Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 115, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND MULTIPLICITY OF HOMOCLINIC SOLUTIONS FOR A DIFFERENCE EQUATION

SHAPOUR HEIDARKHANI, FARIBA GHAREHGAZLOUEI, MAURIZIO IMBESI

Abstract. The aim of this article is to obtain homoclinic solutions for a discrete problem involving p-Laplacian. We prove the existence of at least one, two and three solutions for the problem. Our approach is based on variational methods.

1. Introduction

In this article, we study the nonlinear second-order difference equation, depending on a real parameter $\lambda > 0$,

$$\begin{gathered} -\Delta(a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(k,u(k)) + h(u(k)) \quad \forall k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as } |k| \to \infty. \end{gathered} \tag{1.1}$$

Here $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$, and $a, b : \mathbb{Z} \to (0, +\infty)$, where $b(k) \geq \alpha > 0$ for all $k \in \mathbb{Z}$, and $b(k) \to +\infty$ as $|k| \to +\infty$. The function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous, while $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous of order $p-1$ with Lipschitzian constant $L \geq 0$, i.e.

$$|h(\xi_1) - h(\xi_2)| \leq L|\xi_1 - \xi_2|^{p-1}, \quad \forall \xi_1, \xi_2 \in \mathbb{R},$$

and such that $h(0) = 0$. The forward difference operator is

$$\Delta u(k-1) = u(k) - u(k-1) \quad \forall k \in \mathbb{Z}.$$

A solution $u = \{u(k)\}$ of problem (1.1) is called *homoclinic* if $\lim_{|k|\to\infty} u(k) = 0$.

Discrete boundary value problems have been extensively studied in the previous decade. Modeling of certain nonlinear problems from biological neural net-works, economics, optimal control, electrical circuit analysis, dynamical systems, and other areas of study have led to the rapid progression of the theory of difference equations; see [1, 2, 8, 9, 11]. Most of the classical methods used in differential equations can be used for difference equations. Variational methods are powerful tools in such problems. Critical point theory has been used for proving the existence and multiplicity solutions of discrete nonlinear problems.

The issue of finding solutions on unbounded intervals is more delicate. For a study such problems by variational methods, see [13, 16]. Variational methods for difference equations consist in seeking solutions as critical points for a suitable energy functional defined on a convenient Banach space. In the first approaches

2010 *Mathematics Subject Classification.* 39A10, 47J30, 35B38.
Key words and phrases. Difference equations; discrete p-Laplacian; variational methods.

Submitted January 14, 2020. Published November 23, 2020.

to the issue, the variational methods are applied to boundary value problems on bounded discrete intervals, which lead to the study of an energy functional defined on a finite-dimensional Banach space.

In the case of difference equations on unbounded discrete intervals (typically, on the whole set of integers $\mathbb{Z}$) solutions are sought in a subspace of the space ℓ^p which is still infinite-dimensional but compactly embedded into ℓ^p (see [12, 13, 16]). A standard way to deal with problems on unbounded domains consists in introducing coercive weight functions. This method goes back to a celebrated result of Omana and Willem [17] on homoclinic orbits for a class of Hamiltonian systems.

There is an increasing interest in the existence and multiplicity of homoclinic solutions to discrete nonlinear problems. The existence and multiplicity of homoclinic solutions have been investigated using various methods by many authors; see [3, 10, 13, 14, 15, 19, 20, 21, 22] and the references therein. For example, Iannizzotto and Tersian [13] used critical point theory, and proved the existence of at least two nontrivial homoclinic solutions for the nonlinear second-order difference equation

$$\begin{gathered} -\Delta(\phi_p(\Delta u(k-1))) + a(k)\phi_p(u(k)) = \lambda f(k, u(k)) \quad \forall k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as } |k| \to \infty \end{gathered} \tag{1.2}$$

where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$, $a : \mathbb{Z} \to (0, +\infty)$ is a positive and coercive weight function and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Kong [15] applied the variational method and a variant of the fountain theorem to find new conditions under which the problem (1.1), in the case $h \equiv 0$, has infinitely many solutions. Sun and Mai [22] employed Nehari manifold methods and critical point theory to study the existence of nontrivial homoclinic solutions of discrete p-Laplacian equations with a coercive weight function and superlinear nonlinearity. Without assuming the classical Ambrosetti-Rabinowitz condition and without any periodicity assumptions, they proved the existence and multiplicity results of problem (1.2).

Stegliński [21] by using both the general variational principle of Ricceri and the direct method introduced by Faraci and Kristály [9] obtained infinitely many solutions for the problem (1.1), when $h \equiv 0$. In [3] using variational methods and critical point theory, sufficient conditions for the existence of at least one homoclinic solution for the problem (1.1), in the case $h \equiv 0$ have been presented.

We want to point out the main novelties of our results: we investigate the existence of homoclinic solutions of problem (1.1) when the nonlinearity f has subcritical growth. We use variational methods and critical point theory to study the existence of at least one, two and three weak solutions whenever the parameter λ belongs to a precise positive interval. We note that the existence of three solutions to difference the equations in the set $\mathbb{Z}$ of integers has rarely been studied. The main tools are critical point theorems established in [4, 5, 7]. Examples are presented to demonstrate the applicability of our results.

The rest of this article is organized as follows. Section 2 includes some preliminary results. Section 3 contains the main results, their proofs, and some applications.

2. Preliminaries

In this section, we introduce some definitions and notation which will be used later. For all $1 \leq p < \infty$, denote ℓ^p the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_p^p = \sum_{k\in\mathbb{Z}} |u(k)|^p < +\infty.$$

Let ℓ^∞ be the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_\infty = \sup_{k\in\mathbb{Z}} |u(k)| < +\infty.$$

Set

$$X = \big\{u : \mathbb{Z} \to \mathbb{R} : \sum_{k\in\mathbb{Z}} (a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p) < \infty\big\}$$

equipped with the norm

$$\|u\| = \big(\sum_{k\in\mathbb{Z}} (a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p)\big)^{1/p}.$$

Clearly, we have

$$\|u\|_\infty \le \|u\|_p \le \alpha^{-1/p}\|u\| \quad \forall u \in X. \tag{2.1}$$

As is shown in [13, Proposition 3], $(X, \|\cdot\|)$ is a reflexive Banach space, and the embedding $X \hookrightarrow \ell^p$ is compact. We define

$$\Phi(u) := \frac{1}{p}\|u\|^p - \sum_{k\in\mathbb{Z}} H(u(k)) \quad \forall u \in X, \tag{2.2}$$

$$\Psi(u) := \sum_{k\in\mathbb{Z}} F(k, u(k)) \quad \forall\ u \in l^p \tag{2.3}$$

where $F(k,t) = \int_0^t f(k,\xi)d\xi$ for $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, $H(t) = \int_0^t h(\xi)d\xi$ for $t \in \mathbb{R}$. Let $I_\lambda : X \to \mathbb{R}$ be the energy functional associated to the problem (1.1) defined by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

We suppose that the Lipschitz constant $L > 0$ of the function h satisfies the condition $L < \alpha$.

Proposition 2.1. *Let $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

$$\sup_{|t|\le T} |f(\cdot, t)| \in \ell^1 \quad \forall T > 0. \tag{2.4}$$

Then,

1. *$\Psi \in C^1(\ell^p)$ and $\Psi \in C^1(X)$;*
2. *$\Phi \in C^1(X)$;*
3. *$I_\lambda \in C^1(X)$ and every critical point $u \in X$ of I_λ is a homoclinic solution of problem* (1.1)*;*
4. *I_λ is sequentially weakly lower semicontinuous functional on X.*

Proof. Arguing as in the proof of [21, Proposition 2.2], Part (1) follows from [21, Lemma 2.1]. Parts (2) and (3) can be proved essentially by the same way as [13, Propositions 5 and 7], where $H(u(k)) = 0$, $a(k) = 1$ on $\mathbb{Z}$ and the norm on X is slightly different. See also [14, Lemmas 2.4 and 2.6]. The proof of Part (4) is based on the facts $\Psi \in C(\ell^p)$ and the compactness of $X \to \ell^p$, then it is standard. $\square$

Definition 2.2. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(PS)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(1) $\{I(u_n)\}$ is bounded;
(2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$;
(3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$

has a convergent subsequence.

The following theorems are the main tools for proving our results.

Theorem 2.3 ([5, Theorem 2.3], [4, Theorem 5.1]). *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{v} \in X$, with $0 < \Phi(\bar{v}) < r$ such that*

(1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$;
(2) *for all* $\lambda \in \Lambda := \big(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\big)$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *satisfies the* $(PS)^{[r]}$ *condition.*

Then, for each $\lambda \in \Lambda$ there is $u_{0,\lambda} \in \Phi^{-1}(0, r)$ such that $I'_\lambda(u_{0,\lambda}) = \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) < I_\lambda(u)$ for all $u \in \Phi^{-1}(0, r)$.

Theorem 2.4 ([5, Theorem 3.2]). *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \Big(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\Big)$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the (PS) condition and it is unbounded from below. Then, for each

$$\lambda \in \Big(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\Big)$$

the functional I_λ admits two distinct critical points.

Theorem 2.5 ([7, Theorem 3.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive and continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$;
(2) *for all* $\lambda \in \Lambda_r := \Big(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\Big)$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorems 2.3–2.5 have been successfully applied to the existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent in [6].

3. Main results

We first discuss the existence of one solution for problem (1.1) .

Theorem 3.1. *Assume that there exist two real constants $\tau > 0$ and η and also assume that $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function and $F(l, \eta) > 0$ for some $l \in \mathbb{Z}$, such that*

(1) $\frac{\alpha+L}{\alpha-L}\frac{2a(l)+b(l)}{\alpha p}|\eta|^p < \tau^p$;

(2) $\frac{\sum_{k\in\mathbb{Z}}\max_{|t|\le\tau} F(k,t)}{(\alpha-L)\tau^p} < \frac{\alpha p F(l,\eta)}{(2a(l)+b(l))(\alpha+L)|\eta|^p}$.

Then, for each

$$\lambda \in \Lambda := \Big(\frac{(2a(l)+b(l))(\alpha+L)|\eta|^p}{\alpha p^2 F(l,\eta)}, \frac{(\alpha-L)\tau^p}{p\sum_{k\in\mathbb{Z}}\max_{|t|\le\tau} F(k,t)}\Big), \tag{3.1}$$

problem (1.1) *admits at least one nontrivial solution $u_\lambda \in X$ such that $\|u_\lambda\|_\infty \le \tau$.*

Proof. Our goal is to apply Theorem 2.3 to problem (1.1). To this end, take the real Banach space X with the norm as defined in Section 2, and and let Φ, Ψ be the functionals defined in (2.2) and (2.3). Taking into account that h is a $(p-1)$-Lipschitz continuous function with Lipschizian constant $L > 0$ and $h(0) = 0$, we have

$$\begin{aligned}
\frac{\alpha-L}{\alpha p}\|u\|^p &\le \frac{1}{p}\|u\|^p - \frac{L}{p}\sum_{k\in\mathbb{Z}}|u(k)|^p \\
&\le \Phi(u) \\
&\le \frac{1}{p}\|u\|^p + \frac{L}{p}\sum_{k\in\mathbb{Z}}|u(k)|^p \\
&\le \frac{\alpha+L}{\alpha p}\|u\|^p,
\end{aligned}$$

namely

$$\frac{\alpha-L}{\alpha p}\|u\|^p \le \Phi(u) \le \frac{\alpha+L}{\alpha p}\|u\|^p. \tag{3.2}$$

From the first inequality of (3.2) it follows that Φ is coercive. From Proposition 2.1, we observe that $\Phi, \Psi \in C^1(X,\mathbb{R})$. Moreover, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then $u_n(x)$ converges uniformly to $u(x)$ on $\mathbb{Z}$ as $n \to \infty$; see [23]. Since f is continuous in $\mathbb{R}$ for every $k \in \mathbb{Z}$, we have

$$f(k, u_n(k)) \to f(k, u(k)),$$

as $n \to \infty$. Thus $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Hence, we proved that Ψ' is a compact operator by [23, Proposition 26.2]. This ensures that the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$ condition for each $r > 0$ [4, see Proposition 2.1]. Our aim is to apply Theorem 2.3 to the functional I_λ. We have $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Now, it remains to verify condition (1) in Theorem 2.3. To this aim, put $r := \frac{\alpha-L}{p}\tau^p$. For any $l \in \mathbb{Z}$, we define $e_l \in X$ with $e_l(k) = \delta_{lk}$ for all $k \in \mathbb{Z}$ ($\delta_{lk} = 1$ if $l = k$, $\delta_{lk} = 0$ if $l \ne k$). Set

$$w(k) = \eta e_l(k), \quad \forall k \in \mathbb{Z}. \tag{3.3}$$

Simple calculations show that

$$\|w\|^p = (\frac{2a(l)+b(l)}{p})|\eta|^p,$$

and from (3.2) it follows that

$$\frac{(\alpha-L)(2a(k)+b(k))}{\alpha p^2}|\eta|^p \le \Phi(w) \le \frac{(\alpha+L)(2a(k)+b(k))}{\alpha p^2}|\eta|^p. \tag{3.4}$$

The right hand side of (3.4) in conjunction with the condition (1) in Theorem 3.1 yields $0 < \Phi(w) < r$. From the definition of Φ and (2.1), the estimate $\Phi(u) \le r$ implies that

$$|u(k)|^p \le \|u\|_\infty^p \le \frac{1}{\alpha}\|u\|^p \le \frac{p}{\alpha-L}\Phi(u) \le \frac{p}{\alpha-L} r = \tau^p, \quad \forall k \in \mathbb{Z},$$

from which it follows that

$$\Phi^{-1}(-\infty, r] = \{u \in X; \Phi(u) \le r\} \subseteq \{u \in X; |u| \le \tau\}.$$

Accordingly, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r]} \sum_{k\in\mathbb{Z}} F(k,u(k)) \le \sum_{k\in\mathbb{Z}} \max_{|t|\le\tau} F(k,t). \tag{3.5}$$

In view of (3.4) and (3.5), taking into account (2) of Theorem 3.1 (A_2), we obtain

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r} &\le \frac{p\sum_{k\in\mathbb{Z}}\max_{|t|\le\tau} F(k,t)}{(\alpha-L)\tau^p} \\ &< \frac{\alpha p^2 F(l,\eta)}{(2a(l)+b(l))(\alpha+L)|\eta|^p} \\ &\le \frac{\Psi(w)}{\Phi(w)}, \end{aligned} \tag{3.6}$$

which means that, condition (1) of Theorem 2.3 is satisfied. Hence, applying Theorem 2.3, for each $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$ the functional I_λ admits at least one critical point u_λ such that

$$0 < \Phi(u_\lambda) < r, \quad \text{and} \quad \|u_\lambda\|_\infty \le \tau$$

which is a nontrivial solution of the problem (1.1). $\square$

The following example is an application of Theorem 3.1.

Example 3.2. Consider the problem

$$\begin{gathered} -\Delta(\phi_5(\Delta u(k-1))) + (\frac{|k|+1}{2})\phi_5(u(k)) = \lambda f(k,u(k)) + sin^4(\frac{u(k)}{2}) \quad \forall k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as } |k| \to \infty. \end{gathered}$$

For all $(k,t) \in \mathbb{Z}\times\mathbb{R}$ put

$$f(k,t) = \frac{1}{k^2+1}.$$

By the expression of f we have

$$F(k,t) = \frac{t}{k^2+1} \quad \forall (k,t) \in \mathbb{Z}\times\mathbb{R}.$$

By choosing $\eta = 1$, $\tau = 2$ and $l = 1$, through simple calculations we obtain

$$\frac{\alpha+L}{\alpha-L}\frac{2a(l)+b(l)}{\alpha p}|\eta|^p = \frac{54}{35} < 32$$

and

$$\begin{aligned}\frac{\sum_{k\in\mathbb{Z}}\max_{|t|\le\tau}F(k,t)}{(\alpha-L)\tau^p}&=\frac{1}{7}(\sum_{k\in\mathbb{Z}}\frac{1}{k^2+1})=\frac{1}{7}(1+2\sum_{k=1}^{\infty}\frac{1}{k^2+1})\\&=\frac{1}{7}(\pi\coth(\pi))=0.45\\&<\frac{\alpha pF(l,\eta)}{(2a(l)+b(l))(\alpha+L)|\eta|^p}=\frac{20}{27}=0.74.\end{aligned}$$

Obviously, all assumptions of Theorem 3.1 are satisfied. Hence, it follows that for each $\lambda\in(0.27,0.89)$ the above problem admits at least one nontrivial solution $u_\lambda\in X$ such that $\|u_\lambda\|_\infty\le 2$.

Now, we discuss the existence of two solutions for (1.1) applying Theorem 2.4.

Theorem 3.3. *Let $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Moreover, assume that*

(1) *there exist $\mu>p$ and $R>0$ such that*
$$0<\mu F(k,t)<tf(k,t)$$
for all $k\in\mathbb{Z}$ and $|t|\ge R$.

Then, for each
$$\lambda\in\Lambda:=\Big(0,\frac{(\alpha-L)\tau^p}{p\sum_{k\in\mathbb{Z}}\max_{|t|<\tau}F(k,t)}\Big),$$
problem (1.1) *admits at least two nontrivial solutions.*

Proof. Let Φ,Ψ be the functionals defined in Theorem 3.1 which satisfy all regularity assumptions requested in Theorem 2.4. Arguing as in the proof of Theorem 3.1, choosing $r=\frac{\alpha-L}{p}\tau^p$, for each $\lambda\in\Lambda$ we obtain

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}\le\frac{p\sum_{k\in\mathbb{Z}}\max_{|t|\le\tau}F(k,t)}{(\alpha-L)\tau^p}<\frac{1}{\lambda}.$$

Now, from the condition (1), by standard computations, there is a positive constant m such that

$$F(k,t)\ge m|t|^\mu\quad\text{for all}\quad k\in\mathbb{Z}. \tag{3.7}$$

Hence, for every $\lambda\in\Lambda$, $u\in X\setminus\{0\}$ and $t>1$, we obtain

$$I_\lambda(tu(k))=\Phi(tu(k))-\lambda\sum_{k\in\mathbb{Z}}F(k,tu(k))\le\frac{\alpha+L}{p}t^p\|u\|^p-m\lambda t^\mu\sum_{k\in\mathbb{Z}}|u(k)|^\mu.$$

Since $\mu>p$, this condition guarantees that I_λ is unbounded from below. We recall that I_λ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u\in X$ is the functional $I'_\lambda(u)\in X^*$ given by

$$\begin{aligned}I'_\lambda(u)(v)&=\sum_{k\in\mathbb{Z}}a(k)\phi_p(\Delta u(k-1))\Delta v(k-1)+\sum_{k\in\mathbb{Z}}b(k)\phi_p(u(k))v(k)\\&\quad-\sum_{k\in\mathbb{Z}}h(u(k))v(k)-\lambda\sum_{k\in\mathbb{Z}}f(k,u(k))v(k),\end{aligned}$$

for every $v \in X$. Finally, we verify that I_λ satisfies the (PS)-condition. Indeed, if $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_\lambda(u_n) \to 0$ in X^* as $n \to +\infty$. Then, there exists a positive constant s_0 such that

$$|I_\lambda(u_n)| \le s_0, \quad \|I'_\lambda(u_n)\| \le s_0 \quad \forall n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ we put

$$\begin{gathered} \varphi(u_n(k)) := f(k,u_n(k))u_n(k) - \mu F(k,u_n(k)), \\ K_n := \{k \in \mathbb{Z} : |u_n(k)| > R\}. \end{gathered}$$

Using also conditions (1) and the definition of I'_λ, we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} s_0 + s_1\|u_n\| &\ge \mu I_\lambda(u_n) - I'_\lambda(u_n)u_n \\ &\ge (\frac{\mu}{p} - 1)(1 - \frac{L}{\alpha})\|u_n\|^p + \lambda\sum_{k\in\mathbb{Z}}\varphi(u_n(k)) \\ &= (\frac{\mu}{p} - 1)(1 - \frac{L}{\alpha})\|u_n\|^p + \lambda\sum_{k\in K_n}\varphi(u_n(k)) + \lambda\sum_{k\in\mathbb{Z}\setminus K_n}\varphi(u_n(k)) \\ &\ge (\frac{\mu}{p} - 1)(1 - \frac{L}{\alpha})\|u_n\|^p - \lambda s_2, \end{aligned}$$

for some $s_1, s_2 > 0$. Since $\mu > p$ and $\alpha > L$ it follows $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, since X is a reflexive Banach space, up to a subsequence, we have

$$u_n \rightharpoonup u \quad \text{in } X.$$

By $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in X we observe that

$$(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \to 0.$$

From the continuity of f and h we have

$$\begin{gathered} \sum_{k\in\mathbb{Z}}(f(k,u_n(k)) - f(k,u(k)))(u_n(k) - u(k)) \to 0, \quad \text{as } n \to +\infty, \\ \sum_{k\in\mathbb{Z}}(h(u_n(k)) - h(u(k)))(u_n(k) - u(k)) \to 0, \quad \text{as } n \to +\infty. \end{gathered}$$

An easy computation shows that

$$\begin{aligned} &(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \\ &= \sum_{k\in\mathbb{Z}}\big(a(k)(\phi_p(\Delta u_n(k-1)) - \phi_p(\Delta u_n(k-1)))(\Delta u_n(k-1) - \Delta u(k-1))\big) \\ &\quad + \sum_{k\in\mathbb{Z}} b(k)(\phi_p(u_n(k)) - \phi_p(u(k)))(u_n(k) - u(k)) \\ &\quad - \sum_{k\in\mathbb{Z}}(h(u_n(k) - h(u(k))(u_n(k) - u(k)) \\ &\quad - \lambda\sum_{k\in\mathbb{Z}}(f(k,u_n(k) - f(k,u(k)))(u_n(k) - u(k)) \\ &\ge (1 - \frac{L}{\alpha})\|u_n - u\|^p. \end{aligned}$$

Thus, since $\alpha > L$, this implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in X. Therefore, I_λ satisfies the (PS)-condition, and so all hypotheses of Theorem 2.4 are verified. Hence, applying Theorem 2.4, for each $\lambda \in \Lambda$ the function I_λ

admits at least two distinct critical points that are the solutions of the problem (1.1). □

Remark 3.4. Assume that f satisfies the condition

(1) there exist $m_1, m_2 \in [0,+\infty)$ such that for $1 < q < p$,
$$|F(k,t)| \le m_1 + m_2|t|^{q-1} \forall (k,t) \in \mathbb{Z}\times\mathbb{R}.$$

Setting $r = 1$. Indeed, using condition (1) in Theorem 3.3. there exist $m_3 > 0$ and $m_4 > 0$ such that
$$F(k,t) \ge m_3|t|^{\mu} - m_4 \quad \text{for all } (k,t) \in \mathbb{Z}\times\mathbb{R}.$$
For fixed $\bar u \in X \setminus \{0_X\}$, for each $t > 1$ one has
$$I_\lambda(t\bar u(k)) \le \frac{\alpha+L}{p} t^p \|\bar u\|^p - \lambda \sum_{k\in\mathbb{Z}} (m_3 t^{\mu} |\bar u(k)|^{\mu} - m_4).$$
Since $\mu > p$, this condition guarantees that I_λ is unbounded from below. Therefore, Theorem 2.4 ensures the conclusion. As a general example of application of this result we consider the function f satisfying the condition (1) by fixing $p < \mu < q$ and $r > \max\{(\frac{(\mu-1)m_1}{(\mu-q)m_2})^{\frac{1}{q-1}}, (\frac{m_1}{m_2})^{\frac{1}{q-1}}\}$, so that, for this type of f, the problem (1.1) admits at least two nontrivial solutions.

Now, we discuss the existence of at least three solutions for problem (1.1).

Theorem 3.5. *Assume that there exist two real constants $\tau > 0$ and η, and assume that $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function and $F(l,\eta) > 0$ for some $l \in \mathbb{Z}$, such that*

(1) $\frac{2a(l)+b(l)}{\alpha p}|\eta|^p > \tau^p$;
(2) $\limsup_{|t|\to+\infty} \frac{F(k,t)}{|t|^p} \le 0$ *for all* $k \in \mathbb{Z}$.

Suppose that (2) in Theorem 3.1 holds. Then for every
$$\lambda \in \Lambda_r := \Big(\frac{(2a(l)+b(l))(\alpha+L)|\eta|^p}{\alpha p^2 F(l,\eta)}, \frac{(\alpha-L)\tau^p}{p\sum_{k\in\mathbb{Z}} \max_{|t|\le\tau} F(k,t)} \Big),$$
problem (1.1) *admits at least three solutions.*

Proof. Our aim is to apply Theorem 2.5. We consider the functionals Φ and Ψ as before, which satisfy the regularity assumptions requested in Theorem 2.5. Now, arguing as in the proof of Theorem 3.1, putting $w(k)$ as in (3.4) and $r = \frac{\alpha-L}{p}\tau^p$, bearing in mind (1), we derive
$$\Phi(w) > r > 0.$$
Therefore, (3.6) holds, and the assumption (1) of Theorem 2.5 is satisfied. Now, we prove that, for each $\lambda \in \Lambda_r$ the functional I_λ is coercive. By using the condition (2), for all $0 < \varepsilon < \frac{\alpha-L}{\lambda p}$ there exists $T > 0$ such that
$$F(k,t) \le \varepsilon |t|^p \quad \text{for all } k \in \mathbb{Z}, |t| > T.$$
In addition, by (2.4) there exists $w_1 \in \ell^1$ such that
$$F(k,t) \le w_1(k) \quad \forall k \in \mathbb{Z},\ |t| \le T.$$
For each $u \in X$ we have
$$I_\lambda(u) \ge \frac{\alpha-L}{\alpha p}\|u\|^p - \lambda \sum_{|u(k)|\le T} F(k,u(k)) - \lambda \sum_{|u(k)|>T} F(k,u(k))$$

$$
\begin{aligned}
&\geq \frac{\alpha - L}{\alpha p}\|u\|^p - \lambda\|w_1\|_1 - \lambda\varepsilon\|u\|_p^p \\
&\geq (\frac{\alpha - L}{\alpha p} - \frac{\lambda\varepsilon}{\alpha})\|u\|^p - \lambda\|w_1\|_1,
\end{aligned}
$$

in which $I_\lambda \to +\infty$ as $\|u\| \to +\infty$. Hence the functional I_λ is coercive, also condition the (2) of theorem 2.5 holds. So, for each $\lambda \in \Lambda_r$, Theorem 2.5 implies that the functional I_λ admits at least three critical points in X that are solutions of problem (1.1). □

Now, we present an example to illustrate Theorem 3.5.

Example 3.6. Consider the problem

$$
\begin{gathered}
-\Delta(\phi_{31}(\Delta u(k-1))) + (|k|+2)\phi_{31}(u(k)) = \lambda f(k, u(k)) + sin^{30}(u(k)) \quad \forall k \in \mathbb{Z}, \\
u(k) \to 0 \quad \text{as } |k| \to \infty.
\end{gathered}
$$

For $(k, t) \in \mathbb{Z} \times \mathbb{R}$, we put

$$
f(k, t) = 30t^{29} \sin(\frac{k^2\pi}{4}).
$$

From the definition of f we have

$$
F(k, t) = t^{30} \sin(\frac{k^2\pi}{4}), \quad \forall(k, t) \in \mathbb{Z} \times \mathbb{R}.
$$

Clearly, F satisfies assumption (2) of Theorem 3.5. By choosing $\eta = 2$, $\tau = 1$ and $l = 1$, by simple calculations we obtain $\frac{2a(l)+b(l)}{\alpha p}|\eta|^p > 1$ and

$$
\frac{\sum_{k\in\mathbb{Z}} \max_{|t|\leq\tau} F(k, t)}{(\alpha - L)\tau^p} = \sum_{k\in\mathbb{Z}} \sin(\frac{k^2\pi}{4}) = \sqrt{2} < \frac{\alpha p F(l, \eta)}{(2a(l) + b(l))(\alpha + L)|\eta|^p} = \frac{31\sqrt{2}}{30}.
$$

Thus, all assumptions of Theorem 3.5 are satisfied. It follows that for each $\lambda \in (0.0220, 0.0228)$ the above problem admits at least three solutions.

We finish the discussion by pointing out a simple consequence of Theorem 3.5.

Theorem 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that $F(\eta) > 0$ for some $\eta > 0$ and $F(\xi) \geq 0$ in $[0, \eta]$ and*

$$
\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^p} = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 0.
$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem

$$
\begin{gathered}
-\Delta(a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(u(k)) + h(u(k)) \quad \forall k \in \mathbb{Z}, \\
u(k) \to 0 \quad \text{as } |k| \to \infty,
\end{gathered} \tag{3.8}
$$

admits at least three nontrivial solutions.

Proof. Our aim is to use Theorem 3.5 with $b(l) = (|l| + 1)/2$ and $a(l) = 1/4$ for some $l \in \mathbb{Z}$. We fix

$$
\lambda > \lambda^* := \frac{(\frac{|l|+2}{2})(\alpha + L)|\eta|^p}{\alpha p F(\eta)}.
$$

Recalling that $\liminf_{\xi\to 0} F(\xi)/\xi^p = 0$, there is a sequence $\{\theta_n\} \subset (0, +\infty)$ such that $\lim_{n\to\infty} \theta_n = 0$ and

$$
\lim_{n\to\infty} \frac{\sup_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^p} = 0.
$$

Indeed, one has

$$\lim_{n\to\infty} \frac{\sup_{|\xi|\le\theta_n} F(\xi)}{\theta_n^p} = \lim_{n\to\infty} \frac{\sup_{|\xi|\le\theta_n} F(\xi\theta_n)}{\xi^p\theta_n} \frac{\xi^p\theta_n}{\theta_n^p} = 0,$$

where $F(\xi\theta_n) = \sup_{|\xi|\le\theta_n} F(\xi)$. Hence, there exists $\bar\theta > 0$ such that

$$\frac{\sup_{|\xi|\le\bar\theta} F(\xi)}{\bar\theta^p} < \min\Big\{\frac{(\alpha-L)F(\eta)}{(\frac{|l|+2}{2})(\alpha+L)|\eta|^P}, \frac{\alpha-L}{\lambda\alpha p}\Big\}$$

and $\bar\theta < \eta$. The conclusion follows from Theorem 3.5. □

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Shapour Heidarkhani
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
Email address: s.heidarkhani@razi.ac.ir

Fariba Gharehgazlouei
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
Email address: f.gharehgazloo@yahoo.com

Maurizio Imbesi
Department of Mathematical and Computer Sciences, Physical and Earth Sciences, University of Messina, Viale F. Stagno d'Alcontres 31, 98166 Messina, Italy
Email address: imbesim@unime.it