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EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS TO PARABOLIC PROBLEMS WITH NONSTANDARD GROWTH AND CROSS DIFFUSION

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Abstract. We establish the existence and uniqueness of weak solutions to the parabolic system with nonstandard growth condition and cross diffusion,

$$\partial_t u - \operatorname{div} a(x,t,\nabla u)) = \operatorname{div} |F|^{p(x,t)-2}F),$$
$$\partial_t v - \operatorname{div} a(x,t,\nabla v)) = \delta \Delta u,$$

where $\delta \geq 0$ and $\partial_t u,\ \partial_t v$ denote the partial derivative of u and v with respect to the time variable t, while ∇u and ∇v denote the one with respect to the spatial variable x. Moreover, the vector field $a(x,t,\cdot)$ satisfies certain nonstandard $p(x,t)$ growth, monotonicity and coercivity conditions.

1. Introduction

The study of parabolic problems, i.e. equations and systems, like reaction-diffusion systems or evolutionary equations is motivated amongst others by several applications. For instance, such equations and systems are important for the modeling of space- and time-dependent problems, e.g. problems from physics or biology. In particular, evolutionary equations and systems can be used to model physical processes like heat conduction or diffusion processes, see [9, 25]. One example is the Navier-Stokes equation, the basic equation in fluid mechanics. In addition, applications also include climate modeling and climatology [15]. Furthermore, an interesting aspect of this paper is the nonstandard growth setting, which arises for instance by studying certain classes of non-Newtonian fluids such as electro–rheological fluids or fluids with viscosity depending on the temperature. Some properties of solutions to systems of such modified Navier-Stokes equation are studied in [4]. In general, electro–rheological fluids are of high technological interest, because of their ability to change their mechanical properties under the influence of an exterior electromagnetic field [16, 30]. Many electro-rheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field [31]. Most of the known results concern the stationary case with $p(x)$ growth condition, see [2, 3, 18]. Furthermore, for the restoration in image processing one also uses some diffusion models with nonstandard growth condition [1, 14, 27, 28]. In the context of parabolic

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problems with $p(x,t)$ growth applications are flows in porous media [6] or nonlinear parabolic obstacle problems [19, 22, 23]. Moreover, in the last years parabolic problems with $p(x,t)$ growth arouse more and more interest in mathematics, see [7, 8, 11, 24, 26, 29, 32, 35, 37]. A further aspect of our paper is the effect of a cross diffusion term. Parabolic nonstandard growth problem with cross diffusion is a new and very interesting topic, since the interaction between the species often leads to cross diffusion effects, which may show unexpected behavior, see [13], i.e. the forward of the special issue "Advances in Reaction-Cross-Diffusion Systems" [12]. For instance, in our case the cross diffusion term $\delta\Delta u$, $\delta \geq 0$ requires that the growth exponent $p(x,t)$ is greater or equal to two. Only in case $\delta = 0$ we may assume that $\frac{2n}{n+2} < p(x,t)$, $n \geq 2$. In addition, parabolic systems with cross diffusion play a crucial role in biological applications like epidemic diseases, chemotaxis phenomena, cancer growth and population development.

In this article, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain of dimension $n \geq 2$ and we write $\Omega_T := \Omega \times (0,T)$ for the space-time cylinder over Ω of height $T > 0$. Here, u_t or $\partial_t u$ respectively denote the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x. Moreover, we denote by $\partial_{\mathcal{P}}\Omega_T = (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T))$ the parabolic boundary of Ω_T and we write $z = (x,t)$ for points in $\mathbb{R}^{n+1}$.

The aim of our investigation is to establish the existence of a (weak) solution to the following inhomogeneous parabolic Dirichlet problem with nonstandard growth condition and cross diffusion term $\delta\Delta u$, $\delta \geq 0$:

$$\begin{gathered}
\partial_t u - \operatorname{div} a(x,t,\nabla u)) = \operatorname{div} |F|^{p(x,t)-2}F), \quad \text{in } \Omega_T, \\
\partial_t v - \operatorname{div} a(x,t,\nabla v)) = \delta\Delta u, \quad \text{in } \Omega_T, \\
u = v = 0, \quad \text{on } \partial\Omega \times (0,T), \\
u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, \quad \text{on } \Omega \times \{0\},
\end{gathered} \tag{1.1}$$

where the vector field $a(x,t,\cdot)$ satisfies certain nonstandard $p(x,t)$ growth, monotonicity and coercivity conditions, which we will specify in the next paragraph. Furthermore, we will specify the regularity assumption on the inhomogeneity F and the conditions which are supposed for the supercritical growth exponent function $p : \Omega_T \to [2,\infty)$ later.

1.1. **General assumptions.** The vector fields $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be Carathéodory functions — i.e. $a(z,w)$ is measurable in the first argument for every $w \in \mathbb{R}^n$ and continuous in the second one for a.e. $z \in \Omega_T$ — and satisfy the following nonstandard growth, monotonicity and coercivity properties, for some growth exponent $p : \Omega_T \to [2,\infty)$ and structure constants $0 < \nu \leq 1 \leq L$:

$$|a(z,w)| \leq L(1+|w|)^{p(z)-1}, \tag{1.2}$$

$$(a(z,w) - a(z,w_0)) \cdot (w - w_0) \geq 0, \tag{1.3}$$

$$a(z,w) \cdot w \geq \nu |w|^{p(z)}, \tag{1.4}$$

for all $z \in \Omega_T$ and $w, w_0 \in \mathbb{R}^n$. Further, the growth exponent $p : \Omega_T \to [2,\infty)$ satisfies the following conditions: There exist constants γ_1 and γ_2, such that

$$2 \leq \gamma_1 \leq p(z) \leq \gamma_2 < \infty \quad \text{and} \quad |p(z_1) - p(z_2)| \leq \omega(d_{\mathcal{P}}(z_1,z_2)) \tag{1.5}$$

hold for any choice of $z_1,\ z_2 \in \Omega_T$, where $\omega : [0,\infty) \to [0,1]$ denotes a modulus of continuity. More precisely, we assume that $\omega(\cdot)$ is a concave, non-decreasing function with $\lim_{\rho\downarrow 0}\omega(\rho) = 0 = \omega(0)$. Moreover, the parabolic distance is given by $d_{\mathcal{P}}(z_1, z_2) := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$ for $z_1 = (x_1, t_1),\ z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$. In addition, for the modulus of continuity $\omega(\cdot)$ we assume the weak logarithmic continuity condition

$$\limsup_{\rho\downarrow 0} \omega(\rho) \log\big(\frac{1}{\rho}\big) < \infty. \tag{1.6}$$

1.2. **Function spaces.** The spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces, while the nonstandard $p(z)$ Lebesgue space $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ is defined as the set of those measurable functions $v : \Omega_T \to \mathbb{R}^k$ for $k \in \mathbb{N}$, which satisfy $|v|^{p(z)} \in L^1(\Omega_T, \mathbb{R}^k)$, i.e.

$$L^{p(z)}(\Omega_T, \mathbb{R}^k) := \big\{v : \Omega_T \to \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |v|^{p(z)} \mathrm{d}z < +\infty\big\}.$$

The set $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ equipped with the Luxemburg norm

$$\|v\|_{L^{p(z)}(\Omega_T)} := \inf\big\{\lambda > 0 : \int_{\Omega_T} |\frac{v}{\lambda}|^{p(z)} \mathrm{d}z \le 1\big\}$$

becomes a Banach space. This space is separable and reflexive, see [5, 17]. At this stage, we are able to specify the regularity assumption on the inhomogeneity, i.e. we suppose that $F \in L^{p(z)}(\Omega_T, \mathbb{R}^n)$. For elements of $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ the generalized Hölder's inequality holds in the form: If $f \in L^{p(z)}(\Omega_T, \mathbb{R}^k)$ and $g \in L^{p'(z)}(\Omega_T, \mathbb{R}^k)$, where $p'(z) = \frac{p(z)}{p(z)-1}$, we have

$$\big|\int_{\Omega_T} fg \mathrm{d}z\big| \le \Big(\frac{1}{\gamma_1} + \frac{\gamma_2 - 1}{\gamma_2}\Big)\|f\|_{L^{p(z)}(\Omega_T)}\|g\|_{L^{p'(z)}(\Omega_T)}, \tag{1.7}$$

see also [5]. Moreover, the norm $\|\cdot\|_{L^{p(z)}(\Omega_T)}$ can be estimated as follows

$$-1 + \|v\|^{\gamma_1}_{L^{p(z)}(\Omega_T)} \le \int_{\Omega_T} |v|^{p(z)} \mathrm{d}z \le \|v\|^{\gamma_2}_{L^{p(z)}(\Omega_T)} + 1. \tag{1.8}$$

We will use also the abbreviation $p(\cdot)$ for the exponent $p(z)$. Next, we introduce nonstandard Sobolev spaces for fixed $t \in (0, T)$. From assumption (1.5) we know that $p(\cdot, t)$ satisfies $|p(x_1, t) - p(x_2, t)| \le \omega(|x_1 - x_2|)$ for any choice of $x_1, x_2 \in \Omega$ and for every $t \in (0, T)$. Then, we define for every fixed $t \in (0, T)$ the Banach space

$$W^{1,p(\cdot,t)}(\Omega) := \{u \in L^{p(\cdot,t)}(\Omega, \mathbb{R}) \mid \nabla u \in L^{p(\cdot,t)}(\Omega, \mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot,t)}(\Omega)} := \|u\|_{L^{p(\cdot,t)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot,t)}(\Omega)}.$$

In addition, we define $W_0^{1,p(\cdot,t)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot,t)}(\Omega)$ and we denote by $W^{1,p(\cdot,t)}(\Omega)'$ its dual. For every $t \in (0, T)$ the inclusion $W_0^{1,p(\cdot,t)}(\Omega) \subset W_0^{1,\gamma_1}(\Omega)$ holds true. Furthermore, we denote by $W_g^{p(\cdot)}(\Omega_T)$ the Banach space

$$W_g^{p(\cdot)}(\Omega_T) := \{u \in [g + L^1(0, T; W_0^{1,1}(\Omega))] \cap L^{p(\cdot)}(\Omega_T) : \nabla u \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)\}$$

equipped with the norm $\|u\|_{W^{p(\cdot)}(\Omega_T)} := \|u\|_{L^{p(\cdot)}(\Omega_T)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega_T)}$. In the case $g = 0$ we write $W_0^{p(\cdot)}(\Omega_T)$ instead of $W_g^{p(\cdot)}(\Omega_T)$. Here, it is worth to mention that the notion $(u - g) \in W_0^{p(\cdot)}(\Omega_T)$ or $u \in g + W_0^{p(\cdot)}(\Omega_T)$ respectively indicates that u agrees with g on the lateral boundary of the cylinder Ω_T, i.e. $u \in W_g^{p(\cdot)}(\Omega_T)$. In addition, we denote by $W^{p(\cdot)}(\Omega_T)'$ the dual of the space $W_0^{p(\cdot)}(\Omega_T)$. Note that if $v \in W^{p(\cdot)}(\Omega_T)'$, then there exist functions $v_i \in L^{p'(\cdot)}(\Omega_T)$, $i = 0, 1, \dots, n$, such that

$$\langle\langle v, w\rangle\rangle_{\Omega_T} = \int_{\Omega_T} \Big(v_0 w + \sum_{i=1}^n v_i \nabla_i w\Big) \mathrm{d}z \tag{1.9}$$

for all $w \in W_0^{p(\cdot)}(\Omega_T)$. Furthermore, if $v \in W^{p(\cdot)}(\Omega_T)'$, we define the norm

$$\|v\|_{W^{p(\cdot)}(\Omega_T)'} := \sup\{\langle\langle v, w\rangle\rangle_{\Omega_T} : w \in W_0^{p(\cdot)}(\Omega_T),\ \|w\|_{W_0^{p(\cdot)}(\Omega_T)} \le 1\}.$$

Notice, whenever (1.9) holds, we can write $v = v_0 - \sum_{i=1}^n \nabla_i v_i$, where $\nabla_i v_i$ has to be interpreted as a distributional derivative. By

$$w \in W(\Omega_T) := \{w \in W^{p(\cdot)}(\Omega_T) : w_t \in W^{p(\cdot)}(\Omega_T)'\}$$

we mean that there exists $w_t \in W^{p(\cdot)}(\Omega_T)'$, such that

$$\langle\langle w_t, \varphi\rangle\rangle_{\Omega_T} = -\int_{\Omega_T} w \cdot \varphi_t \mathrm{d}z \quad \text{for all } \varphi \in C_0^\infty(\Omega_T),$$

see also [17]. The previous equality makes sense due to the inclusions

$$W^{p(\cdot)}(\Omega_T) \hookrightarrow L^2(\Omega_T) \cong (L^2(\Omega_T))' \hookrightarrow W^{p(\cdot)}(\Omega_T)'$$

which allow us to identify w as an element of $W^{p(\cdot)}(\Omega_T)'$. Finally, we are in a position to give the definition of a weak solution to the parabolic problem (1.1).

Definition 1.1. We call $u, v \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ a (weak) solution to the parabolic Dirichlet problem (1.1), if

$$\begin{gathered} \int_{\Omega_T} [u \cdot \varphi_t - a(z, \nabla u) \cdot \nabla\varphi] \mathrm{d}z = \int_{\Omega_T} |F|^{p(x,t)-2} F \cdot \nabla\varphi \mathrm{d}z, \\ \int_{\Omega_T} [v \cdot \zeta_t - a(z, \nabla v) \cdot \nabla\zeta] \mathrm{d}z = \int_{\Omega_T} \delta \nabla u \cdot \nabla\zeta \mathrm{d}z, \end{gathered} \tag{1.10}$$

whenever $\varphi, \zeta \in C_0^\infty(\Omega_T)$, $\delta \ge 0$, the boundary condition $u = v = 0$ on $\partial\Omega \times \{0\}$ and initial conditions $u(\cdot, 0) = u_0 \in L^2(\Omega)$, $v(\cdot, 0) = v_0 \in L^2(\Omega)$ a.e. on Ω, i.e.

$$\frac{1}{h}\int_0^h \int_\Omega |u - u_0|^2 \mathrm{d}x\mathrm{d}t \to 0 \quad \text{and} \quad \frac{1}{h}\int_0^h \int_\Omega |v - v_0|^2 \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } h \downarrow 0. \tag{1.11}$$

are satisfied.

We will also use the notation

$$(u, v) \in (C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T))^2$$

instead of $u, v \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ and similarly we will use $(u_0, v_0) \in (L^2(\Omega))^2$, which means the same as $u_0, v_0 \in L^2(\Omega)$.

1.3. **Statement of results.** The main result of this manuscript reads as follows.

Theorem 1.2. *Let $\delta \ge 0$, $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and the exponent function $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies* (1.5) *and* (1.6). *Furthermore, suppose that $F \in L^{p(z)}(\Omega_T, \mathbb{R}^n)$ and the vector field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the growth condition* (1.2), *the monotonicity condition* (1.3) *and the coercivity condition* (1.4). *Moreover, let $u_0, v_0 \in L^2(\Omega)$. Then, there exists a* *unique weak solution* $(u,v) \in (C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T))^2$ *with* $(\partial_t u, \partial_t v) \in (W^{p(\cdot)}(\Omega_T)')^2$ *of problem* (1.1) *and satisfies the energy estimate*

$$\sup_{0\le t\le T}\Big(\int_\Omega |u(\cdot,t)|^2 \mathrm{d}x + \int_\Omega |v(\cdot,t)|^2\mathrm{d}x\Big) + \int_{\Omega_T} |\nabla u|^{p(\cdot)} + |\nabla v|^{p(\cdot)} \le c\mathcal{X}, \tag{1.12}$$

where

$$\mathcal{X} := \|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)} + \int_{\Omega_T} |F|^{p(\cdot)} + 1\mathrm{d}z \tag{1.13}$$

with $u(\cdot,0)=u_0$, $v(\cdot,0)=v_0$ *and a constant* $c=c(\nu,\delta,\gamma_1,\gamma_2,L)$.

To prove the main result, we need some preliminaries. First of all, we will need [20, Lemma 3.1], which reads as follows.

Lemma 1.3. *Let* $n\ge 2$. *Assume that the exponent function* $p:\Omega_T\to[\gamma_1,\gamma_2]$ *satisfies* (1.5)-(1.6). *Then* $W(\Omega_T)$ *is contained in* $C^0([0,T];L^2(\Omega))$. *Moreover, if* $u\in W_0(\Omega_T) := \{u \in W_0^{p(\cdot)}(\Omega_T) | u_t \in W^{p(\cdot)}(\Omega_T)'\}$ *then* $t\mapsto \|u(\cdot,t)\|^2_{L^2(\Omega)}$ *is absolutely continuous on* $[0,T]$,

$$\frac{\mathrm{d}d}{\mathrm{d}t}\int_\Omega |u(\cdot,t)|^2\mathrm{d}x = 2\langle \partial_t u(\cdot,t), u(\cdot,t)\rangle,$$

for a.e. $t\in[0,T]$, *where* $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $W^{1,p(\cdot,t)}(\Omega)'$ *and* $W_0^{1,p(\cdot,t)}(\Omega)$. *Moreover, there is a constant* c *such that* $\|u\|_{C^0([0,T];L^2(\Omega))} \le c\|u\|_{W(\Omega_T)}$ *for every* $u\in W_0(\Omega_T)$.

Moreover, we need the following Poincaré type estimate from [21, Lemma 3.9].

Lemma 1.4. *Let* $\Omega\subset\mathbb{R}^n$ *a bounded Lipschitz domain and* $\gamma_2 := \sup_{\Omega_T} p(\cdot)$. *Assume that* $u\in C^0([0,T];L^2(\Omega))\cap W_0^{p(\cdot)}(\Omega_T)$ *and the exponent* $p(\cdot)$ *satisfies the conditions* (1.5)-(1.6). *Then, there exists a constant* $c=c(n,\gamma_1,\gamma_2,\mathrm{diam}(\Omega),\omega(\cdot))$, *such that the following two versions of the Poincaré type estimate are valid:*

$$\int_{\Omega_T} |u|^{p(\cdot)}\mathrm{d}z \le c\Big(\|u\|^{\frac{4\gamma_2}{n+2}}_{L^\infty(0,T;L^2(\Omega))} + 1\Big)\Big(\int_{\Omega_T}|\nabla u|^{p(\cdot)} + 1\mathrm{d}z\Big), \tag{1.14}$$

$$\|u\|^{\gamma_1}_{L^{p(z)}(\Omega_T)} \le c\Big(\|u\|^{\frac{4\gamma_2}{n+2}}_{L^\infty(0,T;L^2(\Omega))} + 1\Big)\Big(\int_{\Omega_T}|\nabla u|^{p(\cdot)} + 1\mathrm{d}z\Big). \tag{1.15}$$

Also we need the Aubin-Lions type Theorem [20, Theorem 1.3], since it implies the strong convergence in $p(z)$-Lebesgue spaces.

Theorem 1.5. *Let* $\Omega\subset\mathbb{R}^n$ *an open, bounded Lipschitz domain with* $n\ge 2$ *and* $p(\cdot) > \frac{2n}{n+2}$ *satisfying* (1.5) *and* (1.6). *Furthermore, define* $\hat{p}(\cdot) := \max\{2,p(\cdot)\}$. *Then, the inclusion* $W(\Omega_T)\hookrightarrow L^{\hat{p}(\cdot)}(\Omega_T)$ *is compact.*

2. Proof of the main result

In this section, we will prove the existence of a unique weak solution to the Dirichlet problem (1.1).

Proof of Theorem 1.2. The proof is divided into several steps.

Step 1: Construction of a sequence of Galerkin's approximations. We start by constructing a sequence of Galerkin's approximations, where the limit of this sequence is equal to the solution of (1.1). Therefore, we consider $\{\phi_i(x)\}_{i=1}^\infty \subset W_0^{1,\gamma_2}(\Omega)$ and $\{\tilde{\phi}_i(x)\}_{i=1}^\infty \subset W_0^{1,\gamma_2}(\Omega)$, which are orthonormal basis in $L^2(\Omega)$. Since,

$W_0^{1,\gamma_2}(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\phi_k\} \subset W_0^{1,\gamma_2}(\Omega)$ and $\{\tilde{\phi}_k\} \subset W_0^{1,\gamma_2}(\Omega)$. Moreover, we have the dense embedding $W_0^{1,\gamma_2}(\Omega) \subset L^2(\Omega)$ for any $\gamma_2 \geq 2$, cf. [33, 34]. Thus, without loss of generality, we may assume that these systems form orthonormal basis of $L^2(\Omega)$. Now, fix a positive integer m and define the approximate solution to (1.1) as follows

$$u^{(m)}(z) := \sum_{i=1}^{m} c_i^{(m)}(t)\phi_i(x) \quad \text{and} \quad v^{(m)}(z) := \sum_{i=1}^{m} \tilde{c}_i^{(m)}(t)\tilde{\phi}_i(x)$$

where the coefficients $c_i^{(m)}(t)$ and $\tilde{c}_i^{(m)}(t)$ are defined via the identities

$$\begin{gathered}\int_\Omega \Big(u_t^{(m)}\phi_i(x) + \Big(a(x,t,\nabla u^{(m)}) + |F|^{p(x,t)-2}F\Big)\cdot\nabla\phi_i(x)\Big)\mathrm{d}x = 0,\\ \int_\Omega \Big(v_t^{(m)}\tilde{\phi}_i(x) + \Big(a(x,t,\nabla v^{(m)}) + \delta\nabla u^{(m)}\Big)\cdot\nabla\tilde{\phi}_i(x)\Big)\mathrm{d}x = 0,\end{gathered} \tag{2.1}$$

for $i = 0,\dots,m$ and $t \in (0,T)$ with the initial conditions

$$\begin{aligned} c_i^{(m)}(0) &= \int_\Omega u_0\phi_i\mathrm{d}x,\\ \tilde{c}_i^{(m)}(0) &= \int_\Omega v_0\tilde{\phi}_i\mathrm{d}x,\end{aligned} \tag{2.2}$$

for $i = 1,\dots,m$. Then, system (2.1), with these initial condition, generates a system of $2m$ ordinary differential equations

$$\begin{aligned} \big(c_i^{(m)}\big)'(t) &= F_i\Big(t, c_1^{(m)}(t),\dots,c_m^{(m)}(t),\tilde{c}_1^{(m)}(t),\dots,\tilde{c}_m^{(m)}(t)\Big),\\ c_i^{(m)}(0) &= \int_\Omega u_0\phi_i\mathrm{d}x\\ \big(\tilde{c}_i^{(m)}\big)'(t) &= \tilde{F}_i\Big(t, c_1^{(m)}(t),\dots,c_m^{(m)}(t),\tilde{c}_1^{(m)}(t),\dots,\tilde{c}_m^{(m)}(t)\Big),\\ \tilde{c}_i^{(m)}(0) &= \int_\Omega v_0\tilde{\phi}_i\mathrm{d}x\end{aligned} \tag{2.3}$$

for $i = 1,\dots,m$, since $\{\phi_i(x)\}$ and $\{\tilde{\phi}_i(x)\}$ are orthonormal in $L^2(\Omega)$. By [36, Theorem 1.44, p. 25] we know that, there is for every finite system (2.3) a solution $(c_i^{(m)}(t), \tilde{c}_i^{(m)}(t))$, $i = 1,\dots,m$ on the interval $(0,T_m)$ for some $T_m > 0$. Therefore, we multiply the first equation of system (2.1) by the coefficients $c_i^{(m)}(t)$, $i = 1,\dots,m$ and the second equation by $\tilde{c}_i^{(m)}(t)$, $i = 1,\dots,m$. Then, integrating the resulting equations over $(0,\tau)$ for an arbitrarily $\tau \in (0,T_m)$ and summing them over $i = 1,\dots,m$, yields

$$\begin{gathered}\int_{\Omega_\tau} \partial_t u^{(m)}\cdot u^{(m)} + \Big(a(x,t,\nabla u^{(m)}) + |F|^{p(x,t)-2}F\Big)\cdot\nabla u^{(m)}\mathrm{d}z = 0\\ \int_{\Omega_\tau} \partial_t v^{(m)}\cdot v^{(m)} + \Big(a(x,t,\nabla v^{(m)}) + \delta\nabla u^{(m)}\Big)\cdot\nabla v^{(m)}\mathrm{d}z = 0\end{gathered} \tag{2.4}$$

for a.e. $\tau \in (0,T_m)$.

Step 2: Energy estimate for the approximated solution. We derive the needed energy estimate. Therefore, we use that

$$\int_{\Omega_\tau} \partial_t u^{(m)}\cdot u^{(m)}\mathrm{d}z \geq \frac{1}{2}\int_\Omega |u^{(m)}(\cdot,\tau)|^2\mathrm{d}x - \frac{1}{2}\int_\Omega |u_0|^2\mathrm{d}x$$

$$\int_{\Omega_\tau} \partial_t v^{(m)} \cdot v^{(m)} \mathrm{d}z \geq \frac{1}{2} \int_\Omega |v^{(m)}(\cdot,\tau)|^2 \mathrm{d}x - \frac{1}{2} \int_\Omega |v_0|^2 \mathrm{d}x$$

for a.e. $\tau \in (0, T_m)$, since $u_0, v_0 \in L^2(\Omega)$, $\{\phi_i\}_{i=1}^\infty \subset L^2(\Omega)$ and $\{\tilde{\phi}_i\}_{i=1}^\infty \subset L^2(\Omega)$, cf. [20]. Then, we arrive at

$$\begin{aligned} &\frac{1}{2} \int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} a(x,t,\nabla u^{(m)}) \cdot \nabla u^{(m)} \mathrm{d}z \\ &\leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 + \int_{\Omega_\tau} |F|^{p(x,t)-1} |\nabla u^{(m)}| \mathrm{d}z \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\frac{1}{2} \int_\Omega |v^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} a(x,t,\nabla v^{(m)}) \cdot \nabla v^{(m)} \mathrm{d}z \\ &\leq \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 + \delta \int_{\Omega_\tau} |\nabla u^{(m)}| |\nabla v^{(m)}| \mathrm{d}z \end{aligned} \tag{2.6}$$

for a.e. $\tau \in (0, T_m)$. Using the coercivity condition (1.4) on the left-hand side of (2.5) and (2.6) yields

$$\begin{aligned} \frac{1}{2} \int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \nu \int_{\Omega_\tau} |\nabla u^{(m)}|^{p(\cdot)} \mathrm{d}z &\leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 + \int_{\Omega_\tau} |F|^{p(\cdot)-1} |\nabla u^{(m)}| \mathrm{d}z, \\ \frac{1}{2} \int_\Omega |v^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \nu \int_{\Omega_\tau} |\nabla v^{(m)}|^{p(\cdot)} \mathrm{d}z &\leq \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 + \delta \int_{\Omega_\tau} |\nabla u^{(m)}| |\nabla v^{(m)}| \mathrm{d}z. \end{aligned}$$

These estimates holds for a.e. $\tau \in (0, T_m)$. Applying Young's inequality with $1/p(x,t) + 1/p'(x,t) = 1$ to the last term of the second last equation with $0 \leq \varepsilon \leq 1$ and Cauchy's inequality with $0 \leq \tilde{\varepsilon} \leq 1$ to the last term the last equation, we obtain

$$\int_{\Omega_\tau} |F|^{p(x,t)-1} |\nabla u^{(m)}| \mathrm{d}z \leq c(\gamma_1, \gamma_2, \varepsilon) \int_{\Omega_\tau} |F|^{p(\cdot)} \mathrm{d}z + \varepsilon c(\gamma_1, \gamma_2) \int_{\Omega_\tau} |\nabla u^{(m)}|^{p(\cdot)} \mathrm{d}z$$

and

$$\begin{aligned} \delta \int_{\Omega_\tau} |\nabla u^{(m)}| |\nabla v^{(m)}| \mathrm{d}z &\leq c(\delta, \tilde{\varepsilon}) \int_{\Omega_\tau} |\nabla u^{(m)}|^2 \mathrm{d}z + \frac{\tilde{\varepsilon}}{2} \int_{\Omega_\tau} |\nabla v^{(m)}|^2 \mathrm{d}z \\ &\leq c(\gamma_1, \gamma_2, \delta, \tilde{\varepsilon}) \int_{\Omega_\tau} |\nabla u^{(m)}|^{p(\cdot)} + 1 \mathrm{d}z \\ &\quad + \tilde{\varepsilon} c(\gamma_1, \gamma_2) \int_{\Omega_\tau} |\nabla v^{(m)}|^{p(\cdot)} + 1 \mathrm{d}z. \end{aligned}$$

Choosing $\varepsilon \leq \nu/(2c(\gamma_1, \gamma_2))$ and $\tilde{\varepsilon} \leq \nu/(2c(\gamma_1, \gamma_2))$, we can conclude that

$$\int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} |\nabla u^{(m)}|^{p(\cdot)} \mathrm{d}z \leq c_1 \|u_0\|_{L^2(\Omega)}^2 + c_1 \int_{\Omega_\tau} |F|^{p(\cdot)} \mathrm{d}z,$$

$$\begin{aligned} &\int_\Omega |v^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} |\nabla v^{(m)}|^{p(\cdot)} \mathrm{d}z \\ &\leq c_2 \|v_0\|_{L^2(\Omega)}^2 + c_2 \int_{\Omega_\tau} |\nabla u^{(m)}|^{p(\cdot)} + 1 \mathrm{d}z \\ &\leq c_2 \Big(\|v_0\|_{L^2(\Omega)}^2 + \|u_0\|_{L^2(\Omega)}^2 + \int_{\Omega_\tau} |F|^{p(\cdot)} + 1 \mathrm{d}z \Big), \end{aligned}$$

where we used the second last estimate to derive the last estimate with constants $c_1 = c_1(\nu, \gamma_1, \gamma_2)$ and $c_2 = c_2(\nu, \delta, \gamma_1, \gamma_2)$. Finally, the Poincaré type estimate (1.15) in combination with the previous two estimates yields

$$\|u^{(m)}\|_{L^{p(\cdot)}(\Omega_{T_m})} \le c \quad \text{and} \quad \|v^{(m)}\|_{L^{p(\cdot)}(\Omega_{T_m})} \le c$$

with $c = c(n, \nu, \delta, \gamma_1, \gamma_2, \operatorname{diam}(\Omega), \omega(\cdot), \mathcal{X})$, where $\mathcal{X}$ is defined in (1.13). Therefore, we have shown that $u^{(m)}$ and $v^{(m)}$ are uniformly bounded in $W^{p(\cdot)}(\Omega_{T_m})$ and $L^\infty(0, T_m; L^2(\Omega))$ independently of m. Thus, the solution of system (2.3) can be continued to the maximal interval $(0, T)$ and we obtain the estimate

$$\begin{aligned} &\sup_{0\le\tau\le T} \Big(\int_\Omega |u^{(m)}(\cdot, \tau)|^2 \mathrm{d}x + \int_\Omega |v^{(m)}(\cdot, \tau)|^2 \mathrm{d}x \Big) \\ &+ \int_{\Omega_T} |\nabla u^{(m)}|^{p(\cdot)} + |\nabla v^{(m)}|^{p(\cdot)} \mathrm{d}z \\ &\le c\Big(\|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)} + \int_{\Omega_T} |F|^{p(\cdot)} + 1 \mathrm{d}z \Big) = c\mathcal{X} \end{aligned} \tag{2.7}$$

with $c = c(\nu, \delta, \gamma_1, \gamma_2)$.

Step 3: Uniform bounds for $\partial_t u^{(m)}$ and $\partial_t v^{(m)}$. We want to derive an uniform bound for $\partial_t u^{(m)}$ in $W^{p(\cdot)}(\Omega_T)'$. Therefore we define a subspace of the set of admissible test functions

$$\mathcal{W}_m(\Omega_T) := \big\{ \eta : \eta = \sum_{i=1}^m d_i \phi_i, \ d_i \in C^1([0, T]) \big\} \subset W_0^{p(\cdot)}(\Omega_T).$$

Then, we choose a test function

$$\varphi(z) = \sum_{i=1}^m d_i(t) \phi_i(x) \in \mathcal{W}_m(\Omega_T) \quad \text{with } d_i(0) = d_i(T) = 0.$$

Note that $\partial_t \varphi$ exists, since the coefficients $d_i(t)$ lie in $C^1([0, T])$. Moreover, we know that $C^1([0, T], W_0^{1,\gamma_2}(\Omega_T)) \subset W_0^{p(\cdot)}(\Omega_T)$ and therefore, we have also $\varphi \in W_0^{p(\cdot)}(\Omega_T)$. Thus, we can conclude by the definition of $u^{(m)}$ and the first equation of (2.1) that

$$-\int_{\Omega_T} u^{(m)} \varphi_t \mathrm{d}z = \int_{\Omega_T} u_t^{(m)} \varphi \mathrm{d}z = -\int_{\Omega_T} \Big(a(z, \nabla u^{(m)}) + |F|^{p(x,t)-2} F \Big) \cdot \nabla \varphi \mathrm{d}z.$$

Then, we derive by utilizing the growth condition (1.2) and the generalized Hölder's inequality (1.7) the estimate

$$\begin{aligned} \Big| \int_{\Omega_T} u_t^{(m)} \varphi \mathrm{d}z \Big| &\le \int_{\Omega_T} \Big(|a(z, \nabla u^{(m)})| + |F|^{p(\cdot)-1} \Big) \cdot |\nabla \varphi| \mathrm{d}z \\ &\le \int_{\Omega_T} \Big(|a(z, \nabla u^{(m)})| + |F|^{p(\cdot)-1} \Big) \cdot (|\nabla \varphi| + |\varphi|) \,\mathrm{d}z \\ &\le c \big[\|(1 + |\nabla u^{(m)}|^{p(\cdot)-1} + |F|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(\Omega_T)} \big] \times \|\varphi\|_{W^{p(\cdot)}(\Omega_T)}, \end{aligned}$$

where $c = c(\gamma_1, \gamma_2, L)$. Applying (1.8) and (2.7) to the last estimate, we have for every $\varphi \in \mathcal{W}_m(\Omega_T) \subset W_0^{p(\cdot)}(\Omega_T)$ and any m the estimate

$$\Big| \int_{\Omega_T} u_t^{(m)} \varphi \mathrm{d}z \Big| \le c \|\varphi\|_{W^{p(\cdot)}(\Omega_T)}$$

with a constant $c = c(\gamma_1, \gamma_2, \nu, L, \mathcal{X})$, which is independent of m. This shows that $u_t^{(m)} \in W^{p(\cdot)}(\Omega_T)'$ with $\|u_t^{(m)}\|_{W^{p(\cdot)}(\Omega_T)'} \le c(\gamma_1, \gamma_2, \nu, L, \mathcal{X})$. Similarly, one can conclude that $v_t^{(m)} \in W^{p(\cdot)}(\Omega_T)'$ with $\|v_t^{(m)}\|_{W^{p(\cdot)}(\Omega_T)'} \le c(\gamma_1, \gamma_2, \nu, L, \mathcal{X})$.

Step 4: Compactness and passage to the limit. Now, we have the needed uniform bounds of $u^{(m)}$, $v^{(m)}$, $u_t^{(m)}$ and $v_t^{(m)}$ and it follows that

$$\begin{gathered} u^{(m)}, v^{(m)} \in W_0^{p(\cdot)}(\Omega_T) \subseteq L^{\gamma_1}(0, T; W_0^{1,\gamma_1}(\Omega)) \\ u_t^{(m)}, v_t^{(m)} \in W^{p(\cdot)}(\Omega_T)' \subseteq L^{\gamma_2'}(0, T; W^{-1,\gamma_2'}(\Omega)) \end{gathered}$$

are bounded. This implies the following weak convergence for the sequences $\{u^{(m)}\}$ and $\{v^{(m)}\}$ (up to a subsequence):

$$\begin{gathered} u^{(m)} \rightharpoonup^* u \text{ and } v^{(m)} \rightharpoonup^* v \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \nabla u^{(m)} \rightharpoonup \nabla u \text{ and } \nabla v^{(m)} \rightharpoonup \nabla v \text{ weakly in } L^{p(\cdot)}(\Omega_T, \mathbb{R}^n), \\ u_t^{(m)} \rightharpoonup u_t \text{ and } v_t^{(m)} \rightharpoonup v_t \text{ weakly in } W^{p(\cdot)}(\Omega_T)'. \end{gathered}$$

Moreover, by Theorem 1.5 we can conclude that the sequences $\{u^{(m)}\}$ and $\{v^{(m)}\}$ (up to a subsequence) converges strongly in $L^{p(\cdot)}(\Omega_T)$ to some function $u, v \in W(\Omega_T)$. Thus, we obtain the desired convergences

$$\begin{gathered} u^{(m)} \to u \text{ and } v^{(m)} \to v \text{ strongly in } L^{p(\cdot)}(\Omega_T), \\ u^{(m)} \to u \text{ and } v^{(m)} \to v \text{ a.e. in } \Omega_T. \end{gathered}$$

In addition, the growth assumption of $a(z, \cdot)$ and the estimate (2.7) imply that the sequences $\{a(z, \nabla u^{(m)})\}_{m\in\mathbb{N}}$ and $\{a(z, \nabla v^{(m)})\}_{m\in\mathbb{N}}$ are bounded in $L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$. Consequently, after passing to a subsequence once more, we can find limit maps $A_0, A_0^* \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$ with

$$\begin{aligned} a(z, \nabla u^{(m)}) &\to A_0 \quad \text{as } m \to \infty, \\ a(z, \nabla v^{(m)}) &\to A_0^* \quad \text{as } m \to \infty. \end{aligned} \tag{2.8}$$

Our next aim is to show that $A_0 = a(z, \nabla u)$ for almost every $z \in \Omega_T$. We will only show that $A_0 = a(z, \nabla u)$ for almost every $z \in \Omega_T$, but one can easily show that $A_0^* = a(z, \nabla v)$ for almost every $z \in \Omega_T$ using the same approach. First of all, we should mention that each of $u^{(m)}$ satisfies the first equation of the identity (2.1) with a test function $\varphi \in \mathcal{W}_m(\Omega_T)$. This follows by the method of construction, cf. [7]. Then, we fix an arbitrary $m \in \mathbb{N}$ and we have for every $s \le m$ the equation

$$-\int_{\Omega_T} u_t^{(m)}\varphi + \big(a(z, \nabla u^{(m)}) + |F|^{p(\cdot)-2}F\big)\nabla\varphi \mathrm{d}z = 0$$

for all test functions $\varphi \in \mathcal{W}_s(\Omega_T)$. Passing to the limit $m \to \infty$, we can conclude that for all test functions $\varphi \in \mathcal{W}_s(\Omega_T)$ we have

$$-\int_{\Omega_T} u_t\varphi + \big(A_0 + |F|^{p(\cdot)-2}F\big)\nabla\varphi \mathrm{d}z = 0 \tag{2.9}$$

with an arbitrary $s \in \mathbb{N}$, by the convergence from above. Therefore, it follows that the identity (2.9) holds for every $\varphi \in W_0^{p(\cdot)}(\Omega_T)$. According to monotonicity assumption (1.3), we know that for every $w \in \mathcal{W}_s(\Omega_T)$ and every $s \leq m$ the following holds

$$\int_{\Omega_T} [a(z, \nabla u^{(m)}) - a(z, \nabla w)]\nabla(u^{(m)} - w)\mathrm{d}z \geq 0. \tag{2.10}$$

Moreover, it follows from the first equation of (2.1), the conclusion from above and the choice of an admissible test function $\varphi = u^{(m)} - w$ with $w \in \mathcal{W}_s(\Omega_T)$ that

$$-\int_{\Omega_T} u_t^{(m)}\varphi + \big(a(z, \nabla u^{(m)}) + |F|^{p(\cdot)-2}F\big)\nabla\varphi\mathrm{d}z = 0. \tag{2.11}$$

Adding (2.10) and (2.11), we have

$$-\int_{\Omega_T} u_t^{(m)}\varphi + [a(z, \nabla u^{(m)}) + |F|^{p(\cdot)-2}F]\nabla\varphi - [a(z, \nabla u^{(m)}) - a(z, \nabla w)]\nabla\varphi\mathrm{d}z \geq 0$$

with a test function $\varphi = u^{(m)} - w$. This yields

$$-\int_{\Omega_T} u_t^{(m)}(u^{(m)} - w) + [a(z, \nabla w) + |F|^{p(\cdot)-2}F]\nabla(u^{(m)} - w)\mathrm{d}z \geq 0.$$

Then, we test equation (2.9) with $\varphi = u^{(m)} - w$, subtract the resulting equation from the last estimate and finally passing to the limit $m \to \infty$ yields

$$-\int_{\Omega_T} [A_0 - a(z, \nabla w)]\nabla(u - w)\mathrm{d}z \geq 0$$

for all $w \in \mathcal{W}_s(\Omega_T)$. Since, $\mathcal{W}_s(\Omega_T) \subset W_0^{p(\cdot)}(\Omega_T)$ is dense, we are allowed to choose $w \in W_0^{p(\cdot)}(\Omega_T)$. Hence, we choose $w = u \pm \varepsilon\xi$ with an arbitrary $\xi \in W_0^{p(\cdot)}(\Omega_T)$. This yields

$$-\varepsilon\int_{\Omega_T} [A_0 - a(z, \nabla(u \pm \varepsilon\xi))]\nabla\xi\mathrm{d}z \geq 0.$$

Then, passing to the limit $\varepsilon \downarrow 0$, we conclude that

$$\int_{\Omega_T} [A_0 - a(z, \nabla u)]\nabla\xi\mathrm{d}z = 0$$

for all $\xi \in W_0^{p(\cdot)}(\Omega_T)$. This shows that

$$A_0 = a(z, \nabla u) \quad \text{for almost every } z \in \Omega_T.$$

Similarly, we can show that $A_0^* = a(z, \nabla v)$ for almost every $z \in \Omega_T$.

Step 5: Initial values. Moreover, we have to show that $u(\cdot, 0) = u_0$ and $v(\cdot, 0) = v_0$. We prove that $u(\cdot, 0) = u_0$ and the conclusion $v(\cdot, 0) = v_0$ follows in the same way. From (2.9) we obtain by using integration by parts that

$$\int_{\Omega_T} u\varphi_t - \Big(a(z, \nabla u) + |F|^{p(\cdot)-2}F\Big)\nabla\varphi\mathrm{d}z = \int_{\Omega}(u \cdot \varphi)(\cdot, 0)\mathrm{d}x$$

for all $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ with $\varphi(\cdot, T) = 0$. Similarly, we can conclude that

$$\int_{\Omega_T} v\zeta_t - \Big(a(z, \nabla v) + |F|^{p(\cdot)-2}F\Big)\nabla\zeta\mathrm{d}z = \int_{\Omega}(v \cdot \zeta)(\cdot, 0)\mathrm{d}x$$

for all $\zeta \in W_0^{p(\cdot)}(\Omega_T)$ with $\zeta(\cdot,T)=0$. Here, we will only show that $u(\cdot,0)=u_0$, since the conclusion $v(\cdot,0)=v_0$ is then easily to derive. Furthermore, from (2.11) — similar to the previous estimates — we obtain that

$$\int_{\Omega_T} u^{(m)}\varphi_t - \Big(a(z,\nabla u^{(m)}) + |F|^{p(\cdot)-2}F\Big)\nabla\varphi \mathrm{d}z = \int_{\Omega}(u^{(m)}\cdot\varphi)(\cdot,0)\mathrm{d}x$$

for all $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ with $\varphi(\cdot,T)=0$. Passing to the limit $m\to\infty$ and using the convergences from above, we obtain

$$\int_{\Omega_T} u\varphi_t - \Big(a(z,\nabla u) + |F|^{p(\cdot)-2}F\Big)\nabla\varphi \mathrm{d}z = \int_{\Omega} u_0\cdot\varphi(\cdot,0)\mathrm{d}x,$$

where $u^{(m)}(\cdot,0)\to u_0$ as $m\to\infty$, cf. [20]. In addition, $\varphi(\cdot,0)$ is arbitrary and hence, we can conclude that $u(\cdot,0)=u_0$. This together with the conclusion $v(\cdot,0)=v_0$ shows that there exists a weak solution to the Dirichlet problem (1.1).

Step 6: Uniqueness. The final aim is to prove the uniqueness of the weak solution to the Dirichlet problem (1.1). To this end, we assume that there exist two pairs of weak solutions (u,v) and $(u_*,v_*)\in (C^0([0,T];L^2(\Omega))\cap W_0^{p(\cdot)}(\Omega_T))^2$ with $(\partial_t u,\partial_t v),(\partial_t u_*,\partial_t v_*)\in (W^{p(\cdot)}(\Omega_T)')^2$ to the Dirichlet problem (1.1). Thus, we have the following weak formulations

$$\begin{gathered}\int_{\Omega_T}[u\cdot\varphi_t - a(z,\nabla u)\cdot\nabla\varphi]\,\mathrm{d}z = \int_{\Omega_T}|F|^{p(x,t)-2}F\cdot\nabla\varphi\mathrm{d}z,\\ \int_{\Omega_T}[v\cdot\zeta_t - a(z,\nabla v)\cdot\nabla\zeta]\,\mathrm{d}z = \int_{\Omega_T}\delta\nabla u\cdot\nabla\zeta\mathrm{d}z,\end{gathered}$$

and

$$\begin{gathered}\int_{\Omega_T}[u_*\cdot\varphi_t - a(z,\nabla u_*)\cdot\nabla\varphi]\,\mathrm{d}z = \int_{\Omega_T}|F|^{p(x,t)-2}F\cdot\nabla\varphi\mathrm{d}z,\\ \int_{\Omega_T}[v_*\cdot\zeta_t - a(z,\nabla v_*)\cdot\nabla\zeta]\,\mathrm{d}z = \int_{\Omega_T}\delta\nabla u_*\cdot\nabla\zeta\mathrm{d}z,\end{gathered}$$

with the admissible test functions $\varphi = u-u_*\in W_0^{p(\cdot)}(\Omega_T)$ and $\zeta = v-v_*\in W_0^{p(\cdot)}(\Omega_T)$, since $W_0^{p(\cdot)}(\Omega_T)'$ is the dual of $W_0^{p(\cdot)}(\Omega_T)$. Hence, we can conclude using integration by parts that

$$\begin{gathered}\int_{\Omega_T}(u-u_*)_t(u-u_*) + (a(z,\nabla u)-a(z,\nabla u_*))\nabla(u-u_*)\mathrm{d}z = 0,\\ \begin{aligned}&\int_{\Omega_T}(v-v_*)_t(v-v_*) + (a(z,\nabla v)-a(z,\nabla v_*))\nabla(v-v_*)\mathrm{d}z\\ &= -\delta\int_{\Omega_T}\nabla(u-u_*)\cdot\nabla(v-v_*)\mathrm{d}z.\end{aligned}\end{gathered}$$

Using the monotonicity condition (1.3), we arrive at

$$\begin{gathered}0 \ge \int_{\Omega_T}(u-u_*)_t(u-u_*)\mathrm{d}z = \frac{1}{2}\int_{\Omega_T}\partial_t(u-u_*)^2\mathrm{d}z,\\ -\delta\int_{\Omega_T}\nabla(u-u_*)\nabla(v-v_*)\mathrm{d}z \ge \int_{\Omega_T}(v-v_*)_t(v-v_*)\mathrm{d}z = \frac{1}{2}\int_{\Omega_T}\partial_t(v-v_*)^2\mathrm{d}z.\end{gathered}$$

Therefore, $0 \geq \frac{1}{2}\|u(t)-u_*(t)\|^2_{L^2(\Omega)} \geq 0$ for every $t \in (0,T)$, since $u(\cdot,0) = u_*(\cdot,0) = u_0$. In addition, the uniqueness of u implies also that

$$0 \geq \int_{\Omega_T} (v-v_*)(v-v_*)_t \mathrm{d}z = \frac{1}{2}\int_{\Omega_T} \partial_t (v-v_*)^2 \mathrm{d}z$$

and $0 \geq \frac{1}{2}\|v(t)-v_*(t)\|^2_{L^2(\Omega)} \geq 0$ for every $t \in (0,T)$, since $v(\cdot,0) = v_*(\cdot,0) = v_0$. This completes the proof of the Theorem. □

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