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EXISTENCE OF RATIONAL SOLUTIONS FOR q-DIFFERENCE PAINLEVÉ EQUATIONS

HONG YAN XU, JIN TU

Abstract. This article studies properties of meromorphic solutions for several types of q-difference Painlevé equations. We obtain conditions for the existence, and the form of rational solutions for two classes of q-difference Painlevé equations. Also for a solution f we obtain results about the fixed points, the exponents of convergence of poles of $f, \Delta_q f, (\Delta_q f)/f$. Our results extend previous theorems given in the references.

1. Introduction and statement of main results

Painlevé equations have been an important research subject in the field of the mathematics and physics, and they occur in many physical situations: plasma physics, statistical mechanics, nonlinear waves, etc. They appear as differential Painlevé equation, discrete Painlevé equation, difference Painlevé, and so on; see [3, 7, 8].

Around 2006, with the development of Nevanlinna theory, Halburd-Korhonen [11] and Chiang-Feng [5] established independently important results about the complex difference and difference operators. By utilizing these results, Halburd-Korhonen [10, 11, 12] discussed the equation

$$f(z+1)+f(z-1)=R(z,f), \tag{1.1}$$

where $R(z,f)$ is rational in f and meromorphic in z. They pointed out that this equation can be transformed into difference Painlevé I equations

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.2}$$

$$f(z+1)+f(z)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.3}$$

and into difference Painlevé II equations

$$f(z+1)+f(z-1)=\frac{(az+b)f(z)+c}{1-f(z)^2}. \tag{1.4}$$

In 2010, Ronkainen [21] further investigated the meromorphic solutions of the equation

$$f(z+1)f(z-1)=R(z,f) \tag{1.5}$$

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where $R(z,f)$ is a rational and irreducible in f and meromorphic in z. He proved that either f satisfies the difference Riccati equation

$$f(z+1)=\frac{A(z)f(z)+B(z)}{f(z)+C(z)},$$

or equation (1.5) can be transformed to one of the following equations

$$\begin{gathered}f(z+1)f(z-1)=\frac{\eta(z)f(z)^2-\lambda(z)f(z)+\mu(z)}{(f(z)-1)(f(z)-\upsilon(z))},\\ f(z+1)f(z-1)=\frac{\eta(z)f(z)^2-\lambda(z)f(z)}{f(z)-1},\\ f(z+1)f(z-1)=\frac{\eta(z)(f(z)-\lambda(z))}{(f(z)-1)},\\ f(z+1)f(z-1)=h(z)f(z)^m,\end{gathered}$$

where $\eta(z),\lambda(z),\upsilon(z)$ satisfy certain conditions. Generally speaking, the above four equation can be called as the difference Painlevé III equations.

In the past two decades, many mathematicians paid consideration attention to the value distribution of solutions for complex difference equations, and obtained lots of important results on the properties of solutions for difference Painlevé I-III equations (see [2, 3, 10, 11, 12, 18, 19, 33]). In 2010, Chen-Shon [4] considered the difference Painlevé I equation (1.2) and obtained the following theorem.

Theorem 1.1 (see [4, Theorem 4]). *Let a,b,c be constants, where a,b are not both equal to zero. Then*

(i) *if $a\neq 0$, then (1.2) has no rational solution;*
(ii) *if $a=0$, and $b\neq 0$, then (1.2) has a nonzero constant solution $w(z)=A$, where A satisfies $2A^2-cA-b=0$.*

The other rational solution is $w(z)=\frac{P(z)}{Q(z)}+A$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P<\deg Q$.

In 2014, Zhang-Yang [30] studied the difference Painlevé III equations with the constant coefficients, and obtained the following result.

Theorem 1.2 ([30]). *If f is a transcendental finite-order meromorphic solution of*

$$f(z+1)f(z-1)(f(z)-1)=\eta w(z) \text{ or } f(z+1)f(z-1)(f(z)-1)=f(z)^2-\lambda w(z),$$

where $\eta(\neq 0),\lambda(\neq 0,1)$ are constants, then

(i) *$\lambda(f)=\sigma(f)$;*
(ii) *f has at most one non-zero Borel exceptional value for $\sigma(f)>0$.*

The Logarithmic Derivative Lemma on q-difference operators was established by Barnett, Halburd, Korhonen and Morgan [1] in 2007. Then the interest in studying the properties on the existence and value distribution of solutions has increased considerably for some q-difference equation which are formed by replacing the q-difference $f(qz)$, $q\in\mathbb{C}\setminus\{0,1\}$ with $f(z+c)$ of meromorphic function in some expression concerning complex difference equations; see [6, 9, 14, 15, 16, 20, 22, 23, 24, 25, 26, 29, 31, 32].

In 2015, Qi-Yang [20] considered the equations

$$f(qz)+f\big(\frac{z}{q}\big)=\frac{az+b}{f(z)}+c, \tag{1.6}$$

which can be seen as q-difference analogues of (1.2), and obtained the following result.

Theorem 1.3 ([20, Theorem 1.1]). *Let $f(z)$ be a transcendental meromorphic solution with zero order of equation* (1.6), *and let a, b, c be constants such that a, b cannot vanish simultaneously. Then*

(i) *$f(z)$ has infinitely many poles.*
(ii) *If $a \neq 0$ and any $d \in \mathbb{C}$, then $f(z) - d$ has infinitely many zeros.*
(iii) *If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.*

In 2018, Liu-Zhang [17] studied the difference equation

$$Y(\omega z) + Y(z) + Y(\frac{z}{\omega}) = \frac{V(z)}{Y(z)} + c, \tag{1.7}$$

which is a q-difference analogues of (1.3), and obtained the following result.

Theorem 1.4 ([17, Thereom 1.2]). *Let $c \in \mathbb{C} \setminus \{0\}$, $|\omega| \neq 1$, and $V(z) = \frac{X(z)}{B(z)}$ be an irreducible rational function, where $X(z)$ and $B(z)$ are polynomials with $\deg X(z) = x$ and $\deg B(z) = b$.*

(i) *Suppose that $x \geq b$ and $x - b$ is zero or an even number. If* (1.7) *has an irreducible rational solution $Y(z) = \frac{I(z)}{J(z)}$, where $I(z)$ and $J(z)$ are polynomials with $\deg I(z) = i$ and $\deg J(z) = j$, then $i - j = \frac{x-b}{2}$.*
(ii) *Suppose that $x < b$. If* (1.7) *has an irreducible rational solution $Y(z) = \frac{I(z)}{J(z)}$, then $Y(z)$ satisfies one of the following two cases:*
  (1) *$Y(z) = \frac{I(z)}{J(z)} = \frac{c}{3} + \frac{T(z)}{D(z)}$, where $T(z)$ and $D(z)$ are polynomials with $\deg T(z) = t$ and $\deg D(z) = d$, and $b - x = d - t$.*
  (2) *$i - j = x - b$.*

Motivated by the idea [17] and [20, 30], we investigate some properties of meromorphic solutions of the following two equations

$$f(qz)f(\frac{z}{q})f(z)(f(z) - 1) = \mu, \tag{1.8}$$

$$f(qz)f(\frac{z}{q})(f(z) - 1)^2 = (f(z) - \lambda)^2, \tag{1.9}$$

which can be seen as q-difference Painlevé III equations.

Before stating our main theorems, let us introduce some basic notation in the theory of Nevanlinna value distribution (see Hayman [13], Yang [27] and Yi and Yang [28]). We denote $\sigma(f), \lambda(f)$ and $\lambda(\frac{1}{f})$ by the order, the exponent of convergence of zeros and the exponent of convergence of poles of meromorphic function $f(z)$, respectively, and $\tau(f)$ by the exponent of convergence of fixed points of $f(z)$, which is defined as

$$\tau(f) = \limsup_{r \to +\infty} \frac{\log N(r, \frac{1}{f(z) - z})}{\log r}.$$

In addition, let $S(r, f)$ be any quantity satisfying $S(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, the logarithmic density of a set F is defined as

$$\limsup_{r \to \infty} \frac{1}{\log r} \int_{[1, r] \cap F} \frac{1}{t} dt.$$

Our main results in this paper are the following.

Theorem 1.5. *Let $q(\neq 0) \in \mathbb{C}$, $|q| \neq 1$, and $\mu(\neq 0) \in \mathbb{C}$, and suppose that $f(z)$ is a nonconstant rational solution of equation* (1.8)*. Then $f(z)$ can be represented in the form*
$$f(z) = \frac{a(z^n + b)^2}{(z^n + q^{-n}b)(z^n + q^n b)},$$
and
$$a = \frac{q^{2n} + q^n + 1}{(q^n + 1)^2}, \quad \mu = a^3(a-1) = -\frac{q^n(q^{2n} + q^n + 1)^3}{(q^n + 1)^8},$$
where b is an any nonzero constant and $n \in \mathbb{N}_+$;

Example 1.6. Let
$$f(z) = \frac{7(z+1)^2}{9(2z+1)(\frac{z}{2}+1)},$$
then $f(z)$ satisfies the equation
$$f(2z)f(\frac{z}{2})f(z)(f(z)-1) = -2\frac{7^3}{9^4}.$$
This example shows that our conclusion about the form of rational solutions for equation (1.8) is sharp.

Theorem 1.7. *Let $q \in \mathbb{C} - \{0,1\}$ and $\mu(\neq 0) \in \mathbb{C}$, and suppose that $f(z)$ is a transcendental meromorphic solution with zero order of equation* (1.8)*. Then*

(i) *$f(\eta z)$ has infinitely many fixed-points and $\tau(f(\eta z)) = \sigma(f)$ for any $\eta \in \mathbb{C} - \{0,1\}$;*
(ii) *$f(z)$ has infinitely many zeros and poles, and $\Delta_q f, (\Delta_q f)/f$ have infinitely many poles, and*
$$\lambda(f) = \lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta_q f}) = \lambda(\frac{1}{(\Delta_q f)/f}).$$

Theorem 1.8. *Let $q, \lambda \in \mathbb{C} - \{0,1\}$ and $|q| \neq 1$. If* (1.9) *has a nonconstant rational solution*
$$f(z) = R(z) = \frac{P(z)}{Q(z)} = \frac{a_p z^p + a_{p-1}z^{p-1} + \dots + a_1 z + a_0}{b_t z^t + b_{t-1}z^{t-1} + \dots + b_1 z + b_0},$$
then $p = t$ and $\lambda = a^2$, where $a = R(\infty) = a_p/b_p$.

Example 1.9. Let $a = 1/9$, $\lambda = 1/81$ and
$$f(z) = \frac{1}{9}(\frac{z+1}{z-1})^2,$$
then $f(z)$ satisfies the difference equation
$$f(2z)f(\frac{z}{2})[f(z)-1]^2 = \big[f(z) - \frac{1}{81}\big]^2.$$
This example shows that our conclusion about the form of rational solutions for equation (1.9) is sharp to a certain extent.

Theorem 1.10. *Let $q, \lambda \in \mathbb{C} - \{0,1\}$. Suppose that $f(z)$ is a nonconstant meromorphic solution with zero order of equation* (1.9)*. Then*

(i) *$f(\eta z)$ has infinitely many fixed-points and $\tau(f(\eta z)) = \sigma(f)$ for any $\eta \in \mathbb{C} - \{0,1\}$;*

(ii) *$f(z)$ has infinitely many zeros and poles, and $\Delta_q f, \frac{\Delta_q f}{f}$ have infinitely many poles, and*

$$\lambda(f) = \lambda\big(\frac{1}{f}\big) = \lambda\big(\frac{1}{\Delta_q f}\big) = \lambda\Big(\frac{1}{(\Delta_q f)/f}\Big).$$

2. Proof of Theorem 1.5

Proof. Let $f(z) = P(z)/Q(z)$ be a nonconstant rational solution of (1.8), where $P(z), Q(z)$ are relatively prime polynomials with degrees p and t respectively. In view of (1.8), it follows that

$$\frac{P(qz)}{Q(qz)}\frac{P(\frac{z}{q})}{Q(\frac{z}{q})}\frac{P(z)}{Q(z)}\frac{P(z)-Q(z)}{Q(z)} = \mu. \tag{2.1}$$

Without loss of generality, we assume that the coefficients of the highest degree terms of $P(z)$ and $Q(z)$ are $a(\neq 0)$ and 1 respectively, and set $s = p - t$.

If $s > 0$, then $P(z)/Q(z) = az^s(1 + o(1))$ as $|z| = r \to \infty$. Thus, by virtue of (2.1), it follows that

$$a^3 z^{3s}(1 + o(1))(az^s(1 + o(1)) - 1) = \mu, \quad r \to \infty,$$

this is impossible for $a \neq 0$.

If $s < 0$, then as $r \to \infty$, it follows that $\frac{P(z)}{Q(z)} = o(1)$ and

$$\frac{P(qz)}{Q(qz)} = o(1), \quad \frac{P(\frac{z}{q})}{Q(\frac{z}{q})} = o(1).$$

Substituting these into (2.1), we get $o(1) = \mu$ as $r \to \infty$, this is a contradiction for $\mu \neq 0$. Thus, it yields that $s = 0$ and $p = t$. From the assumptions of this theorem, we know that the zeros of $Q(z)$ are not the zeros of $P(z)$ and $P(z) - Q(z)$. Hence, in view of (2.2), it follows that all the zeros of $Q^2(z)$ are the zeros of $P(qz)P(\frac{z}{q})$. Since $\deg_z[Q(z)^2] = \deg_z[P(qz)P(\frac{z}{q})] = 2p$, then it yields from (2.1) that

$$P(qz)P(\frac{z}{q}) = a^2 Q(z)^2, \tag{2.2}$$

$$P(z)(P(z) - Q(z)) = a(a-1)Q(qz)Q(\frac{z}{q}). \tag{2.3}$$

Next, we confirm that the orders of all the zeros of $P(z)$ are even. Let z_0 be a zero of $P(z)$ with the order k. If $z_0 \neq 0$ and k is an odd integer. Then $P(z)$ has the term $(z - z_0)^k$, and $P(qz)P(\frac{z}{q})$ has the term

$$(z - qz_0)^k\Big(z - \frac{z_0}{q}\Big)^k. \tag{2.4}$$

It means that qz_0 and $\frac{z_0}{q}$ are both zeros of $P(qz)P(\frac{z}{q})$ with the order at least k.

In addition, since $P(z)$ and $Q(z)$ are relatively prime polynomials, in view of (2.3), it follows that $Q(qz)Q(\frac{z}{q})$ has the term $(z - z_0)^k$. Suppose that $Q(qz)$ and $Q(\frac{z}{q})$ have the terms $(z - z_0)^m$ and $(z - z_0)^l$ respectively, where $m, l \in N$ and $m + l = k$. Obviously, in view of (2.4), we have $m \neq 0$ and $l \neq 0$. Thus, $Q(z)$ has the term $(z - qz_0)^m(z - \frac{z_0}{q})^l$, that is, $Q(z)^2$ has the term $(z - qz_0)^{2m}(z - \frac{z_0}{q})^{2l}$. So, in view of (2.3), it follows that $P(qz)P(\frac{z}{q})$ has the term

$$(z - qz_0)^{2m}(z - \frac{z_0}{q})^{2l}.$$

In view of $m+l=k$ and k is an odd integer, without loss of generality, assume that $m<l$. Thus, $2m<k$ and $2l>k$. Thus, qz_0 is a zero of $P(qz)P(\frac{z}{q})$ with the order $2m<k$, this is a contradiction with (2.4). Thus, any nonzero zeros of $P(z)$ have even orders.

If 0 is a zero of $P(z)$, by combining with (2.2), then 0 is also a zero of $Q(z)$, this is a contradiction with $P(z), Q(z)$ being relatively prime polynomials. Therefore, all the zeros of $P(z)$ are nonzero with even orders.

Let $P(z)=ar(z)^2$, where

$$r(z)=z^n+A_{n-1}z^{n-1}+A_{n-2}z^{n-2}+\cdots+A_1z+A_0,$$

and $A_0, A_1, \dots, A_{n-1}$ are constants. Since 0 is not the zero of $P(z)$, then $A_0\neq 0$. In view of (2.2) and (2.3), it yields $Q(z)=r(qz)r(\frac{z}{q})$ and $ar(z)^2-r(qz)r(\frac{z}{q})=(a-1)r(q^2z)r(q^{-2}z)$. Denote

$$\varphi(z)=ar(z)^2-r(qz)r(\frac{z}{q})-(a-1)r(q^2z)r(q^{-2}z).$$

Thus, $\varphi(z)\equiv 0$. Substituting $r(z)$ into $\varphi(z)$, then we give the coefficients of term $z^{2n-1}, z^{2n-2}, z^{2n-3}, z^{2n-4}, \dots, z^{n+1}$ as follows

$$B_{2n-1}=-A_{n-1}[a(q+q^{-1}+2)-(q+q^{-1}+1)](q+q^{-1}-2), \tag{2.5}$$

$$B_{2n-2}=-A_{n-2}[a(q^2+q^{-2}+2)-(q^2+q^{-2}+1)](q^2+q^{-2}-2), \tag{2.6}$$

$$\begin{aligned} B_{2n-3}=&-A_{n-2}[a(q^2+q^{-2}+2)-(q^2+q^{-2}+1)](q^2+q^{-2}-2)\\ &+A_{n-1}A_{n-2}[a(q+q^{-1}+2)-(q+q^{-1}+1)](q+q^{-1}-2),\end{aligned} \tag{2.7}$$

$$\begin{aligned} B_{2n-4}=&-A_{n-4}\Big[a(q^4+q^{-4}+2)-(q^4+q^{-4}+1)](q^2+q^{-2}-2)\\ &+A_{n-1}A_{n-3}[a(q^2+q^{-2}+2)-(q^2+q^{-2}+1)\Big](q^2+q^{-2}-2),\end{aligned} \tag{2.8}$$

$$\dots$$

$$\begin{aligned} B_{2n-i}=&-A_{n-i}[a(q^i+q^{-i}+2)-(q^i+q^{-i}+1)](q^i+q^{-i}-2)\\ &+A_{n-1}A_{n-i+1}[a(q^{i-2}+q^{-(i-2)}+2)-(q^{i-2}+q^{-(i-2)}+1)]\\ &\times(q^{i-2}+q^{-(i-2)}-2)+\cdots+A_{n-[\frac{i}{2}]}A_{n-[\frac{i}{2}]+1}\big[a(q^2+q^{-2}+2)\\ &-(q^2+q^{-2}+1)\big](q^2+q^{-2}-2),\end{aligned} \tag{2.9}$$

$$\dots$$

$$\begin{aligned} B_{n+1}=&-A_1[a(q^{n-1}+q^{-(n-1)}+2)-(q^{n-1}+q^{-(n-1)}+1)]\\ &\times(q^{n-1}+q^{-(n-1)}-2)+A_{n-1}A_2[a(q^{n-3}+q^{-(n-3)}+2)\\ &-(q^{n-3}+q^{-(n-3)}+1)](q^{n-3}+q^{-(n-3)}-2)+\dots.\end{aligned} \tag{2.10}$$

Note that if i is an even integer in (2.9), then $A_{n-[\frac{i}{2}]}A_{n-[\frac{i}{2}]+1}$ should be replaced by $A_{n-\frac{i}{2}-1}A_{n-\frac{i}{2}+1}$. In view of (2.5)-(2.10), we conclude that there are at most one of $A_1, A_2, \dots, A_{n-1}$ can be equal to 0. Otherwise, if there exist two integers $i,j\in\mathbb{N}_+$ such that $i\neq j$, $A_j\neq 0, A_i\neq 0$ and $A_t=0$ for $t=1,2,\dots,n-1, t\neq i, t\neq j$. From (2.5)-(2.10), we have

$$\begin{gathered} a(q^i+q^{-i}+2)-(q^i+q^{-i}+1)\equiv 0,\\ a(q^j+q^{-j}+2)-(q^j+q^{-j}+1)\equiv 0.\end{gathered}$$

This is impossible as $|q| \neq 1$. Thus, without loss of generality, we assume that $A_{n-1} \neq 0$ and $A_i = 0$ for $j = 1, 2, \dots, n-2; j \neq n-1$. Then, in view of (2.5), it follows that

$$a = \frac{q^2 + q + 1}{(q+1)^2}. \tag{2.11}$$

Thus, $r(z), P(z)$ can be represented in the form

$$r(z) = z^n + A_{n-1}z^{n-1} + A_0, \quad P(z) = a[z^n + A_{n-1}z^{n-1} + A_0]^2, \tag{2.12}$$

where $n \neq 1$. It leads to

$$Q(z) = (q^n z^n + A_{n-1}q^{n-1}z^{n-1} + A_0)(q^{-n}z^n + A_{n-1}q^{-n+1}z^{n-1} + A_0). \tag{2.13}$$

Substituting (2.12) and (2.13) into $\varphi(z)$, and analyzing the coefficients of the term z^n, we have

$$B_n = -A_0[a(q^n + q^{-n} + 2) - (q^n + q^{-n} + 1)](q^n + q^{-n} - 2). \tag{2.14}$$

In view of (2.11), $|q| \neq 1$ and $n \neq 1$, it follows that

$$a(q^n + q^{-n} + 2) - (q^n + q^{-n} + 1) \neq 0.$$

Thus, by combining with $A_0 \neq 0$, this is a contradiction with $\varphi(z) \equiv 0$. Therefore, $A_1 = A_2 = \dots = A_{n-1} \equiv 0$; that is,

$$r(z) = z^n + b, \quad P(z) = a(z^n + b)^2, \quad Q(z) = (q^n z^n + b)(q^{-n}z^n + b), \tag{2.15}$$

where b is an any nonzero constant. Substituting (2.15) into $\varphi(z)$, we have

$$a = \frac{q^{2n} + q^n + 1}{(q^n+1)^2}. \tag{2.16}$$

Thus, substituting (2.15) and (2.16) into (2.1), we obtain

$$\mu = a^3(a-1) = -\frac{q^n(q^{2n} + q^n + 1)^3}{(q^n+1)^8}.$$

This completes the proof of Theorem 1.5. $\square$

3. Proof of Theorem 1.7

The following lemmas are necessary.

Lemma 3.1 ([1, Theorem 2.5]). *Let f be a nonconstant zero-order meromorphic solution of $P_q(z, f) = 0$, where $P_q(z, f)$ is a q-difference polynomial in $f(z)$. If $P_q(z, a) \not\equiv 0$ for slowly moving target $a(z)$, then*

$$m\Big(r, \frac{1}{f-a}\Big) = S(r, f).$$

Lemma 3.2 ([29, Theorem 1.1 and 1.3]). *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z)), \quad N(r, f(qz)) = (1 + o(1))N(r, f(z)),$$

on a set of lower logarithmic density 1.

Lemma 3.3 ([15, Theorem 2.5]). *Let f be a transcendental meromorphic solution of order zero of a q-difference equation of the form*
$$U_q(z,f)P_q(z,f)=Q_q(z,f),$$
where $U_q(z,f), P_q(z,f)$ and $Q_q(z,f)$ are q-difference polynomials such that the total degree $\deg\, U_q(z,f)=n$ *in $f(z)$ and its q-shifts, whereas* $\deg Q_q(z,f)\le n$. *Moreover, we assume that $U_q(z,f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then*
$$m(r,P_q(z,f))=S(r,f).$$

Remark 3.4. For $q\in\mathbb{C}\backslash\{0,1\}$, a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz)$, …, $f(q^nz)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r,f))$ on a set F of logarithmic density 1, can be called as a q-difference polynomial of f.

Lemma 3.5 (Valiron-Mohon'ko [28]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*
$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$
with meromorphic coefficients $a_i(z),b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies
$$T(r,R(z,f(z)))=dT(r,f)+O(\Psi(r)),$$
where $d=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Lemma 3.6 ([1, Theorem 1.1]). *Let $f(z)$ be a nonconstant zero order meromorphic function and $q\in\mathbb{C}\setminus\{0\}$. Then*
$$m\Big(r,\frac{f(qz)}{f(z)}\Big)=S(r,f).$$

Proof of Theorem 1.7. (i) Let $f(z)$ be a transcendental meromorphic function of zero order. For any $\eta\in\mathbb{C}-\{0,1\}$, substituting ηz into (1.8), we have
$$f(q\eta z)f(\frac{\eta z}{q})f(\eta z)(f(\eta z)-1)=\mu. \tag{3.1}$$
Denoting $g(z)=f(\eta z)$, equation (3.1) can be represented as
$$g(qz)g(\frac{z}{q})g(z)(g(z)-1)=\mu.$$
Let
$$P_1(z,g):=g(qz)g\big(\frac{z}{q}\big)g(z)(g(z)-1)-\mu=0.$$
It follows that
$$P_1(z,z)=z^3(z-1)-\mu\not\equiv 0.$$
In view of $P_1(z,z)\not\equiv 0$, by Lemma 3.1 we have
$$m\Big(r,\frac{1}{g(z)-z}\Big)=S(r,g).$$
Since f is of zero order, from Lemma 3.2, it follows that
$$\begin{aligned}N\Big(r,\frac{1}{f(\eta z)-z}\Big)&=N\Big(r,\frac{1}{g(z)-z}\Big)=T(r,g)+S(r,g)\\&=T(r,f(\eta z))+S(r,f(\eta z))=T(r,f)+S(r,f).\end{aligned}$$

Therefore, $f(\eta z)$ has infinitely many fixed points, and $\tau(f(\eta z)) = \sigma(f)$ for any $\eta \in \mathbb{C} - \{0, 1\}$.

(ii) Since f is a transcendental meromorphic solution of zero order. In view of $\mu \neq 0$, by Lemmas 3.2-3.6,

$$\begin{aligned}
4T(r,f) &= T\Big(r, \frac{\mu}{f^3(f-1)}\Big) + O(1) \\
&= T\Big(r, \frac{f(qz)f(\frac{z}{q})}{f(z)^2}\Big) + O(1) \\
&\leq T\Big(r, \frac{f(qz)}{f(z)}\Big) + T\Big(r, \frac{f(\frac{z}{q})}{f(z)}\Big) + O(1) \\
&\leq 2T\Big(r, \frac{f(qz)}{f(z)}\Big) + S(r,f) = 2T\Big(r, \frac{\Delta_q f}{f}\Big) + S(r,f);
\end{aligned}$$

that is,

$$2T(r,f) \leq T\Big(r, \frac{\Delta_q f}{f}\Big) + S(r,f). \tag{3.2}$$

Thus, from Lemma 3.6 and (3.2), we conclude that

$$N\Big(r, \frac{\Delta_q f}{f}\Big) = T\Big(r, \frac{\Delta_q f}{f}\Big) - m\Big(r, \frac{\Delta_q f}{f}\Big) \geq 2T(r,f) + S(r,f).$$

This means that $\frac{\Delta_q f}{f}$ has infinitely many poles, and $\lambda\big(\frac{1}{\frac{\Delta_q f}{f}}\big) = \sigma(f)$.

Also, we can rewrite equation (1.8) as

$$f(qz)f(\frac{z}{q}) = (\Delta_q f + f)(\Delta_{q^{-1}} f + f) = \frac{\mu}{f(f-1)};$$

that is,

$$\Delta_q f \Delta_{q^{-1}} f + (\Delta_q f + \Delta_{q^{-1}} f)f = \frac{\mu - f^4 + f^3}{f(f-1)}. \tag{3.3}$$

Thus, in view of Lemmas 3.2 and 3.5, it follows that

$$\begin{aligned}
4T(r,f) &= T\Big(r, \frac{\mu - f^4 + f^3}{f(f-1)}\Big) + O(1) \\
&= T(r, \Delta_q f \Delta_{q^{-1}} f) + (\Delta_q f + \Delta_{q^{-1}} f)f) + O(1) \\
&\leq T(r,f) + 2T(r, \Delta_q f) + 2T(r, \Delta_{q^{-1}} f) + O(1) \\
&\leq T(r,f) + 4T(r, \Delta_q f) + S(r,f);
\end{aligned}$$

that is,

$$\frac{3}{4}T(r,f) \leq T(r, \Delta_q f) + S(r,f). \tag{3.4}$$

On the other hand, (1.8) can be represented as

$$f(qz)f(\frac{z}{q})f(z)^2 = \mu + f(qz)f(\frac{z}{q})f(z).$$

By Lemma 3.3, we obtain $m(r,f) = S(r,f)$. Thus, we can conclude from Lemma 3.6 that

$$\begin{aligned}
N(r, \Delta_q f) &= T(r, \Delta_q f) - m(r, \Delta_q f) \\
&\geq T(r, \Delta_q f) - \Big[m(r,f) + m\Big(r, \frac{\Delta_q f}{f}\Big)\Big]
\end{aligned}$$

$$\geq \frac{1}{2}T(r,f) + S(r,f),$$

which implies that $\Delta_q f$ has infinitely many poles and $\lambda(\frac{1}{\Delta_q f}) = \sigma(f)$. Since

$$m\Big(r, \frac{1}{f}\Big) = m\Big(r, \frac{f(qz)f(\frac{z}{q})(f-1)}{\mu}\Big) = m\Big(r, \frac{f(qz)f(\frac{z}{q})}{f^2}\frac{f^2(f-1)}{\mu}\Big),$$

by combining with $m(r,f) = S(r,f)$ and Lemma 3.6, it follows that

$$m\Big(r, \frac{1}{f}\Big) = S(r,f),$$

which yields

$$N\big(r, \frac{1}{f}\big) = T(r,f) + S(r,f), \quad N(r,f) = T(r,f) + S(r,f),$$

which implies that f has infinitely many poles and zeros, and $\lambda(f) = \lambda(\frac{1}{f}) = \sigma(f)$. This completes the proof □

4. Proof of Theorem 1.8

Suppose that $f(z) = P(z)/Q(z)$ is a nonconstant rational solution of equation (2.2), where $P(z), Q(z)$ are relatively prime polynomials with $\deg_z P(z) = p$ and $\deg_z Q(z) = t$. Substituting this into (1.9), we have

$$\frac{P(qz)}{Q(qz)}\frac{P(\frac{z}{q})}{Q(\frac{z}{q})}\Big(\frac{P(z)}{Q(z)} - 1\Big)^2 = \Big(\frac{P(z)}{Q(z)} - \lambda\Big)^2. \tag{4.1}$$

By using the same argument as in the proof of Theorem 1.5 (i), we obtain $p = t$.

Suppose that $\lambda \neq a^2$. Without loss of generality we assume that the coefficients of the highest degree terms of $P(z), Q(z)$ are a and 1, respectively. In view of (1.9), letting $|z| = r \to +\infty$ yields

$$a^2(a-1)^2 = (a-\lambda)^2. \tag{4.2}$$

Since $\lambda \neq 0, 1$, then $a \neq 0, 1, \lambda$. Now, we rewrite (1.9) in the form

$$\frac{P(qz)}{Q(qz)}\frac{P(\frac{z}{q})}{Q(\frac{z}{q})} = \Big(\frac{P(z) - \lambda Q(z)}{P(z) - Q(z)}\Big)^2.$$

Since $\deg_z[P(z) - \lambda Q(z)] = \deg_z[P(z) - Q(z)] = p$, it follows that

$$(a-\lambda)^2 P(qz)P\big(\frac{z}{q}\big) = a^2(P(z) - \lambda Q(z))^2, \tag{4.3}$$

$$(a-1)^2 Q(qz)Q\big(\frac{z}{q}\big) = (P(z) - Q(z))^2. \tag{4.4}$$

In view of (4.3) and (4.4), it is easy to see that 0 is not the zero of $P(z), Q(z)$. Otherwise, if 0 is a zero of $P(z)$, from (4.3), we can get that 0 is also a zero of $Q(z)$, this is a contradiction with the hypothesis of $P(z), Q(z)$ being relatively prime polynomials; if 0 is a zero of $Q(z)$, from (4.4), we can also get a contradiction.

Now, suppose that $z_0(\neq 0)$ is a zero of $P(z)$ with order k, and k is an odd integer. Then z_0/q is a zero of $P(qz)$ with order k. However, in view of (4.3), it yields that the orders of the zeros of $P(qz)P(z/q)$ are all even integers, thus, z_0/q must be a zero of $P(z/q)$ with order l, and l is an odd integer. Hence, z_0/q^2 is a zero of $P(z)$ with the odd order l. Thus, continue this process, we obtain that z_0/q^m are the zeros of $P(z)$ for any integer m. This is impossible as $\deg_z P(z) = p$ and $|q| \neq 1$.

Therefore, all the zeros of $P(z)$ have even orders. Similarly, all the zeros of $Q(z)$ have even orders.

Thus, set $P(z) = a\alpha(z)^2$ and $Q(z) = \beta(z)^2$, where

$$\alpha(z) = z^n + A_{n-1}z^{n-1} + \dots + A_1 z + A_0, \tag{4.5}$$

$$\beta(z) = z^n + B_{n-1}z^{n-1} + \dots + B_1 z + B_0, \tag{4.6}$$

and $A_0, A_1, \dots, A_{n-1}, B_0, B_1, \dots, B_{n-1}$ are constants. Obviously, A_0, B_0 can not be equal to 0 simultaneously. Then, in view of (4.3) and (4.4), we have

$$(a-\lambda)\alpha(qz)\alpha\big(\frac{z}{q}\big) = a\alpha(z)^2 - \lambda\beta(z)^2, \tag{4.7}$$

$$(a-1)\beta(qz)\beta\big(\frac{z}{q}\big) = a\alpha(z)^2 - \beta(z)^2. \tag{4.8}$$

Substituting (4.5) and (4.6) into the above equations, and analyzing the coefficients of terms $z^{2n-1}, z^{2n-2}, \dots$, we can deduce that

$$(a-\lambda)(q+q^{-1})A_{n-1} = 2aA_{n-1} - 2\lambda B_{n-1}, \tag{4.9}$$

$$(a-1)(q+q^{-1})B_{n-1} = 2aA_{n-1} - 2B_{n-1}, \tag{4.10}$$

$$(a-\lambda)[(q^2+q^{-2})A_{n-2} + A_{n-1}^2] = a(2A_{n-2} + A_{n-1}^2) - \lambda(2B_{n-2} + B_{n-1}^2), \tag{4.11}$$

$$(a-1)[(q^2+q^{-2})B_{n-2} + B_{n-1}^2] = a(2A_{n-2} + A_{n-1}^2) - (2B_{n-2} + B_{n-1}^2), \tag{4.12}$$

$$\dots,$$

$$\begin{aligned}
&(a-\lambda)[(q^i+q^{-i})A_{n-i} + (q^{i-2}+q^{-(i-2)})A_{n-1}A_{n-i+1} + \dots \\
&+ (q+q^{-1})A_{n-\frac{i-1}{2}}A_{n-\frac{i+1}{2}}] \\
&= a(2A_{n-i} + 2A_{n-1}A_{n-i+1} + \dots + 2A_{n-\frac{i-1}{2}}A_{n-\frac{i+1}{2}}) \\
&\quad - \lambda(2B_{n-i} + 2B_{n-1}B_{n-i+1} + \dots + 2B_{n-\frac{i-1}{2}}B_{n-\frac{i+1}{2}}),
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&(a-1)[(q^i+q^{-i})B_{n-i} + (q^{i-2}+q^{-(i-2)})B_{n-1}B_{n-i+1} + \dots \\
&+ (q+q^{-1})B_{n-\frac{i-1}{2}}B_{n-\frac{i+1}{2}}] \\
&= a(2A_{n-i} + 2A_{n-1}A_{n-i+1} + \dots + 2A_{n-\frac{i-1}{2}}A_{n-\frac{i+1}{2}}) \\
&\quad - (2B_{n-i} + 2B_{n-1}B_{n-i+1} + \dots + 2B_{n-\frac{i-1}{2}}B_{n-\frac{i+1}{2}}),
\end{aligned} \tag{4.14}$$

$$\dots,$$

$$\begin{aligned}
&(a-\lambda)[(q^j+q^{-j})A_{n-j} + (q^{j-2}+q^{-(j-2)})A_{n-1}A_{n-j+1} + \dots + A_{n-\frac{j}{2}}^2] \\
&= a(2A_{n-j} + 2A_{n-1}A_{n-j+1} + \dots + A_{n-\frac{j}{2}}^2) \\
&\quad - \lambda(2B_{n-1}B_{n-j+1} + \dots + 2B_{n-\frac{j}{2}}^2),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&(a-1)[(q^j+q^{-j})B_{n-j} + (q^{j-2}+q^{-(j-2)})B_{n-1}B_{n-j+1} + \dots + B_{n-\frac{j}{2}}^2] \\
&= a(2A_{n-j} + 2A_{n-1}A_{n-j+1} + \dots + A_{n-\frac{j}{2}}^2) \\
&\quad - (2B_{n-1}B_{n-j+1} + \dots + 2B_{n-\frac{j}{2}}^2),
\end{aligned} \tag{4.16}$$

$$\dots,$$

where i is an odd integer, and j is an even integer.

Assume that there exist a positive integer $i \in \mathbb{N}_+, i < n$ satisfying $A_{n-i} \neq 0$ and $A_{n-j} = 0$ for any non-negative integer $j < i$. Without loss of generality, we let

$i=1$, that is, $A_{n-1}\neq 0$, thus, $B_{n-1}\neq 0$. Otherwise, if $B_{n-1}=0$, then by (4.10), it follows $A_{n-1}=0$, a contradiction.

In view of (4.9)-(4.10), it yields

$$(A_{n-1}-B_{n-1})[(a-\lambda)aA_{n-1}-\lambda(a-1)B_{n-1}]=0, \tag{4.17}$$

which leads to either $A_{n-1}=B_{n-1}$ or $(a-\lambda)aA_{n-1}-\lambda(a-1)B_{n-1}=0$. If $A_{n-1}=B_{n-1}$, we can deduce from (4.10) that $q^1+q^{-1}=2$, which implies a contradiction with $|q|\neq 1$. Thus, it yields that $A_{n-1}\neq B_{n-1}$ and

$$(a-\lambda)aA_{n-1}=\lambda(a-1)B_{n-1}. \tag{4.18}$$

Further, suppose that $A_{n-2}\neq 0$. Then $B_{n-2}\neq 0$. Indeed, if $A_{n-2}=0$, then from (4.11) and (4.12), we have $B_{n-2}=0$. Similarly, if $B_{n-2}=0$, then $A_{n-2}=0$. In view of (4.11) and (4.12), we have

$$[2(A_{n-2}-B_{n-2})+(A_{n-1}^2-B_{n-1}^2)][(a-\lambda)aA_{n-2}-\lambda(a-1)B_{n-2}]=0. \tag{4.19}$$

From (4.19), either $2(A_{n-2}-B_{n-2})+(A_{n-1}^2-B_{n-1}^2)=0$ or $(a-\lambda)aA_{n-2}-\lambda(a-1)B_{n-2}=0$. If $2(A_{n-2}-B_{n-2})+(A_{n-1}^2-B_{n-1}^2)=0$, we can deduce from (4.12) that $q^2+q^{-2}=2$, which implies a contradiction with $|q|\neq 1$. Therefore

$$(a-\lambda)aA_{n-2}=\lambda(a-1)B_{n-2}. \tag{4.20}$$

It follows from (4.18) and (4.20) that

$$(a-\lambda)^2a^2=\lambda^2(a-1)^2.$$

Combining this with (4.2) yields $\lambda=a^2$, a contradiction. Hence, $A_{n-2}=0$ and $B_{n-2}=0$. As in the above argument, it follows that $A_{n-3}=\dots=A_1=0$ and $B_{n-3}=\dots=B_1=0$. Thus, $A_{n-3}=\dots=A_1=0$ and $B_{n-3}=\dots=B_1=0$. Since 0 is not the zero of $P(z),Q(z)$, it follows that $A_0\neq 0$ and $B_0\neq 0$. By analyzing the coefficients of the term z^n, we deduce that

$$(a-\lambda)aA_0=\lambda(a-1)B_0. \tag{4.21}$$

Thus, in view of (4.18),(4.21) and (4.2), it yields $\lambda=a^2$, a contradiction. Hence, we conclude $A_1=A_2=\dots=A_{n-1}=0$ and $B_1=B_2=\dots=B_{n-1}=0$. In view of (4.9)-(4.16), it is easy to deduce that $B_1=B_2=\dots=B_{n-1}=0$. Thus,

$$\alpha(z)=z^n+A_0,\quad \beta(z)=z^n+B_0. \tag{4.22}$$

Hence, substituting α,β into (4.7) and (4.8), by comparing the coefficients of the terms z^n and constant, we have

$$\begin{gathered}(a-\lambda)(q^n+q^{-n})A_0=2aA_0-2\lambda B_0,\\ (a-1)(q^n+q^{-n})B_0=2aA_0-2B_0,\\ (a-\lambda)A_0^2=aA_0^2-\lambda B_0^2,\\ (a-1)B_0^2=aA_0^2-B_0^2.\end{gathered}$$

Then it follows that $A_0=B_0$ or $A_0=-B_0$. If $A_0=B_0$, then $q^n+q^{-n}=2$ is a contradiction. If $A_0=-B_0$, then $\lambda(a-1)B_0-a(a-\lambda)A_0=0$. Thus, with a view of $A_0\neq 0$, this yields $\lambda=a^2$, a contradiction. This completes the proof of Theorem 1.8.

For the proof of Theorem 1.10 we use the same argument as in the proof of Theorem 1.7 and the conclusion follows easily.

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Hong Yan Xu
School of Mathematics and Computer Science, Shangrao Normal University, Shangrao Jiangxi 334001, China
Email address: `xhyhhh@126.com`

Jin Tu
Department of Mathematics, Jiangxi Normal University, Nanchan, Jiangxi 330022, China
Email address: `tujin2008@sina.com`