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HEAT AND LAPLACE TYPE EQUATIONS WITH COMPLEX SPATIAL VARIABLES IN WEIGHTED FOCK SPACES

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Abstract. In a recent book co-authored by the authors of this article, we studied by semigroup theory methods several classical evolution equations, including the heat and Laplace equations, with real time variable and complex spatial variable, under the hypothesis that the boundary function belongs to the space of analytic functions in the open unit disk and continuous in the closed unit disk, endowed with the uniform norm. Also, in a subsequent paper, the authors have extended the results for the heat and Laplace equations in weighted Bergman spaces on the unit disk. The purpose of this article is to show that the semigroup theory methods work for these two evolution equations of complex spatial variables, under the hypothesis that the boundary function belongs to the weighted Fock space on $\mathbb{C}$, $F^p_\alpha(\mathbb{C})$, with $1 \le p < +\infty$, endowed with the L^p-norm. Also, the case of several complex variables is considered. The proofs use the Jensen's inequality, Fubini's theorem for integrals and the L^p-integral modulus of continuity.

1. Introduction

Extending the method of semigroups of operators in solving the evolution equations of real spatial variable, a way of "complexifying" the spatial variable in the classical evolution equations is to "complexify" their solution semigroups of operators, as it was summarized in the book [4]. In the cases of heat and Laplace equations, the results obtained can be summarized as follows.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $A(D) = \{f : \overline{D} \to \mathbb{C}; f \text{ is analytic on } D, \text{ continuous on } \overline{D}\}$, endowed with the uniform norm $\|f\| = \sup\{|f(z)|; z \in \overline{D}\}$. It is well-known that $(A(D), \|\cdot\|)$ is a Banach space. Let $f \in A(D)$ and consider the operator

$$W_t(f)(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(ze^{-iu}) e^{-u^2/(2t)} du, \quad z \in \overline{D}. \tag{1.1}$$

In [3] (see also [4, Chapter 2], for more details) it was proved that $(W_t, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on $A(D)$ and that the unique

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solution $u(t,z)$ (that belongs to $A(D)$, for each fixed $t \geq 0$) of the Cauchy problem

$$\frac{\partial u}{\partial t}(t,z) = \frac{1}{2}\frac{\partial^2 u}{\partial \varphi^2}(t,z), \quad (t,z) \in (0,+\infty) \times D,\ z = re^{i\varphi},\ z \neq 0, \tag{1.2}$$

$$u(0,z) = f(z), \quad z \in \overline{D},\ f \in A(D), \tag{1.3}$$

is exactly

$$u(t,z) = W_t(f)(z). \tag{1.4}$$

In the same contribution [3], setting

$$Q_t(f)(z) := \frac{t}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{u^2+t^2}\, du, \quad z \in \overline{D}, \tag{1.5}$$

we proved that $(Q_t, t \geq 0)$ is a (C_0)-contraction semigroup of linear operators on $A(D)$. Consequently, the unique solution $u(t,z)$ (that belongs to $A(D)$, for each fixed t) of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t,z) + \frac{\partial^2 u}{\partial \varphi^2}(t,z) = 0, \quad (t,z) \in D \times (0,+\infty),\ z = re^{i\varphi},\ z \neq 0, \tag{1.6}$$

$$u(0,z) = f(z), \quad z \in \overline{D},\ f \in A(D), \tag{1.7}$$

is exactly

$$u(t,z) = Q_t(f)(z). \tag{1.8}$$

In a recent paper [2] we proved that the well-posedness of the above problems in the space $A(D)$, can be replaced by well-posedness in (larger) weighted Bergman spaces defined as follows. For $0 < p < +\infty$, $1 < \alpha < +\infty$ and $\rho_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$, the weighted Bergman space $B^p_\alpha(D)$, is the space of all analytic functions in D, such that

$$\Big[\int_D |f(z)|^p dA_\alpha(z)\Big]^{1/p} < +\infty,$$

where $dA_\alpha(z) = \rho_\alpha(z) dA(z)$, with

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} r\, dr\, d\theta, z = x + iy = re^{i\theta},$$

the normalized Lebesgue area measure on the unit disk of the complex plane.

The goal of this note is to show that the well-posedness of the above problems in the space $A(D)$, can be replaced by well-posedness in weighted Fock spaces too, defined as follows.

Definition 1.1 (see, e.g. [7, p. 36]). *Let $0 < p < \infty$ and $\alpha > 0$. The Fock space $F^p_\alpha(\mathbb{C})$ is defined as the space of all entire functions in $\mathbb{C}$ with the property that*

$$\frac{\alpha p}{2\pi}\int_{\mathbb{C}} \big|f(z)e^{-\alpha|z|^2|/2}\big|^p dA(z) < +\infty,$$

where $dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} r\, dr\, d\theta$, $z = x + iy = re^{i\theta}$, is the area measure in the complex plane.

Remark 1.2. Endowed with

$$\|f\|^p_{p,\alpha} = \frac{\alpha p}{2\pi}\int_{\mathbb{C}} \big|f(z)e^{-\alpha|z|^2/2}\big|^p dA(z),$$

it is known (see, e.g., [7, p. 36]) that F^p_α is a Banach space for $1 \le p < \infty$, and a complete metric space for $\|\cdot\|^p_{p,\alpha}$ with $0 < p < 1$. Also, if $p = +\infty$, then endowed with

$$\|f\|_{\infty,\alpha} = \operatorname{ess\,sup}\{|f(z)|e^{-\alpha|z|^2|/2} : z \in \mathbb{C}\},$$

F^∞_α is a Banach space.

It is worth mentioning that the Fock spaces have been introduced in quantum mechanics via tensor products, to describe the quantum states space of variables belonging to a Hilbert space. Later one, it was observed that in fact this description corresponds to the spaces of holomorphic functions of several variables which are square integrable with respect to a Gaussian measure. These spaces are involved in harmonic analysis on the Heisenberg group, PDE, infinite dimensional analysis and free analysis. In more detail, these spaces are related with the white noise space, with the theory of stochastic distributions, see [6], and the parametrized Berezin transform on the Fock space provides a solution to the initial value problem on the complex plane for the heat equation, while weighted translation operators give rise to a unitary representation of the Heisenberg group on the Fock space.

For other details in the theory of Fock spaces one can consult, for example, the book [7].

The results obtained in this paper can be considered as a kind of complex analogues of those for the classical heat and Laplace equations in $L^p(\mathbb{R})$ spaces (see, e.g., [5, p. 23]). In Section 2, we reconsider (1.1), (1.2), (1.3), (1.4) assuming that the boundary function $f \in F^p_\alpha(\mathbb{C})$ with $1 \le p < +\infty$. Section 3 treats (1.5), (1.6), (1.7), (1.8) under the same hypothesis for the boundary function f. It is worth mentioning that since the uniform norm used in the case of the space $A(D)$ is now replaced with the L^p-type norm in the Fock space $F^p_\alpha(\mathbb{C})$, the proofs of these results require new tools, like the Jensen's inequality, the Fubini's theorem for integrals and the L^p-integral modulus of continuity.

2. Heat-type equations with complex spatial variables

The first main result of this section is concerned with the heat equation of complex spatial variable.

Theorem 2.1. *Let $1 \le p < +\infty$ and consider $W_t(f)(z)$ given by* (1.1)*, for $z \in \mathbb{C}$. Then, $(W_t, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on $F^p_\alpha(\mathbb{C})$ and the unique solution $u(t,z)$ that belongs to $F^p_\alpha(\mathbb{C})$ for each fixed t, of the Cauchy problem* (1.2) *(with D replaced there with $\mathbb{C}$) with the initial condition*

$$u(0,z) = f(z), \quad z \in \mathbb{C},\ f \in F^p_\alpha(\mathbb{C}),$$

is given by $u(t,z) = W_t(f)(z)$.

Proof. Since f is entire function, reasoning exactly as in [3, Theorem 2.1] (see also [4, Theorem 2.2.1, p. 27]), (with the unit disk D replaced by $\mathbb{C}$), we obtain that $W_t(f)(z)$ is entire as function of z and for all $t, s \ge 0$, we have

$$W_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-k^2 t/2} z^k$$

and $W_{t+s}(f)(z) = W_t[W_s(f)](z)$.

Next, we apply the following well-known Jensen type inequality for integrals: if $\int_{-\infty}^{+\infty} G(u)du = 1$, $G(u) \geq 0$ for all $u \in \mathbb{R}$ and $\varphi(t)$ is a convex function over the range of the measurable function of real variable F, then

$$\varphi\Big(\int_{-\infty}^{+\infty} F(u)G(u)du\Big) \leq \int_{-\infty}^{+\infty} \varphi(F(u))G(u)du.$$

Now, since $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/(2t)}du = 1$, by the above Jensen's inequality with

$$\varphi(t) = t^p, \quad F(u) = |f(ze^{-iu})|, \quad G(u) = \frac{1}{\sqrt{2\pi t}} e^{-u^2/(2t)}du,$$

we find

$$|W_t(f)(z)|^p \leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} |f(ze^{-iu})|^p e^{-u^2/(2t)}du.$$

Multiplying this inequality by $\frac{\alpha p}{2\pi}[e^{-\alpha|z|^2/2}]^p$, integrating on $\mathbb{C}$ with respect to $dA(z)$ (the normalized Lebesgue's area measure) and taking into account the Fubini's theorem, we obtain

$$\begin{aligned}
&\frac{\alpha p}{2\pi} \int_{\mathbb{C}} \big[|W_t(f)(z)|e^{-\alpha|z|^2/2}\big]^p dA(z) \\
&\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \Big[\frac{\alpha p}{2\pi} \int_{\mathbb{C}} \big[|f(ze^{-iu})|e^{-\alpha|z|^2/2}\big]^p dA(z)\Big] e^{-u^2/(2t)}du.
\end{aligned}$$

But writing $z = re^{i\theta}$ in polar coordinates and taking into account that $dA(z) = \frac{1}{\pi} r\, dr\, d\theta$, some simple calculations lead to the equality

$$\int_{\mathbb{C}} \big[|f(ze^{-iu})|e^{-\alpha|z|^2/2}\big]^p dA(z) = \int_{\mathbb{C}} \big[|f(z)|e^{-\alpha|z|^2/2}\big]^p dA(z), \tag{2.1}$$

for all $u \in \mathbb{R}$. This immediately implies

$$\|W_t(f)\|_{p,\alpha} \leq \|f\|_{p,\alpha}.$$

So $W_t(f) \in F^p_\alpha(D)$ and W_t is a contraction. Next, for $f \in F^p_\alpha(D)$ let us introduce the integral modulus of continuity of the form (see, e.g., [1])

$$\omega_1(f;\delta)_{F^p_\alpha} = \sup_{0\leq |h|\leq \delta} \Big(\int_{\mathbb{C}} \big[|f(ze^{ih}) - f(z)|e^{-\alpha|z|^2/2}\big]^p dA(z)\Big)^{1/p}.$$

We obtain

$$|W_t(f)(z) - f(z)|^p = \Big|\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} [f(ze^{-iu}) - f(z)]e^{-u^2/(2t)}du\Big|^p$$

and repeating the reasonings as above, for $|W_t(f)(z)|^p$, we obtain

$$\begin{aligned}
&\int_{\mathbb{C}} \big[|W_t(f)(z) - f(z)|e^{-\alpha|z|^2/2}\big]^p dA(z) \\
&\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \Big[\int_{\mathbb{C}} \big[|f(ze^{-iu}) - f(z)|e^{-\alpha|z|^2/2}\big]^p dA(z)\Big] e^{-u^2/(2t)}du \\
&\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f;|u|)^p_{F^p_\alpha} e^{-u^2/(2t)}\, du \\
&\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f;\sqrt{t})^p_{F^p_\alpha} (\frac{|u|}{\sqrt{t}} + 1)^p e^{-u^2/(2t)}\, du
\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{\sqrt{t}}{\sqrt{2\pi t}}\int_0^{\infty}\omega_1(f;\sqrt{t})^p_{F^p_\alpha}(v+1)^p e^{-v^2/2}dv\\
&= C_p\omega_1(f;\sqrt{t})^p_{F^p_\alpha}.
\end{aligned}
$$

Consequently,

$$
\|W_t(f)-f\|_{p,\alpha} \le C_p'\omega_1(f;\sqrt{t})_{F^p_\alpha},
$$

and so it yields

$$
\lim_{t\searrow 0}\|W_t(f)-f\|_{p,\alpha}=0.
$$

Since by [7, Corollary 2.8, p. 38], we obtain

$$
|f(z)-W_t(f)(z)| \le e^{\alpha|z|^2/2}\|f-W_t(f)\|_{p,\alpha}, \quad \text{for all } z\in\mathbb{C},
$$

combining this with the previous limit, it easily follows the uniform convergence on any compact in $\mathbb{C}$ of $W_t(f)$ to f, that is

$$
\lim_{t\searrow 0} W_t(f)(z)=f(z), \quad \text{for all } z\in\mathbb{C}.
$$

It is worth mentioning here that this last limit property can be proved in a different way by the same method used in the proof of next Theorem 3.1 for proving that

$$
\lim_{t\searrow 0} Q_t(f)(z)=f(z), \quad \text{for all } z\in\mathbb{C}.
$$

Now, let $s\in(0,+\infty)$, V_s be a small neighborhood of s, both fixed, and take an arbitrary $t\in V_s$, $t\ne s$. Applying the reasonings in the proof of [3, Theorem 2.1, (iii)] (see also [4, Theorem 2.2.1, (iii)]), we obtain

$$
\begin{aligned}
|W_t(f)(z)-W_s(f)(z)| &\le \int_{-\infty}^{+\infty}|f(ze^{-iu})|\Big|\frac{e^{-u^2/(2t)}}{\sqrt{2\pi t}}-\frac{e^{-u^2/(2s)}}{\sqrt{2\pi s}}\Big|du\\
&\le \frac{1}{\sqrt{2\pi}}|t-s|\int_{-\infty}^{+\infty}|f(ze^{-iu})|e^{-u^2/c^2}\big|\frac{2u^2}{c^4}-\frac{1}{c^2}\big|\,du,
\end{aligned}
$$

where c depends on s (and not on t).

Denoting

$$
K_s=\int_{-\infty}^{+\infty}e^{-u^2/c^2}\big|\frac{2u^2}{c^4}-\frac{1}{c^2}\big|\,du,
$$

where $0<K_s<+\infty$, by the proof of [4, Theorem 2.2.1, (iii)], it follows that

$$
\begin{aligned}
&\int_{\mathbb{C}}\big[|W_t(f)(z)-W_s(f)(z)|e^{-\alpha|z|^2/2}\big]^p dA(z)\\
&\le\Big(\frac{1}{\sqrt{2\pi}}\Big)^p|t-s|^pK_s^p\int_{\mathbb{C}}\Big(\int_{-\infty}^{+\infty}|f(ze^{-iu})|\frac{1}{K_s}e^{-u^2/c^2}\big|\frac{2u^2}{c^4}-\frac{1}{c^2}\big|\,du\Big)^p\\
&\quad\times e^{-p\alpha|z|^2/2}dA(z)
\end{aligned}
$$

and reasoning exactly as at the beginning of the proof, we obtain

$$
\int_{\mathbb{C}}\big[|W_t(f)(z)-W_s(f)(z)|e^{-\alpha|z|^2/2}\big]^p dA(z)\le\Big(\frac{1}{\sqrt{2\pi}}\Big)^p|t-s|^pK_s^p\|f\|^p_{p,\alpha}.
$$

This implies

$$
\|W_t(f)-W_s(f)\|_{p,\alpha}\le\frac{1}{\sqrt{2\pi}}|t-s|K_s\|f\|_{p,\alpha}.
$$

Therefore, $(W_t, t \geq 0)$ is a (C_0)-contraction semigroup of linear operators on $F^p_\alpha(\mathbb{C})$. Also, since the series representation for $W_t(f)(z)$ is uniformly convergent in any compact disk included in $\mathbb{C}$, it can be differentiated term by term with respect to t and φ. We then easily obtain that $W_t(f)(z)$ satisfies the Cauchy problem in the statement of the theorem. We also note that in equation we must take $z \neq 0$ simply because $z = 0$ has no polar representation, namely $z = 0$ cannot be represented as function of φ. This completes the proof. $\square$

The partial differential equation (1.2) can equivalently be expressed in terms of t and z as follows.

Corollary 2.2. *Let $1 \leq p < +\infty$. For each $f \in F^p_\alpha(\mathbb{C})$, the initial value problem*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Big(z\frac{\partial u}{\partial z} + z^2\frac{\partial^2 u}{\partial z^2}\Big) = 0, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{C}\backslash\{0\},$$
$$u(0, z) = f(z), \quad z \in \mathbb{C},$$

is well-posed and its unique solution is $W_t(f) \in C^\infty(\mathbb{R}_+; F^p_\alpha(\mathbb{C}))$.

Proof. By [3, Theorem 2.1, (i)], we can compute the generator associated with $W_t(f)$, to obtain

$$\begin{aligned}
\Big(\frac{d}{dt}W_t(f)(z)\Big)\Big|_{t=0} &= -\sum_{k=0}^{\infty}\frac{k^2}{2}a_k z^k \\
&= -\sum_{k=0}^{\infty}\big(\frac{k(k-1)}{2} + \frac{k}{2}\big)a_k z^k \\
&= -\frac{z^2}{2}f''(z) - \frac{z}{2}f'(z).
\end{aligned}$$

Therefore, the statement is an immediate consequence of a classical result of Hill (see, e.g., [4, Theorem 1.2.1, p. 8]). $\square$

Remark 2.3. Theorem 2.1 can be easily extended to functions of several complex variables, as follows. For $0 < p < \infty$, let $f : \mathbb{C}^n \to \mathbb{C}$, $f(z_1, z_2, \dots, z_n)$ be entire function with respect to each variable $z_i \in \mathbb{C}$, $i = 1, \dots, n$, such that

$$\int_{\mathbb{C}^n}\big[|f(z_1, \dots, z_n)|e^{-\alpha|z_1|^2/2}\cdots e^{-\alpha|z_n|^2/2}\big]^p dA(z_1)\cdots dA(z_n) < \infty.$$

We write this by $f \in F^p_\alpha(\mathbb{C}^n)$ and

$$\|f\|_{F^p_\alpha(\mathbb{C}^n)} = \Big(\int_{\mathbb{C}^n}\big[|f(z_1, \dots, z_n)|e^{-\alpha|z_1|^2/2}\cdots e^{-\alpha|z_n|^2/2}\big]^p dA(z_1)\cdots dA(z_n)\Big)^{1/p}$$

becomes a norm on $F^p_\alpha(\mathbb{C}^n)$. We call the later the weighted Fock space in several complex variables. Following now the model for the semigroup attached to the real multivariate heat equation (see, e.g., [5, pg. 69]), for $z_1, z_2, \dots, z_n \in \mathbb{C}$, we may define the integral of Gauss-Weierstrass type by

$$H_t(f)(z_1, \dots, z_n) = (2\pi t)^{-n/2}\int_{\mathbb{R}^n} f(z_1e^{-iu_1}, \dots, z_ne^{-iu_n})e^{-|u|^2/(2t)}du_1\cdots du_n,$$

where $u = (u_1, \dots, u_n)$ and $|u| = \sqrt{u_1^2 + \cdots + u_n^2}$. One can reason exactly as in the proof of Theorem 2.1 to deduce that $(H_t,\ t \geq 0)$ is (C_0)-contraction semigroup of linear operators on $F^p_\alpha(\mathbb{C}^n)$ and that

$$u(t, z_1, \dots, z_n) = H_t(f)(z_1, \dots, z_n)$$

is the unique solution of the Cauchy problem

$$\begin{gathered}\frac{\partial u}{\partial t}(t, z_1, \dots, z_n) = \frac{1}{2}\big[\frac{\partial^2 u}{\partial {\varphi_1}^2}(t, z_1, \dots, z_n) + \dots + \frac{\partial^2 u}{\partial {\varphi_n}^2}(t, z_1, \dots, z_n)\big], \quad t > 0, \\ z_1 = r_1 e^{i\varphi_1}, \dots, z_n = r_n e^{i\varphi_n} \in \mathbb{C}, \quad z_1, \dots, z_n \neq 0, \\ u(0, z_1, \dots, z_n) = f(z_1, \dots, z_n).\end{gathered}$$

Remark 2.4. Reasoning as in the proof of Corollary 2.2 and calculating

$$\Big(\frac{d}{dt} H_t(f)(z_1, \dots, z_n)\Big)\Big|_{t=0},$$

we easily find that the initial value problem

$$\begin{gathered}\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{k=1}^{n}\Big(z_k\frac{\partial u}{\partial z_k} + z_k^2\frac{\partial^2 u}{\partial z_k^2}\Big) = 0, \quad (t, z_1, \dots, z_n) \in \mathbb{R}_+ \times (\mathbb{C}\setminus\{0\})^n, \\ u(0, z_1, \dots, z_n) = f(z_1, \dots, z_n), \quad z_1, \dots, z_n \in \mathbb{C},\end{gathered}$$

is well-posed and its unique solution is $H_t(f) \in C^\infty(\mathbb{R}_+; F_\alpha^p(\mathbb{C}^n))$.

3. Laplace-type equations with complex spatial variables

The first main result of this section is concerned with the Laplace equation of a complex spatial variable.

Theorem 3.1. *Let $1 \le p < +\infty$ and consider $Q_t(f)(z)$ given by (1.5), for $z \in \mathbb{C}$. Then $(Q_t, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on $F_\alpha^p(\mathbb{C})$ and the unique solution $u(t, z)$ that belongs to $F_\alpha^p(\mathbb{C})$ for each fixed t, of the Cauchy problem (1.6) (with D replaced there by $\mathbb{C}$) with the initial condition*

$$u(0, z) = f(z), \quad z \in \mathbb{C},\ f \in F_\alpha^p(\mathbb{C}),$$

is given by $u(t, z) = Q_t(f)(z)$.

Proof. Since f is entire function, reasoning exactly as in [3, Theorem 3.1] (see also [4, Theorem 2.3.1, p. 27]) (with the unit disk D replaced by $\mathbb{C}$), $Q_t(f)(z)$ is entire function as function of z and for all $t, s \ge 0$ we have

$$Q_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-kt} z^k, \quad Q_{t+s}(f)(z) = Q_t[Q_s(f)](z).$$

Now, since $\frac{t}{\pi}\int_{-\infty}^{+\infty} \frac{1}{u^2+t^2} du = 1$, by Jensen's inequality, we obtain

$$|Q_t(f)(z)|^p \le \frac{t}{\pi}\int_{-\infty}^{+\infty} |f(ze^{-iu})|^p \frac{1}{u^2+t^2} du,$$

which multiplied on both sides by $[e^{-\alpha|z|^2/2}]^p$, then integrated on $\mathbb{C}$ with respect to the Lebesgue's area measure $dA(z)$ and applying the Fubini's theorem, gives

$$\begin{aligned}&\int_{\mathbb{C}} [|Q_t(f)(z)|e^{-\alpha|z|^2/2}]^p dA(z) \\ &\le \frac{t}{\pi}\int_{-\infty}^{+\infty}\Big[\int_{\mathbb{C}}\big[|f(ze^{-iu})|e^{-\alpha|z|^2/2}\big]^p dA(z)\Big]\frac{1}{u^2+t^2}\, du.\end{aligned}$$

As in the proof of Theorem 3.1, writing $z = re^{i\theta}$ (in polar coordinates) and taking into account that $dA(z) = \frac{1}{\pi} r\, dr\, d\theta$, some simple calculations lead to the same equality (2.1). Hence, we obtain $\|Q_t(f)\|_{p,\alpha} \le \|f\|_{p,\alpha}$. This implies that $Q_t(f) \in F^p_\alpha(\mathbb{C})$ and that Q_t is a contraction.

To prove that

$$\lim_{t \searrow 0} Q_t(f)(z) = f(z),$$

for any $f \in F^p_\alpha(\mathbb{C})$ and $z \in \mathbb{C}$, let $f = U + iV$, $z = re^{ix}$ be fixed with $0 < r$ and denote

$$F(v) = U[r\cos(v), r\sin(v)], G(v) = V[r\cos(v), r\sin(v)].$$

We can write

$$Q_t(f)(z) = \frac{t}{\pi}\int_{-\infty}^{+\infty} F(x-u)\frac{1}{t^2+u^2}du + i\frac{t}{\pi}\int_{-\infty}^{+\infty} G(x-u)\frac{1}{t^2+u^2}\,du.$$

From the maximum modulus principle, passing to limit as $t \to 0^+$ and taking into account the property in the real case (see, e.g., [5, p. 23, Exercise 2.18.8]), we find

$$\begin{aligned}\lim_{t\searrow 0} |Q_t(f)(z) - f(z)| &\le \lim_{t\searrow 0}\Big|\frac{t}{\pi}\int_{-\infty}^{+\infty} F(x-u)\frac{1}{t^2+u^2}du - F(x)\Big| \\ &\quad + \lim_{t\searrow 0}\Big|\frac{t}{\pi}\int_{-\infty}^{+\infty} G(x-u)\frac{1}{t^2+u^2}du - G(x)\Big| = 0,\end{aligned}$$

which holds uniformly with respect to $|z| \le r$. Consequently,

$$\lim_{t\searrow 0} Q_t(f)(z) = f(z),$$

uniformly in any compact subset of $\mathbb{C}$.

Now, let $s \in (0, +\infty)$, V_s be a small neighborhood of s, both fixed, and take an arbitrary $t \in V_s$, $t \ne s$. Applying the same reasoning as in the proof of [3, Theorem 3.1, (ii)] (see also [4, Theorem 2.3.1, (ii)]), we obtain

$$\begin{aligned}|Q_t(f)(z) - Q_s(f)(z)| &\le \frac{1}{\pi}\int_{-\infty}^{+\infty} |f(ze^{-iu})|\Big|\frac{t}{t^2+u^2} - \frac{s}{s^2+u^2}\Big|\,du \\ &= \frac{1}{\pi}|t-s|\int_{-\infty}^{+\infty} |f(ze^{-iu})|\Big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\Big|\,du.\end{aligned}$$

Setting

$$K_s = \int_{-\infty}^{+\infty}\Big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\Big|\,du,$$

i.e.,

$$1 = \frac{1}{K_s}\int_{-\infty}^{+\infty}\Big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\Big|\,du$$

(where $0 < K_s < +\infty$), by the proof of [4, Theorem 2.3.1, (ii)], it follows that

$$\begin{aligned}&\int_{\mathbb{C}}[|Q_t(f)(z) - Q_s(f)(z)|e^{-\alpha|z|^2/2}]^p dA(z) \\ &\le \big(\frac{1}{\pi}\big)^p|t-s|^p K_s^p\int_{\mathbb{C}}\Big(\int_{-\infty}^{+\infty}|f(ze^{-iu})|\frac{1}{K_s}\Big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\Big|\,du\Big)^p \\ &\quad\times e^{-p\alpha|z|^2/2}dA(z).\end{aligned}$$

Applying Jensen's inequality and then Fubini's theorem, we obtain

$$\int_{\mathbb{C}}\big[|Q_t(f)(z) - Q_s(f)(z)|e^{-\alpha|z|^2/2}\big]^p dA(z) \le \big(\frac{1}{\pi}\big)^p|t-s|^p K_s^p\|f\|^p_{p,\alpha}$$

and so
$$\|Q_t(f) - Q_s(f)\|_{p,\alpha} \le \frac{1}{\pi}|t-s|K_s\|f\|_{p,\alpha}\,.$$
Then $(Q_t, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on $F^p_\alpha(\mathbb{C})$. Also, since the series representation for $Q_t(f)(z)$ is uniformly convergent in any compact disk included in $\mathbb{C}$, it can be differentiated term by term, with respect to t and φ. We then easily obtain that $Q_t(f)(z)$ satisfies the Cauchy problem in the statement of the theorem. We also note that in equation we must take $z \ne 0$ simply because $z = 0$ has no polar representation. This completes the proof. □

Reasoning exactly as in the proof of [4, Theorem 2.3.1, (v), pp. 53-54], we immediately get the following.

Corollary 3.2. *Let $1 \le p < +\infty$. For each $f \in F^p_\alpha(\mathbb{C})$, the initial value problem*
$$\begin{gathered}\frac{\partial u}{\partial t} + z\frac{\partial u}{\partial z} = 0, \quad (t,z) \in \mathbb{R}_+ \times \mathbb{C}\backslash\{0\},\\ u(0,z) = f(z), \quad z \in \mathbb{C},\end{gathered}$$
is well-posed and its unique solution $Q_t(f)$ belongs to $C^\infty(\mathbb{R}_+; F^p_\alpha(\mathbb{C}))$.

Remark 3.3. The above results can be easily extended to several complex variables. We may define the complex Poisson-Cauchy integral by
$$\begin{aligned}P_t(f)(z_1,\dots,z_n) &= \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}t\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(z_1e^{-iu_1},\dots,z_ne^{-iu_n})\\ &\quad\times \frac{1}{(t^2+u_1^2+\dots+u_n^2)^{(n+1)/2}}\,du_1\cdots du_n.\end{aligned}$$
Using similar arguments (following now the model for the semigroup attached to the real multivariate Laplace equation, see, e.g., [5, p. 69]), as in the univariate complex case we can prove that the unique solution of the Cauchy problem
$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}(t,z_1,\dots,z_n) + \frac{\partial^2 u}{\partial {\varphi_1}^2}(t,z_1,\dots,z_n) + \dots + \frac{\partial^2 u}{\partial {\varphi_n}^2}(t,z_1,\dots,z_n) = 0, \quad t>0,\\ z_1 = r_1e^{i\varphi_1}, \quad \dots, \quad z_n = r_ne^{i\varphi_n} \in \mathbb{C}, \quad z_1,\dots,z_n \ne 0,\\ u(0,z_1,\dots,z_n) = f(z_1,\dots,z_n), \quad \text{for } z_1,\dots,z_n \in \mathbb{C},\ f \in F^p_\alpha(\mathbb{C}^n)\end{gathered}$$
is given by
$$u(t,z_1,\dots,z_n) = P_t(f)(z_1,\dots,z_n).$$

Remark 3.4. Reasoning as in the proof of Corollary 3.2 and calculating
$$\Big(\frac{d}{dt}P_t(f)(z_1,\dots,z_n)\Big)\Big|_{t=0},$$
we easily obtain that for each $f \in F^p_\alpha(\mathbb{C}^n)$, the initial value problem
$$\begin{gathered}\frac{\partial u}{\partial t} + \sum_{k=1}^n\Big(z_k\frac{\partial u}{\partial z_k}\Big) = 0, \quad (t,z_1,\dots,z_n) \in \mathbb{R}_+ \times (\mathbb{C}\backslash\{0\})^n,\\ u(0,z_1,\dots,z_n) = f(z_1,\dots,z_n), \quad z_1,\dots,z_n \in \mathbb{C},\end{gathered}$$
is well-posed and its unique solution $Q_t(f)$ belongs to $C^\infty(\mathbb{R}_+; F^p_\alpha(\mathbb{C}))$.

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