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STABILIZATION OF COUPLED THERMOELASTIC KIRCHHOFF PLATE AND WAVE EQUATIONS

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Abstract. We consider a coupled system consisting of a Kirchhoff thermoelastic plate and an undamped wave equation. It is known that the Kirchhoff thermoelastic plate is exponentially stable. The coupling is weak. First, we show that the coupled system is not exponentially stable. Afterwards, we prove that the coupled system is polynomially stable, and provide an explicit polynomial decay rate of the associated semigroup. Our proof relies on a combination of the frequency domain method and the multipliers technique.

1. Introduction

Since the pioneering work of Dafermos [14] on the stability of the thermoelasticity equations in the late sixties, followed by the book of Lagnese [23] on the boundary stabilization of thin elastic plates, there has been a tremendous amount of activity involving the stabilization of thermoelastic systems, especially since the nineties. It was known since the work of Dafermos that the semigroup generated by the infinitesimal operator of the thermoelasticity equations is not even strongly stable, except for certain geometric configurations. It then made sense for Lagnese to tackle the exponential stability problem for a thermoelastic plate by adding mechanical damping mechanisms on a suitable portion of the boundary [23, Chap. 7]. Then arose the natural question of whether for thermoelastic plates, the presence of mechanical damping on the boundary was necessary or not. In other words, could one dispense of the extra mechanical damping, and still exponentially stabilize a thermoelastic plate by relying solely on the dissipation induced by the heat component of the system? A first answer to that challenging question was provided by Kim, who proved that no mechanical damping was necessary to ensure the exponential stability of a clamped plate [21]. Later on, Liu and Renardy [32] proved that the semigroup associated with a clamped or hinged thermoelastic plate is analytic, which is a stronger notion than exponential stability for strongly stable semigroups. Then followed many other works in the same vein by, e.g. Liu and Liu [30], Lasiecka and Triggiani [24, 25, 26], Lasiecka and Avalos [6, 7, 8], Zuazua and collaborators [11, 37, 48], Munõz Rivera and collaborators [15, 33, 34]. As those stabilization works on thermoelastic plates were being carried out, Lebeau and Zuazua returned to the stability of the thermoelasticity equations, and they

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proved exponential decay to a finite dimensional subspace and polynomial stability under certain geometric constraints [28]. Other closely related works include, e.g. [4, 13, 42].

In the present work, we are interested in answering the following question: Knowing that the clamped Kirchhoff thermoelastic plate is exponentially stable, e.g. [7, 15, 20, 30, 42, 46], what type of decay should we expect when it is stacked to a membrane? Such a system is weakly coupled and falls within the general framework of the indirect stabilization of weakly coupled elastic systems, which has quite a rich literature, e.g. [1, 2, 3, 16, 18, 35, 41, 43, 45]. Unlike the works just cited dealing with the indirect stabilization of weakly coupled elastic systems, where one system is mechanically damped and the other one undamped, we are dealing here with a different type of indirect stabilization problem; more precisely, we are dealing with a doubly indirect stabilization problem in the sense that we are relying on the dissipation induced by the heat component of the system to strongly stabilize the coupled system. Given the weak coupling between the thermoelastic plate and the wave equations, uniform or exponential stability is not to be expected, thanks to a result of Triggiani [47]. Therefore, we will be focusing our attention on establishing a polynomial stability of the coupled system.

Now, we shall introduce some notations, and formulate our problem. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open bounded set of $\mathbb{R}^d$ with smooth enough boundary Γ. Let α and β be two nonzero real numbers with the same sign. Consider the coupled thermoelastic Kirchhoff plate/wave system

$$\begin{gathered}
y_{tt} - \gamma\Delta y_{tt} + a\Delta^2 y + \alpha\Delta\theta + \mu z = 0 \quad \text{in } \Omega \times (0,+\infty), \\
\theta_t - \sigma\Delta\theta - \beta\Delta y_t = 0 \text{ in } \Omega \times (0,+\infty), \\
z_{tt} - \eta\Delta z + \mu y = 0 \quad \text{in } \Omega \times (0,+\infty), \\
y = \partial_\nu y = 0, \quad \theta = z = 0, \quad \text{on } \Gamma \times (0,+\infty), \\
y(x,0) = y^0, \quad y_t(x,0) = y^1, \quad \theta(x,0) = \theta^0 \quad \text{in } \Omega, \\
z(x,0) = z^0, \quad z_t(x,0) = z^1 \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where a, η, γ, σ are positive physical constants representing respectively, the flexural stiffness of the plate, wave speed, rotational force constant, and thermal conductivity, while μ denotes the coupling parameter, and is a nonzero real number. From a physical point of view, the parameters α, β and μ should be positive with $\alpha = \beta$. Further, we assume that the coupling parameter μ satisfies

$$|\mu| < \lambda_0\mu_0\sqrt{a\eta}, \tag{1.2}$$

where λ_0^2 is the first eigenvalue of the operator "$-\Delta$" with Dirichlet boundary conditions, and μ_0^2 is first eigenvalue of the operator Δ^2 with clamped boundary conditions. This smallness condition on μ ensures that the right hand side of (1.3) below is positive for nonzero elements in the energy space, and thereby defines a norm indeed, which, thanks to Poincaré and Rellich inequalities, is equivalent to the natural norm in the energy space.

We introduce the Hilbert space over the field $\mathbb{C}$ of complex numbers

$$\mathcal{H}_\gamma := H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega),$$

equipped with the norm

$$\|U\|_{\mathcal{H}_\gamma}^2 := a\|\Delta u\|^2 + \gamma\|\nabla v\|^2 + \|v\|^2 + \frac{\alpha}{\beta}\|\theta\|^2 + \eta\|\nabla y\|^2 + \|z\|^2 + 2\mu\int_\Omega \operatorname{Re}(u\overline{y})d\Omega, \tag{1.3}$$

for all $U=(u,v,\theta,y,z)\in\mathcal{H}_\gamma$. We also define the linear differential operator

$$\mathcal{A}_\gamma U = \begin{pmatrix} v \\ -aP_\gamma^{-1}\Delta^2 u - \alpha P_\gamma^{-1}\Delta\theta - \mu P_\gamma^{-1} y \\ \beta\Delta v + \sigma\Delta\theta \\ z \\ -\mu u + \eta\Delta y \end{pmatrix}$$

where $P_\gamma = I - \gamma\Delta$ which is an isomorphism of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ and $U = (u,v,\theta,y,z)$.

The operator $\mathcal{A}_\gamma$ is unbounded in $\mathcal{H}_\gamma$, and (thanks to elliptic regularity) its domain is

$$D(\mathcal{A}_\gamma) = [H^3(\Omega)\cap H_0^2(\Omega)]\times H_0^2(\Omega)\times[H^2(\Omega)\cap H_0^1(\Omega)]\times[H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega).$$

The system (1.1) can be recast as an abstract evolution system,

$$\dot{U} = \mathcal{A}_\gamma U, \quad U(0) = U^0 = (u^0, v^0, \theta^0, y^0, z^0).$$

For the well-posedness of the system (1.1), we have the following result.

Theorem 1.1. *Operator $\mathcal{A}_\gamma$ is the infinitesimal generator of a C_0-semigroup of contractions $(S_\gamma(t))_{t\geq 0}$ on the Hilbert space $\mathcal{H}_\gamma$.*

Proof. To prove that $\mathcal{A}_\gamma$ generates a C_0-semigroup of contractions, we shall show that the conditions of the Lumer-Phillips theorem are satisfied [36, Theorem 4.3]; since the domain of $\mathcal{A}_\gamma$ is dense in $\mathcal{H}_\gamma$, we shall demonstrate here that $\mathcal{A}_\gamma$ is maximal dissipative.

Let $U=(u,v,\theta,y,z)\in D(\mathcal{A}_\gamma)$. We have

$$\begin{aligned}
&(\mathcal{A}_\gamma U, U)\\
&= a(\Delta v, \Delta u)_{L^2(\Omega)} + (P_\gamma^{1/2}\left(-aP_\gamma^{-1}\Delta^2 u - \alpha P_\gamma^{-1}\Delta\theta - \mu P_\gamma^{-1} y\right), P_\gamma^{1/2} v)_{L^2(\Omega)}\\
&\quad + \frac{\alpha}{\beta}(\beta\Delta v + \sigma\Delta\theta, \theta)_{L^2(\Omega)} + \eta(\nabla z, \nabla y)_{L^2(\Omega)} + (-\mu u + \eta\Delta y, z)_{L^2(\Omega)}\\
&\quad + \mu\int_\Omega (v\overline{y} + \overline{u}z)\,dx\\
&= a(\Delta u, \Delta v)_{L^2(\Omega)} - a\left(\Delta^2 u, v\right)_{H^{-2}(\Omega),H_0^2(\Omega)} - \alpha\left(\Delta\theta, v\right)_{L^2(\Omega)} - \mu\left(y, v\right)_{L^2(\Omega)}\\
&\quad + \frac{\alpha}{\beta}(\beta\Delta v + \sigma\Delta\theta, \theta)_{L^2(\Omega)} + \eta(\nabla z, \nabla y)_{L^2(\Omega)} + (-\mu u + \eta\Delta y, z)_{L^2(\Omega)}\\
&\quad + \mu\int_\Omega (v\overline{y} + \overline{u}z)\,dx
\end{aligned}$$

Using Green's formula, we obtain

$$\operatorname{Re}(\mathcal{A}_\gamma U, U) = -\sigma\frac{\alpha}{\beta}\|\nabla\theta\|^2 \leq 0.$$

Now let $F = (f_1, f_2, f_3, f_4, f_5)\in\mathcal{H}_\gamma$. We look for an element $U=(u,v,\theta,y,z)\in D(\mathcal{A}_\gamma)$ such that

$$(I - \mathcal{A}_\gamma)U = F.$$

Equivalently, we consider the system

$$u - v = f_1, \quad y - z = f_4, \tag{1.4}$$

$$P_\gamma u + a\Delta^2 u + \alpha\Delta\theta + \mu y = P_\gamma(f_1 + f_2), \tag{1.5}$$

$$\theta - \beta\Delta u - \sigma\Delta\theta = -\beta\Delta f_1 + f_3, \tag{1.6}$$

$$y - \eta\Delta y + \mu u = f_4 + f_5. \tag{1.7}$$

Taking $\phi \in H_0^2(\Omega)$, $\varphi \in H_0^1(\Omega)$ and $\psi \in H_0^1(\Omega)$, multiplying (1.5) by $\overline{\phi}$, (1.6) by $\overline{\varphi}$ and (1.7) by $\overline{\psi}$, we obtain the variational problem

$$\mathcal{B}((u,\theta,y),(\phi,\varphi,\psi)) = \mathcal{L}((\phi,\varphi,\psi)),$$

where

$$\begin{aligned}\mathcal{B}((u,\theta,y),(\phi,\varphi,\psi)) = \int_\Omega \Big\{ & P_\gamma^{1/2} u P_\gamma^{1/2}\overline{\phi} + a\Delta u\Delta\overline{\phi} - \alpha\nabla\theta\nabla\overline{\phi} + \frac{\alpha}{\beta}\theta\overline{\varphi} \\ & + \alpha\nabla u\nabla\overline{\varphi} + \sigma\frac{\alpha}{\beta}\nabla\theta\nabla\overline{\varphi} + y\overline{\psi} + \eta\nabla y\nabla\overline{\psi} + \mu(y\overline{\phi} + u\overline{\psi})\Big\} dx,\end{aligned}$$

and

$$\mathcal{L}((\phi,\varphi,\psi)) = \int_\Omega \Big\{ P_\gamma^{1/2}(f_1 + f_2) P_\gamma^{1/2}\overline{\phi} + \frac{\alpha}{\beta}(-\beta\Delta f_1 + f_3)\overline{\varphi} + (f_4 + f_5)\overline{\psi}\Big\} dx.$$

We can easily check that $\mathcal{B}$ is a sesquilinear continuous and coercive map in $[H_0^2(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)]^2$ and $\mathcal{L}$ is a linear continuous form in $H_0^2(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$. Thanks to Lax-Milgram Lemma, the above variational problem admits a unique solution $(u,\theta,y) \in H_0^2(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$, which shows that the operator $I - \mathcal{A}_\gamma$ is onto. ☐

Now, we shall analyze the asymptotic behavior of system (1.1).

Theorem 1.2. *Let $\gamma > 0$.*

1. *The semigroup $(S_\gamma(t))_{t\geq 0}$ is strongly stable on $\mathcal{H}_\gamma$,*
$$\lim_{t\to+\infty} \|S_\gamma(t)U^0\|_{\mathcal{H}_\gamma} = 0, \quad \forall U^0 \in \mathcal{H}_\gamma.$$
2. *The semigroup $(S_\gamma(t))_{t\geq 0}$ is not exponentially stable.*

Proof. (1) To prove the strong stability of the semigroup, it suffices to check that the imaginary axis is included in the resolvent set, viz., $i\mathbb{R} \subset \rho(\mathcal{A}_\gamma)$, where $\rho(\mathcal{A}_\gamma)$ is the resolvent set of $\mathcal{A}_\gamma$.

Before going forward, let us note that by the regularity theory for linear elliptic operators, if (u,v,θ,y,z) belongs to $D(\mathcal{A}_\gamma)$, then (u,v,θ,y,z) lies in $H^3(\Omega) \times H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) \times H_0^1(\Omega)$; so, in particular, the operator $\mathcal{A}_\gamma$ has a compact resolvent. Therefore, its spectrum is discrete.

It is easy to check that $0 \in \rho(\mathcal{A}_\gamma)$. Now we prove that $i\mathbb{R}\backslash\{0\} \subset \rho(\mathcal{A}_\gamma)$. Suppose that there exist $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, and $U = (u,v,\theta,y,z) \in D(\mathcal{A}_\gamma)$ with

$$i\lambda U - \mathcal{A}_\gamma U = 0. \tag{1.8}$$

We shall prove that $U = \mathbf{0} = (0,0,0,0,0)$. Equivalently, we consider the system

$$i\lambda u - v = 0, \quad i\lambda y - z = 0, \tag{1.9}$$

$$i\lambda P_\gamma v + a\Delta^2 u + \alpha\Delta\theta + \mu y = 0, \tag{1.10}$$

$$i\lambda\theta - \beta\Delta v - \sigma\Delta\theta = 0, \tag{1.11}$$

$$i\lambda z + \mu u - \eta\Delta y = 0. \tag{1.12}$$

Taking the inner product with U on both sides of (1.8), and taking the real parts, we immediately find $\theta = 0$. Therefore (1.11) and the fact that $v = 0$ on Γ yield $v = 0$ in Ω. Then we obtain $u = 0$ by (1.9), since $\lambda \neq 0$. Next, we derive $y = 0$ from (1.10) and $z = 0$ by (1.9). Hence $U = \mathbf{0}$. Finally, $\mathcal{A}_\gamma$ has no purely imaginary eigenvalue, and so $(S_\gamma(t))_{t\geq 0}$ is strongly stable, thanks to the semigroup strong stability criterion of Benchimol [10], or Arendt-Batty [5].

(2) We shall show that the semigroup $(S_\gamma(t))_{t\geq 0}$ is not exponentially stable. We will use a result of Triggiani [47] on compact perturbations of semigroups. Let

$$\mathcal{C}_\mu U = \begin{pmatrix} 0 \\ \mu P_\gamma^{-1} y \\ 0 \\ 0 \\ \mu u \end{pmatrix}$$

for all $U \in \mathcal{H}_\gamma$ and $\mathcal{A}_\gamma^0$ be the operator obtained from $\mathcal{A}_\gamma$ by setting $\mu = 0$. Therefore, $\mathcal{A}_\gamma^0 = \mathcal{A}_\gamma + \mathcal{C}_\mu$. It is clear that $\mathcal{A}_\gamma^0$ is a compact perturbation of $\mathcal{A}_\gamma$.

We consider a nonzero real number c and $w \in H_0^1(\Omega)$ such that $-\Delta w = \frac{c^2}{\eta} w$. Let

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w \\ icw \end{pmatrix}.$$

Then $\mathcal{A}_\gamma^0 V = icV$, so $i\mathbb{R} \not\subset \rho(\mathcal{A}_\gamma^0)$, which shows that the semigroup generated by $\mathcal{A}_\gamma^0$ is not strongly stable, hence not exponentially stable. Therefore, applying Triggiani's result, we find that the semigroup $(S_\gamma(t))_{t\geq 0}$ is not exponentially stable. $\square$

Theorem 1.3. *The semigroup $(S_\gamma(t))_{t\geq 0}$ is polynomially stable i.e. for all nonzero μ small enough, there exists $C > 0$ such that*

$$\|S_\gamma(t)Z^0\|_\gamma \leq \frac{C}{(1+t)^{\frac{1}{6}}}\|Z^0\|_{\mathcal{D}(\mathcal{A}_\gamma)}, \quad \forall t \geq 0,\ \forall Z^0 \in \mathcal{D}(\mathcal{A}_\gamma)$$

Before proving the above theorem, we want to compare the polynomial estimate obtained here with the one established in [18, Theorem 3.1].

Remark 1.4. In [18, Section 3], the authors consider a mechanically damped Kirchhoff plate weakly coupled to an undamped wave equation, and prove that the corresponding semigroup satisfies for every positive integer m, $\gamma > 0$, and every nonzero α, there exists $C_{\alpha,\gamma,m} > 0$ such that

$$\|\widehat{S}_{\alpha,\gamma}(t)Z^0\|_{\alpha,\gamma} \leq \frac{C_{\alpha,\gamma,m}\|Z^0\|_{D(\widehat{\mathcal{A}}^m_{\alpha,\gamma})}}{(1+t)^{\frac{m}{8}}}, \quad \forall t \geq 0,\ \forall Z^0 \in D(\widehat{\mathcal{A}}^m_{\alpha,\gamma}).$$

Thus the decay of the semigroup in the case of a mechanically damped plate is $O(t^{-1/8})$ when $m = 1$, while our polynomial stability result shows that, in the case of a thermoelastic plate, the decay rate of the semigroup is $O(t^{-\frac{1}{6}})$. Thus, our result shows that the decay of the semigroup in the case of a thermally damped plate is faster than in the case of a mechanically damped plate. We can also invoke

[9, Proposition 3.1], to derive from our theorem that, for every positive integer m, every $\gamma > 0$, and every nonzero constant μ, the semigroup satisfies the following decay estimate: there exists $C > 0$ such that

$$\|S_\gamma(t)Z^0\|_\gamma \leq \frac{C\|Z^0\|_{D(\mathcal{A}_\gamma^m)}}{(1+t)^{m/6}}, \quad \forall t \geq 0,\ \forall Z^0 \in D(\mathcal{A}_\gamma^m).$$

Proof of Theorem 1.3. Thanks to a result of Borichev-Tomilov [12], it suffices to prove the resolvent estimate

$$\|(ibI - \mathcal{A}_\gamma)^{-1}\|_{\mathcal{L}(\mathcal{H}_\gamma)} = O(|b|^6), \quad \text{as } |b| \nearrow +\infty. \tag{1.13}$$

To prove that resolvent estimate, we shall show that there exists $C_0 > 0$ such that for every $U \in \mathcal{H}_\gamma$, one has

$$\|(ibI - \mathcal{A}_\gamma)^{-1}U\|_\gamma \leq C_0|b|^6\|U\|_\gamma, \quad \forall b \in \mathbb{R},\ \text{with } |b| \geq 1.$$

Now, let $U \in \mathcal{H}_\gamma$ and let b a real number with $|b| \geq 1$. There exists $Z \in \mathcal{A}_\gamma$ such that

$$ibZ - \mathcal{A}_\gamma Z = U. \tag{1.14}$$

We note $Z = (u, v, \theta, y, z)$, and $U = (f, g, h, k, l)$. Taking the inner product with Z on both sides of (1.14), then taking the real parts, we immediately obtain

$$|\nabla\theta|_2^2 \leq C\|U\|_\gamma\|Z\|_\gamma, \tag{1.15}$$

where, hereafter, $|q|_2$ stands for $\|q\|_{L^2(\Omega)}$ and C denotes a generic positive constant that depends on the parameters of the system, but is independent of b. This constant varies from an inequality to another and it can vary even in the same line.

With the notation above, equation (1.14) can be rewritten as

$$ibu - v = f, \tag{1.16}$$

$$ibv + aP_\gamma^{-1}\Delta^2 u + \alpha P_\gamma^{-1}\Delta\theta + \mu P_\gamma^{-1}y = g, \tag{1.17}$$

$$ib\theta - \beta\Delta v - \sigma\Delta\theta = h, \tag{1.18}$$

$$iby - z = k, \tag{1.19}$$

$$ibz + \mu u - \eta\Delta y = l. \tag{1.20}$$

By applying the operator P_γ in equation (1.17), we obtain

$$ibP_\gamma v + a\Delta^2 u + \alpha\Delta\theta + \mu y = P_\gamma g. \tag{1.21}$$

Now, multiplying (1.18) by $\overline{v}$ and integrating over Ω, we derive

$$\beta|\nabla v|_2^2 = \int_\Omega \{-ib\theta + h\}\,\overline{v}\,dx - \int_\Omega \nabla\theta\nabla\overline{v}\,dx. \tag{1.22}$$

Using Cauchy-Schwarz inequality, Poincaré inequality, Young inequality and (1.15) yields

$$\begin{aligned} |\nabla v|_2^2 &\leq C(b^2|\theta|_2^2 + |\nabla\theta|^2 + |h|^2) \\ &\leq C(b^2\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2) \\ &\leq C(b^2\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2). \end{aligned} \tag{1.23}$$

Then by (1.16) and (1.23), we have

$$b^2|\nabla u|_2^2 \leq 2(|\nabla v|^2 + |\nabla f|^2) \leq C(b^2\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2). \tag{1.24}$$

and thanks Poincaré inequality,

$$b^2|u|_2^2 \leq C(b^2\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2). \tag{1.25}$$

Now we estimate each term in $\|Z\|_\gamma$.
Estimation of $|\Delta u|_2$: Substituting (1.16) in (1.21), we obtain

$$-b^2P_\gamma u + a\Delta^2 u + \alpha\Delta\theta + \mu y = P_\gamma g + ibP_\gamma f. \tag{1.26}$$

Multiplying (1.26) by $\overline{u}$, integrating the resulting equation over Ω and using Green's formula, we have

$$\begin{aligned}|\Delta u|_2^2 = \frac{b^2}{a}|P_\gamma^{1/2}u|_2^2 + \frac{\alpha}{a}\int_\Omega \nabla\theta\nabla\overline{u}\,dx - \frac{\mu}{a}\int_\Omega y\overline{u}\,dx\\ + \frac{1}{a}\int_\Omega \{P_\gamma g + ibP_\gamma f\}\,\overline{u}\,dx.\end{aligned} \tag{1.27}$$

Using the Cauchy-Schwarz inequality, (1.15) and (1.24) we obtain

$$|\frac{\alpha}{a}\int_\Omega \nabla\theta\nabla\overline{u}\,dx| \le \frac{|\alpha|}{a}|\nabla\theta|_2|\nabla u|_2 \le C\big(\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2}\big). \tag{1.28}$$

By (1.19), we have

$$|by|_2 \le (|z|_2 + |k|_2) \le \big(\|Z\|_\gamma + \|U\|_\gamma\big), \tag{1.29}$$

which, together with (1.25), yield

$$\begin{aligned}&|-\frac{\mu}{a}\int_\Omega y\overline{u}\,dx|\\ &\le \frac{|\mu|}{a}|y|_2|u|_2 \le \frac{|\mu|}{a}|b|^{-2}|by|_2|bu|_2\\ &\le C\Big(b^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + b^{-2}\|U\|_\gamma\|Z\|_\gamma + b^{-1}\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + b^{-2}\|U\|_\gamma^2\Big),\end{aligned} \tag{1.30}$$

and

$$\begin{aligned}|\int_\Omega\{P_\gamma g + ibP_\gamma f\}\overline{u}\,dx| &\le \frac{1}{2}\big(|P_\gamma^{1/2}f|_2^2 + |P_\gamma^{1/2}g|_2^2\big) + b^2|P_\gamma^{1/2}u|_2^2\\ &\le C\|U\|_\gamma^2 + b^2|P_\gamma^{1/2}u|_2^2.\end{aligned} \tag{1.31}$$

Now, using (1.28), (1.30) and (1.31) in (1.27), we have

$$\begin{aligned}|\Delta u|_2^2 \le C\Big(&b^2|P_\gamma^{1/2}u|_2^2 + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma\\ &+ b^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma^2\Big).\end{aligned} \tag{1.32}$$

In the sequel we will use the estimate

$$|\Delta u|_2^2 \le C\Big(b^2|P_\gamma^{1/2}u|_2^2 + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2 + |u|_2|y|_2\Big). \tag{1.33}$$

Estimation of $b^2|P_\gamma^{1/2}u|^2$. By (1.16), we have

$$b^2|P_\gamma^{1/2}u|_2^2 \le 2\Big(|P_\gamma^{1/2}v|_2^2 + |P_\gamma^{1/2}f|_2^2\Big). \tag{1.34}$$

Then it suffices to estimate $|P_\gamma^{1/2}v|^2$. To this end, we use some multiplier techniques developed in [6, 42]. Multiply both sides of (1.18) by $GP_\gamma\overline{v}$, where $G = (-\Delta)^{-1}$ with $-\Delta$ considered with Dirichlet boundary conditions. Integrating over Ω and using Green's formula, we derive

$$ib\int_\Omega G\theta P_\gamma\overline{v}\,dx + \beta|P_\gamma^{1/2}v|_2^2 + \sigma\int_\Omega P_\gamma^{1/2}\theta P_\gamma^{1/2}v\,dx = \int_\Omega P_\gamma^{1/2}(Gh)P_\gamma^{1/2}v\,dx. \tag{1.35}$$

Thanks to Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned}|P_\gamma^{1/2}v|^2 &\le |\frac{ib}{\beta}\int_\Omega G\theta P_\gamma \overline{v}\,dx| + \frac{\sigma}{|\beta|}|P_\gamma^{1/2}\theta|_2|P_\gamma^{1/2}v|_2 + \frac{1}{|\beta|}|P_\gamma^{1/2}(Gh)|_2|P_\gamma^{1/2}v|_2\\ &\le |\frac{ib}{\beta}\int_\Omega G\theta P_\gamma \overline{v}\,dx| + \frac{\sigma}{|\beta|}|P_\gamma^{1/2}\theta|_2^2 + \frac{1}{|\beta|}|P_\gamma^{1/2}(Gh)|_2^2 + \frac{1}{2}|P_\gamma^{1/2}v|_2^2.\end{aligned}$$

Then, by (1.15) we have

$$|P_\gamma^{1/2}v|_2^2 \le C\Big(|ib\int_\Omega G\theta P_\gamma \overline{v}\,dx| + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2\Big). \tag{1.36}$$

It remains to estimate the first term in the right hand side of (1.36). Multiply (1.21) by $G\overline{\theta}$ and apply the Green's formula to obtain

$$\begin{aligned}&ib\int_\Omega P_\gamma v G\overline{\theta}\,dx + a\int_\Omega \nabla u.\nabla\overline{\theta}\,dx - a\int_\Gamma \Delta u\partial_\nu(G\overline{\theta})\,d\Gamma - \alpha|\theta|^2 + \mu\int_\Omega yG\overline{\theta}\,dx\\ &= \int_\Omega P_\gamma g G\overline{\theta}.\end{aligned}$$

Using the Cauchy-Schwarz inequality leads to the estimate

$$\begin{aligned}\big|ib\int_\Omega P_\gamma vG\overline{\theta}\,dx\big| &\le C\Big(|\nabla u|_2|\nabla\theta|_2 + |\Delta u|_{L^2(\Gamma)}|\partial_\nu G\theta|_{L^2(\Gamma)} + |\alpha|\|\theta|_2^2\\ &\quad + \mu|y|_2|G\theta|_2 + |P_\gamma^{1/2}g|_2|P_\gamma^{1/2}G\theta|_2\Big).\end{aligned} \tag{1.37}$$

Now, $\partial_\nu \in \mathcal{L}(H^2(\Omega), L^2(\Gamma))$, $G \in \mathcal{L}(L^2(\Omega), H^2(\Omega)\cap H_0^1(\Omega))$. Therefore, this fact, (1.15) and (1.24) yield

$$\begin{aligned}\big|ib\int_\Omega P_\gamma vG\overline{\theta}\,dx\big| &\le C\Big(\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{\frac{3}{2}}\|Z\|_\gamma^{1/2}\Big)\\ &\quad + C|y|_2|\theta|_2 + C|\Delta u|_{L^2(\Gamma)}|\theta|_2.\end{aligned} \tag{1.38}$$

The combination of (1.36) and (1.38) leads to

$$\begin{aligned}&|P_\gamma^{1/2}v|^2\\ &\le C\Big(\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{\frac{3}{2}}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^2\Big) + C|y|_2|\theta|_2 + C|\Delta u|_{L^2(\Gamma)}|\theta|_2.\end{aligned} \tag{1.39}$$

Estimation of $|\Delta u|_{L^2(\Gamma)}$. Let ξ be a positive constant to be specified later. Let $q \in \big[C^2(\overline{\Omega})\big]^d$ be a vector field satisfying $q = \nu$ on Γ, see for example [22, 29]. Multiply (1.26) by $\xi\overline{u} + 2q.\nabla\overline{u}$ and integrate the result over Ω to obtain

$$\begin{aligned}&-\xi b^2|P_\gamma^{1/2}u|_2^2 - 2b^2\operatorname{Re}\int_\Omega P_\gamma uq\cdot\nabla\overline{u}\,dx + 2a\operatorname{Re}\int_\Omega \Delta^2 uq\cdot\nabla\overline{u}\,dx\\ &+ a\xi|\Delta u|_2^2\\ &= \operatorname{Re}\int_\Omega\{-\alpha\Delta\theta - \mu y + P_\gamma g + ibP_\gamma f\}(\xi\overline{u} + 2q\cdot\nabla\overline{u})\,dx.\end{aligned} \tag{1.40}$$

Now, applying Green's formula, we find

$$\begin{aligned}2\operatorname{Re}\int_\Omega \Delta^2 u(q\cdot\nabla\overline{u})\,dx &= -\int_\Omega div(q)|\Delta u|^2\,dx + 2\operatorname{Re}\int_\Omega \Delta q_k\frac{\partial u}{\partial x_k}\Delta\overline{u}\,dx\\ &\quad + 4\operatorname{Re}\int_\Omega \nabla q_k\cdot\nabla\big(\frac{\partial u}{\partial x_k}\big)\Delta\overline{u}\,dx + \int_\Gamma q\cdot\nu|\Delta u|^2 d\Gamma\end{aligned}$$

$$- 2\operatorname{Re}\int_\Gamma \Delta u\partial_\nu(q\cdot\nabla\overline{u})\,d\Gamma + 2\operatorname{Re}\int_\Gamma q\cdot\nabla u\partial_\nu(\Delta\overline{u})\,d\Gamma.$$

Thanks to the boundary conditions on u, one checks that $\partial_\nu(q\cdot\nabla\overline{u}) = q\cdot\nu\Delta\overline{u}$ on Γ. Hence

$$\begin{aligned} 2\operatorname{Re}\int_\Omega \Delta^2 u(q\cdot\nabla\overline{u})\,dx &= -\int_\Omega div(q)|\Delta u|^2\,dx + 2\int_\Omega \Delta q_k\frac{\partial u}{\partial x_k}\Delta\overline{u}\,dx \\ &\quad + 4\int_\Omega \nabla q_k\cdot\nabla\big(\frac{\partial u}{\partial x_k}\big)\Delta\overline{u}\,dx - \int_\Gamma |\Delta u|^2\,d\Gamma. \end{aligned} \tag{1.41}$$

Proceeding similarly, we obtain

$$\begin{aligned} &\operatorname{Re}\int_\Omega \Delta u(2q\cdot\nabla\overline{u})\,dx \\ &= -\int_\Omega 2\operatorname{Re}(\nabla u\cdot\nabla(q_k)\partial_k\overline{u} + q_k\nabla u\cdot\nabla(\partial_k\overline{u}))\,dx + 2\int_\Gamma |\partial_\nu u|^2\,d\Gamma \\ &= -\int_\Omega 2\operatorname{Re}(\nabla u\cdot\nabla(q_k)\partial_k\overline{u}) + q_k\partial_k(|\nabla u|^2)\,dx \\ &= -\int_\Omega 2\operatorname{Re}(\nabla u\cdot\nabla(q_k)\partial_k\overline{u})\,dx + \int_\Omega \operatorname{div}(q)|\nabla u|^2\,dx, \end{aligned} \tag{1.42}$$

and

$$\operatorname{Re}\int_\Omega u(2q\cdot\nabla\overline{u})\,dx = -\int_\Omega div(q)|u|^2\,dx, \tag{1.43}$$

which gives

$$\begin{aligned} -2b^2\operatorname{Re}\int_\Omega P_\gamma u q\cdot\nabla\overline{u}\,dx &= -b^2\gamma\int_\Omega 2\operatorname{Re}(\nabla u\cdot\nabla(q_k)\partial_k\overline{u})\,dx \\ &\quad + b^2\int_\Omega \operatorname{div}(q)(|u|_2^2 + \gamma|\nabla u|_2^2)\,dx. \end{aligned} \tag{1.44}$$

Reporting (1.41) and (1.44) in (1.40), we find

$$\begin{aligned} &a\int_\Gamma |\Delta u|^2 d\Gamma + \xi b^2|P_\gamma^{1/2}u|_2^2 - b^2\int_\Omega \operatorname{div}(q)(P_\gamma^{1/2}u|^2)\,dx \\ &+ 2\gamma b^2\int_\Omega \operatorname{Re}(\nabla u\cdot\nabla(q_k)\partial_k\overline{u})\,dx \\ &= \xi a|\Delta u|_2^2 - \int_\Omega div(q)|\Delta u|^2\,dx + 2\int_\Omega \Delta q_k\frac{\partial u}{\partial x_k}\Delta\overline{u}\,dx \\ &\quad + 4\int_\Omega \nabla q_k\cdot\nabla\big(\frac{\partial u}{\partial x_k}\big)\Delta\overline{u}\,dx \\ &\quad + \operatorname{Re}\int_\Omega \{\alpha\Delta\theta + \mu y - P_\gamma g - ibP_\gamma f\}(\xi\overline{u} + 2q\cdot\nabla\overline{u})\,dx \\ &\le C|\Delta u|_2^2 + \operatorname{Re}\int_\Omega \{\alpha\Delta\theta + \mu y - P_\gamma g - ibP_\gamma f\}\,(\xi\overline{u} + 2q\cdot\nabla\overline{u})\,dx. \end{aligned} \tag{1.45}$$

Now we estimate the last integral in the right hand side of (1.45). For that purpose, an application of Green's formula yields

$$\begin{aligned} &\int_\Omega \Delta\theta(\xi\overline{u} + 2q.\nabla\overline{u})\,dx \\ &= 2\int_\Omega \nabla\theta\cdot\nabla(q\cdot\nabla\overline{u})\,dx - \xi\int_\Omega \nabla\theta\cdot\nabla\overline{u}\,dx \end{aligned}$$

$$= 2\int_\Omega \nabla\theta\cdot\nabla(q_k)\partial_k\overline{u}\,dx + 2\int_\Omega q_k\nabla u\cdot\nabla(\partial_k\overline{u}))\,dx - \xi\int_\Omega \nabla\theta\cdot\nabla\overline{u}\,dx.$$

Applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
\Big|\int_\Omega \Delta\theta(\xi\overline{u}+2q.\nabla\overline{u})\,dx\Big| &\le C(|\nabla\theta|_2\|\nabla u|_2 + |\nabla\theta|_2|\Delta u|_2)\\
&\le C|\nabla\theta|_2|\Delta u|_2 \le C(|\nabla\theta|_2^2+|\Delta u|_2^2)\\
&\le C\|U\|_\gamma\|Z\|_\gamma + C|\Delta u|_2^2.
\end{aligned}\tag{1.46}$$

Similarly, and keeping in mind (1.25), we derive

$$\begin{aligned}
&\Big|\int_\Omega\{\mu y - P_\gamma g - ibP_\gamma f\}(\xi\overline{u}+2q\cdot\nabla\overline{u})\,dx\Big|\\
&\le C(b^{-1}|by|_2|u|_2 + b^{-1}|by|_2|\nabla u|_2 + |P_\gamma^{1/2}g|_2|\Delta u|_2) + |b\|\Delta f|_2|P_\gamma^{1/2}u|_2\\
&\le C_\epsilon\Big(|b|^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma\|Z\|_\gamma + |b|^{-1}\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^2\Big)\\
&\quad + \epsilon b^2|P_\gamma^{1/2}u|^2
\end{aligned}\tag{1.47}$$

Using (1.46) and (1.47) in (1.45) where ξ is chosen with $\xi \ge 2(\|\operatorname{div}(q)\|_{L^\infty(\Omega)} + 2\|\nabla q\|_{L^\infty(\Omega)})$, we obtain

$$\begin{aligned}
&\int_\Gamma |\Delta u|^2 d\Gamma + \frac{\xi}{2}b^2|P_\gamma^{1/2}u|_2^2\\
&\le \epsilon b^2|P_\gamma^{1/2}u|_2^2 + C_\epsilon\Big(|b|^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma\|Z\|_\gamma\\
&\quad + |b|^{-1}\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^2\Big) + C|\Delta u|_2^2\,.
\end{aligned}$$

Choosing $\epsilon = \xi/3$, we obtain

$$\begin{aligned}
\int_\Gamma |\Delta u|^2 d\Gamma \le C\Big(&|b|^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma\|Z\|_\gamma\\
&+ |b|^{-1}\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^2\Big) + C|\Delta u|_2^2.
\end{aligned}\tag{1.48}$$

Then by (1.15) and (1.48), the Poincaré inequality and the Young inequality, we have

$$\begin{aligned}
C|\Delta u|_{L^2(\Gamma)}|\theta|_2 \le C\Big(&\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4}\\
&+ \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4}\Big) + \frac{|\Delta u|_2^2}{4}.
\end{aligned}\tag{1.49}$$

Finally from (1.33), (1.34), (1.39) and (1.49) we have

$$\begin{aligned}
|\Delta u|_2^2 \le C\Big(&\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4} + \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4}\\
&+ \|U\|_\gamma^2\Big) + C(|u|_2|y|_2 + |y|_2|\theta|_2).
\end{aligned}\tag{1.50}$$

Thanks (1.15) and (1.29), we have

$$|y|_2|\theta|_2 = b^{-1}|by|_2|\theta|_2 \le b^{-1}\Big(\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2}\Big).\tag{1.51}$$

Using (1.51) and (1.30) in (1.50), we obtain

$$\begin{aligned}|\Delta u|_2^2 \le C\Big(&\|U\|_\gamma^2 + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4} + \|U\|_\gamma\|Z\|_\gamma \\ &+ \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4} + |b|^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2}\Big).\end{aligned} \tag{1.52}$$

Estimation of $|\nabla y|_2$. Reporting (1.19) in (1.20), multiplying the result by $\overline{y}$ and integrating over Ω, we obtain

$$\eta|\nabla y|_2^2 = b^2|y|^2 + \operatorname{Re}\int_\Omega (ibk - \mu u + l)\overline{y}\,dx. \tag{1.53}$$

Using Hölder and Young inequalities and (1.25), we obtain

$$\begin{aligned}\Big|\int_\Omega (ibk - \mu u + l)\overline{y}\,dx\Big| &\le |b||k|_2|y|_2 + |\mu||u|_2|y|_2 + |l|_2|y|_2 \\ &\le b^2|y|_2^2 + C(|u|_2^2 + |l|_2^2 + |k|_2^2) \\ &\le b^2|y|_2^2 + C(\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2)\end{aligned} \tag{1.54}$$

Combining (1.53) and (1.54), we have

$$|\nabla y|_2^2 \le C(b^2|y|_2^2 + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2). \tag{1.55}$$

Estimation of $b^2|y|_2^2$: Multiplying (1.21) by $\frac{1}{\mu}\overline{y}$, integrating over Ω and using Green's formula lead to

$$\begin{aligned}|y|_2^2 = &-\frac{ib}{\mu}\int_\Omega P_\gamma v\overline{y}\,dx - \frac{a}{\mu}\int_\Omega \Delta u\Delta\overline{y}\,dx \\ &+ \frac{a}{\mu}\int_\Gamma \Delta u\partial_\nu\overline{y}\,d\Gamma + \frac{\alpha}{\mu}\int_\Omega \nabla\theta\cdot\nabla\overline{y}\,dx + \frac{1}{\mu}\int_\Omega P_\gamma g\overline{y}\,dx.\end{aligned} \tag{1.56}$$

Multiplying the conjugate of (1.20) by $\frac{\Delta u}{\eta}$ and integrating over Ω, we derive

$$\int_\Omega \Delta u\Delta\overline{y}\,dx = \frac{1}{\eta}\int_\Omega(-ib\overline{z} + \mu\overline{u} - \overline{l})\Delta u\,dx. \tag{1.57}$$

Therefore,

$$\begin{aligned}b^2|y|_2^2 = &\frac{b^2}{\mu}\operatorname{Re}\int_\Omega(-ibP_\gamma v + P_\gamma g)\overline{y}\,dx + \frac{ab^2}{\mu\eta}\operatorname{Re}\int_\Omega(ib\overline{z} - \mu\overline{u} + \overline{l})\Delta u\,dx \\ &+ \frac{ab^2}{\mu}\operatorname{Re}\int_\Gamma \Delta u\partial_\nu\overline{y}\,d\Gamma + \frac{\alpha b^2}{\mu}\operatorname{Re}\int_\Omega \nabla\theta\cdot\nabla\overline{y}\,dx.\end{aligned} \tag{1.58}$$

To simplify notations, we denote

$$b^2|y|_2^2 = I_1 + I_2 + I_3 + I_4, \tag{1.59}$$

where I_i corresponds to the ith integral in the right-hand side in (1.58). Thus, we estimate each integral I_i. By the Poincaré inequality, we have

$$|P_\gamma^{1/2}y|_2^2 = |y|_2^2 + \gamma|\nabla y|_2^2 \le C|\nabla y|_2^2. \tag{1.60}$$

Then, using Cauchy-Schwarz inequality and (1.60), we have

$$
\begin{aligned}
|I_1| &\le \Big|\frac{b^2}{\mu}\int_\Omega -ibP_\gamma v\overline{y}\,dx\Big| + \Big|\frac{b^2}{\mu}\int_\Omega P_\gamma g\overline{y}\,dx\Big| \\
&\le \Big|\frac{b^2}{\mu}\int_\Omega -ibP_\gamma v\overline{y}\,dx\Big| + \frac{b^2}{|\mu|}|P_\gamma^{1/2}g|_2|P_\gamma^{1/2}y|_2 \\
&\le \Big|\frac{b^2}{\mu}\int_\Omega -ibP_\gamma v\overline{y}\,dx\Big| + Cb^2\|U\|_\gamma\|Z\|_\gamma.
\end{aligned} \tag{1.61}
$$

On the other hand,

$$
\int_\Omega -ibP_\gamma v\overline{y}\,dx = -ib\int_\Omega v\overline{y}\,dx + i\gamma b\int_\Omega \Delta v\overline{y}\,dx. \tag{1.62}
$$

By the Poincaré inequality, (1.23) and (1.29), we have

$$
\begin{aligned}
&\Big| -ib\int_\Omega v\overline{y}\,dx\Big| \\
&\le |v|_2|by|_2 \\
&\le C|\nabla v|_2|by|_2 \\
&\le C\Big(|b|\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + |b|\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2\Big)
\end{aligned} \tag{1.63}
$$

Now multiplying (1.18) by $\overline{y}$ and integrating over Ω, we obtain

$$
\int_\Omega \Delta v\overline{y}\,dx = \frac{1}{\beta}\int_\Omega \{ib\theta - \sigma\Delta\theta - h\}\overline{y}\,dx. \tag{1.64}
$$

Thanks to Green's formula in (1.64), Cauchy-Scwharz inequality and (1.29), we obtain

$$
\begin{aligned}
&|i\gamma b\int_\Omega \Delta v\overline{y}\,dx| \\
&\le \frac{\gamma}{|\beta|}\big(b^2|\theta|_2|y|_2 + \sigma|b||\nabla\theta|_2|\nabla y|_2 + |b||h|_2|y|_2\big) \\
&\le \frac{\gamma}{|\beta|}\big(|b||\theta|_2|by|_2 + \sigma|b||\nabla\theta|_2|\nabla y|_2 + |h|_2|by|_2\big) \\
&\le \frac{\gamma}{|\beta|}\Big((1+\sigma)|b|\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + |b|\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2\Big)
\end{aligned} \tag{1.65}
$$

Combining (1.62), (1.63) and (1.65), we have

$$
\begin{aligned}
&\Big|\int_\Omega -ibP_\gamma v\overline{y}\,dx\Big| \\
&\le C\Big(|b|\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + |b|\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^2\Big)
\end{aligned} \tag{1.66}
$$

Using (1.66) in (1.61), we obtain

$$
|I_1| \le C\Big(|b|^3\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + |b|^3\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + b^2\|U\|_\gamma\|Z\|_\gamma + b^2\|U\|_\gamma^2\Big) \tag{1.67}
$$

For the second integral I_2 in (1.59), using Cauchy-Schwarz inequality and (1.25), we find

$$\begin{aligned} |I_2| &\le |\frac{ab^2}{\mu\eta}\int_\Omega ib\overline{z}\Delta u\,dx| + \frac{ab^2}{|\mu|\eta}\,(|u|_2|\Delta u|_2 + |l|_2|\Delta u|_2) \\ &\le \frac{ab^2}{|\mu|\eta}|\int_\Omega ib\overline{z}\Delta u\,dx| + \frac{a}{|\mu|\eta}\big(|b||bu|_2|\Delta u|_2 + b^2|l|_2|\Delta u|_2\big) \\ &\le \frac{ab^2}{|\mu|\eta}|\int_\Omega ib\overline{z}\Delta u\,dx| + Cb^2\big(\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma\|Z\|_\gamma\big). \end{aligned} \tag{1.68}$$

Now, using (1.16) in (1.18) and multiplying the resulting equation by $\overline{z}$, we obtain

$$\int_\Omega ib\overline{z}\Delta u\,dx = \int_\Omega \big\{\frac{ib}{\beta}\theta + \Delta f - \frac{1}{\beta}h\big\}\overline{z}\,dx + \frac{\sigma}{\beta}\int_\Omega \nabla\theta\nabla\overline{z}\,dx. \tag{1.69}$$

From (1.19), we have the estimate $|\nabla z|_2 \le \sqrt{2}(|b||\nabla y|_2 + |\nabla k|_2)$. Then, using Cauchy-Schwarz inequality, (1.15) in (1.69), we obtain

$$\begin{aligned} &|\int_\Omega ib\overline{z}\Delta u\,dx| \\ &\le \Big(\frac{|b|}{|\beta|}|\theta|_2 + |\Delta f|_2 + \frac{1}{|\beta|}|h|_2\Big)|z|_2 + \frac{\sqrt{2}\sigma}{|\beta|}|\nabla\theta|_2(|b||\nabla y|_2 + |\nabla k|_2) \\ &\le C\Big(|b|\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma\Big). \end{aligned} \tag{1.70}$$

Combining (1.70) and (1.68), we have

$$|I_2| \le C\Big(|b|^3\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + b^2\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + b^2\|U\|_\gamma\|Z\|_\gamma\Big). \tag{1.71}$$

For the integral I_4 in (1.59), using (1.15) and Young inequality, we have the estimate

$$|I_4| = |\frac{\alpha b^2}{\mu}\int_\Omega \nabla\theta\cdot\nabla\overline{y}\,dx| \le Cb^2|\nabla\theta|_2|\nabla y|_2 \le Cb^2\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2}. \tag{1.72}$$

Now it remains to estimate the boundary integral I_3 in (1.59). To that end, we have to estimate $|\partial_\nu y|_{L^2(\Gamma)}$ which can be estimated in the same way as [43, pp. 8-9]. Thus, we have

$$\begin{aligned} |\partial_\nu y|_{L^2(\Gamma)} &\le C\Big(|b||y|_2 + \|U\|_\gamma^{1/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\Big) \\ &\le C\Big(\|U\|_\gamma^{1/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma + \|Z\|_\gamma\Big). \end{aligned} \tag{1.73}$$

Since by (1.48) and (1.52), we have

$$\begin{aligned} |\Delta u|_{L^2(\Gamma)}^2 \le C\Big(&\|U\|_\gamma^2 + \|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4} + \|U\|_\gamma\|Z\|_\gamma \\ &+ \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4} + |b|^{-1}\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2}\Big), \end{aligned} \tag{1.74}$$

it then follows from (1.73) and (1.74) that

$$\begin{aligned}|I_3| &\le \frac{a}{|\mu|} b^2 |\Delta u|_{L^2(\Gamma)} |\partial_\nu y|_{L^2(\Gamma)} \\ &\le Cb^2\Big((\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4} + \|U\|_\gamma^{9/8}\|Z\|_\gamma^{7/8} \\ &\quad + \|U\|_\gamma^{7/8}\|Z\|_\gamma^{9/8}\Big) \\ &\quad + Cb^2\Big(\|U\|_\gamma^{7/4}\|Z\|_\gamma^{1/4} + \|U\|_\gamma^{13/8}\|Z\|_\gamma^{3/8} + \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4} + \|U\|_\gamma^{5/8}\|Z\|_\gamma^{11/8}\Big) \\ &\quad + Cb^2\Big(\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + \|U\|_\gamma^{3/8}\|Z\|_\gamma^{13/8} + |b|^{-1/2}\|U\|_\gamma^{1/4}\|Z\|_\gamma^{7/4} + \|U\|_\gamma^2\Big)\end{aligned} \tag{1.75}$$

Finally, reporting (1.63), (1.71), (1.72) and (1.75) in (1.59), we find

$$\begin{aligned}&b^2|y|^2 \\ &\le C|b|^3\Big(\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2}\Big) + Cb^2\Big(\|U\|_\gamma^{3/8}\|Z\|_\gamma^{13/8} + \|U\|_\gamma^2\Big) \\ &\quad + Cb^2\Big(\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4} + \|U\|_\gamma^{9/8}\|Z\|_\gamma^{7/8} + \|U\|_\gamma^{7/8}\|Z\|_\gamma^{9/8}\Big) \\ &\quad + Cb^2\Big(\|U\|_\gamma^{7/4}\|Z\|_\gamma^{1/4} + \|U\|_\gamma^{13/8}\|Z\|_\gamma^{3/8} + \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4} \\ &\quad + \|U\|_\gamma^{5/8}\|Z\|_\gamma^{11/8}\Big) + C|b|^{3/2}\|U\|_\gamma^{1/4}\|Z\|_\gamma^{7/4}.\end{aligned} \tag{1.76}$$

Substituting (1.76) in (1.55) and combining (1.19) and (1.76), we find that $|\nabla y|_2^2$ and $|z|_2^2$ are bounded from above by the right hand side of (1.76). Since by the Cauchy-Schwarz inequality and (1.25), we have

$$\big|2\mu\int_\Omega \mathrm{Re}(u\overline{y})\,dx\big| \le 2|\mu||u|_2|y|_2 \le C\Big(\|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2} + |b|^{-1}\|U\|_\gamma\|Z\|_\gamma\Big) \tag{1.77}$$

it follows from (1.15), (1.52), (1.76) and (1.77) that

$$\begin{aligned}&\|Z\|_\gamma^2 \\ &\le C|b|^3\Big(\|U\|_\gamma^{3/2}\|Z\|_\gamma^{1/2} + \|U\|_\gamma^{1/2}\|Z\|_\gamma^{3/2}\Big) + Cb^2\Big(\|U\|_\gamma^{3/8}\|Z\|_\gamma^{13/8} + \|U\|_\gamma^2\Big) \\ &\quad + Cb^2\Big(\|U\|_\gamma\|Z\|_\gamma + \|U\|_\gamma^{5/4}\|Z\|_\gamma^{3/4} + \|U\|_\gamma^{9/8}\|Z\|_\gamma^{7/8} + \|U\|_\gamma^{7/8}\|Z\|_\gamma^{9/8}\Big) \\ &\quad + Cb^2\Big(\|U\|_\gamma^{7/4}\|Z\|_\gamma^{1/4} + \|U\|_\gamma^{13/8}\|Z\|_\gamma^{3/8} + \|U\|_\gamma^{3/4}\|Z\|_\gamma^{5/4} \\ &\quad + \|U\|_\gamma^{5/8}\|Z\|_\gamma^{11/8}\Big) + C|b|^{3/2}\|U\|_\gamma^{1/4}\|Z\|_\gamma^{7/4}.\end{aligned}$$

Now, applying Young inequality several times and successively, one derives

$$\|Z\|_\gamma^2 \le Cb^{12}\|U\|_\gamma^2.$$

Hence

$$\|(ibI - \mathcal{A}_\gamma)^{-1}U\|_\gamma \le Cb^6\|U\|_\gamma, \quad \forall U \in \mathcal{H}_\gamma,\ \forall b \in \mathbb{R},\ \ |b| \ge 1,$$

By applying the Borichev-Tomilov result [12, Theorem 2.4], we complete the proof. □

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