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FREE BOUNDARY VALUE PROBLEM FOR COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. In this article we consider a free boundary value problem for barotropic compressible magnetohydrodynamic equations with density-dependent viscosity coefficients. Under certain assumptions imposed on the initial data, there exists a unique global strong solution which is strictly positive after a finite time. Furthermore, the free boundaries propagate along the particle path and the domain expands outwards at an algebraic rate.

1. Introduction

The three-dimensional barotropic compressible magnetohydrodynamic equation with density-dependent viscosity coefficients read

$$\begin{gathered} \rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \\ (\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P(\rho) \\ = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mu \Delta \mathbf{U} + (\mu + \lambda(\rho)) \nabla \operatorname{div} \mathbf{U}, \\ \mathbf{H}_t - \nabla \times (\mathbf{U} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \\ \operatorname{div} \mathbf{H} = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T], \end{gathered} \tag{1.1}$$

where $\rho(\mathbf{x},t) > 0$, $\mathbf{U}(\mathbf{x},t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ stand for the flow density, velocity and pressure respectively. $\mathbf{H}(\mathbf{x},t)$ is the magnetic field with $\mathbf{x} = (x, y, z)$. The shear viscosity coefficient $\mu > 0$ is a positive constant, and the bulk viscosity coefficient is $\lambda(\rho) = \rho^\beta$ with $\beta > 0$. The constant $\nu > 0$ is the resistivity coefficient which is inversely proportional to the electrical conductivity constant.

In this article we focus on the free boundary value problem for one-dimensional barotropic compressible Magnetohydrodynamic equations with density-dependent viscosity coefficients. The existence, regularity and dynamical behavior of global strong solution will be discussed. For $\gamma > 1$ and $\beta > 0$, we show that the free boundary value problem with regular initial data admits a unique global strong solution which is strictly positive from a finite time and decays pointwise to zero at an algebraic time-rate. also the domain expands outwards at an algebraic rate. See Theorem 2.1.

The rest of this article is arranged as follows. In section 2, the main results about existence and dynamical behavior of global strong solutions for compressible

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Magnetohydrodynamic equations are stated. In section 3, a priori estimates will be given. Then, the main results are proven in section 4.

2. Main results

Consider a magnetic flow which is moving in the x-direction and uniform in the transverse direction (y,z) under a planar magnetic field. Let $\rho(\mathbf{x},t)=\rho(x,t)$, $\mathbf{U}(\mathbf{x},t)=(u(x,t),0,0)$ and $\mathbf{H}(\mathbf{x},t)=(0,H_2(x,t),H_3(x,t))$. This means the longitudinal velocity is $u(x,t)$ and the transverse velocity is $(0,0)$. The longitudinal magnetic field is 0 and the transverse magnetic field is $(H_2(x,t),H_3(x,t))$. We assume $H_2(x,t)=H(x,t)$, $H_3(x,t)=kH(x,t)$ and the constant k is in $[0,+\infty)$.

We investigate the existence and dynamics of of a global solution of the free boundary value problem for the planar magnetohydrodynamic equations with density-dependent viscosity coefficient,

$$\begin{gathered}
\rho_t+(\rho u)_x=0,\quad x\in(a(t),b(t)),\ t>0,\\
(\rho u)_t+(\rho u^2)_x+(\rho^\gamma)_x=-(1+k^2)HH_x+((2\mu+\rho^\beta)u_x)_x,\\
x\in(a(t),b(t)),\ t>0,\\
H_t+(uH)_x=\nu H_{xx},\quad x\in(a(t),b(t)),\, t>0,\\
(\rho^\gamma-(2\mu+\rho^\beta)u_x)(a(t),t)=0,\quad (\rho^\gamma-(2\mu+\rho^\beta)u_x)(b(t),t)=0,\ t>0,\\
H(a(t),t)=H(b(t),t)=0,\quad t>0,\\
(\rho,u,H)(x,0)=(\rho_0,u_0,H_0),\quad x\in[a_0,b_0],
\end{gathered}\tag{2.1}$$

where $x=a(t)$ and $x=b(t)$ are the free boundaries defined by

$$\begin{gathered}
\frac{d}{dt}a(t)=u(a(t),t),\quad a(0)=a_0,\\
\frac{d}{dt}b(t)=u(b(t),t),\quad b(0)=b_0,\ t>0.
\end{gathered}\tag{2.2}$$

The initial data satisfies

$$\begin{gathered}
\inf_{[a_0,b_0]}\rho_0\ge\underline{\rho}>0,\quad \rho_0\in L^1([a_0,b_0]),\quad \rho_{0x}\in L^2([a_0,b_0]),\\
(\rho_0^\gamma-(2\mu+\rho_0^\beta)u_{0x})(a_0)=0,\quad (\rho_0^\gamma-(2\mu+\rho_0^\beta)u_{0x})(b_0)=0,\\
H_0(a_0)=H_0(b_0)=0,\quad u_0\in H^2([a_0,b_0]),\quad H_0\in H^1([a_0,b_0]),
\end{gathered}\tag{2.3}$$

where $\underline{\rho}$ is a positive constant.

Without loss of generality, the total initial mass can be renormalized to be one. By the conservation of mass,

$$\int_{a(t)}^{b(t)}\rho(x,t)dx=\int_{a_0}^{b_0}\rho_0(x)dx:=1.\tag{2.4}$$

Then, we have the existence of a global solution and time-asymptotical behavior of strong solution as follows.

Theorem 2.1. *Let $\gamma > 1$, $\beta > 0$ and $T > 0$. Assume that the initial data satisfies (2.3). Then, there exists a global strong solution (ρ, u, H, a, b) to (2.1) satisfying*

$$\begin{gathered}
(\rho, u) \in C^0([0,T] \times [a(t), b(t)]), \\
c_T \le \rho \in L^\infty(0,T; H^1([a(t), b(t)])), \\
\rho_t \in L^\infty(0,T; L^2([a(t), b(t)])), \\
u \in L^\infty(0,T; H^2([a(t), b(t)])) \cap L^2(0,T; H^3([a(t), b(t)])), \\
u_t \in L^\infty(0,T; L^2([a(t), b(t)])) \cap L^2(0,T; H^1([a(t), b(t)])), \\
H \in L^\infty(0,T; H^1([a(t), b(t)])), \\
a(t), b(t) \in H^2([0,T]), \\
\rho^\gamma - (2\mu + \rho^\beta) u_x \in C^0([0,T] \times ([a(t), b(t)])),
\end{gathered} \tag{2.5}$$

where $c_T > 0$ is a constant depending on time.

As $\beta > 0$, the domain expands outwards in time as

$$\begin{aligned}
D_M(t) &:= \sup_{\tau \in [0,t]} (b(\tau) - a(\tau)) \\
&\ge \begin{cases} C(1+t)^{\frac{\gamma-1}{\gamma}}, & 1 < \gamma < 2, \\ C(1+t)^{1/\gamma}(1 + \ln(1+t))^{-1/\gamma}, & \gamma \ge 2\,. \end{cases}
\end{aligned} \tag{2.6}$$

Furthermore,

$$b(t) - a(t) \ge C(1+t)^{1/\gamma}(1 + \ln(1+t))^{-1/\gamma}, \quad \gamma \ge 2, \tag{2.7}$$

where $C > 0$ is a constant independent of time.

In particular, if $0 < \beta \le 1$, we have

$$\int_{a(t)}^{b(t)} \rho^\gamma(x,t)dx + \int_{a(t)}^{b(t)} H^2(x,t)dx \le C(1+t)^{-\eta}, \quad 0 < \beta \le 1. \tag{2.8}$$

where $0 < \eta \le \min\{\gamma - 1, \beta\}$ denotes a positive constant.

Remark 2.2. If $H_3(x,t) \ne kH_2(x,t)$, the system becomes

$$\begin{gathered}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + (\rho^\gamma)_x = -H_2 H_{2x} - H_3 H_{3x} + ((2\mu + \rho^\beta) u_x)_x, \\
H_{2t} + (uH_2)_x = \nu H_{2xx}, \\
H_{3t} + (uH_3)_x = \nu H_{3xx},
\end{gathered} \tag{2.9}$$

using the some method as in Theorem 2.1, the well-posedness of the solutions to the free boundary value problem with initial finite mass can also be proved.

Remark 2.3. conditions (2.6) and (2.7) imply that as time approaches infinity, the lower bound approaches infinity.

3. A priori estimates

In this section, we deduce a priori estimates for the solution (ρ, u, H) to the (2.1).

Lemma 3.1. *Under the assumptions of Theorem 2.1, for every strong solution (ρ, u, H) of (2.1) satisfies*

$$\begin{aligned}
&\int_{a(t)}^{b(t)} \big(\frac{1}{2}\rho u^2 + \frac{1+k^2}{2}H^2 + \frac{1}{\gamma-1}\rho^\gamma\big)dx + \int_0^t \int_{a(s)}^{b(s)} (2\mu+\rho^\beta)u_x^2\,dx\,ds \\
&+ \nu \int_0^t \int_{a(s)}^{b(s)} H_x^2\,dx\,ds \\
&= \int_{a_0}^{b_0} \big(\frac{1}{2}\rho_0 u_0^2 + \frac{1+k^2}{2}H_0^2 + \frac{1}{\gamma-1}\rho_0^\gamma\big)dx, \quad t\in[0,T].
\end{aligned} \tag{3.1}$$

Proof. Taking the product of $(2.1)_2$ and $(2.1)_3$ with u and H respectively, and integrating on $[a(t), b(t)]$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{a(t)}^{b(t)} \big(\frac{1}{2}\rho u^2 + \frac{1+k^2}{2}H^2 + \frac{1}{\gamma-1}\rho^\gamma\big)dx + \int_{a(t)}^{b(t)} (2\mu+\rho^\beta)u_x^2 dx \\
&+ \nu \int_{a(t)}^{b(t)} H_x^2 dx = 0,
\end{aligned} \tag{3.2}$$

which leads to (3.1) after the integrating with respect to $t\in[0,T]$. □

Lemma 3.2. *Under the assumptions of Theorem 2.1, we have*

$$c_T \le \rho(x,t) \le C, \quad (x,t)\in[a(t),b(t)]\times[0,T], \quad T>0, \tag{3.3}$$

where C is a positive constant independent of time, and c_T is also a positive constant but dependent of time.

Proof. Firstly, denote the effective viscous flux by

$$F = (2\mu+\lambda(\rho))u_x - \rho^\gamma - \frac{1+k^2}{2}H^2. \tag{3.4}$$

Then, we can rewrite $(2.1)_2$ as

$$\rho\dot{u} = F_x, \tag{3.5}$$

where $\dot{u} = u_t + uu_x$. Define

$$\zeta(x,t) = \int_{a(t)}^{x} \rho u(x,t)dx, \tag{3.6}$$

$$\eta(x,t) = \rho u^2(x,t) - \rho u^2(a(t),t). \tag{3.7}$$

Integrating $(2.1)_2$ from $a(t)$ to x, and using (3.6) and (3.7), we have

$$\zeta_t + \eta - F = -\rho u^2(a(t),t). \tag{3.8}$$

Define

$$\theta(\rho) = \int_1^\rho \frac{2\mu+\lambda(s)}{s}ds = 2\mu\ln\rho + \frac{1}{\beta}(\rho^\beta - 1). \tag{3.9}$$

Multiplying $(2.1)_1$ by $\theta'(\rho)$, we have

$$\theta(\rho)_t + u\theta(\rho)_x + (2\mu+\lambda(\rho))u_x = 0, \tag{3.10}$$

which together with (3.9) gives

$$\theta(\rho)_t + u\theta(\rho)_x + F + \rho^\gamma + \frac{1+k^2}{2}H^2 = 0. \tag{3.11}$$

Using that

$$\eta - u\zeta_x = -\rho u^2(a(t), t), \tag{3.12}$$

we obtain

$$(\zeta + \theta(\rho))_t + u(\zeta + \theta(\rho))_x + \rho^\gamma + \frac{1+k^2}{2} H^2 = 0. \tag{3.13}$$

Define the particle path $\mathbf{X}(\tilde{t}; x, t)$ through the point $(x, t) \in [a(t), b(t)]$ as

$$\begin{gathered} \frac{d}{d\tilde{t}} \mathbf{X}(\tilde{t}; x, t) = u(\mathbf{X}(\tilde{t}; x, t), \tilde{t}), \\ \mathbf{X}(\tilde{t}; x, t)|_{\tilde{t}=t} = x, \end{gathered} \tag{3.14}$$

which together with (3.13) gives

$$\frac{d}{d\tilde{t}}(\zeta + \theta(\rho))(\mathbf{X}(\tilde{t}; x, t), \tilde{t}) + \rho^\gamma(\mathbf{X}(\tilde{t}; x, t), \tilde{t}) + \frac{1+k^2}{2} H^2(\mathbf{X}(\tilde{t}; x, t), \tilde{t}) = 0. \tag{3.15}$$

Integrating (3.15) over $[0, t]$, we have

$$\begin{aligned} &2\mu \ln \frac{\rho(x,t)}{\rho_0(\mathbf{X}_0)} + \frac{1}{\beta}(\rho^\beta(x,t) - \rho_0^\beta(\mathbf{X}_0)) + \zeta(x,t) - \zeta(\mathbf{X}_0, 0) \\ &+ \int_0^t \rho^\gamma(x,s) ds + \frac{1+k^2}{2} \int_0^t H^2(x,s) ds = 0, \end{aligned} \tag{3.16}$$

where $\mathbf{X}_0 = \mathbf{X}(\tilde{t}; x, t)|_{\tilde{t}=0} \in [a_0, b_0]$. Using (3.1) and Hölder's inequality, we have

$$\begin{aligned} |\zeta(x,t)| &= \Big| \int_{a(t)}^x \rho u dx \Big| = \Big| \int_{a(t)}^x \sqrt{\rho}\sqrt{\rho} u dx \Big| \\ &\le \Big(\int_{a(t)}^{b(t)} \rho dx \Big)^{1/2} \Big(\int_{a(t)}^{b(t)} \rho u^2 dx \Big)^{1/2} \le C, \end{aligned} \tag{3.17}$$

where C is a positive constants independent of time. Then, if $\rho > 1$, we obtain

$$\sup_{t\in[0,T]} \|\rho\|_{L^\infty([a(t),b(t)])} \le C. \tag{3.18}$$

If $\rho \le 1$, then

$$2\mu \ln \frac{1}{\rho} \le C_T + \frac{1+k^2}{2} \int_0^t \|H\|_{L^\infty}^2 ds \le C_T, \tag{3.19}$$

from which, we obtain the positive lower bound of the density

$$\rho(x,t) \ge c_T. \tag{3.20}$$

□

Remark 3.3. From (3.1) and (3.3), we obtain that

$$\begin{aligned} b(t) - a(t) &= b_0 - a_0 + \int_0^t (b'(s) - a'(s)) ds \\ &\le b_0 - a_0 + C(1+t)^{1/2} \Big(\int_0^t (b'(s)^2 + a'(s)^2) ds \Big)^{1/2}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \int_0^t (b'(s)^2 + a'(s)^2) ds &= \int_0^t (u(b(s), s)^2 + u(a(s), s)^2) ds \\ &\le 2 \int_0^t \|u\|_{L^\infty}^2 dx \le C_T, \end{aligned} \tag{3.22}$$

where C_T is the positive constant depending on time.

Lemma 3.4. *Under the assumptions of Theorem 2.1, we have*

$$\int_{a(t)}^{b(t)} H_x^2 dx + \int_0^t \int_{a(s)}^{b(s)} H_{xx}^2 \,dx\,ds \le C_T, \quad t \in [0,T], \tag{3.23}$$

where C_T is a positive constant depending on time.

Proof. Multiplying $(2.1)_3$ by H_{xx} and integrating on $(a(t),b(t))$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{a(t)}^{b(t)} H_x^2 dx + \nu \int_{a(t)}^{b(t)} H_{xx}^2 dx \\
&= \frac{1}{2}\int_{a(t)}^{b(t)} u_x H_x^2 dx + \int_{a(t)}^{b(t)} u_x H H_{xx} dx + 2\int_{a(t)}^{b(t)} u H_x H_{xx} dx \\
&\le \frac{\nu}{2}\int_{a(t)}^{b(t)} H_{xx}^2 dx + C_T \int_{a(t)}^{b(t)} H_x^2 dx \int_{a(t)}^{b(t)} u_x^2 dx \\
&\quad + C\Big(\|H\|_{L^\infty}^2 \int_{a(t)}^{b(t)} u_x^2 dx + \|u\|_{L^\infty}^2 \int_{a(t)}^{b(t)} H_x^2 dx\Big) \\
&\le \frac{\nu}{2}\int_{a(t)}^{b(t)} H_{xx}^2 dx + C_T \int_{a(t)}^{b(t)} H_x^2 dx \Big(1 + \int_{a(t)}^{b(t)} u_x^2 dx\Big) \\
&\quad + C\int_{a(t)}^{b(t)} H^2 dx \int_{a(t)}^{b(t)} u_x^2 dx \\
&\le \frac{\nu}{2}\int_{a(t)}^{b(t)} H_{xx}^2 dx + C_T\big(1 + \int_{a(t)}^{b(t)} u_x^2 dx\big)\Big(1 + \int_{a(t)}^{b(t)} H_x^2 dx\Big),
\end{aligned} \tag{3.24}$$

which together with Gronwall's inequality gives (3.23). □

Lemma 3.5. *Under the assumptions of Theorem 2.1, we have*

$$\int_{a(t)}^{b(t)} F^2 dx + \int_0^t \int_{a(s)}^{b(s)} \rho \dot{u}^2 \,dx\,ds \le C_T, \quad t \in [0,T], \tag{3.25}$$

where C_T is a positive constant depending on time.

Proof. After a straight calculation, we deduce that

$$\begin{aligned}
(\dot{u})_x &= u_{tx} + u u_{xx} + u_x^2 \\
&= \Big(\frac{F + \rho^\gamma + \frac{1+k^2}{2} H^2}{2\mu + \rho^\beta}\Big)_t + u\Big(\frac{F + \rho^\gamma + \frac{1+k^2}{2} H^2}{2\mu + \rho^\beta}\Big)_x + u_x^2 \\
&= D_t\big(\frac{F}{2\mu + \rho^\beta}\big) + D_t\big(\frac{\rho^\gamma}{2\mu + \rho^\beta}\big) + \frac{1}{2} D_t\big(\frac{H^2}{2\mu + \rho^\beta}\big) + u_x^2,
\end{aligned} \tag{3.26}$$

where $D_t f = f_t + u f_x$. Multiplying (3.26) by F, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{a(t)}^{b(t)} \frac{F^2}{2\mu+\rho^\beta}dx + \int_{a(t)}^{b(t)} \rho\dot{u}^2 dx \\
&= \frac{1}{2}\int_{a(t)}^{b(t)} F^2 u_x\big(\rho(\frac{1}{2\mu+\rho^\beta})' - \frac{1}{2\mu+\rho^\beta}\big)dx \\
&\quad + \int_{a(t)}^{b(t)} F u_x\big(\rho(\frac{\rho^\gamma}{2\mu+\rho^\beta})' - \frac{\rho^\gamma}{2\mu+\rho^\beta}\big)dx \\
&\quad - (1+k^2)\int_{a(t)}^{b(t)} \frac{FH(H_t+uH_x)}{2\mu+\rho^\beta}dx \\
&\quad + \frac{1+k^2}{2}\int_{a(t)}^{b(t)} FH^2 u_x\big(\rho(\frac{1}{2\mu+\rho^\beta})' - \frac{1}{2\mu+\rho^\beta}\big)dx \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.27}
$$

Using (3.1), (3.3) and (3.23), we deduce that

$$
\begin{aligned}
I_1 &\le C_T\|F\|_{L^4}^2\|u_x\|_{L^2} \le C_T\|F\|_{L^\infty}^2\|u_x\|_{L^2} \\
&\le C_T\|F_x\|_{L^2}\|F\|_{L^2}\|u_x\|_{L^2} \le \frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C_T\|\frac{F}{\sqrt{2\mu+\rho^\beta}}\|_{L^2}^2\|u_x\|_{L^2}^2,
\end{aligned} \tag{3.28}
$$

$$
I_2 \le C_T\|F\|_{L^2}\|u_x\|_{L^2} \le C_T\|\frac{F}{\sqrt{2\mu+\rho^\beta}}\|_{L^2}^2 + C_T\|u_x\|_{L^2}^2, \tag{3.29}
$$

$$
\begin{aligned}
I_3 &\le C_T\|FH_t\|_{L^1} + C_T\|FuH_x\|_{L^1} \\
&\le C_T\|F\|_{L^2}^2 + C_T\|H_t\|_{L^2}^2 + C_T\|F\|_{L^2}^2 + C_T\|uH_x\|_{L^2}^2 \\
&\le C_T\|\frac{F}{\sqrt{2\mu+\rho^\beta}}\|_{L^2}^2 + C_T\|H_t\|_{L^2}^2 + C_T(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2)\|H_x\|_{L^2}^2 \\
&\le C_T\|\frac{F}{\sqrt{2\mu+\rho^\beta}}\|_{L^2}^2 + C_T\|H_t\|_{L^2}^2 + C_T\|u_x\|_{L^2}^2,
\end{aligned} \tag{3.30}
$$

$$
I_4 \le C_T\|F\|_{L^2}\|u_x\|_{L^2} \le C_T\|\frac{F}{\sqrt{2\mu+\rho^\beta}}\|_{L^2}^2 + C_T\|u_x\|_{L^2}^2. \tag{3.31}
$$

Substituting (3.28)-(3.31) into (3.27), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{a(t)}^{b(t)} \frac{F^2}{2\mu+\rho^\beta}dx + \int_{a(t)}^{b(t)} \rho\dot{u}^2 dx \\
&\le C_T(1+\|\frac{F}{\sqrt{2\mu+\rho^\beta}}\|_{L^2}^2)(1+\|u_x\|_{L^2}^2) + C_T\|H_t\|_{L^2}^2,
\end{aligned} \tag{3.32}
$$

which together with Gronwall's inequality gives (3.25). □

Lemma 3.6. *Under the assumptions of Theorem 2.1, we have*

$$
\int_0^T \|u_x\|_{L^\infty([a(t),b(t)])}^2 dt \le C_T, \tag{3.33}
$$

where C_T is a positive constant depending on time.

Proof. From (3.1), (3.3) and (3.23), we have

$$
\begin{aligned}
&\|u_x\|_{L^\infty}\\
&\le C\|(2\mu+\rho^\beta)u_x-\rho^\gamma-\frac{1+k^2}{2}H^2\|_{L^\infty}+C\|\rho^\gamma\|_{L^\infty}+C\|H^2\|_{L^\infty}\\
&\le C\|(2\mu+\rho^\beta)u_x-\rho^\gamma-\frac{1+k^2}{2}H^2\|_{L^2}^{1/2}\\
&\quad\times\big\|((2\mu+\rho^\beta)u_x-\rho^\gamma-\frac{1+k^2}{2}H^2)_x\big\|_{L^2}^{1/2}+C_T\\
&\le C_T(\|\sqrt{2\mu+\rho^\beta}u_x\|_{L^2}+1)^{1/2}(\|\sqrt{\rho}u_t\|_{L^2}+\|u_x\|_{L^\infty}\|\sqrt{\rho}u\|_{L^2})^{1/2}+C_T\\
&\le C_T\big(\|\sqrt{2\mu+\rho^\beta}u_x\|_{L^2}+1\big)^{1/2}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\\
&\quad+(\|\sqrt{2\mu+\rho^\beta}u_x\|_{L^2}+1)^{1/2}\|u_x\|_{L^\infty}^{1/2}+C_T\\
&\le\frac12\|u_x\|_{L^\infty}+C_T\|\sqrt{2\mu+\rho^\beta}u_x\|_{L^2}+C_T\|\sqrt{\rho}u_t\|_{L^2}+C_T.
\end{aligned}
\tag{3.34}
$$

Then, it holds that

$$
\|u_x\|_{L^\infty}^2\le C_T\|\sqrt{2\mu+\rho^\beta}u_x\|_{L^2}^2+C_T\|\sqrt{\rho}u_t\|_{L^2}^2+C_T,\tag{3.35}
$$

which implies (3.33) after the integration with respect to t . □

Lemma 3.7. *Under the assumptions of Theorem 2.1, we have*

$$
\int_{a(t)}^{b(t)}\rho_x^2dx\le C_T,\tag{3.36}
$$

where C_T is a positive constant depending on time.

Proof. Differentiating $(2.1)_1$ with respect to x, we have

$$
\rho_{tx}+\rho_{xx}u+2\rho_xu_x+\rho u_{xx}=0.\tag{3.37}
$$

Multiplying (3.37) by ρ_x, it holds

$$
\begin{aligned}
\frac12\frac{d}{dt}\int_{a(t)}^{b(t)}\rho_x^2dx&=-\frac32\int_{a(t)}^{b(t)}\rho_x^2u_xdx-\int_{a(t)}^{b(t)}\rho\rho_xu_{xx}dx\\
&\le C\|u_x\|_{L^\infty}\|\rho_x\|_{L^2}^2+C\|\rho\|_{L^\infty}\|\rho_x\|_{L^2}\|u_{xx}\|_{L^2}.
\end{aligned}
\tag{3.38}
$$

Using (3.34) and that

$$
\begin{aligned}
&\|u_{xx}\|_{L^2}\\
&\le C(\|\rho u_t\|_{L^2}+\|\rho uu_x\|_{L^2}+\|(\rho^\gamma)_x\|_{L^2}+\|(H^2)_x\|_{L^2}+\|(\rho^\beta)_xu_x\|_{L^2})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|u_x\|_{L^\infty}+\|(\rho^\gamma)_x\|_{L^2}+\|(H^2)_x\|_{L^2}+\|u_x\|_{L^\infty}\|(\rho^\beta)_x\|_{L^2})\\
&\le C_T(1+\|\sqrt{\rho}u_t\|_{L^2}+\|(\rho^\gamma)_x\|_{L^2}+(1+\|\sqrt{\rho}u_t\|_{L^2})\|(\rho^\beta)_x\|_{L^2}),
\end{aligned}
\tag{3.39}
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{a(t)}^{b(t)}\rho_x^2dx\\
&\le C_T(1+\|\sqrt{\rho}u_t\|_{L^2})\|\rho_x\|_{L^2}^2+C_T\Big(1+\|\sqrt{\rho}u_t\|_{L^2}+\|(\rho^\gamma)_x\|_{L^2}\\
&\quad+\big(1+\|\sqrt{\rho}u_t\|_{L^2}\big)\|(\rho^\beta)_x\|_{L^2}\Big)\\
&\le C_T(1+\|\sqrt{\rho}u_t\|_{L^2}^2)\|\rho_x\|_{L^2}^2+C_T(\|(\rho^\gamma)_x\|_{L^2}^2+\|(\rho^\beta)_x\|_{L^2}^2)\\
&\quad+C_T(1+\|\sqrt{\rho}u_t\|_{L^2}^2)\\
&\le C_T(1+\|\sqrt{\rho}u_t\|_{L^2}^2)\|\rho_x\|_{L^2}^2+C_T(1+\|\sqrt{\rho}u_t\|_{L^2}^2),
\end{aligned}\tag{3.40}
$$

which together with Gronwall's inequality gives (3.36). □

Lemma 3.8. *Under the assumptions of Theorem 2.1, we have*

$$
\int_{a(t)}^{b(t)}\rho\dot{u}^2dx+\int_0^t\int_{a(s)}^{b(s)}(2\mu+\rho^\beta)(\dot{u})_x^2\,dx\,ds\le C_T,\quad t\in[0,T],\tag{3.41}
$$

where C_T is a positive constant depending on time.

Proof. Differentiating (3.5) with respect to t, we have

$$
\rho\dot{u}_t+\rho_t\dot{u}=F_{xt}.\tag{3.42}
$$

Multiplying (3.42) by $\dot{u}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{a(t)}^{b(t)}\rho\dot{u}^2dx+\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot{u})_x^2dx\\
&=(\rho u\dot{u})(a(t),t)-\int_{a(t)}^{b(t)}\rho u\dot{u}(\dot{u})_xdx-\beta\int_{a(t)}^{b(t)}\rho^{\beta-1}\rho_tu_x(\dot{u})_xdx\\
&\quad+\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)u_x^2(\dot{u})_xdx+\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)uu_{xx}(\dot{u})_xdx\\
&\quad+\gamma\int_{a(t)}^{b(t)}\rho^{\gamma-1}\rho_t(\dot{u})_xdx+\int_{a(t)}^{b(t)}HH_t(\dot{u})_xdx\\
&:=J_1+J_2+J_3+J_4+J_5+J_6+J_7.
\end{aligned}\tag{3.43}
$$

Using (3.1), (3.3), (3.23), (3.33) and (3.36), we have

$$
J_1+J_2\le C_T\int_{a(t)}^{b(t)}\rho\dot{u}^2dx+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot{u})_x^2dx,\tag{3.44}
$$

$$
\begin{aligned}
J_3&=\beta\int_{a(t)}^{b(t)}\rho^{\beta-1}\rho_xuu_x(\dot{u})_xdx+\beta\int_{a(t)}^{b(t)}\rho^{\beta-1}\rho u_x^2(\dot{u})_xdx\\
&\le C_T\|u_x\|_{L^\infty}^2\Big(\int_{a(t)}^{b(t)}\rho_x^2dx+\int_{a(t)}^{b(t)}u_x^2dx\Big)+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot{u})_x^2dx\\
&\le C_T\|u_x\|_{L^\infty}^2+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot{u})_x^2dx,
\end{aligned}\tag{3.45}
$$

$$\begin{aligned} J_4 &\le C_T\|u_x\|_{L^\infty}^2\int_{a(t)}^{b(t)} u_x^2dx+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx\\ &\le C_T\|u_x\|_{L^\infty}^2+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx, \end{aligned} \tag{3.46}$$

$$\begin{aligned} J_5 &\le C_T\int_{a(t)}^{b(t)} u_{xx}^2dx+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx\\ &\le C_T+C_T\int_{a(t)}^{b(t)}\rho u_t^2dx+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx, \end{aligned} \tag{3.47}$$

$$\begin{aligned} J_6 &= -\gamma\int_{a(t)}^{b(t)}\rho^{\gamma-1}\rho_x u(\dot u)_xdx-\gamma\int_{a(t)}^{b(t)}\rho^{\gamma-1}\rho u_x(\dot u)_xdx\\ &\le C_T\Big(\int_{a(t)}^{b(t)}\rho_x^2dx+\int_{a(t)}^{b(t)}u_x^2dx\Big)+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx\\ &\le C_T+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx, \end{aligned} \tag{3.48}$$

$$J_7\le C_T\int_{a(t)}^{b(t)} H_t^2dx+\frac{1}{8}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx. \tag{3.49}$$

Then

$$\begin{aligned} &\frac{d}{dt}\int_{a(t)}^{b(t)}\rho\dot u^2dx+\frac{1}{4}\int_{a(t)}^{b(t)}(2\mu+\rho^\beta)(\dot u)_x^2dx\\ &\le C_T+C_T\int_{a(t)}^{b(t)}\rho\dot u^2dx+C_T\|u_x\|_{L^\infty}^2+C_T\int_{a(t)}^{b(t)}\rho u_t^2dx+C_T\int_{a(t)}^{b(t)}H_t^2dx\\ &\le C_T+C_T\int_{a(t)}^{b(t)}\rho\dot u^2dx+C_T\int_{a(t)}^{b(t)}H_{xx}^2dx\\ &\quad +C_T\int_{a(t)}^{b(t)}u^2H_x^2dx+C_T\int_{a(t)}^{b(t)}H^2u_x^2dx\\ &\le C_T+C_T\int_{a(t)}^{b(t)}\rho\dot u^2dx+C_T\int_{a(t)}^{b(t)}H_{xx}^2dx+C_T\|u\|_{L^\infty}^2\int_{a(t)}^{b(t)}H_x^2dx\\ &\quad +C_T\|H\|_{L^\infty}^2\int_{a(t)}^{b(t)}u_x^2dx\\ &\le C_T+C_T\int_{a(t)}^{b(t)}\rho\dot u^2dx+C_T\int_{a(t)}^{b(t)}H_{xx}^2dx\\ &\quad +C\Big(\int_{a(t)}^{b(t)}u^2dx+\int_{a(t)}^{b(t)}u_x^2dx\Big)\int_{a(t)}^{b(t)}H_x^2dx\\ &\quad +C\Big(\int_{a(t)}^{b(t)}H^2dx+\int_{a(t)}^{b(t)}H_x^2dx\Big)\int_{a(t)}^{b(t)}u_x^2dx\\ &\le C_T+C_T\int_{a(t)}^{b(t)}\rho\dot u^2dx+C_T\int_{a(t)}^{b(t)}H_{xx}^2dx \end{aligned}$$

$$
+ C_T\Big(1+\int_{a(t)}^{b(t)} u_x^2 dx\Big)\Big(1+\int_{a(t)}^{b(t)} H_x^2 dx\Big),
$$

which together with (3.23), (3.25), (3.33), (3.35) and Gronwall's inequality gives (3.41). □

4. Proof of the main results

Proof of Theorem 2.1. The existence of a global strong solution to (2.1) is established in terms of the short time existence carried out as in [6], the uniform a-priori estimates and the analysis of regularities which indeed follow from Lemmas 3.1-3.8.

Next, we will give the large time behaviors of the strong solution to the free boundary value problem (2.1) as follows. Firstly, we prove (2.6). Define the following energy functional

$$
\begin{aligned}
L(t) := &\int_{a(t)}^{b(t)} (x-(1+t)u)^2 \rho dx + \frac{2}{\gamma-1}(1+t)^2 \int_{a(t)}^{b(t)} \rho^\gamma dx \\
&+ (1+k^2)(1+t)^2 \int_{a(t)}^{b(t)} H^2 dx.
\end{aligned}
\tag{4.1}
$$

After a direct calculation, we have

$$
\begin{aligned}
L'(t) &= \frac{2(3-\gamma)}{\gamma-1}(1+t)\int_{a(t)}^{b(t)} \rho^\gamma dx + (1+k^2)(1+t)\int_{a(t)}^{b(t)} H^2 dx \\
&\quad + 2(1+t)\int_{a(t)}^{b(t)} (2\mu+\rho^\beta) u_x dx - 2(1+t)^2 \int_{a(t)}^{b(t)} (2\mu+\rho^\beta) u_x^2 dx \\
&\quad - 2\nu(1+k^2)(1+t)^2 \int_{a(t)}^{b(t)} H_x^2 dx \\
&\le \frac{2(3-\gamma)}{\gamma-1}(1+t)\int_{a(t)}^{b(t)} \rho^\gamma dx + (1+k^2)(1+t)\int_{a(t)}^{b(t)} H^2 dx \\
&\quad + (1+t)\int_{a(t)}^{b(t)} (2\mu+\rho^\beta) dx,
\end{aligned}
\tag{4.2}
$$

where C is a positive constant independent of time.

If $\gamma \ge 3$, we deduce from (4.2) that

$$
L'(t) \le \frac{1}{1+t} L(t) + C(b(t)-a(t)),
\tag{4.3}
$$

which leads to

$$
L(t) \le C(1+t)\{1+\int_0^t \frac{b(\tau)-a(\tau)}{1+\tau} d\tau\}.
\tag{4.4}
$$

Thus, we have

$$
\int_{a(t)}^{b(t)} \rho^\gamma dx \le C(1+t)^{-1}\{1+\int_0^t \frac{b(\tau)-a(\tau)}{1+\tau} d\tau\}, \quad \gamma \ge 3.
\tag{4.5}
$$

Similarly, from (4.2), we have

$$
L'(t) \le \frac{2(2-\gamma)}{\gamma-1}(1+t)\int_{a(t)}^{b(t)} \rho^\gamma dx + \frac{L(t)}{1+t} + C(b(t)-a(t))
$$

$$\le \begin{cases} \frac{L(t)}{1+t} + C(b(t)-a(t)), & 2 \le \gamma < 3, \\ (3-\gamma)\frac{L(t)}{1+t} + C(b(t)-a(t)), & 1 < \gamma < 2, \end{cases}$$

which together with Gronwall's inequality yields

$$\begin{aligned} \int_{a(t)}^{b(t)} \rho^\gamma dx &\le \frac{CL(t)}{(1+t)^2} \\ &\le \begin{cases} C(1+t)^{-1}\{1+\int_0^t \frac{b(\tau)-a(\tau)}{1+\tau} d\tau\}, & 2 \le \gamma < 3, \\ C(1+t)^{1-\gamma}\{1+\int_0^t \frac{b(\tau)-a(\tau)}{(1+\tau)^{3-\gamma}} d\tau\}, & 1 < \gamma < 2. \end{cases} \end{aligned} \tag{4.6}$$

Note that

$$1 = \int_{a_0}^{b_0} \rho_0 dx = \int_{a(t)}^{b(t)} \rho dx \le (b(t)-a(t))^{\frac{\gamma-1}{\gamma}} \Big(\int_{a(t)}^{b(t)} \rho^\gamma dx\Big)^{1/\gamma}, \tag{4.7}$$

which, combined with (4.5) and (4.6), implies

$$\begin{gathered} (b(t)-a(t))^{\gamma-1}\{1+\int_0^t \frac{b(\tau)-a(\tau)}{1+\tau} d\tau\} \ge C(1+t), \quad \gamma \ge 2, \\ (b(t)-a(t))^{\gamma-1}\{1+\int_0^t \frac{b(\tau)-a(\tau)}{(1+\tau)^{3-\gamma}} d\tau\} \ge C(1+t)^{\gamma-1}, \quad 1 < \gamma < 2. \end{gathered} \tag{4.8}$$

From (4.8), we can obtain

$$D_M(t) := \sup_{\tau\in[0,t]} (b(\tau)-a(\tau)) \ge \begin{cases} C(1+t)^{\frac{1}{\gamma}}(1+\ln(1+t))^{-1/\gamma}, & \gamma \ge 2, \\ C(1+t)^{\frac{\gamma-1}{\gamma}}, & 1 < \gamma < 2. \end{cases} \tag{4.9}$$

Next, we prove (2.7). Using a similar argument as to (4.2), we obtain

$$\begin{aligned} L'(t) &\le \frac{2(3-\gamma)}{\gamma-1}(1+t)\int_{a(t)}^{b(t)} \rho^\gamma dx + (1+k^2)(1+t)\int_{a(t)}^{b(t)} H^2 dx \\ &\quad + \int_{a(t)}^{b(t)} \rho^\beta dx + 4\mu(1+t)\int_{a(t)}^{b(t)} u_x dx \\ &\le \frac{2(3-\gamma)}{\gamma-1}(1+t)\int_{a(t)}^{b(t)} \rho^\gamma dx + (1+k^2)(1+t)\int_{a(t)}^{b(t)} H^2 dx \\ &\quad + 4\mu(1+t)\frac{d}{dt}(a(t)-b(t)) + C \\ &\le \frac{L(t)}{1+t} + 4\mu(1+t)\frac{d}{dt}(a(t)-b(t)) + C, \quad \text{if } \gamma \ge 2 \text{ and } \beta \ge 1, \end{aligned} \tag{4.10}$$

which implies

$$L(t) \le C(1+t)(b(t)-a(t)+1+\ln(1+t)) \le C(1+t)(b(t)-a(t))(1+\ln(1+t)). \tag{4.11}$$

Thus, we have

$$\int_{a(t)}^{b(t)} \rho^\gamma dx \le C(1+t)^{-1}(b(t)-a(t))(1+\ln(1+t)), \quad \gamma \ge 2, \tag{4.12}$$

which, combined with (4.7), leads to

$$b(t)-a(t) \ge C(1+t)^{1/\gamma}(1+\ln(1+t))^{-1/\gamma}, \quad \gamma \ge 2. \tag{4.13}$$

Finally, to prove (2.8), we define the Lagrange coordinates transformation

$$\xi = \int_{a(t)}^{x} \rho(y,t)dy, \quad \tau = t. \tag{4.14}$$

Since the conservation of total mass holds, the boundaries $x = a(t)$ and $x = b(t)$ are transformed into $\xi = 0$ and $\xi = 1$ respectively. The domain $[a(t), b(t)]$ is transformed into $[0, 1]$. The FBVP (2.1) is reformed into

$$\begin{gathered}
\rho_\tau + \rho^2 u_\xi = 0, \quad \xi \in (0,1),\ \tau > 0, \\
u_\tau + (\rho^\gamma)_\xi = -(1+k^2)HH_\xi + (\rho(2\mu + \rho^\beta)u_\xi)_\xi, \quad \xi \in (0,1),\ \tau > 0, \\
H_\tau + \rho H u_\xi = \nu\rho(\rho H_\xi)_\xi, \quad \xi \in (0,1),\ \tau > 0, \\
(\rho^\gamma - \rho(2\mu + \rho^\beta)u_\xi)(0,\tau) = 0, \quad (\rho^\gamma - \rho(2\mu + \rho^\beta)u_\xi)(1,\tau) = 0, \quad \tau \ge 0, \\
H(0,\tau) = H(1,\tau) = 0, \quad \tau \ge 0, \\
(\rho_0, u_0, H_0) = (\rho_0, u_0, H_0)(\xi), \quad \xi \in [0,1],
\end{gathered} \tag{4.15}$$

where the initial data satisfies

$$\begin{gathered}
\inf_{[0,1]} \rho_0 \ge \underline{\rho} > 0, \quad \rho_0 \in H^1([0,1]), \quad u_0 \in H^2([0,1]), \quad H_0 \in H^1([0,1]), \\
(\rho_0^\gamma - \rho_0(2\mu + \rho_0^\beta)u_{0x})(0) = 0, \quad (\rho_0^\gamma - \rho_0(2\mu + \rho_0^\beta)u_{0x})(1) = 0,
\end{gathered} \tag{4.16}$$

and the consistencies between initial data and boundary conditions hold.

From $(4.15)_2$ we find that

$$\frac{d}{d\tau} \int_0^1 u(\xi,\tau)d\xi = 0, \tag{4.17}$$

and without loss of generality, we can renormalize $\int_0^1 u_0(\xi)d\xi$ to be zero, then we denote

$$w = u - \frac{1}{1+\tau} \int_0^\xi \frac{1}{\rho} d\zeta + \frac{1}{1+\tau} \int_0^1 \int_0^\xi \frac{1}{\rho} d\zeta d\xi. \tag{4.18}$$

Applying (4.17), we obtain

$$w_\xi = u_\xi - \frac{1}{(1+\tau)\rho} = \Big(\frac{1}{\rho}\Big)_\tau - \frac{1}{(1+\tau)\rho}\,, \tag{4.19}$$

$$w_\tau + \frac{w}{1+\tau} = u_\tau. \tag{4.20}$$

Then, the system (4.15) becomes

$$
\begin{gathered}
\rho_\tau+\rho^2 w_\xi+\frac{\rho}{1+\tau}=0,\\
w_\tau+\frac{w}{1+\tau}+(\rho^\gamma)_\xi=-(1+k^2)HH_\xi+(\rho(2\mu+\rho^\beta)w_\xi+\frac{\rho^\beta}{1+\tau})_\xi,\\
H_\tau+\rho H(w_\xi+\frac{1}{(1+\tau)\rho})=\nu\rho(\rho H_\xi)_\xi,\\
(\rho^\gamma-\rho(2\mu+\rho^\beta)(w_\xi+\frac{1}{(1+\tau)\rho}))(0,\tau)=0,\\
(\rho^\gamma-\rho(2\mu+\rho^\beta)(w_\xi+\frac{1}{(1+\tau)\rho}))(1,\tau)=0,\\
H(0,\tau)=H(1,\tau)=0,\quad \tau\in[0,T],\\
(\rho_0,H_0,w_0)=(\rho_0,H_0,u_0-\int_0^\xi\frac{1}{\rho_0}d\zeta+\int_0^1\int_0^\xi\frac{1}{\rho_0}d\zeta d\xi)(\xi).
\end{gathered}
\tag{4.21}
$$

Multiplying $(4.21)_2$ by w and $(4.21)_3$ by $\frac{H}{\rho}$, integrating the result equations over $(0,1)$, after a straightforward calculation, we have

$$
\begin{aligned}
&\frac12\frac{d}{d\tau}\int_0^1 w^2d\xi+\frac{1+k^2}{2}\frac{d}{d\tau}\int_0^1\frac{H^2}{\rho}d\xi+\frac{1}{1+\tau}\int_0^1 w^2d\xi\\
&+\int_0^1\rho(2\mu+\rho^\beta)w_\xi^2d\xi+\nu(1+k^2)\int_0^1\rho H_\xi^2d\xi+\frac{1+k^2}{2(1+\tau)}\int_0^1\frac{H^2}{\rho}d\xi\\
&=-\frac{1}{1+\tau}\int_0^1\rho^\beta w_\xi d\xi+\int_0^1\rho^\gamma w_\xi d\xi.
\end{aligned}
\tag{4.22}
$$

For $0<\beta<1$, from (4.19) it holds

$$
\begin{aligned}
-\frac{1}{1+\tau}\int_0^1\rho^\beta w_\xi d\xi&=-\frac{1}{1+\tau}\int_0^1\rho^\beta\Big\{\Big(\frac{1}{\rho}\Big)_\tau-\frac{1}{(1+\tau)\rho}\Big\}d\xi\\
&=\frac{1}{(\beta-1)(1+\tau)}\int_0^1(\rho^{\beta-1})_\tau d\xi+\frac{1}{(1+\tau)^2}\int_0^1\rho^{\beta-1}d\xi,
\end{aligned}
\tag{4.23}
$$

and

$$
\begin{aligned}
\int_0^1\rho^\gamma w_\xi d\xi&=\int_0^1\rho^\gamma\Big\{\Big(\frac{1}{\rho}\Big)_\tau-\frac{1}{(1+\tau)\rho}\Big\}d\xi\\
&=\frac{1}{1-\gamma}\int_0^1(\rho^{\gamma-1})_\tau d\xi-\frac{1}{1+\tau}\int_0^1\rho^{\gamma-1}d\xi,
\end{aligned}
\tag{4.24}
$$

which together with (4.22) leads to

$$
\begin{aligned}
&\frac12\frac{d}{d\tau}\int_0^1 w^2d\xi+\frac{1}{\gamma-1}\frac{d}{d\tau}\int_0^1\rho^{\gamma-1}d\xi+\frac{1+k^2}{2}\frac{d}{d\tau}\int_0^1\frac{H^2}{\rho}d\xi\\
&+\frac{1}{(1-\beta)(1+\tau)}\frac{d}{d\tau}\int_0^1\rho^{\beta-1}d\xi+\frac{1}{1+\tau}\int_0^1 w^2d\xi\\
&+\frac{1}{1+\tau}\int_0^1\rho^{\gamma-1}d\xi+\int_0^1\rho(2\mu+\rho^\beta)w_\xi^2d\xi+\nu(1+k^2)\int_0^1\rho H_\xi^2d\xi\\
&+\frac{1+k^2}{2(1+\tau)}\int_0^1\frac{H^2}{\rho}d\xi-\frac{1}{(1+\tau)^2}\int_0^1\rho^{\beta-1}d\xi=0.
\end{aligned}
\tag{4.25}
$$

Multiplying (4.25) by $(1+\tau)^\eta$ for some $0<\eta<1$ to be determined later, we have

$$\begin{aligned}
&\frac{d}{d\tau}\int_0^1\Big(\frac{(1+\tau)^\eta}{2}w^2+\frac{(1+\tau)^\eta}{\gamma-1}\rho^{\gamma-1}+\frac{(1+k^2)(1+\tau)^\eta}{2}\frac{H^2}{\rho}+\frac{(1+\tau)^{\eta-1}}{1-\beta}\rho^{\beta-1}\Big)d\xi\\
&+(1-\frac{\eta}{2})(1+\tau)^{\eta-1}\int_0^1 w^2d\xi+\frac{\gamma-1-\eta}{\gamma-1}(1+\tau)^{\gamma-1}\int_0^1\rho^{\gamma-1}d\xi\\
&+(1+\tau)^\eta\int_0^1\rho(2\mu+\rho^\beta)w_\xi^2d\xi+\nu(1+k^2)(1+\tau)^\eta\int_0^1\rho H_\xi^2d\xi\\
&+\frac{1}{2}(1+k^2)(1-\eta)(1+\tau)^{\eta-1}\int_0^1\frac{H^2}{\rho}d\xi+\frac{\beta-\eta}{1-\beta}(1+\tau)^{\eta-2}\int_0^1\rho^{\beta-1}d\xi=0.
\end{aligned}\tag{4.26}$$

If $0<\eta\le\min\{\gamma-1,\beta\}$, from(4.26), we obtain

$$\int_0^1\rho^{\gamma-1}d\xi+\int_0^1\frac{H^2}{\rho}d\xi\le C(1+\tau)^{-\eta}.\tag{4.27}$$

For $\beta=1$ and $0<\eta\le\min\{\gamma-1,1\}$, from (4.22) it holds

$$\begin{aligned}
&\frac{d}{d\tau}\int_0^1\Big(\frac{(1+\tau)^\eta}{2}w^2+\frac{(1+\tau)^\eta}{\gamma-1}\rho^{\gamma-1}+\frac{(1+k^2)(1+\tau)^\eta}{2}\frac{H^2}{\rho}\Big)d\xi\\
&+(1-\frac{\eta}{2})(1+\tau)^{\eta-1}\int_0^1 w^2d\xi\\
&+\frac{\gamma-1-\eta}{\gamma-1}(1+\tau)^{\eta-1}\int_0^1\rho^{\gamma-1}d\xi+(1+\tau)^\eta\int_0^1\rho(2\mu+\rho)w_\xi^2d\xi\\
&+\nu(1+k^2)(1+\tau)^\eta\int_0^1\rho H_\xi^2d\xi+\frac{1}{2}(1+k^2)(1-\eta)(1+\tau)^{\eta-1}\int_0^1\frac{H^2}{\rho}d\xi\\
&=\frac{d}{d\tau}\Big((1+\tau)^{\eta-1}\int_0^1\ln\rho d\xi\Big)+(1-\eta)(1+\tau)^{\eta-2}\int_0^1\ln\rho d\xi+(1+\tau)^{\eta-2}.
\end{aligned}\tag{4.28}$$

Integrating (4.28) over $[0,\tau]$ and using

$$\int_0^1\ln\rho d\xi\le\int_0^1\rho d\xi\le C,\tag{4.29}$$

we have

$$\int_0^1\rho^{\gamma-1}d\xi+\int_0^1\frac{H^2}{\rho}d\xi\le C(1+\tau)^{-\eta}.\tag{4.30}$$

Thus, the proof of Theorem 2.1 is complete. □

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