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POSITIVE AND NODAL SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS PARAMETRIC NEUMANN PROBLEMS

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Abstract. We consider a parametric Neumann problem driven by a nonlinear nonhomogeneous differential operator plus an indefinite potential term. The reaction term is superlinear but does not satisfy the Ambrosetti-Rabinowitz condition. First we prove a bifurcation-type result describing in a precise way the dependence of the set of positive solutions on the parameter $\lambda > 0$. We also show the existence of a smallest positive solution. Similar results hold for the negative solutions and in this case we have a biggest negative solution. Finally using the extremal constant sign solutions we produce a smooth nodal solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous Neumann problem

$$\begin{gathered} -\operatorname{div} a(\nabla u(z)) + [\xi(z) + \lambda]u(z)^{p-1} = f(z, u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,\ u > 0,\ \lambda > 0,\ 1 < p < +\infty. \end{gathered} \tag{1.1}$$

In this problem the map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator is strictly monotone and continuous, thus maximal monotone too. Also it satisfies certain other regularity and growth conditions listed in hypotheses (H1) (see Section 2). These conditions are not restrictive and incorporate in our framework many differential operators of interest such as the p-Laplacian ($1 < p < +\infty$) and the (p, q)-Laplacian ($1 < q < p < +\infty$), that is, the sum of a p-Laplacian and of a q-Laplacian. The differential operator of (1.1) is not homogeneous and this is a source of difficulties in the analysis of problem (1.1). There is also a parametric potential term $u \to [\xi(z) + \lambda]u^{p-1}$ with the potential function $\xi \in L^\infty(\Omega)$ being indefinite (that is, sign-changing). Hence the left hand side of (1.1) is not in general coercive and this is another feature of problem (1.1) that complicates our arguments. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that for a.a. $z \in \Omega$ the function $x \to f(z, x)$ is $(p-1)$-superlinear near $+\infty$. However, the superlinearity of $f(z, \cdot)$ is not expressed via the usual for such

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problems Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead we employ an alternative less restrictive condition which permits the consideration of $(p-1)$-superlinear nonlinearities with "slower" growth near $+\infty$. Near 0^+ we assume that $f(z,\cdot)$ is $(q-1)$-superlinear with $1<q<p$.

Using variational tools from the critical point theory together with suitable truncation, perturbation and comparison techniques, we prove a bifurcation-type result which describes the dependence on the parameter $\lambda>0$ of the set of positive solutions of problem (1.1). More precisely, we show that there exists a critical parameter value $\lambda_*>0$ such that

- for all $\lambda>\lambda_*$ problem (1.1) has at least two positive solutions;
- for $\lambda=\lambda_*$ problem (1.1) has at least a positive solution;
- for all $\lambda\in(0,\lambda_*)$ problem (1.1) has no positive solution.

In addition we show that for every $\lambda\in\mathcal{L}=[\lambda_*,+\infty)$, problem (1.1) has a smallest positive solution $\overline{u}_\lambda$ and we examine the monotonicity and continuity properties of the map $\lambda\to\overline{u}_\lambda$.

With the conditions valid on the negative semiaxis $\mathbb{R}_-=(-\infty,0]$, we can have analogous results for the negative solutions. In particular we can produce a biggest negative solution $\overline{v}_\lambda$ for problem (1.1). When the conditions are bilateral (that is, valid for all $x\in\mathbb{R}$ and not only on the semiaxes), then using the two extremal constant sign solutions $\overline{u}_\lambda$ and $\overline{v}_\lambda$, we produce a nodal (sign-changing) solution for problem (1.1). Our work here continues and extends the ones by Motreanu-Motreanu-Papageorgiou [8], Averna-Papageorgiou-Tornatore [1] and Papageorgiou-Rădulescu [11]. In [8] the differential operator is also nonhomogeneous but the conditions on the map $a(\cdot)$ are more restrictive excluding, for example, the important case of the (p,q)-Laplacian. Also $\xi\equiv 0$ and the authors do not prove the precise dependence on $\lambda>0$ of the set of positive solutions (bifurcation-type result). In [1] the differential operator is the p-Laplacian and $\xi\equiv 0$. The authors do not prove the existence of nodal solutions. Finally in [11] the equation is semilinear driven by the Laplacian, but the boundary condition is Robin. It is an interesting open problem whether we can extend our work here to Robin boundary value problems.

2. Mathematical Background - Hypotheses

In the analysis of problem (1.1) we will use the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\|=[\|u\|_p^p+\|\nabla u\|_p^p]^{1/p}\quad\text{for all } u\in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+=\{u\in C^1(\overline{\Omega}): u(z)\geq 0 \text{ for all } z\in\overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+=\{u\in C^1(\overline{\Omega}): u(z)>0 \text{ for all } z\in\overline{\Omega}\}.$$

We will also consider another order cone for $C^1(\overline{\Omega})$, namely the cone

$$\widehat{C}_+=\big\{u\in C^1(\overline{\Omega}): u(z)\geq 0 \text{ for all } z\in\overline{\Omega},\ \frac{\partial u}{\partial n}\big|_{\partial\Omega\cap u^{-1}(0)}\leq 0\big\}.$$

This cone too has a nonempty interior

$$\operatorname{int}\widehat{C}_+=\big\{u\in \widehat{C}_+: u(z)>0 \text{ for all } z\in\overline{\Omega},\ \frac{\partial u}{\partial n}\big|_{\partial\Omega\cap u^{-1}(0)}<0\big\}.$$

Given $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. For any measurable function $u : \Omega \to \mathbb{R}^N$, we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. If $u \in W^{1,p}(\Omega)$, then $u^{\pm} \in W^{1,p}(\Omega)$. If $u, v \in W^{1,p}(\Omega)$ and $v \leq u$, we define

$$[v, u] = \{h \in W^{1,p}(\Omega) : v(z) \leq h(z) \leq u(z) \quad \text{for a.a. } z \in \Omega\},$$
$$[v) = \left\{h \in W^{1,p}(\Omega) : v(z) \leq h(z) \quad \text{for a.a. } z \in \Omega\right\}.$$

By $\mathrm{int}_{C^1(\overline{\Omega})}[v, u]$ we denote the interior in $C^1(\overline{\Omega})$ of $[v, u] \cap C^1(\overline{\Omega})$.

Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We define

$$K_\varphi = \{x \in X : \varphi'(x) = 0\} \quad \text{(the critical set of } \varphi\text{)},$$
$$\varphi^c = \{x \in X : \varphi(x) \leq c\} \quad \text{(the sublevel set of } \varphi \text{ at } c\text{)}.$$

Let (A, B) be a topological pair such that $B \subseteq A \subseteq X$. By $H_k(A, B)$, $k \in \mathbb{N}_0$, we denote the k^{th}-relative singular homology group for the pair (A, B) with integer coefficients. If $u \in K_\varphi$ is isolated and $\varphi(u) = c$, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that this definition is independent of the isolating neighborhood.

Let X^* be the topological dual of X and denote by $\langle \cdot, \cdot \rangle$ the duality brackets of the pair (X^*, X). A map $A : X \to X^*$ is said to be of type $(S)_+$ if it has the following property:

$$u_n \xrightarrow{w} u \text{ in } X \text{ and } \limsup_{n \to +\infty} \langle A(u_n), u_n - u \rangle \leq 0 \ \Rightarrow\ u_n \to u \text{ in } X.$$

Also, we say that $\varphi \in C^1(X, \mathbb{R})$ satisfies the "C-condition", if the following property holds:

> Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence.

If $h_1, h_2 \in L^\infty(\Omega)$, then we write $h_1 \preceq h_2$ when we have

$$h_1(z) \leq h_2(z) \quad \text{for a.a. } z \in \Omega$$

and the above inequality is strict on a set of positive measure.

Finally for any measurable function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, by $N_f(\cdot)$ we denote the Nemytskii operator corresponding to f, that is,

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \text{ for every } u : \Omega \to \mathbb{R} \text{ measurable},$$

and by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Let $\vartheta \in C^1(0, +\infty)$ with $\vartheta(t) > 0$ for all $t > 0$. We assume that

$$0 < \widehat{c} \leq \frac{\vartheta'(t)t}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2[t^{s-1} + t^{p-1}]$$

for all $t > 0$, with $1 \leq s < p < +\infty$, $c_1, c_2 > 0$.

The hypotheses on the map $a(\cdot)$ are as follows:

(H1) $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, and
  (i) $a_0 \in C^1(0, +\infty)$, $t \to a_0(t)t$ is strictly increasing on $(0, +\infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$ and $\lim_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1$;

(ii) there exists $c_3 > 0$ such that $|\nabla a(y)| \le c_3 \frac{\vartheta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \frac{\vartheta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, $\xi \in \mathbb{R}^N$;

(iv) If $G_0(t) = \int_0^t a_0(s)s ds$, then there exist $q \in (1,p)$ and $c^*, c_4 > 0$ such that

$$\limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} \le c^*, \quad t \to G_0(t^{1/q}) \text{ is convex,}$$
$$c_4 t^p \le a_0(t)t^2 - qG_0(t) \quad \text{for all } t > 0,$$
$$0 \le pG_0(t) - a_0(t)t^2 \quad \text{for all } t > 0.$$

Remark 2.1. Hypotheses (H1)(i)(ii)(iii) are dictated by the nonlinear regularity theory of Lieberman [7] and the nonlinear maximum principle of Pucci-Serrin [15]. Hypothesis (H1)(iv) is motivated by the particular needs of our problem. However, as the examples below illustrate, it is not restrictive and it is satisfied in all cases of interest.

From the above hypotheses we see that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then $G(\cdot)$ is convex and

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$. This fact and the convexity of $G(\cdot)$ imply that

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.1}$$

Hypotheses (H1) lead to the following lemma summarizing the main properties of the map $y \to a(y)$ (see Papageorgiou-Rădulescu [9]).

Lemma 2.2. *If hypotheses* (H1)(i)(ii)(iii) *hold, then*

(a) $a(\cdot)$ *is continuous, strictly monotone, hence maximal monotone;*
(b) *there exists* $c_5 > 0$ *such that* $|a(y)| \le c_5[|y|^{s-1} + |y|^{p-1}]$ *for all* $y \in \mathbb{R}^N$*;*
(c) $(a(y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

This lemma and (2.1) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.3. *If hypotheses* (H1)(i)(ii)(iii) *hold, then there exists* $c_6 > 0$ *such that* $\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_6[1 + |y|^p]$ *for all* $y \in \mathbb{R}^N$.

The following examples show that the framework provided by hypotheses (H1) is broad.

Example 2.4. The following maps satisfy hypotheses (H1) (see [9]):

(a) $a(y) = |y|^{p-2}y$ with $1 < p < +\infty$. This map corresponds to the p-Laplacian differential operator.
(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < +\infty$. This map corresponds to the (p,q)-Laplacian differential operator, that is, the sum of a p-Laplacian and of a q-Laplacian. Such operators arise in many problems of mathematical physics and correspond to the so-called double phase equations. In this direction we mention the works of Cherfils-Il'yasov [2] (reaction-diffusion systems) and of Zhikov [16] (problems in elasticity theory).
(c) $a(y) = [1 + |y|^2]^{\frac{p-2}{2}}y$ with $1 < p < +\infty$. This map corresponds to the extended capillary differential operator.

(d) $a(y)=[1+\frac{1}{1+|y|^p}]|y|^{p-2}y$ with $1<p<+\infty$. This map corresponds to a differential operator which arises in problems of plasticity theory (see Fuchs-Li [3]).

Let $A: W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u),h\rangle=\int_\Omega (a(\nabla u),\nabla h)_{\mathbb{R}^N}\,dz \quad \text{for all } u,h\in W^{1,p}(\Omega).$$

From Gasiński-Papageorgiou [4] (Problem 2.192, p. 279), we have the following result.

Proposition 2.5. *If hypotheses* (H1) *hold, then the map* $A(\cdot)$ *is continuous, monotone* (*hence maximal monotone too*) *and of type* $(S)_+$.

The following strong comparison principle by Papageorgiou-Rădulescu-Repovš [13], will be useful in our analysis of problem (1.1).

Proposition 2.6. *If hypotheses* (H1) *hold,* $\widehat{\xi}\in L^\infty(\Omega)$ *with* $\widehat{\xi}(z)\geq 0$ *for a.a.* $z\in\Omega$, $h_1,h_2\in L^\infty(\Omega)$ *with* $0<\eta\leq h_2(z)-h_1(z)$ *for a.a.* $z\in\Omega$ *and* $u,v\in C^{1,\alpha}(\overline{\Omega})$ *with* $\alpha\in(0,1]$, $v\leq u$ *and*

$$-\operatorname{div} a(\nabla v(z))+\widehat{\xi}(z)|v(z)|^{p-2}v(z)=h_1(z) \quad \text{for a.a. } z\in\Omega,$$
$$-\operatorname{div} a(\nabla u(z))+\widehat{\xi}(z)|u(z)|^{p-2}u(z)=h_2(z) \quad \text{for a.a. } z\in\Omega,$$

then $u-v\in\operatorname{int}\widehat{C}_+$.

Next we introduce hypotheses on the potential function $\xi(z)$ and on the reaction term $f(z,x)$.

(H2) $\xi\in L^\infty(\Omega)$.

(H3) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and
  (i) $\eta(z)x^{p-1}\leq f(z,x)\leq\alpha(z)[1+x^{r-1}]$ for a.a. $z\in\Omega$, all $x\geq 0$, with $\eta,\alpha\in L^\infty(\Omega)$, $\xi^+\preceq\eta$ and $p<r<p^*=\begin{cases}\frac{Np}{N-p} & \text{if } N>p\\ +\infty & \text{if } p\geq N\end{cases}$;
  (ii) if $F(z,x)=\int_0^x f(z,s)ds$, then $\lim_{x\to+\infty}\frac{F(z,x)}{x^p}=+\infty$ uniformly for a.a. $z\in\Omega$;
  (iii) if $d(z,x)=f(z,x)x-pF(z,x)$, then there exists $e\in L^1(\Omega)$ such that $d(z,x)\leq d(z,y)+e(z)$ for a.a. $z\in\Omega$, all $0\leq x\leq y$ and $d(z,x)\to+\infty$ for a.a. $z\in\Omega$ as $x\to+\infty$;
  (iv) with $q\in(1,p)$ as in hypothesis (H1)(iv), we have $\lim_{x\to 0^+}\frac{f(z,x)}{x^{q-1}}=+\infty$ uniformly for a.a. $z\in\Omega$;
  (v) for each $\rho>0$ there exists $\widehat{\xi}_\rho>0$ such that for a.a. $z\in\Omega$ the function $x\to f(z,x)+\widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 2.7. Since initially (Section 3) our aim is to produce positive solutions for problem (1.1) and all the above conditions of $f(z,\cdot)$ concern the positive semiaxis $\mathbb{R}_+=[0,+\infty)$, without any loss of generality we may assume that

$$f(z,x)=0 \quad \text{for a.a. } z\in\Omega, \text{ all } x\leq 0. \tag{2.2}$$

Hypotheses (H3)(ii)(iii) imply that

$$\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}=+\infty \quad \text{uniformly for a.a. } z\in\Omega.$$

Therefore the reaction $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. Usually such problems are treated using the AR-condition which leads to an easy verification of the C-condition for the energy (Euler) functional of the problem. We recall that the AR-condition (unilateral version due to (2.2)), says that there exist $\vartheta > p$ and $M > 0$ such that

$$0 < \vartheta F(z,x) \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \tag{2.3}$$

$$0 < \operatorname{ess\,inf}_{\Omega} F(\cdot, M). \tag{2.4}$$

Integrating (2.3) and using (2.4), we obtain the following weaker condition

$$\begin{gathered} c_7 x^{\vartheta} \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \text{ some } c_7 > 0, \\ \Rightarrow\ c_7 x^{\vartheta-1} \le f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \text{ (see (2.3))}. \end{gathered} \tag{2.5}$$

From (2.5) we see that the AR-condition implies that $f(z,\cdot)$ has at least $(\vartheta-1)$-polynomial growth. In this work, we replace the AR-condition by the quasimonotonicity condition on $d(z,\cdot)$ stated in hypothesis (H3)(iii). This hypothesis is a slight generalization of a condition used by Li-Yang [6]. This condition is satisfied if there exists $M > 0$ such that for a.a. $z \in \Omega$ the function $x \to \frac{f(z,x)}{x^{p-1}}$ is nondecreasing on $[M, +\infty)$. Hence from (2.5) we infer that the quasimonotonicity condition on $d(z,\cdot)$ is more general than the AR-condition. It permits the consideration of superlinear nonlinearities with "slower" growth near $+\infty$. To see this, consider the following function

$$f(z,x) = \begin{cases} \eta(z)(x^+)^{q-1} & \text{if } x \le 1, \\ x^{p-1}\ln x + \eta(z)x^{\tau-1} & \text{if } 1 < x, \end{cases}$$

with $\eta \in L^\infty(\Omega)$, $\xi^+ \preceq \eta$ and $1 < \tau, q < p$. This function satisfies hypotheses (H3) but fails to satisfy the AR-condition (see (2.3), (2.4)).

In what follows $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\gamma(u) = \int_\Omega pG(\nabla u)dz + \int_\Omega \xi(z)|u|^p dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

3. Positive Solutions

In this section we study the dependence on the parameter $\lambda > 0$ of the set of positive solutions. So, we introduce the following two sets:

$$\mathcal{L} = \{\lambda > 0 : \text{ problem (1.1) has a positive solution}\},$$

$$S_\lambda = \text{ set of positive solutions of (1.1)}.$$

We start with the following result about these two sets.

Proposition 3.1. *If hypotheses* (H1)–(H3) *hold, then* $\mathcal{L} \neq \emptyset$ *and, for every* $\lambda \in \mathcal{L}$, $\emptyset \neq S_\lambda \subseteq D_+$.

Proof. Let $\mu > \|\xi\|_\infty$ (see hypothesis (H2)) and consider the following auxiliary Neumann problem

$$\begin{gathered} -\operatorname{div} a(\nabla u(z)) + [\xi(z)+\mu]u(z)^{p-1} = 1 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

Using Lemma 2.2, Proposition 2.5 and the fact that $\mu > \|\xi\|_\infty$, we see that the left hand side of (3.1) is continuous, strictly monotone and coercive. Therefore problem (3.1) admits a unique solution $\overline{u} \in W^{1,p}(\Omega)$, $\overline{u} \neq 0$. Moreover, the nonlinear regularity theory (see [7]) and the nonlinear maximum principle (see [15]), imply that $\overline{u} \in D_+$. We set

$$\begin{gathered} M_0 = \|N_f(\overline{u})\|_\infty \quad \text{(see hypothesis } (H3)\,(i)), \\ m_0 = \min_{\overline{\Omega}} \overline{u} > 0 \quad \text{(recall that } \overline{u} \in D_+), \\ \overline{\lambda} = \mu + \frac{M_0}{m_0^{p-1}} > 0. \end{gathered}$$

We have

$$\begin{aligned} &- \operatorname{div} a(\nabla \overline{u}(z)) + [\xi(z) + \overline{\lambda}]\overline{u}(z)^{p-1} \\ &= - \operatorname{div} a(\nabla \overline{u}(z)) + [\xi(z) + \mu]\overline{u}(z)^{p-1} + M_0 \big(\frac{\overline{u}(z)}{m_0}\big)^{p-1} \\ &\geq 1 + M_0 \quad \text{(see (3.1) and recall that } \overline{u}(z) \geq m_0 \text{ for all } z \in \overline{\Omega}) \\ &> f(z, \overline{u}(z)) \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{3.2}$$

We introduce the Carathéodory function

$$\widehat{f}(z,x) = \begin{cases} f(z,x^+) & \text{if } x \leq \overline{u}(z), \\ f(z,\overline{u}(z)) & \text{if } \overline{u}(z) < x. \end{cases} \tag{3.3}$$

(see (2.2)).

We set $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)ds$ and consider the C^1-functional $\widehat{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{p}\gamma(u) + \frac{\overline{\lambda}}{p}\|u\|_p^p - \int_\Omega \widehat{F}(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.3) and since $\overline{\lambda} > \mu > \|\xi\|_\infty$, we see that $\widehat{\varphi}(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we show that $\widehat{\varphi}(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}(u_0) = \inf[\widehat{\varphi}(u) : u \in W^{1,p}(\Omega)]. \tag{3.4}$$

Hypotheses (H1)(iv) and (H3)(iv) imply that given $\eta > c_0^* > c^*$, we can find $\delta \in (0, m_0]$ such that

$$\begin{gathered} G(y) \leq \frac{c_0^*}{q}|y|^q \quad \text{for all } |y| \leq \delta, \\ F(z,x) \geq \frac{\eta}{q}x^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \end{gathered} \tag{3.5}$$

Given $u \in D_+$, we choose $t \in (0,1)$ small such that

$$t|\nabla u(z)| \leq \delta \quad \text{and} \quad tu(z) \leq \delta \quad \text{for all } z \in \overline{\Omega}. \tag{3.6}$$

Using (3.5) and (3.6), we have

$$\begin{aligned} \widehat{\varphi}(tu) &\leq \frac{c_0^*}{q}t^q\|\nabla u\|_q^q + \frac{t^p}{p}[\|\xi\|_\infty + \overline{\lambda}]\|u\|_p^p - \frac{\eta}{q}t^q\|u\|_q^q \\ &= t^q[c_8 - \eta c_9] + t^p c_{10} \quad \text{for some } c_8, c_9, c_{10} > 0. \end{aligned}$$

Since $\eta > c_0^*$ is arbitrary, by choosing $\eta > \frac{c_8}{c_9}$ we obtain

$$\widehat{\varphi}(tu) \leq c_{10}t^p - c_{11}t^q \quad \text{for all } t > 0, \text{ and some } c_{11} > 0.$$

Recall that $q < p$. So, by choosing $t \in (0,1)$ even smaller if necessary, we have $\widehat{\varphi}(tu) < 0$ implies $\widehat{\varphi}(u_0) < 0 = \widehat{\varphi}(0)$ (see (3.4)) which in turn implies $u_0 \neq 0$.

From (3.4) we have that $\widehat{\varphi}'(u_0) = 0$ implies

$$\langle A(u_0), h\rangle + \int_\Omega [\xi(z) + \overline{\lambda}]|u_0|^{p-2}u_0 h dz = \int_\Omega \widehat{f}(z, u_0) h dz \quad \forall h \in W^{1,p}(\Omega). \tag{3.7}$$

In (3.7) first we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\frac{c_1}{p-1}\|\nabla u_0^-\|_p^p + \int_\Omega [\xi(z) + \overline{\lambda}](u_0^-)^p dz \le 0 \quad \text{(see Lemma 2.2 and (3.3))},\\
&\Rightarrow\ c_{12}\|u_0^-\|^p \le 0 \quad \text{for some } c_{12} > 0 \text{ (recall } \overline{\lambda} > \mu > \|\xi\|_\infty),\\
&\Rightarrow\ u_0 \ge 0,\ u_0 \ne 0.
\end{aligned}$$

Next in (3.7) we choose $h = (u_0 - \overline{u})^+ \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned}
&\langle A(u_0), (u_0 - \overline{u})^+\rangle + \int_\Omega [\xi(z) + \overline{\lambda}]u_0^{p-1}(u_0 - \overline{u})^+ dz\\
&= \int_\Omega f(z, \overline{u})(u_0 - \overline{u})^+ dz \quad \text{(see (3.3))},\\
&\le \langle A(\overline{u}), (u_0 - \overline{u})^+\rangle + \int_\Omega [\xi(z) + \overline{\lambda}]\overline{u}^{p-1}(u_0 - \overline{u})^+ dz \quad \text{(see (3.2))},
\end{aligned}$$

which implies $u_0 \le \overline{u}$ (see Proposition 2.5 and recall that $\overline{\lambda} > \mu > \|\xi\|_\infty$). So, we have proved that

$$u_0 \in [0, \overline{u}], \quad u_0 \ne 0. \tag{3.8}$$

From (3.3), (3.7) and (3.8) it follows that

$$\begin{gathered}
-\operatorname{div} a(\nabla u_0(z)) + [\xi(z) + \overline{\lambda}]u_0(z)^{p-1} = f(z, u_0(z)) \quad \text{for a.a. } z \in \Omega,\\
\frac{\partial u_0}{\partial n} = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.9}$$

From (3.9) and [10, Proposition 2.10], we have $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [7] implies that $u_0 \in C_+ \setminus \{0\}$. From (3.9) and hypothesis (H3)(i), we have that

$$\operatorname{div} a(\nabla u_0(z)) \le [\|\eta\|_\infty + \|\xi\|_\infty + \overline{\lambda}]u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{3.10}$$

The nonlinear maximum principle of Pucci-Serrin [15, pp. 111, 120] and (3.10) imply that $u_0 \in D_+$. Therefore we conclude that $\overline{\lambda} \in \mathcal{L} \ne \emptyset$ and for all $\lambda \in \mathcal{L}$, $\emptyset \ne S_\lambda \subseteq D_+$. □

In the next proposition, we prove a structural property of the set $\mathcal{L}$, namely we show that $\mathcal{L}$ is an upper half line. In addition we establish a kind of monotonicity property for the solution multifunction $\lambda \to S_\lambda$.

Proposition 3.2. *If hypotheses* (H1)–(H3) *hold,* $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq D_+$ *and* $\eta > \lambda$, *then* $\eta \in \mathcal{L}$ *and there exists* $u_\eta \in S_\eta \subseteq D_+$ *such that* $u_\lambda - u_\eta \in \operatorname{int}\widehat{C}_+$.

Proof. We have

$$\begin{aligned}
&-\operatorname{div} a(\nabla u_\lambda(z)) + [\xi(z) + \lambda]u_\lambda(z)^{p-1}\\
&= f(z, u_\lambda(z))\\
&< -\operatorname{div} a(\nabla u_\lambda(z)) + [\xi(z) + \eta]u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega \text{ (since } \eta > \lambda).
\end{aligned}$$

We introduce the Carathéodory function

$$k(z,x)=\begin{cases} f(z,x^+)+\widehat{\mu}(x^+)^{p-1} & \text{if } x\le u_\lambda(z),\\ f(z,u_\lambda(z))+\widehat{\mu}u_\lambda(z)^{p-1} & \text{if } u_\lambda(z)<x,\end{cases} \tag{3.11}$$

where $\widehat{\mu}\ge\|\xi\|_\infty$. We set $K(z,x)=\int_0^x k(z,s)ds$ and consider the C^1-functional $\widehat{\psi}_\eta: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\psi}_\eta(u)=\frac{1}{p}\gamma(u)+\frac{\eta+\widehat{\mu}}{p}\|u\|_p^p-\int_\Omega K(z,u)dz \quad \text{for all } u\in W^{1,p}(\Omega).$$

From (3.11) and since $\widehat{\mu}\ge\|\xi\|_\infty$ and $\eta>\lambda>0$, we see that $\widehat{\psi}_\eta(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_\eta\in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\eta(u_\eta)=\inf[\widehat{\psi}_\eta(u): u\in W^{1,p}(\Omega)]. \tag{3.12}$$

As before (see the proof of Proposition 3.1) on account of hypotheses (H1)(iv) and (H3)(iv) we have

$$\widehat{\psi}_\eta(u_\eta)<0=\widehat{\psi}_\eta(0) \quad\Rightarrow\quad u_\eta\ne 0.$$

From (3.12) we have that $\widehat{\psi}_\eta'(u_\eta)=0$ implies

$$\langle A(u_\eta),h\rangle+\int_\Omega[\xi(z)+\eta+\mu]|u_\eta|^{p-2}u_\eta h dz=\int_\Omega k(z,u_\eta)h dz \quad \forall h\in W^{1,p}(\Omega). \tag{3.13}$$

In (3.13) we choose $h=-u_\eta^-\in W^{1,p}(\Omega)$ and $h=(u_\eta-u_\lambda)^+\in W^{1,p}(\Omega)$ and as in the proof of Proposition 3.1, we show that

$$u_\eta\in[0,u_\lambda], \quad u_\eta\ne 0. \tag{3.14}$$

From (3.11), (3.13) and (3.14) we infer that

$$\eta\in\mathcal{L} \text{ and } u_\eta\in S_\eta\subseteq D_+ \quad \text{(see Proposition 3.1)}, \quad u_\eta\le u_\lambda.$$

Let $\rho=\|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho>0$ be as postulated by hypothesis (H3)(v). We have

$$\begin{aligned}
&-\operatorname{div} a(\nabla u_\lambda)+[\xi(z)+\eta+\widehat{\xi}_\rho]u_\lambda^{p-1}\\
&=f(z,u_\lambda)+\widehat{\xi}_\rho u_\lambda^{p-1}+(\eta-\lambda)u_\lambda^{p-1} \quad (\text{since } u_\lambda\in S_\lambda)\\
&\ge f(z,u_\eta)+\widehat{\xi}_\rho u_\eta^{p-1}+(\eta-\lambda)u_\eta^{p-1} \quad (\text{see (H3)(v) and recall } u_\eta\le u_\lambda)\\
&> -\operatorname{div} a(\nabla u_\eta)+[\xi(z)+\eta+\widehat{\xi}_\rho]u_\eta^{p-1} \quad \text{for a.a. } z\in\Omega \ (\text{since } u_\eta\in S_\eta).
\end{aligned} \tag{3.15}$$

Let $m_\eta=\min_{\overline{\Omega}} u_\eta>0$ (recall that $u_\eta\in D_+$). We have

$$(\eta-\lambda)u_\eta^{p-1}\ge(\eta-\lambda)m_\eta^{p-1}>0 \quad (\text{since } \eta>\lambda).$$

Then from (3.15) and Proposition 2.6, it follows that $u_\lambda-u_\eta\in\operatorname{int}\widehat{C}_+$. $\square$

Let $\lambda_*=\inf\mathcal{L}$.

Proposition 3.3. *If hypotheses* (H1)–(H3) *hold, then* $\lambda_*>0$.

Proof. Let $\varphi_\lambda: W^{1,p}(\Omega)\to\mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi_\lambda(u)=\frac{1}{p}\gamma(u)+\frac{\lambda}{p}\|u\|_p^p-\int_\Omega F(z,u)dz \quad \text{for all } u\in W^{1,p}(\Omega).$$

Arguing by contradiction, suppose that $\lambda_* = 0$. Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{L}$ such that $\lambda_n \downarrow 0$. We fix $\lambda > \lambda_1$. For every $n \in \mathbb{N}$ and $\widehat{u}_n \in S_{\lambda_n} \subseteq D_+$, on account of Proposition 3.2 and its proof we can find $u_\lambda^n \in S_\lambda \subseteq D_+$ such that $\varphi_\lambda(u_\lambda^n) < 0$, $u_\lambda^n \leq \widehat{u}_n$. We have

$$-\operatorname{div} a(\nabla u_\lambda^{n+1}) + [\xi(z) + \lambda_n] u_\lambda^{n+1} \leq f(z, u_\lambda^{n+1}) \quad \text{for a.a. } z \in \Omega, \tag{3.16}$$

$$-\operatorname{div} a(\nabla \widehat{u}_{n+1}) + [\xi(z) + \lambda_n] \widehat{u}_{n+1}^{p-1} \geq f(z, \widehat{u}_{n+1}) \quad \text{for a.a. } z \in \Omega. \tag{3.17}$$

With $\widehat{\mu} \geq \|\xi\|_\infty$ we introduce the Carathéodory function

$$k_n(z,x) = \begin{cases} f(z, u_\lambda^{n+1}(z)) + \widehat{\mu} u_\lambda^{n+1}(z)^{p-1} & \text{if } x < u_\lambda^{n+1}(z), \\ f(z,x) + \widehat{\mu} x^{p-1} & \text{if } u_\lambda^{n+1}(z) \leq x \leq \widehat{u}_{n+1}(z), \\ f(z, \widehat{u}_{n+1}(z)) + \widehat{\mu} \widehat{u}_{n+1}(z)^{p-1} & \text{if } \widehat{u}_{n+1} < x. \end{cases} \tag{3.18}$$

We set $K_n(z,x) = \int_0^x k_n(z,s)ds$ and consider the C^1-functional $\widetilde{\varphi}_{\lambda_n} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}_{\lambda_n}(u) = \frac{1}{p}\gamma(u) + \frac{\lambda_n + \widehat{\mu}}{p}\|u\|_p^p - \int_\Omega K_n(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega),$$

with $\widehat{\mu} \geq \|\xi\|_\infty$. Evidently $\widetilde{\varphi}_{\lambda_n}(\cdot)$ is coercive (see (3.18)) and sequentially weakly lower semicontinuous and so we can find $u_n \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widetilde{\varphi}_{\lambda_n}(u_n) = \inf[\widetilde{\varphi}_{\lambda_n}(u) : u \in W^{1,p}(\Omega)], \\ &\Rightarrow\ \widetilde{\varphi}'_{\lambda_n}(u_n) = 0, \\ &\Rightarrow\ \langle A(u_n), h\rangle + \int_\Omega [\xi(z) + \lambda_n + \widehat{\mu}]|u_n|^{p-2}u_n h dz = \int_\Omega k_n(z,u_n)h dz \end{aligned} \tag{3.19}$$

for all $h \in W^{1,p}(\Omega)$. Choosing $h = (u_\lambda^n - u_n)^+ \in W^{1,p}(\Omega)$ and $h = (u_n - \widehat{u}_{n+1})^+ \in W^{1,p}(\Omega)$ and using (3.16), (3.17) and (3.18), we show (see also the proof of Proposition 3.1) that

$$u_n \in [u_\lambda^n, \widehat{u}_{n+1}] \cap D_+ \quad \text{(by the nonlinear regularity theory).}$$

We have

$$\begin{aligned} \widetilde{\varphi}_{\lambda_n}(u_\lambda^n) &\leq \frac{1}{p}\gamma(u_\lambda^n) + \frac{\lambda_n}{p}\|u_\lambda^n\|_p^p - \int_\Omega f(z,u_\lambda^n)u_\lambda^n dz \quad \text{(see (3.18))} \\ &\leq \frac{1}{p}\gamma(u_\lambda^n) + \frac{\lambda}{p}\|u_\lambda^n\|_p^p - \int_\Omega pF(z,u_\lambda^n)dz + \|e\|_1 \quad \text{(see (H3)(iii))} \\ &\leq \frac{1}{p}\gamma(u_\lambda^n) + \frac{\lambda}{p}\|u_\lambda^n\|_p^p - \int_\Omega F(z,u_\lambda^n)dz + \|e\|_1 \quad \text{(since } F \geq 0) \\ &= \varphi_\lambda(u_\lambda^n) + \|e\|_1 \\ &< \|e\|_1, \end{aligned}$$

which implies $\widetilde{\varphi}_{\lambda_n}(u_n) < \|e\|_1$ for all $n \in \mathbb{N}$ (see (3.19)). This in turn implies $\varphi_{\lambda_n}(u_n) \leq c_{13}$ for some $c_{13} > 0$ and all $n \in \mathbb{N}$ (see (3.18)).

Therefore we have produced a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$u_n \in S_{\lambda_n} \subseteq D_+ \quad \text{and} \quad \varphi_{\lambda_n}(u_n) \leq c_{13} \quad \text{for all } n \in \mathbb{N}. \tag{3.20}$$

From (3.20) we have

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega [\xi(z) + \lambda_n] u_n^{p-1} h dz \\ &= \int_\Omega f(z, u_n) h dz \quad \text{for all } h \in W^{1,p}(\Omega), \text{ and all } n \in \mathbb{N}, \end{aligned} \tag{3.21}$$

$$\gamma(u_n) + \lambda_n \|u_n\|_p^p - \int_\Omega pF(z, u_n) dz \le pc_{13} \quad \text{for all } n \in \mathbb{N}. \tag{3.22}$$

In (3.21) we choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$-\gamma(u_n) - \lambda_n \|u_n\|_p^p + \int_\Omega f(z, u_n) u_n dz = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.23}$$

We add (3.22) and (3.23) to obtain

$$\int_\Omega [f(z, u_n) u_n - pF(z, u_n)] dz = \int_\Omega d(z, u_n) dz \le pc_{13} \quad \text{for all } n \in \mathbb{N}. \tag{3.24}$$

We will show that $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded. Arguing indirectly, suppose that at least for a subsequence we have

$$\|u_n\| \to +\infty. \tag{3.25}$$

We set $y_n = u_n/\|u_n\|$ for $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^r(\Omega),\ y \ge 0.$$

First, we assume that $y \ne 0$. Let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$. Then $|\Omega_+|_N > 0$ (recall that $y \ge 0$). From (3.25) it follows that $u_n(z) \to +\infty$ for all $z \in \Omega_+$. So, we have $d(z, u_n(z)) \to +\infty$ for a.a. $z \in \Omega$ (see hypothesis (H3)(iii)). This implies

$$\int_{\Omega_+} d(z, u_n) dz \to +\infty \quad \text{(by Fatou's lemma)}. \tag{3.26}$$

From hypothesis (H3)(iii) we have

$$d(z, x) \ge -e(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.27}$$

Then we have

$$\begin{aligned} \int_\Omega d(z, u_n) dz &= \int_{\Omega_+} d(z, u_n) dz + \int_{\Omega\setminus\Omega_+} d(z, u_n) dz \\ &\ge \int_{\Omega_+} d(z, u_n) dz - \|e\|_1 \quad \text{for all } n \in \mathbb{N} \text{ (see (3.27))}, \end{aligned}$$

which implies $\int_\Omega d(z, u_n) dz \to +\infty$ as $n \to +\infty$ (see (3.26)). This contradicts (3.24).

Now we assume that $y = 0$. Let $\tau > 0$ and set $v_n = (p\tau)^{1/p} y_n \in W^{1,p}(\Omega)$ for all $n \in \mathbb{N}$. Let $\overline{\gamma} : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\overline{\gamma}(u) = \frac{c_1}{p-1} \|\nabla u\|_p^p + \int_\Omega \xi(z) |u|^p dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We introduce the C^1-functionals $\overline{\varphi}_{\lambda_n} : W^{1,p}(\Omega) \to \mathbb{R}$, $n \in \mathbb{N}$, defined by

$$\overline{\varphi}_{\lambda_n}(u) = \frac{1}{p} \overline{\gamma}(u) + \frac{\lambda_n}{p} \|u\|_p^p - \int_\Omega F(z, u) dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Let $t_n \in [0,1]$ be such that

$$\overline{\varphi}_{\lambda_n}(t_n u_n) = \max[\varphi_{\lambda_n}(tu_n) : 0 \le t \le 1] \quad \text{for all } n \in \mathbb{N}. \tag{3.28}$$

On account of (3.25), we see that we can find $n_0 \in \mathbb{N}$ such that

$$(p\tau)^{1/p} \frac{1}{\|u_n\|} \le 1 \quad \text{for all } n \ge n_0. \tag{3.29}$$

From (3.28) and (3.29) it follows that

$$\begin{aligned}
&\overline{\varphi}_{\lambda_n}(t_n u_n) \ge \overline{\varphi}_{\lambda_n}(v_n) \\
&\quad = \tau[\overline{\gamma}(y_n) + [\lambda_n + \widehat{\mu}]\|y_n\|_p^p] - \int_\Omega [F(z, v_n) + \frac{\widehat{\mu}}{p} v_n^p] dz \\
&\qquad \text{for all } n \ge n_0, \text{ with } \widehat{\mu} \ge \|\xi\|_\infty \\
&\quad \ge \tau c_{14} - \int_\Omega [F(z, v_n) + \frac{\widehat{\mu}}{p} v_n^p] dz \\
&\qquad \text{for some } c_{14} > 0, \text{ all } n \ge n_0 \text{ (since } \widehat{\mu} \ge \|\xi\|_\infty).
\end{aligned} \tag{3.30}$$

Evidently $\int_\Omega [F(z, v_n) + \frac{\widehat{\mu}}{p} v_n^p] dz \to 0$ as $n \to +\infty$ (recall $y = 0$). Hence from (3.30) it follows that

$$\overline{\varphi}_{\lambda_n}(t_n u_n) \ge \frac{\tau}{2} c_{14} \quad \text{for all } n \ge n_1 \ge n_0.$$

Since $\tau > 0$ is arbitrary, we infer that

$$\overline{\varphi}_{\lambda_n}(t_n u_n) \to +\infty \quad \text{as } n \to +\infty. \tag{3.31}$$

We have

$$\overline{\varphi}_{\lambda_n}(0) = 0 \text{ and } \overline{\varphi}_{\lambda_n}(u_n) \le \varphi_{\lambda_n}(u_n) \le c_{13} \quad \text{for all } n \in \mathbb{N} \text{ (see (3.20))}.$$

Then on account of (3.31), we have

$$t_n \in (0,1) \quad \text{for all } n \ge n_2. \tag{3.32}$$

From (3.28) and (3.32) it follows that

$$\begin{aligned}
&\frac{d}{dt} \overline{\varphi}_{\lambda_n}(tu_n)\big|_{t=t_n} = 0 \quad \text{for all } n \ge n_2, \\
&\Rightarrow \langle \overline{\varphi}'_{\lambda_n}(t_n u_n), t_n u_n \rangle = 0 \quad \text{for all } n \ge n_2 \text{ (by the chain rule)}, \\
&\Rightarrow \frac{c_1}{p-1} \|\nabla(t_n u_n)\|_p^p + \int_\Omega [\xi(z) + \lambda_n](t_n u_n)^p dz = \int_\Omega f(z, t_n u_n)(t_n u_n) dz \\
&\qquad \text{for all } n \ge n_2, \\
&\Rightarrow p\overline{\varphi}_{\lambda_n}(t_n u_n) \le \int_\Omega d(z, t_n u_n) dz \le \int_\Omega d(z, u_n) dz + \|e\|_1 \\
&\qquad \text{for all } n \ge n_2 \text{ (see hypothesis } (H3)\,(iii) \text{ and (3.32))}, \\
&\Rightarrow p\overline{\varphi}_{\lambda_n}(t_n u_n) \le pc_{13} + \|e\|_1 \quad \text{for all } n \ge n_2 \text{ (see (3.24))},
\end{aligned}$$

which contradicts (3.31).

So, we have that $\{u_n\}_{n\ge 1} \subseteq W^{1,p}(\Omega)$ is bounded. We may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^r(\Omega). \tag{3.33}$$

In (3.21) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.33). Then we obtain

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

which implies

$$u_n \to u_* \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.5).} \tag{3.34}$$

So, if in (3.21) we pass to the limit as $n \to +\infty$ and use (3.34) and the fact that $\lambda_n \downarrow 0$ (recall we have assumed that $\lambda_* = 0$), we obtain

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h dz = \int_\Omega f(z, u_*) h dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.35}$$

In (3.35) we choose $h \equiv 1$. Then

$$\int_\Omega \xi(z) u_*^{p-1} dz = \int_\Omega f(z, u_*) dz \geq \int_\Omega \eta(z) u_*^{p-1} dz \quad \text{(see (H3)(i))},$$

which implies

$$\int_\Omega [\eta(z) - \xi(z)] u_*^{p-1} dz \leq 0. \tag{3.36}$$

Note that hypotheses (H3)(i),(iv) imply that we can find $c_{15} > 0$ such that

$$f(z, x) \geq x^{q-1} - c_{15} x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

Evidently we can always assume that $c_{15} > \|\xi\|_\infty$. We consider the following auxiliary Neumann problem

$$\begin{gathered} -\operatorname{div} a(\nabla u(z)) + \xi(z) u(z)^{p-1} = u(z)^{q-1} - c_{15} u(z)^{r-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0, \; u > 0. \end{gathered}$$

From [9, Proposition 3.5] we know that this problem has a unique positive solution $\widetilde{u} \in D_+$. Let $\lambda \in \mathcal{L}$ and $u \in S_\lambda \subseteq D_+$. We introduce the Carathéodory function

$$\beta(z, x) = \begin{cases} (x^+)^{q-1} - c_{15}(x^+)^{r-1} + \widehat{\mu}(x^+)^{p-1} & \text{if } x \leq u(z), \\ u(z)^{q-1} - c_{15} u(z)^{r-1} + \widehat{\mu} u(z)^{p-1} & \text{if } u(z) < x, \end{cases} \tag{3.37}$$

with $\widehat{\mu} \geq \|\xi\|_\infty$.

We set $B(z, x) = \int_0^x \beta(z, s) ds$ and consider the C^1-functional $\widetilde{\sigma}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\sigma}_\lambda(u) = \frac{1}{p}\gamma(u) + \frac{\lambda + \widehat{\mu}}{p}\|u\|_p^p - \int_\Omega B(z, u) dz \quad \text{for } u \in W^{1,p}(\Omega).$$

The direct method of calculus of variations gives $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\widetilde{\sigma}_\lambda(\widetilde{u}_0) = \inf[\widetilde{\sigma}_\lambda(u) : u \in W^{1,p}(\Omega)] < 0 = \widetilde{\sigma}_\lambda(0) \quad \text{(since } q < p).$$

So, $\widetilde{u}_0 \neq 0$ and $\widetilde{u}_0 \in K_{\widetilde{\sigma}_\lambda} \subseteq [0, u] \cap C_+$ (see (3.37) and use the nonlinear regularity theory). Hence from (3.37) we infer that $\widetilde{u}_0 = \widetilde{u} \in D_+$ and so $\widetilde{u} \leq u$ for all $u \in S_\lambda$, all $\lambda \in \mathcal{L}$. It follows that

$$\widetilde{u} \leq u_* \quad \Rightarrow \quad \int_\Omega [\eta(z) - \xi(z)] u_*^{p-1} dz > 0 \quad \text{(since } \xi \prec \eta)$$

which contradicts (3.36). So, we conclude that $\lambda_* > 0$. ∎

Proposition 3.4. *If hypotheses* (H1)–(H3) *hold and* $\lambda \in (\lambda_*, +\infty)$*, then problem* (1.1) *admits at least two positive solutions* $u_0, \widehat{u} \in S_\lambda \subseteq D_+$.

Proof. Let $\lambda_* < \theta < \lambda < \eta$. By Proposition 3.2, we can find $u_\theta \in S_\theta \subseteq D_+$, $u_0 \in S_\lambda \subseteq D_+$ and $u_\eta \in S_\eta \subseteq D_+$ such that

$$\begin{aligned} &u_\theta - u_0 \in \operatorname{int} \widehat{C}_+ \text{ and } u_0 - u_\eta \in \operatorname{int} \widehat{C}_+, \\ &\Rightarrow\ u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[u_\eta, u_\theta]. \end{aligned} \tag{3.38}$$

We introduce the Carathéodory function

$$j(z,x) = \begin{cases} f(z,u_\eta(z)) + \widehat{\mu} u_\eta(z)^{p-1} & \text{if } x \le u_\eta(z), \\ f(z,x) + \widehat{\mu} x^{p-1} & \text{if } u_\eta(z) < x, \end{cases} \tag{3.39}$$

with $\widehat{\mu} \ge \|\xi\|_\infty$. We set $J(z,x) = \int_0^x j(z,s)ds$ and consider the C^1-functional $\widehat{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}\gamma(u) + \frac{\lambda + \widehat{\mu}}{p}\|u\|_p^p - \int_\Omega J(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

In addition, we introduce the following truncation of $j(z,\cdot)$,

$$\widetilde{j}(z,x) = \begin{cases} j(z,x) & \text{if } x \le u_\theta(z), \\ j(z,u_\theta(z)) & \text{if } u_\theta(z) < x. \end{cases} \tag{3.40}$$

This is a Carathéodory function. We set $\widetilde{J}(z,x) = \int_0^x \widetilde{j}(z,s)ds$ and consider the C^1-functional $\widetilde{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\psi}_\lambda(u) = \frac{1}{p}\gamma(u) + \frac{\lambda + \widehat{\mu}}{p}\|u\|_p^p - \int_\Omega \widetilde{J}(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.39), (3.40) and the nonlinear regularity theory of Lieberman [7], we have

$$K_{\widehat{\psi}_\lambda} \subseteq [u_\eta) \cap D_+ \text{ and } K_{\widetilde{\psi}_\lambda} \subseteq [u_\eta, u_\theta] \cap D_+. \tag{3.41}$$

From (3.39), (3.40), (3.41), we see that we may assume that

$$K_{\widetilde{\psi}_\lambda} = \{u_0\}. \tag{3.42}$$

Otherwise we already have a second positive solution of (1.1), distinct from u_0 and the proof is complete.

Clearly $\widetilde{\psi}_\lambda(\cdot)$ is coercive (see (3.40)) and sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widetilde{\psi}_\lambda(\widetilde{u}_0) = \inf[\widetilde{\psi}_\lambda(u) : u \in W^{1,p}(\Omega)] \\ &\Rightarrow\ \widetilde{u}_0 \in K_{\widetilde{\psi}_\lambda} \\ &\Rightarrow\ \widetilde{u}_0 = u_0 \quad \text{(see (3.42))}. \end{aligned}$$

From (3.39) and (3.40) we see that

$$\widehat{\psi}_\lambda\big|_{[u_\eta,u_\theta]} = \widetilde{\psi}_\lambda\big|_{[u_\eta,u_\theta]}.$$

Then from (3.38) it follows that

$$\begin{aligned} &u_0 \in D_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widehat{\psi}_\lambda, \\ &\Rightarrow\ u_0 \in D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}_\lambda \end{aligned} \tag{3.43}$$

(see Papageorgiou-Rădulescu [10]).

From (3.41) we can assume that

$$K_{\widehat{\psi}_\lambda} \text{ is finite.} \tag{3.44}$$

Using (3.43), (3.44) and [14, Theorem 5.7.6, p. 367,], we see that we can find $\rho \in (0,1)$ small such that

$$\widehat{\psi}_\lambda(u_0) < \inf[\widehat{\psi}_\lambda(u) : \|u - u_0\| = \rho] = \widehat{m}_\lambda. \tag{3.45}$$

By (H3)(ii), for $u \in D_+$ we have

$$\widehat{\psi}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.46}$$

Moreover, reasoning as in the proof of Proposition 3.3 (see the part of the proof from (3.20) up to (3.33)), we can show that

$$\widehat{\psi}_\lambda(\cdot) \text{ satisfies the } C\text{-condition.} \tag{3.47}$$

Then (3.45), (3.46), (3.47) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{gathered}
\widehat{u} \in K_{\widehat{\psi}_\lambda} \subseteq [u_\eta) \cap D_+ \text{ (see (3.41)) and } \widehat{m}_\lambda \le \widehat{\psi}_\lambda(\widehat{u}) \text{ (see (3.45)),} \\
\Rightarrow \ \widehat{u} \neq u_0 \text{ (see (3.45)) and } \widehat{u} \in S_\lambda \subseteq D_+ \text{ (see (3.39)).}
\end{gathered}$$

□

Proposition 3.5. *If hypotheses* (H1)–(H3) *hold, then* $\lambda_* \in \mathcal{L}$.

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{L}$ such that $\lambda_n \downarrow \lambda_*$. From the proof of Proposition 3.3, we know that we can find $u_n \in S_{\lambda_n} \subseteq D_+$, $n \in \mathbb{N}$, such that

$$\widetilde{u} \le u_n \text{ and } \varphi_{\lambda_n}(u_n) \le c_{16} \text{ for some } c_{16} > 0, \text{ all } n \in \mathbb{N}.$$

As in the proof of Proposition 3.3, we can show that $u_n \to u_*$ in $W^{1,p}(\Omega)$. Then in the limit as $n \to +\infty$ we have

$$\widetilde{u} \le u_* \text{ and } \langle A(u_*), h\rangle + \int_\Omega [\xi(z) + \lambda_*] u_*^{p-1} h dz = \int_\Omega f(z, u_*) h dz,$$

for all $h \in W^{1,p}(\Omega)$, which implies $u_* \in S_{\lambda_*} \subseteq D_+$, and so $\lambda_* \in \mathcal{L}$. □

Note that Proposition 3.5 implies that $\mathcal{L} = [\lambda_*, +\infty)$. Summarizing our results on the dependence of the set of positive solutions of (1.1) on the parameter $\lambda > 0$, we can state the following bifurcation-type result for big values of $\lambda > 0$.

Theorem 3.6. *If hypotheses* (H1)–(H3) *hold, then there exists a critical parameter value* $\lambda_* > 0$ *such that*

- (a) *for all* $\lambda > \lambda_*$ *problem* (1.1) *has at least two positive solutions* $u_0, \widehat{u} \in D_+$, $u_0 \neq \widehat{u}$;
- (b) *for* $\lambda = \lambda_*$ *problem* (1.1) *has at least one positive solution* $u_* \in D_+$;
- (c) *for all* $\lambda \in (0, \lambda_*)$ *problem* (1.1) *has no positive solutions.*

Next we show that for every $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$, problem (1.1) has a smallest positive solution.

Proposition 3.7. *If hypotheses* (H1)–(H3) *hold and* $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$*, then problem* (1.1) *has a smallest positive solution* $\widehat{u}_\lambda \in D_+$.

Proof. From Papageorgiou-Rădulescu-Repovš [12] (see the proof of Proposition 7), we know that the solution set S_λ is downward directed (that is, if $u_1, u_2 \in S_\lambda$, then we can find $u \in S_\lambda$ such that $u \le u_1$, $u \le u_2$). Then invoking [5, Lemma 3.10, p. 178], we can find a decreasing sequence $\{u_n\}_{n\ge1} \subseteq S_\lambda$ such that

$$\inf S_\lambda = \inf_{n\ge1} u_n \text{ and } 0 \le u_n \le u_1 \text{ for all } n \in \mathbb{N}. \tag{3.48}$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega [\xi(z)+\lambda]u_n^{p-1} h dz = \int_\Omega f(z,u_n) h dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.49}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ in (3.49) and using (3.48), we infer that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Proposition 7 in Papageorgiou-Rădulescu [10] implies that we can find $c_{16} > 0$ such that

$$u_n \in L^\infty(\Omega) \text{ and } \|u_n\|_\infty \leq c_{16} \text{ for all } n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [7] implies that there exist $\alpha \in (0,1)$ and $c_{17} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_{17} \quad \text{for all } n \in \mathbb{N}. \tag{3.50}$$

From (3.50), the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, and the monotonicity of $\{u_n\}_{n\geq 1}$, we have

$$u_n \to \widehat{u}_\lambda \text{ in } C^1(\overline{\Omega}). \tag{3.51}$$

From the proof of Proposition 3.3, we know that

$$\begin{aligned} &\widetilde{u} \leq u_n \quad \text{for all } n \in \mathbb{N}, \\ &\Rightarrow \ \widetilde{u} \leq \widehat{u}_\lambda \quad \text{(see (3.51)), hence } \widehat{u}_\lambda \neq 0. \end{aligned}$$

If in (3.49) we pass to the limit as $n \to +\infty$ and use (3.51), we obtain

$$\begin{aligned} &\langle A(\widehat{u}_\lambda), h\rangle + \int_\Omega [\xi(z)+\lambda]\widehat{u}_\lambda^{p-1} h dz = \int_\Omega f(z,\widehat{u}_\lambda) h dz \quad \text{for all } h \in W^{1,p}(\Omega), \\ &\Rightarrow \ \widehat{u}_\lambda \in S_\lambda \subseteq D_+ \text{ and } \widehat{u}_\lambda = \inf S_\lambda. \end{aligned}$$

□

Next we examine the properties of the map $\mathcal{L} \ni \lambda \to \widehat{u}_\lambda \in C^1(\overline{\Omega})$.

Proposition 3.8. *If hypotheses* (H1)–(H3) *hold, then the map* $\sigma : \mathcal{L} \to C^1(\overline{\Omega})$ *defined by* $\sigma(\lambda) = \widehat{u}_\lambda$ *has the following properties:*

(a) $\sigma(\cdot)$ *is strictly decreasing in the sense that* $\lambda_* \leq \lambda < \eta$ *implies* $\widehat{u}_\lambda - \widehat{u}_\eta \in \operatorname{int} \widehat{C}_+$*;*
(b) $\sigma(\cdot)$ *is right continuous.*

Proof. (a) Let $\widehat{u}_\lambda \in D_+$ be the minimal positive solution of (1.1) ($\lambda \in \mathcal{L}$). According to Proposition 3.2, we can find $u_\eta \in S_\eta \subseteq D_+$ such that

$$\begin{aligned} &\widehat{u}_\lambda - u_\eta \in \operatorname{int} \widehat{C}_+, \\ &\Rightarrow \ \widehat{u}_\lambda - \widehat{u}_\eta \in \operatorname{int} \widehat{C}_+ \quad (\text{since } \widehat{u}_\eta \leq u_\eta) \\ &\Rightarrow \ \sigma(\cdot) \text{ is strictly decreasing.} \end{aligned}$$

(b) Let $\lambda_n \downarrow \lambda \in \mathcal{L}$. As in the proof of Proposition 3.3, we can find $u_n \in W^{1,p}(\Omega)$ such that

$$u_n \in S_{\lambda_n} \subseteq D_+ \text{ and } \varphi_{\lambda_n}(u_n) \leq c_{18} \text{ for some } c_{18} > 0, \text{ all } n \in \mathbb{N}.$$

From this it follows that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded (see the proof of Proposition 3.3). We have

$$\begin{aligned} &0 \leq \widehat{u}_{\lambda_n} \leq u_n \quad \text{for all } n \in \mathbb{N}, \\ &\Rightarrow \ \{\widehat{u}_{\lambda_n}\}_{n\geq 1} \subseteq W^{1,p}(\Omega). \end{aligned}$$

From this and the nonlinear regularity theory of Lieberman [7] (see the proof of Proposition 11), we obtain (at least for a subsequence) that

$$\widehat{u}_{\lambda_n} \to \widetilde{u}_\lambda \text{ in } C^1(\overline{\Omega}). \tag{3.52}$$

If $\widetilde{u}_\lambda \neq \widehat{u}_\lambda$, then we can find $z_0 \in \overline{\Omega}$ such that $\widehat{u}_\lambda(z_0) < \widetilde{u}_\lambda(z_0)$ implies $\widehat{u}_\lambda(z_0) < \widehat{u}_{\lambda_n}(z_0)$ for all $n \geq n_0$ (see (3.52)). This contradicts (a). Therefore by Urysohn's criterion, for the original sequence we have

$$\widehat{u}_{\lambda_n} \to \widehat{u}_\lambda \text{ in } C^1(\overline{\Omega}) \;\Rightarrow\; \sigma(\cdot) \text{ is right continuous.}$$

$\square$

If we impose on $f(z,\cdot)$ similar conditions valid on the negative semiaxis $\mathbb{R}_- = (-\infty, 0]$, we can have analogous results for the negative solutions.

Now the hypotheses on the reaction $f(z,x)$ are as follows

- (H3') $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and
  - (i) $\eta(z)|x|^p \leq f(z,x)x \leq \alpha(z)[1 + |x|^r]$ for a.a. $z \in \Omega$, all $x \leq 0$, with $\eta, \alpha \in L^\infty(\Omega)$, $\xi \preceq \eta$ and $p < r < p^*$;
  - (ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to -\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
  - (iii) if $d(z,x) = f(z,x)x - pF(z,x)$, then there exists $e \in L^1(\Omega)$ such that $d(z,x) \leq d(z,y) + e(z)$ for a.a. $z \in \Omega$, all $y < x \leq 0$ and $d(z,x) \to +\infty$ for a.a. $z \in \Omega$ as $x \to -\infty$;
  - (iv) with $q \in (1,p)$ as in hypothesis $(H1)\,(iv)$, we have $\lim_{x\to 0^-} \frac{f(z,x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;
  - (v) for every $\rho > 0$ there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, 0]$.

We introduce the two sets:

$$\mathcal{L}' = \{\lambda > 0 : \text{ problem (1.1) has a negative solution}\},$$
$$S'_\lambda = \text{ set of negative solutions of (1.1).}$$

Reasoning as we did above for positive solutions, we have the following bifurcation-type result describing the dependence of the set of negative solutions on the parameter $\lambda > 0$.

Theorem 3.9. *If hypotheses (H1), $(H2)$, $(H3')$ hold, then there exists a critical parameter value $\lambda'_* > 0$ such that*

- (a) *for all $\lambda > \lambda'_*$ problem (1.1) has at least two negative solutions $v_0, \widehat{v} \in -D_+$, $v_0 \neq \widehat{v}$;*
- (b) *for $\lambda = \lambda'_*$ problem (1.1) has at least one negative solution $v_* \in -D_+$;*
- (c) *for all $\lambda \in (0, \lambda'_*)$ problem (1.1) has no negative solutions.*

We can generate extremal negative solutions. In this case S'_λ is upward directed, that is, if $v_1, v_2 \in S'_\lambda \subseteq -D_+$, we can find $v \in S'_\lambda$ such that $v_1 \leq v$, $v_2 \leq v$ (see [12]). So, in this case we produce the biggest negative solution for problem (1.1).

Proposition 3.10. *If hypotheses* (H1), (H2), (H3') *hold and $\lambda \in \mathcal{L}' = [\lambda'_*, +\infty)$, then problem* (1.1) *has a biggest negative solution $\widehat{v}_\lambda \in -D_+$ and the map $\sigma' : \mathcal{L}' \to C^1(\overline{\Omega})$ defined by $\sigma'(\lambda) = \widehat{v}_\lambda$ is strictly increasing in the sense that $\lambda'_* \leq \lambda < \eta$ implies $\widehat{v}_\eta - \widehat{v}_\lambda \in \operatorname{int} \widehat{C}_+$ and $\sigma'(\cdot)$ is also right continuous.*

4. Nodal solutions

Let $\widehat{\lambda}_* = \max\{\lambda_*, \lambda'_*\}$. Suppose that the conditions of $f(z,\cdot)$ are bilateral (that is, valid on all of $\mathbb{R}$). Then Proposition 3.8 and 3.10 guarantee that for all $\lambda \geq \widehat{\lambda}_*$ problem (1.1) has a smallest positive solution $\widehat{u}_\lambda \in D_+$ and a biggest negative solution $\widehat{v}_\lambda \in -D_+$. Using these two extremal constant sign solutions of (1.1), we can produce a nodal (sign-changing) solution.

Now the hypotheses on the reaction $f(z,x)$ are as follows:

(H3") $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and
- (i) $\eta(z)|x|^p \leq f(z,x)x \leq \alpha(z)[1 + |x|^r]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\eta, \alpha \in L^\infty(\Omega)$, $\xi \preceq \eta$ and $p < r < p^*$;
- (ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) if $d(z,x) = f(z,x)x - pF(z,x)$, then there exists $e \in L^\infty(\Omega)$ such that $d(z,x) \leq d(z,y) + e(z)$ for a.a. $z \in \Omega$, all $0 \leq x \leq y$ or $y \leq x \leq 0$ and $d(z,x) \to +\infty$ for a.a. $z \in \Omega$ as $x \to \pm\infty$;
- (iv) with $q < p$ as in hypothesis $(H1)\,(iv)$, there exists $\tau \in (1,q)$ and $\delta_0 > 0$ such that $\widehat{c}_0|x|^\tau \leq f(z,x)x \leq \tau F(z,x)$ for a.a. $z \in \Omega$, all $|x| \leq \delta_0$, some $\widehat{c}_0 > 0$;
- (v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

Note that now our condition on $f(z,\cdot)$ near zero is stronger than before.

Proposition 4.1. *If hypotheses* (H1), (H2), (H3") *hold and* $\lambda \geq \widehat{\lambda}_*$, *then problem* (1.1) *admits a nodal solution* $y_\lambda \in C^1(\overline{\Omega})$.

Proof. Using the extremal constant sign solutions $\widehat{u}_\lambda \in D_+$ and $\widehat{v}_\lambda \in -D_+$, we introduce the Carathéodory function

$$\widehat{\gamma}(z,x) = \begin{cases} f(z,\widehat{v}_\lambda(z)) + \widehat{\mu}|\widehat{v}_\lambda(z)|^{p-2}\widehat{v}_\lambda(z) & \text{if } x < \widehat{v}_\lambda(z), \\ f(z,x) + \widehat{\mu}|x|^{p-2}x & \text{if } \widehat{v}_\lambda(z) \leq x \leq \widehat{u}_\lambda(z), \\ f(z,\widehat{u}_\lambda(z)) + \widehat{\mu}\widehat{u}_\lambda(z)^{p-1} & \text{if } \widehat{u}_\lambda(z) < x, \end{cases} \tag{4.1}$$

with $\widehat{\mu} \geq \|\xi\|_\infty$. Also we consider the positive and negative truncations of $\widehat{\gamma}(z,\cdot)$, namely the Carathéodory functions

$$\widehat{\gamma}_\pm(z,x) = \widehat{\gamma}(z,\pm x^\pm). \tag{4.2}$$

We set $\widehat{\Gamma}(z,x) = \int_0^x \widehat{\gamma}(z,s)ds$ and $\widehat{\Gamma}_\pm(z,x) = \int_0^x \widehat{\gamma}_\pm(z,s)ds$ and consider the C^1-functionals $\widehat{\sigma}, \widehat{\sigma}_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\sigma}(u) = \frac{1}{p}\gamma(u) + \frac{\lambda + \widehat{\mu}}{p}\|u\|_p^p - \int_\Omega \widehat{\Gamma}(z,u)dz$$

$$\widehat{\sigma}_\pm(u) = \frac{1}{p}\gamma(u) + \frac{\lambda + \widehat{\mu}}{p}\|u\|_p^p - \int_\Omega \widehat{\Gamma}_\pm(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using (4.1) and (4.2), we can easily show that

$$K_{\widehat{\sigma}} \subseteq [\widehat{v}_\lambda, \widehat{u}_\lambda] \cap C^1(\overline{\Omega}),\ K_{\widehat{\sigma}_+} \subseteq [0, \widehat{u}_\lambda] \cap C_+, K_{\widehat{\sigma}_-} \subseteq [\widehat{v}_\lambda, 0] \cap (-C_+).$$

The extremality of $\widehat{u}_\lambda$ and $\widehat{v}_\lambda$ implies that

$$K_{\widehat{\sigma}} \subseteq [\widehat{v}_\lambda, \widehat{u}_\lambda] \cap C^1(\overline{\Omega}),\ K_{\widehat{\sigma}_+} = \{0, \widehat{u}_\lambda\},\ K_{\widehat{\sigma}_-} = \{0, \widehat{v}_\lambda\}. \tag{4.3}$$

From (4.1) and (4.2) it is clear that $\widehat{\sigma}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{\sigma}_+(\widetilde{u}_\lambda) = \inf[\widehat{\sigma}_+(u) : u \in W^{1,p}(\Omega)]. \tag{4.4}$$

By hypothesis (H3")(iv), we have

$$\begin{aligned} &\widehat{\sigma}_+(\widetilde{u}_\lambda) < 0 = \widehat{\sigma}_+(0), \\ &\Rightarrow \ \widetilde{u}_\lambda \neq 0, \\ &\Rightarrow \ \widetilde{u}_\lambda = \widehat{u}_\lambda \in D_+ \quad \text{(see (4.3), (4.4))}. \end{aligned}$$

Since $\widehat{\sigma}_+\big|_{C_+} = \widehat{\sigma}\big|_{C_+}$ it follows that

$$\begin{aligned} &\widehat{u}_\lambda \in D_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widehat{\sigma}(\cdot), \\ &\Rightarrow \ \widehat{u}_\lambda \in D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\sigma}(\cdot) \end{aligned} \tag{4.5}$$

(see Papageorgiou-Rădulescu [10]). Similarly, using this time $\widehat{\sigma}_-(\cdot)$, we show that

$$\widehat{v}_\lambda \in -D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\sigma}(\cdot). \tag{4.6}$$

We can assume that $\widehat{\sigma}(\widehat{v}_\lambda) \leq \widehat{\sigma}(\widehat{u}_\lambda)$. The reasoning is the same if the opposite inequality holds, using (4.6) instead of (4.5).

By (4.3) and the extremality of $\widehat{u}_\lambda$ and $\widehat{v}_\lambda$, we see that we can assume that

$$K_{\widehat{\sigma}} \subseteq C^1(\overline{\Omega}) \text{ is finite.} \tag{4.7}$$

Otherwise we already have an infinity of smooth nodal solutions and so we are done. By (4.5), (4.7) and [14, Theorem 5.7.6, p. 367,], we can find $\rho \in (0,1)$ small such that

$$\widehat{\sigma}(\widehat{v}_\lambda) \leq \widehat{\sigma}(\widehat{u}_\lambda) < \inf[\widehat{\sigma}(u) : \|u - \widehat{u}_\lambda\| = \rho] = \widehat{m}_\lambda, \ \|\widehat{v}_\lambda - \widehat{u}_\lambda\| > \rho. \tag{4.8}$$

From (4.1) and since $\widehat{\mu} \geq \|\xi\|_\infty$, we see that $\widehat{\sigma}(\cdot)$ is coercive. Hence

$$\widehat{\sigma}(\cdot) \text{ satisfies the } C\text{-condition.} \tag{4.9}$$

Then (4.8) and (4.9) permit the use of the mountain pass theorem. So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &y_\lambda \in K_{\widehat{\sigma}} \text{ and } \widehat{m}_\lambda \leq \widehat{\sigma}(y_\lambda), \\ &\Rightarrow \ y_\lambda \in C^1(\overline{\Omega}) \text{ and } y_\lambda \notin \{\widehat{u}_\lambda, \widehat{v}_\lambda\} \text{(see (4.7), (4.8))}. \end{aligned} \tag{4.10}$$

Also. from Papageorgiou-Rădulescu-Repovš [14, Theorem 6.5.8, p. 431], we have

$$C_1(\widehat{\sigma}, y_\lambda) \neq 0. \tag{4.11}$$

On the other hand by (H3")(iv) and Papageorgiou-Rădulescu [9, Proposition 3.7], we have

$$C_k(\widehat{\sigma}, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.12}$$

Comparing (4.11) and (4.12), we infer that $y_\lambda \neq 0$. Therefore from (4.3) and (4.10), we have that $y_\lambda \in C^1(\overline{\Omega})$ is a nodal solution of (1.1). ☐

Now we can state a multiplicity theorem for problem (1.1).

Theorem 4.2. *If hypotheses* (H1), (H2), (H3") *hold, then there exists a* $\widehat{\lambda}_* > 0$ *such that*

(a) *for all* $\lambda > \widehat{\lambda}_*$ *problem* (1.1) *has at least five nontrivial smooth solutions* $u_0, \widehat{u} \in D_+$, $v_0, \widehat{v} \in -D_+$, $y_\lambda \in C^1(\overline{\Omega})$ *nodal;*

(b) *for* $\lambda = \widehat{\lambda}_*$ *problem* (1.1) *has at least three nontrivial smooth solutions* $u_0 \in D_+$, $v_0 \in -D_+$, $y_\lambda \in C^1(\overline{\Omega})$ *nodal.*

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