^\lambda}$ in [30], the Hardy inequalities in the framework of equalities in, for instance, [18, 20, 25, 35, 36, 37], and the functional inequalities with radial derivation $\mathcal{R} := \frac{x}{|x|} \cdot \nabla$ in [19, 23, 29, 31, 43, 46], we will establish in this paper the L^2-Hardy type identities with weight $\frac{x^A}{|x|^\lambda}$. More precisely, let

$$\begin{gathered} x^A = |x_1|^{A_1} \dots |x_N|^{A_N} \\ A_1 \geq 0, \dots, A_N \geq 0 \\ A = (A_1, \dots, A_N) \end{gathered}$$

and $\mathbb{R}^N_* = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i > 0 \text{ whenever } A_i > 0\}$, $D = N + A_1 + \dots + A_N$. Then, we have the following result.

Theorem 1.2. *Let $\lambda \in \mathbb{R}$. For $u \in C_0^\infty(\mathbb{R}^N_* \setminus \{0\})$, one has*

$$\int_{\mathbb{R}^N_*} |\nabla u|^2 \frac{x^A}{|x|^\lambda} dx - \Big(\frac{D-\lambda-2}{2}\Big)^2 \int_{\mathbb{R}^N_*} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx = \int_{\mathbb{R}^N_*} \frac{|\nabla(|x|^{\frac{D-\lambda-2}{2}} u)|^2}{|x|^{D-\lambda-2}} \frac{x^A}{|x|^\lambda} dx,$$

$$\int_{\mathbb{R}^N_*} |\mathcal{R} u|^2 \frac{x^A}{|x|^\lambda} dx - \Big(\frac{D-\lambda-2}{2}\Big)^2 \int_{\mathbb{R}^N_*} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx = \int_{\mathbb{R}^N_*} \frac{|\mathcal{R}(|x|^{\frac{D-\lambda-2}{2}} u)|^2}{|x|^{D-\lambda-2}} \frac{x^A}{|x|^\lambda} dx.$$

Obviously, our results imply the following Hardy inequalities with non-radial weight $x^A/|x|^\lambda$,

$$\int_{\mathbb{R}^N_*} |\nabla u|^2 \frac{x^A}{|x|^\lambda} dx \geq \int_{\mathbb{R}^N_*} |\mathcal{R} u|^2 \frac{x^A}{|x|^\lambda} dx \geq \Big(\frac{D-\lambda-2}{2}\Big)^2 \int_{\mathbb{R}^N_*} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx. \tag{1.3}$$

Also, we note that with $u = |x|^{-\frac{D-\lambda-2}{2}}$, the integral $\int_{\mathbb{R}^N_*} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx$ diverges. Hence, the constant $(\frac{D-\lambda-2}{2})^2$ is sharp in Theorem 1.2, but is never attained. Nevertheless, we can consider $|x|^{-(D-\lambda-2)/2}$ as the "virtual" optimizer of the Hardy inequalities (1.3).

Another consequence of our Theorem 1.2 is the following Heisenberg-Pauli-Weyl type uncertainty principle

$$\Big(\int_{\mathbb{R}^N_*} |\nabla u|^2 x^A dx\Big)^{1/2} \Big(\int_{\mathbb{R}^N_*} |x|^2 |u|^2 x^A dx\Big)^{1/2} \geq |\frac{D-2}{2}| \int_{\mathbb{R}^N_*} |u|^2 dx.$$

Obviously, when $A = \overrightarrow{0}$, we recover the classical Heisenberg-Pauli-Weyl uncertainty principle that can be stated as follows: for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, we have

$$\frac{N-2}{2} \int_{\mathbb{R}^N} u^2 dx \leq \Big(\int_{\mathbb{R}^N} |x|^2 u^2 dx\Big)^{1/2} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2}. \tag{1.4}$$

The meaning of this inequality in quantum mechanics is that position and momentum of a quantum particle cannot both be sharply localized. Uncertainty principles have long been one of the most famous problems in mathematical physics and classical Fourier analysis alike. They can be translated into the mathematical form that a function and its Fourier transform cannot both be small. See the survey paper of Folland and Sitaram [22] for several mathematical forms of the uncertainty principle.

It is interesting to note that (1.4) is just a special case of the following class of the Caffarelli-Kohn-Nirenberg inequalities (1.1), for $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$,

$$C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx \leq \Big(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx\Big)^{1/2} \Big(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx\Big)^{1/2}. \tag{1.5}$$

It is worth mentioning that if we do not require that the functions u in (1.5) to vanish at the origin, then by [7], it is necessary that $a < N/2$, $b < N/2$ and $a+b < N-1$, for the integrability conditions. However, as observed in [9, 11, 16], if we work on functions $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, then we have no restriction on the parameters a and b.

The sharp constant and optimizers for (1.5) have been investigated in [9, 11]. More exactly, let

$$A_1 = \big\{a < b+1, b \le \frac{N-2}{2}\big\}, \quad A_2 = \big\{a > b+1, b \ge \frac{N-2}{2}\big\}, \quad A = A_1 \cup A_2,$$
$$B_1 = \big\{a > b+1, b \le \frac{N-2}{2}\big\}, \quad B_2 = \{a < b+1,\ b \ge \frac{N-2}{2}\}, \quad B = B_1 \cup B_2\,.$$

Then when $(a,b) \in A$, then $C(N,a,b) = \frac{|N-a-b-1|}{2}$. Also, the optimizers are of the form $D\exp(\frac{s|x|^{b+1-a}}{b+1-a})$ with $s < 0$ for $(a,b) \in A_1$ and $s > 0$ for $(a,b) \in A_2$. When $(a,b) \in B$, $C(N,a,b) = \frac{|N+a-3b-3|}{2}$. The extremal functions are $D|x|^{2(b+1)-N}\exp(\frac{s|x|^{b+1-a}}{b+1-a})$ with $s > 0$ for $(a,b) \in B_1$ and $s < 0$ for $(a,b) \in B_2$.

Motivated by the results in [30], our next aim is to set up the following Caffarelli-Kohn-Nirenberg inequalities with non-radial weights.

Theorem 1.3. *For all $u \in C_0^\infty(\mathbb{R}_*^N \setminus \{0\})$,*

$$\begin{aligned}
C(N,A,a,b)\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{a+b+1}}dx &\le \Big(\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{2a}}dx\Big)\big)^{1/2}\Big(\int_{\mathbb{R}_*^N} |\mathcal{R}u|^2 \frac{x^A}{|x|^{2b}}dx\Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{2a}}dx\Big)^{1/2}\Big(\int_{\mathbb{R}_*^N} |\nabla u|^2 \frac{x^A}{|x|^{2b}}dx\Big)^{1/2},
\end{aligned}$$

where

$$C(N,A,a,b) = \begin{cases} \frac{|a+b+1-D|}{2} & \text{if } (a,b) \in \mathcal{A} \\ \frac{|D+a-3b-3|}{2} & \text{if } (a,b) \in \mathcal{B}. \end{cases}$$

Here

$$\mathcal{A}_1 = \big\{a < b+1, b \le \frac{D-2}{2}\big\}, \quad \mathcal{A}_2 = \big\{a > b+1,\ b \ge \frac{D-2}{2}\big\}, \quad \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2,$$
$$\mathcal{B}_1 = \big\{a > b+1,\ b \le \frac{D-2}{2}\big\}, \quad \mathcal{B}_2 = \big\{a < b+1, b \ge \frac{D-2}{2}\big\}, \quad \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2.$$

As a consequence of Theorem 1.3, we can deduce that all the extremal functions for

$$C(N,A,a,b)\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{a+b+1}}dx \le \Big(\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{2a}}dx\Big)^{1/2}\Big(\int_{\mathbb{R}_*^N} |\nabla u|^2 \frac{x^A}{|x|^{2b}}dx\Big)^{1/2} \tag{1.6}$$

must be radial.

When $A = \overrightarrow{0}$, we obtain the L^2-Caffarelli-Kohn-Nirenberg inequality with radial derivative

$$C(N,a,b)\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx \le \Big(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} \frac{|\mathcal{R}u|^2}{|x|^{2b}}dx\Big)^{1/2}$$

which implies the L^2-Caffarelli-Kohn-Nirenberg inequality

$$C(N,a,b)\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx \le \Big(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}dx\Big)^{1/2}. \tag{1.7}$$

As mention earlier, this L^2-Caffarelli-Kohn-Nirenberg inequality has been investigated in [9]. The approach in [9] is to make a change of variables into the cylinder $\mathbb{S}^{N-1}\times\mathbb{R}$, and then using spherical harmonics to reduce the problem to the one-dimensional case with parameter N. In this article, we will provide an alternative argument for their approach. Our argument is very simple and can be used for more general class of the Caffarelli-Kohn-Nirenberg inequality. See the Proof of Theorem 1.3 for more details.

2. Proofs of main results

Proof of Theorem 1.2. Denoting in the polar coordinate

$$x^A = r^{|A|}\varphi_A(\sigma),$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_*}\frac{|\mathcal{R}(|x|^{\frac{D-\lambda-2}{2}}u)|^2}{|x|^{D-\lambda-2}}\frac{x^A}{|x|^\lambda}dx\\
&=\int_{\partial B_1^*}\varphi_A(\sigma)\int_0^\infty\big|\partial_r\big(r^{\frac{D-\lambda-2}{2}}u(r\sigma)\big)\big|^2 r^{|A|-D+2}r^{N-1}dr\,d\sigma.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^\infty\Big|\partial_r\Big(r^{\frac{D-\lambda-2}{2}}u(r\sigma)\Big)\Big|^2 r^{|A|-D+2}r^{N-1}dr\\
&=\int_0^\infty\Big|\frac{D-\lambda-2}{2}r^{\frac{D-\lambda-4}{2}}u(r\sigma)+r^{\frac{D-\lambda-2}{2}}u_r(r\sigma)\Big|^2 r dr\\
&=\int_0^\infty|u_r(r\sigma)|^2r^{D-\lambda-1}dr+\Big(\frac{D-\lambda-2}{2}\Big)^2\int_0^\infty|u(r\sigma)|^2r^{D-\lambda-3}dr\\
&+\Big(\frac{D-\lambda-2}{2}\Big)\int_0^\infty 2u(r\sigma)u_r(r\sigma)r^{D-\lambda-2}dr.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
\int_0^\infty 2u(r\sigma)u_r(r\sigma)r^{D-\lambda-2}dr&=\int_0^\infty\partial_r\big(|u(r\sigma)|^2\big)r^{D-\lambda-2}dr\\
&=-(D-\lambda-2)\int_0^\infty|u(r\sigma)|^2r^{D-\lambda-3}dr.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^\infty\big|\partial_r\big(r^{\frac{D-\lambda-2}{2}}u(r\sigma)\big)\big|^2 r^{|A|-D+2}r^{N-1}dr\\
&=\int_0^\infty|u_r(r\sigma)|^2r^{D-\lambda-1}dr-\big(\frac{D-\lambda-2}{2}\big)^2\int_0^\infty|u(r\sigma)|^2r^{D-\lambda-3}dr
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N_*}\frac{|\mathcal{R}(|x|^{\frac{D-\lambda-2}{2}}u)|^2}{|x|^{D-\lambda-2}}\frac{x^A}{|x|^\lambda}dx\\
&=\int_{\partial B_1^*}\varphi_A(\sigma)\Big[\int_0^\infty|u_r(r\sigma)|^2r^{D-\lambda-1}dr-\big(\frac{D-\lambda-2}{2}\big)^2\int_0^\infty|u(r\sigma)|^2r^{D-\lambda-3}dr\Big]d\sigma\\
&=\int_{\partial B_1^*}\varphi_A(\sigma)\int_0^\infty|u_r(r\sigma)|^2r^{|A|-\lambda}r^{N-1}\,dr\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \big(\frac{D-\lambda-2}{2}\big)^2 \int_{\partial B_1^*} \varphi_A(\sigma) \int_0^\infty |u(r\sigma)|^2 r^{|A|-\lambda-2} r^{N-1}\, dr\, d\sigma \\
&= \int_{\mathbb{R}_*^N} |\mathcal{R}u|^2 \frac{x^A}{|x|^\lambda} dx - \big(\frac{D-\lambda-2}{2}\big)^2 \int_{\mathbb{R}_*^N} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{\mathbb{R}_*^N} \frac{|\nabla(|x|^{\frac{D-\lambda-2}{2}} u)|^2}{|x|^{D-\lambda-2}} \frac{x^A}{|x|^\lambda} dx \\
&= \int_{\mathbb{R}_*^N} \frac{||x|^{\frac{D-\lambda-2}{2}} \nabla u + \frac{D-\lambda-2}{2} |x|^{\frac{D-\lambda-4}{2}} u \frac{x}{|x|}|^2}{|x|^{D-\lambda-2}} \frac{x^A}{|x|^\lambda} dx \\
&= \int_{\mathbb{R}_*^N} |\nabla u|^2 \frac{x^A}{|x|^\lambda} dx + \Big(\frac{D-\lambda-2}{2}\Big)^2 \int_{\mathbb{R}_*^N} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx \\
&+ \Big(\frac{D-\lambda-2}{2}\Big) \int_{\mathbb{R}_*^N} 2u\mathcal{R}u \frac{1}{|x|} \frac{x^A}{|x|^\lambda} dx.
\end{aligned}
$$

Again, as above, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}_*^N} 2u\mathcal{R}u \frac{1}{|x|} \frac{x^A}{|x|^\lambda} dx \\
&= \int_{\partial B_1^*} \varphi_A(\sigma) \int_0^\infty 2u(r\sigma) u_r(r\sigma) r^{D-\lambda-2}\, dr\, d\sigma \\
&= -(D-\lambda-2) \int_{\mathbb{S}_*^{N-1}} \varphi_A(\sigma) \int_0^\infty |u(r\sigma)|^2 r^{D-\lambda-3}\, dr\, d\sigma \\
&= -(D-\lambda-2) \int_{\mathbb{R}_*^N} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\int_{\mathbb{R}_*^N} \frac{|\nabla(|x|^{\frac{D-\lambda-2}{2}} u)|^2}{|x|^{D-\lambda-2}} \frac{x^A}{|x|^\lambda} dx \\
&= \int_{\mathbb{R}_*^N} |\nabla u|^2 \frac{x^A}{|x|^\lambda} dx - \Big(\frac{D-\lambda-2}{2}\Big)^2 \int_{\mathbb{R}_*^N} \frac{|u|^2}{|x|^2} \frac{x^A}{|x|^\lambda} dx.
\end{aligned}
$$

$\square$

Proof of Theorem 1.3. When u is radial, we have

$$
\begin{aligned}
\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{a+b+1}} dx &= \Big(\int_{\partial B_1^*} \varphi_A(\sigma) d\sigma\Big) \int_0^\infty |u|^2 r^{N-1+|A|-a-b-1} dr, \\
\int_{\mathbb{R}_*^N} |u|^2 \frac{x^A}{|x|^{2a}} dx &= \Big(\int_{\partial B_1^*} \varphi_A(\sigma) d\sigma\Big) \int_0^\infty |u|^2 r^{N-1+|A|-2a} dr, \\
\int_{\mathbb{R}_*^N} |\mathcal{R}u|^2 \frac{x^A}{|x|^{2b}} dx &= \Big(\int_{\partial B_1^*} \varphi_A(\sigma) d\sigma\Big) \int_0^\infty |u_r|^2 r^{N-1+|A|-2b} dr.
\end{aligned}
$$

Using the results in [9, 11], we obtain

$$
\Big(\int_0^\infty |u|^2 r^{N-1+|A|-2a} dr\Big)\Big(\int_0^\infty |u_r|^2 r^{N-1+|A|-2b} dr\Big)
$$

$$\geq C^2(N,A,a,b)\Big(\int_0^\infty |u|^2 r^{N-1+|A|-a-b-1}dr\Big)^2$$

where

$$C(N,A,a,b)=\begin{cases}\frac{|a+b+1-D|}{2} & \text{if } (a,b)\in\mathcal{A}\\ \frac{|D+a-3b-3|}{2} & \text{if } (a,b)\in\mathcal{B}.\end{cases}$$

Now, when u is not radial, we set

$$U(r)=\Big(\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\int_{\partial B_1^*}|u(r\sigma)|^2\varphi_A(\sigma)d\sigma\Big)^{1/2}.$$

Then

$$|U(r)|^2=\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\int_{\partial B_1^*}|u(r\sigma)|^2\varphi_A(\sigma)d\sigma.$$

Hence for all $\lambda\in\mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N_*}|U|^2\frac{x^A}{|x|^\lambda}dx\\
&=\Big(\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma\Big)\int_0^\infty |U|^2 r^{N-1+|A|-\lambda}dr\\
&=\Big(\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma\Big)\int_0^\infty\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\int_{\partial B_1^*}|u(r\sigma)|^2\varphi_A(\sigma)d\sigma r^{N-1+|A|-\lambda}dr\\
&=\int_0^\infty\int_{\partial B_1^*}|u(r\sigma)|^2 r^{|A|}\varphi_A(\sigma)r^{N-1-\lambda}d\sigma dr\\
&=\int_{\mathbb{R}^N_*}|u|^2\frac{x^A}{|x|^\lambda}dx
\end{aligned}$$

Now, we note that

$$\begin{aligned}
|2U(r)U_r(r)|&=\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\Big|\int_{\partial B_1^*}2|u(r\sigma)|u_r(r\sigma)\varphi_A(\sigma)d\sigma\Big|\\
&\leq 2\Big(\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\int_{\partial B_1^*}|u(r\sigma)|^2\varphi_A(\sigma)d\sigma\Big)^{1/2}\\
&\quad\times\Big(\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\int_{\partial B_1^*}|u_r(r\sigma)|^2\varphi_A(\sigma)d\sigma\Big)^{1/2}.
\end{aligned}$$

Hence

$$|U_r(r)|^2\leq\frac{1}{\int_{\partial B_1^*}}\varphi_A(\sigma)d\sigma\int_{\partial B_1^*}|u_r(r\sigma)|^2\varphi_A(\sigma)d\sigma.$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N_*}|\nabla U|^2\frac{x^A}{|x|^{2b}}dx\\
&=\Big(\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma\Big)\int_0^\infty |U_r|^2 r^{N-1+|A|-2b}dr\\
&\leq\Big(\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma\Big)\int_0^\infty\frac{1}{\int_{\partial B_1^*}\varphi_A(\sigma)d\sigma}\int_{\partial B_1^*}|u_r(r\sigma)|^p\varphi_A(\sigma)d\sigma r^{N-1+|A|-2b}dr
\end{aligned}$$

$$= \int_{\mathbb{R}^N_*} |\mathcal{R}u|^2 \frac{x^A}{|x|^{2b}} dx.$$

Hence, we have

$$\begin{gathered}
\int_{\mathbb{R}^N_*} |U|^2 \frac{x^A}{|x|^{a+b+1}} dx = \int_{\mathbb{R}^N_*} |u|^2 \frac{x^A}{|x|^{a+b+1}} dx, \\
\int_{\mathbb{R}^N_*} |U|^2 \frac{x^A}{|x|^{2a}} dx = \int_{\mathbb{R}^N_*} |u|^2 \frac{x^A}{|x|^{2a}} dx, \\
\int_{\mathbb{R}^N_*} |\nabla U|^2 \frac{x^A}{|x|^{2b}} dx \le \int_{\mathbb{R}^N_*} |\mathcal{R}u|^2 \frac{x^A}{|x|^{2b}} dx.
\end{gathered}$$

Using the Caffarelli-Kohn-Nirenberg inequalities for the radial function U, we obtain

$$\begin{aligned}
C(N,A,a,b) \int_{\mathbb{R}^N_*} |u|^2 \frac{x^A}{|x|^{a+b+1}} dx &= C(N,A,a,b) \int_{\mathbb{R}^N_*} |U|^2 \frac{x^A}{|x|^{a+b+1}} dx \\
&\le \Big(\int_{\mathbb{R}^N_*} U|^2 \frac{x^A}{|x|^{2a}} dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N_*} |\nabla U|^2 \frac{x^A}{|x|^{2b}} dx \Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}^N_*} |u|^2 \frac{x^A}{|x|^{2a}} dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N_*} |\mathcal{R}u|^2 \frac{x^A}{|x|^{2b}} dx)^{1/2} \\
&\le \Big(\int_{\mathbb{R}^N_*} |u|^2 \frac{x^A}{|x|^{2a}} dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N_*} |\nabla u|^2 \frac{x^A}{|x|^{2b}} dx)^{1/2}.
\end{aligned}$$

□

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