*Electronic Journal of Differential Equations*, Vol. 2020 (2020), No. 111, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# CONVERGENCE OF SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS TO POWER-TYPE FUNCTIONS

MOHAMMED DAHAN KASSIM, NASSER EDDINE TATAR

Abstract. In this article we study the asymptotic behavior of solutions of some fractional differential equations. We prove convergence to power type functions under some assumptions on the nonlinearities. Our results extend and generalize some existing well-known results on solutions of ordinary differential equations. Appropriate estimations and lemmas such as a fractional version of L'Hopital's rule are used.

## 1. Introduction

We consider the initial value problems

$$\begin{gathered}({}^C\mathfrak{D}_0^\alpha x)'(\tau) = f(\tau, x(\tau), {}^C\mathfrak{D}_0^\beta x(\tau)), \quad 0 < \beta < \alpha < 1,\ \tau > 0 \\ {}^C\mathfrak{D}_0^\alpha x(\tau)\big|_{\tau=0} = b_2, \quad = x(\tau)|_{\tau=0} = b_1, \quad b_1, \quad b_2 \in \mathbb{R},\end{gathered} \tag{1.1}$$

and

$$\begin{gathered}{}^C\mathfrak{D}_0^\alpha x(\tau) = f(\tau, x(\tau), {}^C\mathfrak{D}_0^\beta x(\tau)), \quad 0 \le \beta < \alpha < 1,\ \tau > 0 \\ x(\tau)|_{\tau=0} = b,\end{gathered} \tag{1.2}$$

where ${}^C\mathfrak{D}_0^\alpha$ is the Caputo fractional derivative. The definition of the Caputo fractional derivative is given in the next section. We prove that the solutions of (1.1) approach power type functions and the solutions of (1.2) are bounded. To this end, the fractional differential problems (1.1) and (1.2) are first transformed into equivalent integral equations in appropriate underlying spaces. Various appropriate estimates, comparison theorems and lemmas are used. Moreover, we prove a Caputo fractional version of L'Hopital's rule. Our arguments here are quite different from those used so far in the literature.

The behavior of solutions of various classes of ODEs (ordinary differential equations) has been discussed in fairly a large number of papers in the literature. For example the equation

$$x''(\tau) + f(\tau, x(\tau)) = 0, \tag{1.3}$$

has been studied in [17, 18, 7, 30, 32, 9, 31] and other papers. The authors proved that, under various conditions, all solutions of (1.3) are asymptotic to $c\tau + b$ as $\tau \to \infty$, $c, b \in \mathbb{R}$. For the equation

$$x''(\tau) + f(\tau, x(\tau), x'(\tau)) = 0, \tag{1.4}$$

2010 *Mathematics Subject Classification.* 34E10, 26A33, 34A08.
*Key words and phrases.* Asymptotic behavior; boundedness; fractional differential equation; Caputo fractional derivative; Riemann-Liouville fractional derivative.

Submitted September 15, 2020. Published November 4, 2020.

see, for instance [8, 10, 19, 22, 23, 25, 27, 28]. It is proved that all solutions of (1.4) can be expressed asymptotically as $c\tau + b$ as $\tau \to \infty$, $c, b \in \mathbb{R}$.

Medveď and Pekárková [23], studied the one-dimensional p-Laplacian equation

$$(|x'|^{p-1}x')' = f(\tau, x, x'), \quad p > 1. \tag{1.5}$$

They demonstrated that any solution of (1.5) behaves asymptotically as $b + c\tau$ as $\tau \to \infty$ for some real numbers $b$, $c$.

In [22], the initial value problem

$$\begin{gathered} (\Phi_p(x')\Psi(\tau))' + f(\tau, x, x') = 0, \quad 1 < p < 2, \\ x(\tau_0) = x_0, \quad x'(\tau_0) = x_1, \quad \tau_0 \geq 1, \end{gathered} \tag{1.6}$$

was studied, where $\Phi_p(u) = |u|^{p-2}u$ and $\Psi(\tau)$ is a continuous positive function. Sufficient conditions under which all solutions of (1.6) obey the asymptotic expansion $x(\tau) = b + c\tau$ are established.

In contrast, the fractional case of equations (1.3) and (1.4) have been studied by comparatively a only few researchers; see, for instance [1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15, 20, 21, 24]. In 2009, Băleanu and Mustafa [1] studied the nonlinear fractional differential equation

$$^{C}\mathfrak{D}_0^{\alpha} x(\tau) = f(\tau, x(\tau)), \quad 0 < \alpha < 1, \quad \tau > 0. \tag{1.7}$$

They showed that the solutions of (1.7) are asymptotic to $o(\tau^{c\alpha})$ as $\tau \to \infty$, for some $c$.

In 2012, Medveď [20] studied the problem

$$\begin{gathered} ^{C}\mathfrak{D}_a^{\alpha+1} x(\tau) = f(\tau, x(\tau)), \quad 0 < \alpha < 1, \quad \tau \geq a > 1 \\ x(a) = c_1, \quad x'(a) = c_2. \end{gathered} \tag{1.8}$$

He demonstrated that any solution of (1.8) has the asymptotic property $x(\tau) = b + c\tau$ as $\tau \to \infty$, for some $c, b \in \mathbb{R}$.

Also, in 2013, Medveď [21] discussed the equation

$$^{C}\mathfrak{D}_a^{\alpha+1} x(\tau) = f(\tau, x(\tau), x'(\tau)), \quad 0 < \alpha < 1, \quad \tau \geq a > 1 \tag{1.9}$$

and proved that every solution of (1.9) can be expressed asymptotically as $b + c\tau$ as $\tau \to \infty$, for some $c, b \in \mathbb{R}$.

Brestovanská and Medveď [6] studied the problem

$$\begin{gathered} x''(\tau) + f(\tau, x(\tau), x'(\tau)) + \sum_{i=1}^{m} r_i(\tau) \int_0^{\tau} (\tau - s)^{\alpha_i - 1} f_i(s, x(s), x'(s)) ds = 0 \\ x(1) = b_1, \quad x'(1) = b_2, \quad 0 < \alpha_i < 1,\ i = 1, 2, \dots, m. \end{gathered} \tag{1.10}$$

They showed that any solution enjoys the asymptotic expansion $x(\tau) = b + c\tau$ as $\tau \to \infty$, for some $c, b \in \mathbb{R}$.

In 2015, Medveď and Pospíšil [24] considered the equation

$$^{C}\mathfrak{D}_a^{\alpha} x(\tau) = f(\tau, x(\tau), {}^{C}\mathfrak{D}_0^{\beta} x(\tau)), \quad \tau > a. \tag{1.11}$$

They proved that any solution $x(\tau)$ with $0 < \beta < \alpha < 1$, has the asymptotic property $x(\tau) = c\tau^{\beta} + o(\tau^{\beta})$ as $\tau \to \infty$, for some $c \in \mathbb{R}$. Also they proved that any solution $x(\tau)$ of (1.11), $0 < \beta < 1 < \alpha < 2$, has the asymptotic property $x(\tau) = c\tau + o(\tau)$ as $\tau \to \infty$, for some $c \in \mathbb{R}$. Moreover, they proved that there exists a constant $c \in \mathbb{R}$ such that any global solution $x(\tau)$ of the initial value problem

$$\begin{gathered} {}^C\mathfrak{D}_a^\alpha x(\tau) = f\big(\tau, x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau), {}^C\mathfrak{D}_0^{\beta_1} x(\tau), \dots, {}^C\mathfrak{D}_0^{\beta_m} x(\tau)\big), \\ \tau > a. x^{(n-1)}(a) = c_i, \quad i = 0, 1, \dots, n-1, \end{gathered} \tag{1.12}$$

has the asymptotic property $x(\tau) = c\tau^k + o(\tau^k)$ as $\tau \to \infty$, where $k \in \max\{n-1, \beta_m\}$, $0 < \beta_1 < \dots . < \beta_m < \alpha < n$, and $n, m \in \mathbb{N}$.

In Sections 2 and 3, we prepare some material which will be needed later in our proofs. Sections 4, 5 and 6 are devoted to the main results on the asymptotic behavior results and boundedness of solutions for non-fractional and fractional source terms, respectively.

## 2. Preliminaries

In this section, we introduce some basic definitions, notation, properties and lemmas to be used in our results. We refer the reader to citeKilbas,Podl-1999,Samko for more details.

**Definition 2.1** ([16])**.** We introduce the space

$$C_\eta[a,b] = \{h : (a,b] \to \mathbb{R} : \quad h(\tau)(\tau - a)^\eta \in C[a,b]\}, \quad 1 > \eta \geq 0.$$

**Definition 2.2** ([16])**.** The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$\mathfrak{I}_a^\alpha f(\tau) := \frac{1}{\Gamma(\alpha)} \int_a^\tau \frac{f(s)}{(\tau - s)^{1-\alpha}} ds, \quad \tau > a,$$

provided that the right hand side exists.

**Definition 2.3** ([16])**.** The left-sided Riemann-Liouville fractional derivative of order $\alpha \geq 0$, $n - 1 \leq \alpha < n$, $n = -[-\alpha]$, is defined by

$$\begin{aligned} \mathfrak{D}_a^\alpha f(\tau) &= D^n \mathfrak{I}_a^{n-\alpha} f(\tau) = (\frac{d}{d\tau})^n \mathfrak{I}_a^{n-\alpha} f(\tau) \\ &= \frac{1}{\Gamma(n-\alpha)} \big(\frac{d}{d\tau}\big)^n \int_a^\tau \frac{f(s)}{(\tau - s)^{\alpha - n + 1}} ds, \quad \tau > a. \end{aligned}$$

In particular, when $\alpha = n$ we have $\mathfrak{D}_a^\alpha f = D^n f$, and when $\alpha = 0$, $\mathfrak{D}_a^0 f = f$.

**Definition 2.4** ([16])**.** The left-sided Caputo fractional derivative of order $\alpha \geq 0$, $n - 1 \leq \alpha < n$, $n = -[-\alpha]$, is defined by

$${}^C\mathfrak{D}_a^\alpha f(\tau) = \mathfrak{I}_a^{n-\alpha} f^{(n)}(\tau), \ \tau > a.$$

The fractional integral and fractional derivative of power functions have the same effect as the integer-order integral and derivative. Namely,

**Lemma 2.5** ([16])**.** *If $\beta > 0$ and $\alpha \geq 0$, then*

$$\begin{aligned} \mathfrak{I}_a^\alpha (\tau - a)^{\beta - 1} &= \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (\tau - a)^{\alpha + \beta + 1}, \quad \alpha > 0, \ \tau > a, \\ {}^C\mathfrak{D}_a^\alpha (\tau - a)^{\beta - 1} &= \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (\tau - a)^{\beta - \alpha - 1}, \quad \alpha \geq 0, \ \tau > a. \end{aligned}$$

*If $\beta = 1$, then $({}^C\mathfrak{D}_a^\alpha 1)(\tau) = 0$, $\tau > a$.*

The Riemann-Liouville fractional integral (Definition 2.2) satisfies the following semigroup property.

**Lemma 2.6.** [16]] *Let $0 \le \eta < 1$, $\alpha > 0$ and $\beta > 0$. If $h \in C_\eta[a,b]$, then*

$$\mathfrak{I}_a^\beta \mathfrak{I}_a^\alpha h(\tau) = \mathfrak{I}_a^{\beta+\alpha} h(\tau), \quad \tau > a.$$

The following result provides another composition of the fractional integration operator $\mathfrak{I}_a^\alpha$ with the fractional differentiation operator $\mathfrak{D}_a^\alpha$.

**Lemma 2.7** ([16])**.** *Let $\alpha > 0$, $0 \le \eta < 1$, $n = -[-\alpha]$. If $h \in C_\eta[a,b]$ and $\mathfrak{I}_a^{n-\alpha} h \in C_\eta^n[a,b]$, then*

$$\mathfrak{I}_a^\alpha \mathfrak{D}_a^\alpha h(\tau) = h(\tau) - \sum_{i=1}^{n} \frac{(\mathfrak{D}^{n-i} \mathfrak{I}_a^{n-\alpha} h)(a)}{\Gamma(\alpha - i + 1)} (\tau - a)^{\alpha - i}, \quad \tau > a.$$

**Lemma 2.8** ([16])**.** *Let $\alpha > 0$, $n = -[-\alpha]$. If $h \in C^n[a,b]$ or $h \in AC^n[a,b]$, then*

$$\mathfrak{I}_a^{\alpha\,C} \mathfrak{D}_a^\alpha h(\tau) = h(\tau) - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{k!} (\tau - a)^k, \quad \tau > a.$$

**Lemma 2.9.** *Let $0 < \beta \le \alpha < 1$. If $h \in AC[a,b]$, then*

$${}^C\mathfrak{D}_0^\beta h = \mathfrak{I}_0^{\alpha-\beta\,C} \mathfrak{D}_0^\alpha h.$$

*Proof.* From Definition 2.4 and Lemma 2.6, we have

$${}^C\mathfrak{D}_0^\beta h = \mathfrak{I}_0^{1-\beta} h' = \mathfrak{I}_0^{\alpha-\beta} \mathfrak{I}_0^{1-\alpha} h' = \mathfrak{I}_0^{\alpha-\beta\,C} \mathfrak{D}_0^\alpha h.$$

□

**Lemma 2.10** ([12])**.** *Let $f \in L_1(0,\infty)$. Then*

$$\lim_{\tau\to\infty} \frac{1}{\tau^\alpha} \mathfrak{I}_0^{\alpha+1} f(\tau) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(s)ds = \frac{1}{\Gamma(\alpha+1)} \mathfrak{I}_0^1 f(\infty), \ \alpha > 0.$$

**Lemma 2.11.** *Let $0 < \alpha < 1$ and $0 \le \eta < 1$. Assume that $x \in AC[0,\infty)$ and $\mathfrak{I}_0^{1-\alpha} x' \in C_\eta^1[0,\infty)$. Then*

$$\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = \lim_{\tau\to\infty} \frac{{}^C\mathfrak{D}_0^\alpha x(\tau)}{\Gamma(1+\alpha)}. \tag{2.1}$$

*Proof.* Since $x \in AC[0,\infty) \subset C_\eta[0,\infty)$ and $\mathfrak{I}_0^{1-\alpha} x' \in C_\eta^1[0,\infty)$, we can use Lemma 2.7 to obtain

$$\mathfrak{I}_0^\alpha \mathfrak{D}_0^\alpha x'(\tau) = x'(\tau) - \frac{\mathfrak{I}_0^{1-\alpha} x'(0)}{\Gamma(\alpha)} \tau^{\alpha-1}, \quad \tau > 0. \tag{2.2}$$

Applying $\mathfrak{I}_0^1$ to both sides of (2.2), using Lemma 2.5 (with $\beta = \alpha$) and Lemma 2.6, we obtain

$$x(\tau) = x(0) + \frac{\mathfrak{I}_0^{1-\alpha} x'(0)}{\Gamma(\alpha+1)} \tau^\alpha + \mathfrak{I}_0^{1+\alpha} \mathfrak{D}_0^\alpha x'(\tau), \quad \tau > 0. \tag{2.3}$$

Dividing both sides of (2.3) by $\tau^\alpha$, we obtain

$$\frac{x(\tau)}{\tau^\alpha} = \frac{x(0)}{\tau^\alpha} + \frac{\mathfrak{I}_0^{1-\alpha} x'(0)}{\Gamma(\alpha+1)} + \frac{1}{\tau^\alpha} \mathfrak{I}_0^{1+\alpha} \mathfrak{D}_0^\alpha x'(\tau), \quad \tau > 0.$$

Next, we take the limit as $\tau \to \infty$, we arrive at

$$\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = \frac{\mathfrak{I}_0^{1-\alpha} x'(0)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \lim_{\tau\to\infty} \mathfrak{I}_0^1 \mathfrak{D}_0^\alpha x'(\tau), \tag{2.4}$$

where we used Lemma 2.10. Moreover, we conclude that

$$\mathfrak{I}_0^1\mathfrak{D}_0^\alpha x'(\tau)=\mathfrak{I}_0^1 D\mathfrak{I}_0^{1-\alpha}x'(\tau)=\mathfrak{I}_0^{1-\alpha}x'(\tau)-\mathfrak{I}_0^{1-\alpha}x'(0)={}^C\mathfrak{D}_0^\alpha x(\tau)-{}^C\mathfrak{D}_0^\alpha x(0),\quad \tau>0, \tag{2.5}$$

and (2.1) follows directly from (2.4) and (2.5). □

## 3. Some useful inequalities

First, we define the following special classes of functions

$$H_k=\{h\in L_1(0,\infty): h \text{ is positive and } s^k h\in L_1(1,\infty),\ k>-1\}, \tag{3.1}$$

$$\begin{aligned} M=\big\{&F:(0,\infty)\times\mathbb{R}_+\to\mathbb{R}_+ \text{ where } 0\le F(\tau,s)-F(\tau,r)\le H(\tau)(s-r),\\ &\text{for some continuous function } H \text{ on } \mathbb{R}_+,\ s\ge r\ge 0 \text{ and } \tau>0\big\}. \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Phi=\big\{&\varphi\in C(0,\infty):\varphi \text{ is nondecreasing and positive on } (0,\infty),\\ &\frac{1}{v}\varphi(w)\le\varphi(\frac{w}{v}),\ w>0,\ v\ge 1\big\}. \end{aligned} \tag{3.3}$$

The proofs of the following lemmas are based on an application of the Bihari inequality which is a generalization of the Gronwall inequality.

**Lemma 3.1** ([12]). *Let $g(\tau)$ and $z(\tau)$ be nonnegative continuous functions defined for $\tau\ge 0$, $\varphi\in\Phi$ and $c_i\in\mathbb{R}$, $i=1,2,3$. Then*

$$z(\tau)\le c_1+c_2\tau^\gamma+c_3\tau^\gamma\int_0^\tau g(s)\varphi(z(s))ds,\quad \tau,\quad \gamma\ge 0, \tag{3.4}$$

*implies*

$$z(\tau)\le\begin{cases} E^{-1}(E(|c_1|+|c_2|)+|c_3|\int_0^\tau g(s)ds), & 0\le\tau<1\\ \tau^\gamma E^{-1}(E(A)+|c_3|\int_1^\tau s^\gamma g(s)ds), & \tau\ge 1, \end{cases} \tag{3.5}$$

*where*

$$A=|c_1|+|c_2|+|c_3|\varphi(E^{-1}(C))\int_0^1 g(s)ds,$$

$$C=E(|c_1|+|c_2|)+|c_3|\int_0^1 g(s)ds<\infty,$$

*and $E^{-1}$ is the inverse function of*

$$E(\xi)=\int_{\xi_0}^\xi\frac{ds}{\varphi(s)}.$$

**Lemma 3.2** ([12]). *Let $z(\tau)$ satisfy*

$$z(\tau)\le c_1\tau^\gamma+c_2\tau^\gamma\int_0^\tau[F_1(s,z(s)+c_3)+F_2(s,z(s)+c_4)+h(s)]ds,\quad \tau\ge 0, \tag{3.6}$$

*where $h:C[\mathbb{R}_+,\mathbb{R}_+]$, $F_j\in M$, $j=1,2$ and $\gamma$, $c_i>0$, $i=1,2,3,4$. Then*

$$z(\tau)\le\tau^\gamma f(\tau),\quad \tau>0, \tag{3.7}$$

*where*

$$\begin{aligned} f(\tau)=&\Big(c_1+c_2\int_0^\tau[F_1(s,c_3)+F_2(s,c_4)+h(s)]ds\Big)\\ &\times\exp\Big(c_2\int_0^\tau s^\gamma[N_1(s)+N_2(s)]ds\Big),\quad \tau>0, \end{aligned} \tag{3.8}$$

*with $N_1$ and $N_2$ are as in the definition of $M$ corresponding to $F_1$ and $F_2$, respectively.*

**Remark 3.3.** If $p, q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then for $\alpha > 0$, $p(\alpha-1)+1 > 0 \Longleftrightarrow q\alpha > 1$.

**Lemma 3.4** ([13]). *If $\upsilon, \lambda + 1 > 1/r$, for some $r > 1$, and $g$ is a nonnegative continuous function defined on $[0, \infty)$, then*

$$\int_0^\tau (\tau - s)^{\upsilon-1} s^\lambda g(s)ds \le C\tau^{\upsilon+\lambda-1/r}\Big(\int_0^\tau g^r(s)ds\Big)^{1/r}, \quad \tau > 0, \tag{3.9}$$

*where*

$$C = K_{p(\upsilon-1)+1,p\lambda} = \frac{\Gamma(p\lambda + 1)\Gamma(p(\upsilon - 1) + 1)}{\Gamma(p\lambda + p(\upsilon - 1) + 2)}, \quad \frac{1}{p} + \frac{1}{r} = 1.$$

**Lemma 3.5** ([13]). *Let $h$ and $z$ be nonnegative continuous functions defined on $[0, \infty)$. Let $\varphi_i(z) > 0$ on $(0, \infty)$, $i = 1, 2$, and $\varphi_i(z)$ are continuous nondecreasing functions defined on $[0, \infty)$. If*

$$z(\tau) \le K_1 + K_2\Big(\int_0^\tau h^q(s)\varphi_1^q(z(s))\varphi_2^q(z(s))ds\Big)^{1/q}, \quad q > 1,\ \tau > 0, \tag{3.10}$$

*where $K_i \in \mathbb{R}_+$, $i = 1, 2$, then*

$$z(\tau) \le \Big[E^{-1}\Big(E(2^{q-1}K_1) + 2^{q-1}K_2 \int_0^\tau h^q(s)ds\Big)\Big]^{1/q}, \quad \tau > 0,$$

*where $E^{-1}$ is the inverse of*

$$E(\xi) = \int_{\xi_0}^\xi \frac{ds}{\varphi_1^q(s^{1/q})\varphi_2^q(s^{1/q})}, \quad \xi > \xi_0 > 0.$$

## 4. Source without fractional derivatives

We consider the asymptotic behavior of solutions of the equation

$$({}^C\mathfrak{D}_0^\alpha x)'(\tau) = f(\tau, x(\tau)), \quad 0 < \alpha < 1,\ \tau \ge 0, \tag{4.1}$$

subject to

$$x(0) = b_1, \quad {}^C\mathfrak{D}_0^\alpha x(\tau)|_{\tau=0} = b_2, \tag{4.2}$$

in the space

$$C_{1-\alpha}^{\alpha,1}[0, \infty) = \{x \in AC[0, \infty), ({}^C\mathfrak{D}_0^\alpha x)' \in C_{1-\alpha}[0, \infty)\}. \tag{4.3}$$

We assume the following conditions:

- (C1) $f(\tau, x) \in C[[0, \infty) \times \mathbb{R}, \mathbb{R}]$ is such that $f(\cdot, x(\cdot)) \in C_{1-\alpha}[0, \infty)$ for any $x \in AC[0, \infty)$.
- (C2) There are continuous functions $P, \varphi : [0, \infty) \to [0, \infty)$ such that
$$|f(\tau, x(\tau))| \le \varphi(|x(\tau)|)P(\tau), \quad \tau \ge 0, \tag{4.4}$$
where
$$\int_1^\infty s^\alpha P(s)ds < \infty, \tag{4.5}$$
and $\varphi \in \Phi$.

**Theorem 4.1.** *Suppose $f$ satisfies* (C1), (C2) *and $x \in AC[0, \infty)$ is a solution of* (4.1)-(4.2). *Then*

$$\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = a \in \mathbb{R}, \quad \text{as } \tau \to \infty.$$

*Proof.* Integrating both sides of (4.1), we find

$$ {}^{C}\mathfrak{D}_0^{\alpha}x(\tau) = b_2 + \int_0^{\tau} f(s,x(s))ds = b_2 + \mathfrak{I}_0^{1} f(\tau, x(\tau)). \tag{4.6} $$

Applying $\mathfrak{I}_0^{\alpha}$ to both sides of (4.6), and using Lemmas 2.8 and 2.5, we arrive at

$$ x(\tau) = b_1 + \frac{b_2}{\Gamma(\alpha+1)}\tau^{\alpha} + \frac{1}{\Gamma(\alpha+1)}\int_0^{\tau}(\tau-s)^{\alpha} f(s,x(s))ds, \quad \tau \ge 0. \tag{4.7} $$

By using (4.4) we obtain

$$ |x(\tau)| \le |b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}\tau^{\alpha} + \frac{1}{\Gamma(\alpha+1)}\tau^{\alpha}\int_0^{\tau} P(s)\varphi(|x(s)|)ds, \quad \tau \ge 0. \tag{4.8} $$

Applying Lemma 3.1 to (4.8) we obtain

$$ |x(\tau)| \le \begin{cases} E^{-1}\big(E(|b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}) + \frac{1}{\Gamma(\alpha+1)}\int_0^{\tau} P(s)ds\big), & 0 \le \tau < 1 \\ \tau^{\alpha}E^{-1}\big(E(A) + \frac{1}{\Gamma(\alpha+1)}\int_1^{\tau} s^{\alpha}P(s)ds\big), & \tau \ge 1, \end{cases} $$

where

$$ \begin{gathered} A = |b_1| + \frac{|b_2|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\varphi(E^{-1}(K))\int_0^1 P(s)ds, \\ K = E(|b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}) + \frac{1}{\Gamma(\alpha+1)}\int_0^1 P(s)ds < \infty. \end{gathered} $$

From (4.5) and the continuity of $P$ on $\mathbb{R}_+$, we see that

$$ |x(\tau)| \le \begin{cases} C_1, & 0 \le \tau < 1, \\ \tau^{\alpha}C_2, & \tau \ge 1, \end{cases} \tag{4.9} $$

with

$$ \begin{gathered} C_1 = E^{-1}\Big(E\big(|b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}\big) + \frac{1}{\Gamma(\alpha+1)}\int_0^1 P(s)ds\Big) < \infty, \\ C_2 = E^{-1}\Big(E(A) + \frac{1}{\Gamma(\alpha+1)}\int_1^{\infty} s^{\alpha}P(s)ds\Big) < \infty. \end{gathered} $$

Next, it is clear that

$$ \begin{aligned} \int_0^{\tau}|f(s,x(s))|ds &\le \int_0^{\tau} P(s)\varphi(|x(s)|)ds \\ &\le \int_0^1 P(s)\varphi(|x(s)|)ds + \int_1^{\tau} P(s)\varphi(|x(s)|)ds \\ &\le \int_0^1 P(s)\varphi(|x(s)|)ds + \int_1^{\tau} s^{\alpha}P(s)\varphi(\frac{|x(s)|}{s^{\alpha}})ds, \quad \tau > 0. \end{aligned} \tag{4.10} $$

By (4.9) and (4.10), we have

$$ \lim_{\tau\to\infty}\int_0^{\tau} f(s,x(s))ds < \infty. \tag{4.11} $$

On the other hand, integrating (4.1) yields

$$ {}^{C}\mathfrak{D}_0^{\alpha}x(\tau) = b_2 + \int_0^{\tau} f(s,x(s))ds, \quad \tau > 0. \tag{4.12} $$

From (4.11) and (4.12), we conclude

$$\lim_{\tau\to\infty} {}^C\mathfrak{D}_0^\alpha x(\tau) = c, \quad c \in \mathbb{R}.$$

Further, by Lemma 2.11, we can write

$$\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = \lim_{\tau\to\infty} \frac{{}^C\mathfrak{D}_0^\alpha x(\tau)}{\Gamma(\alpha+1)} = a,$$

for some real number $a$. □

**Example 4.2.** All solutions of

$$({}^C\mathfrak{D}_0^\alpha x)'(\tau) = e^{-\tau} x^r(\tau), \quad 0 < \alpha, r \le 1,\ \tau > 0. \tag{4.13}$$

satisfy $\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = a$, as $\tau \to \infty$, for some real number $a$.

To prove this claim, let $\varphi(\tau) = \tau^r$ and $P(\tau) = e^{-\tau}$. Then

$$\int_1^\infty s^\alpha P(s)ds \le \int_0^\infty s^\alpha e^{-s} ds = \Gamma(\alpha+1) < \infty.$$

Obviously $\varphi$ is a nondecreasing and positive function with

$$u\varphi(v) = uv^r \le (vu)^r = \varphi(vu), \quad u \ge 1,\ v > 0,$$

and

$$\int_0^\infty \frac{ds}{\varphi(s)} = \int_0^\infty \frac{ds}{s^r} = \infty.$$

Then $\varphi \in \Phi$. All the conditions of Theorem 4.1 are satisfied, therefore every solution $x$ of (4.13) has satisfy $\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = a$, , $a \in \mathbb{R}$, as $\tau \to \infty$.

## 5. Equations with fractional source terms

We study problem (1.1) in the space $C^{\alpha,1}_{1-\alpha}[0,\infty)$ defined in (4.3) with the following assumptions:

(C3) $f(\tau, v, w) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is so that $f(\cdot, v(\cdot), w(\cdot)) \in C_{1-\alpha}[0,\infty)$ for every $v, w \in AC[0,\infty)$.

(C4)
$$|f(\tau, u(\tau), v(\tau))| \le F_1(\tau, |u(\tau)|) + F_2(\tau, \tau^\beta |v(\tau)|), \quad \tau \ge 0, \tag{5.1}$$
where $F_i \in M$, $i = 1, 2$.

**Lemma 5.1.** *Suppose that $f$ satisfies* (C3), (4) *and $x \in AC[0,\infty)$ is a solution of* (1.1)*. Then*

$$\max\big\{|x(\tau)|, \tau^\beta |{}^C\mathfrak{D}_0^\beta x(\tau)|\big\} \le |b_1| + z(\tau), \quad \tau > 0, \tag{5.2}$$

*where*

$$\begin{gathered} z(\tau) = C_2\tau^\alpha + C_3\tau^\alpha \int_0^\tau [F_1(s, |x(s)|) + F_2(s, \tau^\beta |{}^C\mathfrak{D}_0^\beta x(s)|)]ds, \quad \tau > 0, \\ C_3 = \max\big\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha-\beta+1)}\big\}, \quad C_2 = |b_2| C_3. \end{gathered} \tag{5.3}$$

*Proof.* Applying $\mathfrak{I}_0^1$ to (1.1), we obtain

$$\begin{aligned} {}^C\mathfrak{D}_0^\alpha x(\tau) &= b_2 + \mathfrak{I}_0^1 f\big(\tau, x(\tau), {}^C\mathfrak{D}_0^\beta x(\tau)\big) \\ &= b_2 + \int_0^\tau f\big(s, x(s), {}^C\mathfrak{D}_0^\beta x(s)\big)ds, \quad \tau > 0. \end{aligned} \tag{5.4}$$

Next, we apply $\mathfrak{I}_0^\alpha$ to both sides of (5.4), using Lemmas 2.6, 2.8 and 2.5, we find

$$\begin{aligned} x(\tau) &= b_1 + \frac{b_2}{\Gamma(\alpha+1)}\tau^\alpha + \mathfrak{I}_0^{1+\alpha} f\big(\tau, x(\tau), {}^C\mathfrak{D}_0^\beta x(\tau)\big) \\ &= b_1 + \frac{b_2}{\Gamma(\alpha+1)}\tau^\alpha + \frac{1}{\Gamma(\alpha+1)}\int_0^\tau (\tau-s)^\alpha f(s, x(s), {}^C\mathfrak{D}_0^\beta x(s))ds, \end{aligned} \tag{5.5}$$

for $\tau > 0$. Thus, from (5.5) and (5.1) we have

$$\begin{aligned} |x(\tau)| &\le |b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}\tau^\alpha + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\int_0^\tau \big|f\big(s, x(s), {}^C\mathfrak{D}_0^\beta x(s)\big)\big|ds \\ &\le |b_1| + C_2\tau^\alpha + C_3\tau^\alpha \int_0^\tau \Big(F_1(s, |x(s)|) + F_2\big(s, s^\beta|{}^C\mathfrak{D}_0^\beta x(s)|\big)\Big)ds, \end{aligned} \tag{5.6}$$

for $\tau > 0$. By Lemma 2.9, we see that

$$ {}^C\mathfrak{D}_0^\beta x(\tau) = \mathfrak{I}_0^{\alpha-\beta}({}^C\mathfrak{D}_0^\alpha x(\tau)). \tag{5.7}$$

Let us insert the expression (5.4) into (5.7), using Lemmas 2.6 and 2.5, we have

$$\begin{aligned} {}^C\mathfrak{D}_0^\beta x(\tau) &= \mathfrak{I}_0^{\alpha-\beta}\Big(b_2 + \mathfrak{I}_0^1 f\big(s, x(s), {}^C\mathfrak{D}_0^\beta x(s)\big)\Big)(\tau) \\ &= \frac{b_2}{\Gamma(\alpha-\beta+1)}\tau^{\alpha-\beta} + \mathfrak{I}_0^{\alpha-\beta+1} f\big(\tau, x(\tau), {}^C\mathfrak{D}_0^\beta x(\tau)\big) \\ &= \frac{b_2}{\Gamma(\alpha-\beta+1)}\tau^{\alpha-\beta} \\ &\quad + \frac{1}{\Gamma(\alpha-\beta+1)}\int_0^\tau (\tau-s)^{\alpha-\beta} f(s, x(s), {}^C\mathfrak{D}_{0^+}^\beta x(s))ds, \quad \tau > 0. \end{aligned}$$

Then from this and (5.1) we obtain the bound

$$\begin{aligned} &\tau^\beta|{}^C\mathfrak{D}_0^\beta x(\tau)| \\ &\le C_3|b_2|\tau^\alpha + C_3\tau^\alpha\int_0^\tau |f(s, x(s), {}^C\mathfrak{D}_{0^+}^\beta x(s))|ds \\ &\le C_2\tau^\alpha + C_3\tau^\alpha\int_0^\tau \big(F_1(s, |x(s)|) + F_2(s, s^\beta|{}^C\mathfrak{D}_0^\beta x(s)|)\big)ds, \quad \tau > 0. \end{aligned} \tag{5.8}$$

Relation (5.2) follows directly from (5.3), (5.6) and (5.8). □

**Theorem 5.2.** *Suppose that $f$ satisfies **(C3)**-**(C4)** and*

$$\int_0^\infty s^\alpha N_i(s)ds < \infty, \quad \int_0^\infty F_i(s, |b_1|)ds < \infty, \ i = 1, 2. \tag{5.9}$$

*Then, every solution $x(\tau)$ of problem (1.1) has the following property*

$$\lim_{\tau\to\infty} \frac{x(\tau)}{\tau^\alpha} = a, \quad a \in \mathbb{R}.$$

*Proof.* By using Lemma 5.1 we have

$$F_1(\tau,|x(\tau)|)\le F_1(\tau,|b_1|+z(\tau)),\quad \tau>0, \tag{5.10}$$

$$F_2\big(\tau,\tau^\beta|{}^C\mathfrak{D}_0^\beta x(\tau)|\big)\le F_2(\tau,z(\tau)+|b_1|),\quad \tau>0. \tag{5.11}$$

Taking into account (5.3), (5.10) and (5.11) we arrive at

$$z(\tau)\le C_2\tau^\alpha+C_3\tau^\alpha\int_0^\tau [F_1(s,|b_1|+z(s))+F_2(s,z(\tau)+|b_1|)]ds,\quad \tau>0.$$

Then, by Lemma 3.2, we find that

$$z(\tau)\le C\tau^\alpha,\quad \tau>0 \tag{5.12}$$

where

$$\begin{aligned} C=&\Big(C_2+C_3\int_0^\infty [F_1(s,|b_1|)+F_2(s,|b_1|)]ds\Big)\\ &\times\exp(C_3\int_0^\infty s^\gamma[N_1(s)+N_2(s)]ds)<\infty. \end{aligned}$$

It follows from Lemma 5.1 and (5.12) that

$$|x(\tau)|\le |b_1|+C\tau^\alpha,\quad \tau^\beta|{}^C\mathfrak{D}_0^\beta x(\tau)|\le |b_1|+C\tau^\alpha,\quad \tau>0. \tag{5.13}$$

Again by hypothesis (5.1) we have

$$\begin{aligned} |\int_0^\tau f(s,x(s),{}^C\mathfrak{D}_0^\beta x(s))ds| &\le \int_0^\tau |f(s,x(s),{}^C\mathfrak{D}_0^\beta x(s))|ds\\ &\le \int_0^\tau [F_1(s,|x(s)|)+F_2(s,s^\beta|{}^C\mathfrak{D}_0^\beta x(s)|)]ds, \end{aligned}$$

for $\tau>0$. From this inequality and (5.13), we obtain

$$\begin{aligned} &|\int_0^\tau f(s,x(s),{}^C\mathfrak{D}_0^\beta x(s))ds|\\ &\le \int_0^\tau [F_1(s,|b_1|+Cs^\alpha)+F_2(s,|b_1|+Cs^\alpha)]ds\\ &=\int_0^\tau \Big\{F_1(s,|b_1|+Cs^\alpha)-F_1(s,|b_1|)+F_1(s,|b_1|)\\ &\quad +F_2(s,|b_1|+Cs^\alpha)-F_2(s,|b_1|)+F_2(s,|b_1|)\Big\}ds,\quad \tau>0. \end{aligned}$$

As the functions $F_i$, $i=1,2$ are in $M$, we obtain

$$\begin{aligned} &\Big|\int_0^\tau f(s,x(s),{}^C\mathfrak{D}_0^\beta x(s))ds\Big|\\ &\le C\int_0^\tau s^\alpha[N_1(s)+N_s(s)]ds+\int_0^\tau [F_1(s,|b_1|)+F_2(s,|b_1|)]ds<\infty, \end{aligned} \tag{5.14}$$

where we have used (5.9). Then

$$\lim_{\tau\to\infty}\int_0^\tau f(s,x(s),{}^C\mathfrak{D}_0^\beta x(s))ds<\infty.$$

By (5.4) we conclude that there is $b\in\mathbb{R}$ such that $\lim_{\tau\to\infty}{}^C\mathfrak{D}_0^\alpha x(\tau)=b$. Further, by Lemma 2.11, we deduce that

$$\lim_{\tau\to\infty}\frac{x(\tau)}{\tau^\alpha}=\lim_{\tau\to\infty}\frac{{}^C\mathfrak{D}_0^\alpha x(\tau)}{\Gamma(\alpha+1)}=a,$$

and the proof is now complete. □

## 6. Boundedness

We consider the fractional differential problem (1.2) in the space

$$C^\alpha[0,\infty) = \{x \in AC[0,\infty) : {}^C\mathfrak{D}_0^\alpha x \in C[0,\infty)\}. \tag{6.1}$$

We assume the following conditions:

(C5) $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is so that $f(\cdot, v(\cdot), w(\cdot)) \in C[0,\infty)$ for every $v$, $w$ in $C[0,\infty)$.

(C6)
$$|f(\tau,u,v)| \le \tau^\gamma h(\tau)\varphi_1(|u(\tau)|)\varphi_2(|v(\tau)|), \quad \tau > 0, \tag{6.2}$$
where $h$, $\varphi_1$, $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions with $\varphi_i$, $i = 1,2$, are nondecreasing functions and $h \in L_q(0,\infty)$ for some $q > \frac{1}{\alpha-\beta}$, $\gamma = \frac{1}{q} - \alpha$.

**Lemma 6.1.** *: Suppose that $f$ satisfies (C5), (C6) and $x \in AC[0,\infty)$ is a solution of* (1.2)*. Then*

$$\max\{|x(\tau)|, |{}^C\mathfrak{D}_0^\beta x(\tau)|\} \le z(\tau), \quad \tau \ge \tau_0 > 0, \tag{6.3}$$

*where*

$$z(\tau) = |b| + K_1\Big(\int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|{}^C\mathfrak{D}_0^\beta x(s)|)ds\Big)^{1/q}, \quad \tau > 0, \tag{6.4}$$

*and*

$$K_1 = \max\Big\{\frac{K^{1/p}_{1+p(\alpha-1),p\gamma}}{\Gamma(\alpha)}, \frac{K^{1/p}_{1+p(\alpha-\beta-1),p\gamma}}{\Gamma(\alpha-\beta)\tau_0^\beta}\Big\},$$
$$K_{\alpha,\beta} = \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

*Proof.* Applying $\mathfrak{I}_0^\alpha$ to (1.2) and taking into account Lemma 2.8, we have

$$x(\tau) = b + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-s)^{\alpha-1} f(s, x(s), {}^C\mathfrak{D}_0^\beta x(s))ds, \quad \tau > 0. \tag{6.5}$$

Using the inequality (6.2), we obtain

$$|x(\tau)| \le |b| + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-s)^{\alpha-1} s^\gamma h(s)\varphi_1(|x(s)|)\varphi_2(|{}^C\mathfrak{D}_0^\beta x(s)|)ds, \tag{6.6}$$

for $\tau > 0$. It follows from the assumptions $\beta < \alpha$, $q > \frac{1}{\alpha-\beta}$ and $\gamma = \frac{1}{q} - \alpha$ that $p(\alpha-1)+1 \ge p(\alpha-\beta-1)+1 > 0$ and $p\gamma + 1 = p(\frac{1}{q}-\alpha)+1 = p(1-\alpha) > 0$. Then, we apply Lemma 3.4, to obtain

$$\begin{aligned}|x(\tau)| &\le |b| + \frac{1}{\Gamma(\alpha)}K^{1/p}_{p(\alpha-1)+1,p\gamma}\tau^{\alpha+\gamma-1/q}\Big(\int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|{}^C\mathfrak{D}_0^\beta x(s)|)ds\Big)^{1/q}\\ &\le |b| + K_1\Big(\int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|{}^C\mathfrak{D}_0^\beta x(s)|)ds\Big)^{1/q}, \quad \tau > 0.\end{aligned} \tag{6.7}$$

Also, by Lemma 2.9, we conclude that

$$\begin{aligned} {}^C\mathfrak{D}_0^\beta x(\tau) &= \mathfrak{I}_0^{\alpha-\beta}\ {}^C\mathfrak{D}^\alpha x(\tau) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^\tau (\tau-s)^{\alpha-\beta-1}\ {}^C\mathfrak{D}_0^\alpha x(s)ds \\ &= \frac{1}{\Gamma(\alpha-\beta)}\int_0^\tau (\tau-s)^{\alpha-\beta-1} f(s,x(s),{}^C\mathfrak{D}_0^\beta x(s))ds, \quad \tau>0. \end{aligned} \tag{6.8}$$

In view of (6.2), we have

$$|{}^C\mathfrak{D}_0^\beta x(\tau)| \le \frac{1}{\Gamma(\alpha-\beta)}\int_0^\tau (\tau-s)^{\alpha-\beta-1}s^\gamma h(s)\varphi_1(|x(s)|)\varphi_2(|{}^C\mathfrak{D}_0^\beta x(s)|)ds,$$

for $\tau>0$. Again, from Lemma 3.4, we find

$$\begin{aligned} &|{}^C\mathfrak{D}_0^\beta x(\tau)| \\ &\le \frac{K^{1/p}_{p(\alpha-\beta-1)+1,p\gamma}}{\Gamma(\alpha-\beta)}\tau^{\alpha-\beta+\gamma-1/q}\Big(\int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|{}^C\mathfrak{D}_0^\beta x(s)|)ds\Big)^{1/q} \\ &\le \frac{K^{1/p}_{p(\alpha-\beta-1)+1,p\gamma}}{\Gamma(\alpha-\beta)}\tau^{-\beta}\big(\int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|{}^C\mathfrak{D}_0^\beta x(s)|)ds\big)^{1/q} \\ &\le K_1(\int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|{}^C\mathfrak{D}_0^\beta x(s)|)ds)^{1/q}, \quad \tau\ge\tau_0>0. \end{aligned} \tag{6.9}$$

Therefore (6.3) follows from (6.4), (6.7) and (6.9). □

**Theorem 6.2.** *Assume that $f$ satisfies* (C5), (C6). *Then, any solution $x$ of* (1.2) *satisfies*

$$|x(\tau)|\le C, \quad |{}^C\mathfrak{D}_0^\beta x(\tau)|<C,$$

*for some positive constant $C,\tau>0$, provided that*

$$\int_{\xi_0}^\infty \frac{ds}{\varphi_1^q(s^{1/q})\varphi_2^q(s^{1/q})}=\infty, \quad \xi_0>0.$$

*Proof.* In view of Lemma 6.1 we have

$$\varphi_1(|x(\tau)|)\le\varphi_1(z(\tau)), \quad {}_2(|{}^C\mathfrak{D}_0^\beta x(\tau)|)\le\varphi_2(z(\tau)), \quad \tau>0. \tag{6.10}$$

From this inequality and (6.4), we obtain

$$z(\tau)\le|b|+K_1\Big(\int_0^\tau h^q(s)\varphi_1^q(z(s))\varphi_2^q(z(s))ds\Big)^{1/q}, \quad \tau>0. \tag{6.11}$$

Therefore, Lemma 3.5 implies

$$z(\tau)\le\Big[E^{-1}\Big(E(2^{q-1}K_1)+2^{q-1}K_2\int_0^\tau h^q(s)ds\Big)\Big]^{1/q}<\infty,$$

because $h\in L_q(0,\infty)$. This completes the proof. □

**Example 6.3.** Consider the problem

$$\begin{gathered} {}^C\mathfrak{D}_0^{2/3}x(\tau)=\tau^{1/q-2/3}e^{-\lambda\tau}(x(\tau))^{3/5}\big({}^C\mathfrak{D}_0^{1/3}x(\tau)\big)^{1/3}\big(\cos({}^C\mathfrak{D}_0^{1/3}x)\big) \quad \tau>0, \\ x(0)=b, \quad q>3, \quad \lambda>0. \end{gathered} \tag{6.12}$$

Let $\varphi_1(\tau) = \tau^{3/5}$, $\varphi_2(\tau) = \tau^{1/3}$ and $h(\tau) = e^{-\lambda\tau}$, $\gamma = 1/q - 2/3$. Then $h \in L_q(0,\infty)$ and

$$\int_{\xi_0}^{\infty} \frac{ds}{\varphi_1^q(s^{\frac{1}{q}})\varphi_2^q(s^{1/q})} = \int_{\xi_0}^{\infty} \frac{ds}{s^{3/5}s^{1/3}} = \int_{\xi_0}^{\infty} \frac{ds}{s^{14/15}} = \infty.$$

Then, by Theorem 6.2, we deduce that any solution $x$ of (6.12) satisfies

$$|x(\tau)| \le C, \quad |{}^C\mathfrak{D}_0^{\beta} x(\tau)| < C,$$

for $\alpha = 2/3$, $\beta = 1/3$, and $\tau > 0$.

**Acknowledgements.** M. D. Kassim wants to thank Imam Abdulrahman Bin Faisal University for its support and facilities. N. E. Tatar is grateful for the financial support and the facilities provided by King Fahd University of Petroleum and Minerals through project number IN181008.

## References

[1] D. Băleanu, O. G. Mustafa; *On the asymptotic integration of a class of sublinear fractional differential equations*, J. Math. Phys., 50 (2009) 123520.

[2] D. Băleanu, O. G. Mustafa, R. P. Agarwal; *On the solution set for a class of sequential fractional differential equations*, J. Phys. A: Math. Theor. 43 (2010) 385209.

[3] D. Băleanu, O. G. Mustafa, R. P. Agarwal; *Asymptotically linear solutions for some linear fractional differential equations*, Abstr. Appl. Anal., (2010). Article ID 865139. http://dx.doi.org/10.1155/2010/865139.

[4] D. Băleanu, O. G. Mustafa, R. P. Agarwal; *Asymptotic integration of $(1+\alpha)$-order fractional differential equations*, Computers Math. Appl., 62 (2011), 1492–1500.

[5] D. Băleanu, R. P. Agarwal, O. G. Mustafa, M. Coşulschi; *Asymptotic integration of some nonlinear differential equations with fractional time derivative*, J. Phys. A: Math. Theor., 44 (2011), 055203 9 pp.

[6] E. Brestovanská M. Medveď; *Asymptotic behavior of solutions to second-order differential equations with fractional derivative perturbations*, Electronic Journal of Differential Equations, 2014 (2014) no. 201, 1-10.

[7] D. S. Cohen; *The asymptotic behavior of a class of nonlinear differential equations*, Proc. Amer. Math. Soc. 18 (1967), 607–609.

[8] A. Constantin; *On the asymptotic behavior of second order nonlinear differential equations*, Rend. Math. Appl., 13 (7) (1993), 627–634.

[9] A. Constantin; *On the existence of positive solutions of second order differential equations*, Annali di Matematica, Vol. 184 (2005), 131-138.

[10] F. M. Dannan; *Integral inequalities of Gronwall-Bellman-Bihari type and asymptotic behavior of certain second order nonlinear differential equations*, J. Math. Anal. Appl., 108 (1985), 151–164.

[11] M. D. Kassim; *Well-posedness for a Cauchy fractional differential problem with Hilfer type fractional derivative*, Ph D thesis, King Fahd University of Petroleum and Minerals, Saudi Arabia (2011).

[12] M. Kassim, K. Furati, N.-E. Tatar; *Asymptotic behavior of solutions to nonlinear fractional differential equations*, Math. Model Anal., 21:5 (2016), 610-629.

[13] M. D. Kassim, K. M. Furati, N.-E. Tatar; *Asymptotic behavior of solutions to nonlinear initial-value fractional differential problems*, Electronic Journal of Differential Equations, 2016 (2016) no. 291, 1-14.

[14] M. D. Kassim, N.-E. Tatar; *Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative*, Abstract and Applied Analysis. Vol. 2013, 1-12.

[15] M. D. Kassim, N.-E. Tatar; *Stability of logarithmic type for a Hadamard fractional differential problem*, J. Pseudo-Differ. Oper. Appl., 11 (2020), 447–466.

[16] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Editor: Jan van Mill, Elsevier, Amsterdam, The Netherlands 2006.

[17] T. Kusano, W. F. Trench; *Global existence of second order differential equations with integrable coefficients*, J. London Math. Soc. 31 (1985), 478–486.
[18] T. Kusano, W. F. Trench; *Existence of global solutions with prescribed asymptotic behavior for nonlinear ordinary differential equations*, Mat. Pura Appl. 142 (1985), 381–392.
[19] O. Lipovan; *On the asymptotic behaviour of the solutions to a class of second order nonlinear differential equations*, Glasg. Math. J. 45 no. 1, (2003), 179–187.
[20] M. Medveď; *On the asymptotic behavior of solutions of nonlinear differential equations of integer and also of non-integer order*, Electron. J. Qual. Th. Diff. Eq., Proc. 9th Coll. QTDE, No. 10 (2012), 1-9.
[21] M. Medveď; *Asymptotic integration of some classes of fractional differential equations*, Tatra Mt. Math. Publ. 54 (2013), 119–132.
[22] M. Medveď, E. Pekárková; *Asymptotic integration of differential equations with singular p-laplacian*, Archivum Mathematicum (Brno), 52 (2016), 13-19.
[23] M. Medveď, E. Pekárková; *Large time behavior of solutions to second-order differential equations with p-Laplacian*, Electronic Journal of Differential Equations, 2008 (2008) no. 108, 1-12.
[24] M. Medveď, M. Pospíšil; *Asymptotic integration of fractional differential equations with integrodifferential right-hand side*, Math. Model. Anal., 20.4 (2015), 471-489.
[25] O. G. Mustafa, Y. V. Rogovchenko; *Global existence of solutions with prescribed asymptotic behavior for second-order nonlinear differential equations*, Nonl. Anal. TMA, 51 (2002), 339–368.
[26] I. Podlubny; *Fractional Differential Equations*, Mathematics in Sciences and Engineering, 198, Academic Press, San-Diego, 1999.
[27] Y. V. Rogovchenko; *On asymptotics behavior of solutions for a class of second order nonlinear differential equations*, Collect. Math., 49 (1) (1998), 113–120.
[28] S. P. Rogovchenko, Y. V. Rogovchenko; *Asymptotics of solutions for a class of second order nonlinear differential equations*, Portugaliae Math., 57 (1) (2000), 17–32.
[29] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, 1987. (Trans. from Russian) 1993.
[30] J. Tong; *The asymptotic behavior of a class of nonlinear differential equations of second order*, Proc. Amer. Math. Soc., 84 (1982), 235–236.
[31] W. F. Trench; *On the asymptotic behavior of solutions of second order linear differential equations*, Proc. Amer. Math. Soc., 54 (1963), 12–14.
[32] P. Waltman; *On the asymptotic behavior of solutions of a nonlinear equation*, Proc. Amer. Math. Soc. 15 (1964), 918-923.

Mohammed Dahan Kassim
Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 31441, Saudi Arabia
*Email address*: mdkassim@iau.edu.sa

Nasser Eddine Tatar
King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran, 31261, Saudi Arabia
*Email address*: tatarn@kfupm.edu.sa