Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 108, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POSITIVE VORTEX SOLUTIONS AND PHASE SEPARATION FOR COUPLED SCHRÖDINGER SYSTEM WITH SINGULAR POTENTIAL

JIN DENG, ALIANG XIA, JIANFU YANG

Abstract. We consider the existence of rotating solitary waves (vortices) for a coupled Schrödinger equations by finding solutions to the singular system

$$
\begin{gathered}
-\Delta u+\lambda_1 u+\frac{u}{|x|^2}=\mu_1 u^3+\beta u v^2, \quad x \in \mathbb{R}^2, \\
-\Delta v+\lambda_2 v+\frac{v}{|x|^2}=\mu_2 v^3+\beta u^2 v, \quad x \in \mathbb{R}^2, \\
u, v \geq 0, \quad x \in \mathbb{R}^2,
\end{gathered}
$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive parameters, $\beta \neq 0$. We show that this system has a positive least energy solution for the cases when either β is negative or β is positive and small or large. Moreover, if $\lambda_1=\lambda_2$, then the solution is unique. We also study the limiting behavior of the least energy solutions in the repulsive case for $\beta \rightarrow-\infty$, and phase separation.

1. Introduction

In this article, we consider solitary wave solutions of the time-dependent coupled nonlinear Schrödinger equations:

$$
\begin{gathered}
-i \frac{\partial \Phi_1}{\partial t}=\Delta \Phi_1+\mu_1|\Phi_1|^2 \Phi_1+\beta|\Phi_2|^2 \Phi_1, \quad x \in \mathbb{R}^N,\ t>0, \\
-i \frac{\partial \Phi_2}{\partial t}=\Delta \Phi_2+\mu_2|\Phi_2|^2 \Phi_1+\beta|\Phi_1|^2 \Phi_2, \quad x \in \mathbb{R}^N,\ t>0, \\
\Phi_j=\Phi_j(x, t) \in \mathbb{C}, \quad j=1,2,
\end{gathered} \tag{1.1}
$$

where i is the imaginary unit, $\mu_1, \mu_2>0$ and $\beta \neq 0$ is a coupling constant. When $N \leq 3$, system (1.1) appears in many physical problems. Especially in nonlinear optics, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media, the positive parameter μ_j is for self-focusing in the j-th component of the beam, see for instance [2]. System (1.1) also arises in the Hartree-Fock theory for a double condensate, that is, a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, see [13]. Physically, Φ_j values are the corresponding condensate amplitudes, μ_j and β are the intraspecies and interspecies scattering lengths. The sign of β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive: the interaction is attractive if $\beta>0$, and the

2010 *Mathematics Subject Classification.* 35J20, 35B08, 35B40.
Key words and phrases. Schödinger equation; singular potential; Nehari manifold.

Submitted March 5, 2020. Published October 30, 2020.

interaction is repulsive if $\beta < 0$, where the two states are in strong competition. If the condensates repel, the spatially separate. This phenomenon is called phase separation and has been described in [28].

By a solitary wave solution of system (1.1), we mean a solution of (1.1) with the form

$$\Phi_1(x,t) = u(x)e^{i\lambda_1 t} \quad \text{and} \quad \Phi_2(x,t) = v(x)e^{i\lambda_2 t}.$$

Then (u,v) satisfies

$$\begin{gathered} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, \quad x \in \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^N, \\ u, v \geq 0, \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.2}$$

When $N \leq 3$, the existence of solutions has received great interest. Particularly, it is considered the existence of a ground state solution in [1, 6, 10, 17, 21, 26], the existence of semiclassical states or singular perturbed settings in [16, 18, 19, 22, 24, 25], and the existence of multiple solutions in [5, 12, 20, 23, 29]. When $N = 4$, this problem becomes critical case, one can find related results in [9, 11] and references therein.

However, if $\psi_0(x) \in \mathbb{R}$, the *angular momentum* of $\psi(t,x) = \psi_0(x)e^{i\omega_0 t}$, $\omega_0 > 0$, is trivial, that is, $M(\psi) = 0$, where

$$M(\psi) = \operatorname{Re} \int_{\mathbb{R}^N} \overline{\partial_t \psi}(x \times \nabla \psi)\,dx. \tag{1.3}$$

Therefore, a solution (u,v) with vortices should be complex valued.

In this article, we are interested in finding a standing wave solution with nontrivial angular momentum in the dimension $N = 2$, that is, a solution with vortices. Making an ansatz of the form

$$\Phi_1(x,t) = u(x)e^{i(k_0\theta(x)+\lambda_1 t)} \quad \text{and} \quad \Phi_2(x,t) = v(x)e^{i(k_0\theta(x)+\lambda_2 t)},$$

where $\theta(x) \in \mathbb{R}/2\pi\mathbb{Z}$, $k_0 \neq 0$, we see that system (1.1) is equivalent to the system

$$\begin{gathered} -\Delta u + \lambda_1 u + k_0^2 u|\nabla\theta|^2 = \mu_1 u^3 + \beta u v^2, \quad x \in \mathbb{R}^N, \\ 2\nabla\theta \cdot \nabla u - u\Delta\theta = 0, \quad x \in \mathbb{R}^N, \\ -\Delta v + \lambda_2 v + k_0^2 v|\nabla\theta|^2 = \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^N, \\ 2\nabla\theta \cdot \nabla v - v\Delta\theta = 0, \quad x \in \mathbb{R}^N, \\ u, v \geq 0, \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.4}$$

If we assume $u(x) = u(|x|)$ and choose the angular coordinate in $\mathbb{R}^2$ as phase function, see [3, 4], that is,

$$\theta(x) := \begin{cases} \arctan \frac{x_2}{x_1}, & \text{if } x_1 > 0, \\ \pi + \arctan \frac{x_2}{x_1}, & \text{if } x_1 < 0, \\ \pi/2, & \text{if } x_1 = 0 \text{ and } x_2 > 0, \\ -\pi/2, & \text{if } x_1 = 0 \text{ and } x_2 < 0, \end{cases} \tag{1.5}$$

we obtain

$$\Delta\theta = 0, \quad \nabla\theta \cdot \nabla u = 0, \quad |\nabla\theta|^2 = \frac{1}{|x|^2},$$

and the system reduces to

$$\begin{gathered}-\Delta u+\lambda_1 u+k_0^2\frac{u}{|x|^2}=\mu_1 u^3+\beta u v^2,\quad x\in\mathbb{R}^2,\\ -\Delta v+\lambda_2 v+k_0^2\frac{v}{|x|^2}=\mu_2 v^3+\beta u^2 v,\quad x\in\mathbb{R}^2,\\ u,v\geq 0,\quad x\in\mathbb{R}^2.\end{gathered}\tag{1.6}$$

Noting that $x\times\nabla\psi=x_1\partial_2\psi-x_2\partial_2\psi$ if $N=2$, we may verify that

$$\begin{gathered}M\big(u(x)e^{i(k_0\theta(x)+\lambda_1 t)}\big)=\lambda_1 k_0\int_{\mathbb{R}^2}u^2dx,\\ M\big(v(x)e^{i(k_0\theta(x)+\lambda_2 t)}\big)=\lambda_2 k_0\int_{\mathbb{R}^2}v^2dx.\end{gathered}$$

We point out that (1.2) is a special case of system (1.6), that is, $k_0=0$. In this article, we consider the case $k_0>0$. For simplicity, we always assume $k_0=1$ in (1.6), and thus we study the following coupled singular Schrödinger system

$$\begin{gathered}-\Delta u+\lambda_1 u+\frac{u}{|x|^2}=\mu_1 u^3+\beta u v^2,\quad x\in\mathbb{R}^2,\\ -\Delta v+\lambda_2 v+\frac{v}{|x|^2}=\mu_2 v^3+\beta u^2 v,\quad x\in\mathbb{R}^2,\\ u,v\geq 0,\quad x\in\mathbb{R}^2.\end{gathered}\tag{1.7}$$

It is well known that solutions of (1.7) are the critical points of the functional $E:\mathcal{H}\to\mathbb{R}$, where

$$\begin{aligned}E(u,v)=&\frac12\int_{\mathbb{R}^2}\Big(|\nabla u|^2+\lambda_1 u^2+\frac{u^2}{|x|^2}+|\nabla v|^2+\lambda_2 v^2+\frac{v^2}{|x|^2}\Big)dx\\ &-\frac14\int_{\mathbb{R}^2}\big(\mu_1 u^4+2\beta u^2v^2+\mu_2 v^4\big)dx,\end{aligned}\tag{1.8}$$

where $\mathcal{H}:=H_{\lambda_1}\times H_{\lambda_2}$ is given in section 2. We call a solution (u,v) *nontrivial* if both $u\not\equiv 0$ and $v\not\equiv 0$; and *semi-trivial* if (u,v) is a type of solution $(u,0)$ or $(0,v)$. A solution (u,v) of (1.7) is *a least energy* solution if (u,v) is nontrivial and $E(u,v)\leq E(\varphi,\psi)$ for any other nontrivial solutions (φ,ψ) of (1.7).

We define

$$\begin{aligned}\mathcal{M}=\Big\{&(u,v)\in\mathcal{H}\setminus\{0,0\}:\\ &\int_{\mathbb{R}^2}\big(|\nabla u|^2+\lambda_1u^2+\frac{u^2}{|x|^2}\big)dx=\int_{\mathbb{R}^2}\big(\mu_1u^4+\beta u^2v^2\big)dx,\\ &\int_{\mathbb{R}^2}\big(|\nabla v|^2+\lambda_2v^2+\frac{v^2}{|x|^2}\big)dx=\int_{\mathbb{R}^2}\big(\mu_2v^4+\beta u^2v^2\big)dx\Big\},\end{aligned}$$

which is the Nehari manifold for system (1.7), and contains all nontrivial solution of (1.7). We consider the minimization problem

$$\begin{aligned}\mathcal{I}&=\inf_{(u,v)\in\mathcal{M}}E(u,v)\\ &=\inf_{(u,v)\in\mathcal{M}}\frac14\int_{\mathbb{R}^2}\Big(|\nabla u|^2+\lambda_1u^2+\frac{u^2}{|x|^2}\Big)+\Big(|\nabla v|^2+\lambda_2v^2+\frac{v^2}{|x|^2}\Big)dx.\end{aligned}\tag{1.9}$$

Let $Q(x)=Q(|x|)$ be the unique positive ground solution of scalar equation

$$-\Delta Q+Q+\frac{Q}{|x|^2}=Q^3,\quad x\in\mathbb{R}^2. \tag{1.10}$$

The function Q is well studied, see [15]. Our first result deals with the case $\lambda_1=\lambda_2$. In this case, the ground state solution of (1.7) can be constructed from the solution of the scalar equation (1.10), and a more explicit expression of positive ground state solutions can be obtained as follows.

Theorem 1.1. *Assume that $\lambda_1=\lambda_2>0$.*

(1) *If $0<\beta<\min\{\mu_1,\mu_2\}$ or $\beta>\max\{\mu_1,\mu_2\}$, then $\mathcal{I}$ is attained at $(\sqrt{k}w_{\lambda_1},\sqrt{l}w_{\lambda_1})$, where $k,l>0$ satisfy*

$$\begin{aligned}\mu_1k+\beta l&=1,\\ \beta k+\mu_2l&=1,\end{aligned} \tag{1.11}$$

and

$$w_{\lambda_1}(x)=\sqrt{\lambda_1}Q\big(\sqrt{\lambda_1}x\big). \tag{1.12}$$

That is, $(\sqrt{k}w_{\lambda_1},\sqrt{l}w_{\lambda_1})$ is a positive least energy solution of (1.7).

(2) *If $\beta\in[\min\{\mu_1,\mu_2\},\max\{\mu_1,\mu_2\}]$ and $\mu_1\neq\mu_2$, then (1.7) does not have a nontrivial nonnegative solution.*

Taking advantage of $\lambda_1=\lambda_2$, we can prove the uniqueness of positive ground state solution of (1.7). Precisely, we have the following result.

Theorem 1.2. *Assume that $\lambda_1=\lambda_2>0$, and let $0<\beta<\min\{\mu_1,\mu_2\}$ or $\beta>\max\{\mu_1,\mu_2\}$. Let (u,v) be any positive least energy solution of (1.7), then $(u,v)=(\sqrt{k}w_{\lambda_1},\sqrt{l}w_{\lambda_1})$, where (k,l) satisfies (1.11) and w_{λ_1} is given in (1.12).*

Next, we consider the general case $\lambda_2\geq\lambda_1>0$ and $\beta\in\mathbb{R}$ which covers all negative value, that is, the repulsive case. For the existence, we have that following result.

Theorem 1.3. *Assume that $\lambda_2\geq\lambda_1>0$. Let χ_0 be the smaller root of the equation*

$$\lambda^{-1/2}\big(2-\lambda^{-1/2}\big)x^2-(\nu_1+\nu_2)x+\nu_1\nu_2=0, \tag{1.13}$$

where

$$\nu_1=\mu_1\lambda^{1/2},\quad \nu_2=\mu_2\lambda^{-1/2},\quad \lambda=\frac{\lambda_2}{\lambda_1}. \tag{1.14}$$

If $-\infty<\beta<\chi_0$, then (1.7) possesses a positive ground state solution.

In Theorem 1.3, the existence of positive radial ground state solution is shown for $\beta<\chi_0$. Indeed we can prove the existence of radial ground state solution for $\beta\in(-\infty,\chi_1)$ where $\chi_1>\chi_0$, see Proposition 2.5 for the definition of χ_1. In next section, we also show that $\chi_0<\chi_1<\min\{\nu_1,\nu_2\}$.

Our last result concerns with the limiting behavior of positive ground state solutions of (1.7) as $\beta\to-\infty$. Denote $\{w>0\}:=\{x\in\mathbb{R}^2:w(x)>0\}$. Then we have the following result.

Theorem 1.4. *Assume that $\lambda_2\geq\lambda_1>0$. Let $\beta_n<0$, $n\in\mathbb{N}$ satisfy $\beta_n\to-\infty$ as $n\to\infty$, and let (u_n,v_n) be the positive least energy solutions of (1.7) with $\beta=\beta_n$, finding by Theorem 1.3. Then, after passing to a subsequence, we have that*

$u_n \to u_\infty$ and $v_n \to v_\infty$ strongly in H, the functions u_∞ and v_∞ are continuous, $u_\infty \geq 0$, $v_\infty \geq 0$, $u_\infty v_\infty \equiv 0$, u_∞ solves the problem

$$-\Delta u + \lambda_1 u + \frac{u^2}{|x|^2} = \mu_1 u^3 \quad \text{in } \{u_\infty > 0\},$$

and v_∞ solves the problem

$$-\Delta v + \lambda_2 v + \frac{v^2}{|x|^2} = \mu_2 v^3 \quad \text{in } \{v_\infty > 0\}.$$

Furthermore, both $\{u_\infty > 0\}$ and $\{v_\infty > 0\}$ are connected domains and $\{u_\infty > 0\} = \mathbb{R}^2 \setminus \overline{\{v_\infty > 0\}}$.

As shown in Theorem 1.4, the components of the limiting profile tend to separate in different regions of $\mathbb{R}^2$, and thus the phenomena of phase separation happens.

The paper is organized as follows. After presenting preliminary results in Section 2, we prove the existence and non-existence results in Section 3. Section 4 is devoted to prove the uniqueness result. Finally, we prove the phenomena of phase separation in Section 5.

2. Preliminary results

In this section, we show some preliminary results for future reference. Let

$$H := \big\{u \in H_r^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} dx < \infty\big\},$$

where $H_r^1(\mathbb{R}^2) = \{u \in H^1(\mathbb{R}^2) : u(x) = u(|x|)\}$. We denote by H_λ the Hilbert spaces H endowed with the norm defined by

$$\|u\|_\lambda^2 := \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + \lambda u^2 + \frac{u^2}{|x|^2}\Big) dx \quad \text{for all } u \in H_\lambda,$$

which is induced by the inner product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^2} \Big(\nabla u \nabla v + \lambda uv + \frac{uv}{|x|^2}\Big) dx \quad \text{for all } u, v \in H_\lambda.$$

Apparently, $H_\lambda \hookrightarrow H_r^1(\mathbb{R}^2)$, and by well known compact embedding of $H_r^1(\mathbb{R}^2)$, one has $H_\lambda \hookrightarrow\hookrightarrow L^4(\mathbb{R}^2)$ is compact. The following proposition shows that minimizers of $\mathcal{I}$ defined by (1.9) are solutions of (1.7).

Proposition 2.1. *If $\mathcal{I}$ is attained by a couple $(u, v) \in \mathcal{M}$, then (u, v) is a solution of (1.7) provided $-\infty < \beta < \sqrt{\mu_1\mu_2}$.*

Proof. We need to show that any minimizer (u, v) of $\mathcal{I}$ satisfies $dE(u, v) = E'(u, v) = 0$.

We write $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, where $\mathcal{M}_i$ is the set of pairs $(u, v) \in \mathcal{H}$ such that $u \not\equiv 0$, $v \not\equiv 0$, $f_i(u, v) = 0$, for $i = 1, 2$, where

$$\begin{aligned} f_1(u, v) &:= \int_{\mathbb{R}^2} \big(|\nabla u|^2 + \lambda_1 u^2 + \frac{u^2}{|x|^2}\big) dx - \int_{\mathbb{R}^2} \big(\mu_1 u^4 + \beta u^2 v^2\big) dx, \\ f_2(u, v) &:= \int_{\mathbb{R}^2} \big(|\nabla v|^2 + \lambda_2 v^2 + \frac{v^2}{|x|^2}\big) dx - \int_{\mathbb{R}^2} \big(\mu_2 v^4 + \beta u^2 v^2\big) dx. \end{aligned}$$

For each $(\varphi,\psi)\in\mathcal{H}$,

$$
\begin{aligned}
&\langle E'(u,v),(\varphi,\psi)\rangle\\
&=\int_{\mathbb{R}^2}\Big(\nabla u\nabla\varphi+\lambda_1 u\varphi+\frac{u\varphi}{|x|^2}\Big)dx+\int_{\mathbb{R}^2}\Big(\nabla v\nabla\psi+\lambda_2 v\psi+\frac{v\psi}{|x|^2}\Big)dx\\
&\quad-\int_{\mathbb{R}^2}\big(\mu_1 u^3\varphi+\beta uv(u\psi+v\varphi)+\mu_2 v^3\psi\big)dx,
\end{aligned}
$$

$$
\begin{aligned}
&\langle f_1'(u,v),\frac{1}{2}(\varphi,\psi)\rangle\\
&=\int_{\mathbb{R}^2}\Big(\nabla u\nabla\varphi+\lambda_1 u\varphi+\frac{u\varphi}{|x|^2}\Big)dx-\int_{\mathbb{R}^2}\big(2\mu_1 u^3\varphi+\beta uv(u\psi+v\varphi)\big)dx,\\
&\langle f_2'(u,v),\frac{1}{2}(\varphi,\psi)\rangle\\
&=\int_{\mathbb{R}^2}\Big(\nabla v\nabla\psi+\lambda_2 v\psi+\frac{v\psi}{|x|^2}\Big)dx-\int_{\mathbb{R}^2}\big(2\mu_2 v^3\psi+\beta uv(u\psi+v\varphi)\big)dx.
\end{aligned}
$$

We can verify that $f_i'(u,v)\neq 0$ for $(u,v)\in\mathcal{M}$, since $u\not\equiv 0$ and $v\not\equiv 0$ in $\mathcal{M}_i$.

Let $(u,v)\in\mathcal{M}$ be a minimizer for E restricted on $\mathcal{M}$, there are two Lagrange multipliers $L_1,L_2\in\mathbb{R}$ such that

$$
E'(u,v)+L_1 f_1'(u,v)+L_2 f_2'(u,v)=0.
$$

Taking into account $f_1(u,v)=0$, from

$$
\langle E'(u,v)+L_1 f_1'(u,v)+L_2 f_2'(u,v),(u,0)\rangle=0, \tag{2.1}
$$

we deduce that

$$
L_1\int_{\mathbb{R}^2}\mu_1 u^4dx+L_2\int_{\mathbb{R}^2}\beta u^2v^2dx=0. \tag{2.2}
$$

Similarly, from

$$
\langle E'(u,v)+L_1 f_1'(u,v)+L_2 f_2'(u,v),(u,0)\rangle=0,
$$

we obtain

$$
L_1\int_{\mathbb{R}^2}\beta u^2v^2dx+L_2\int_{\mathbb{R}^2}\mu_2 v^4dx=0. \tag{2.3}
$$

Using that $f_1(u,v)=f_2(u,v)=0$, if $\beta<0$,

$$
A=\det\begin{pmatrix}\int_{\mathbb{R}^2}\mu_1u^4dx & \int_{\mathbb{R}^2}\beta u^2v^2dx\\ \int_{\mathbb{R}^2}\beta u^2v^2dx & \int_{\mathbb{R}^2}\mu_2v^4dx\end{pmatrix} \tag{2.4}
$$

is diagonally dominant, see [18, Lemma 2.1]. By the Hölder inequality, A is positively definite if $\beta^2<\mu_1\mu_2$. Hence, the only solution of system (2.2)-(2.3) is $L_1=L_2=0$, which implies $E'(u,v)=0$ by (2.1). $\square$

The existence and properties of semi-trivial solutions of (1.7) are well-studied. Let us recall some facts. Consider the minimization problems

$$
S_{\lambda,\mu}=\inf_{u\in H\setminus\{0\}}\frac{\|u\|_\lambda^2}{\big(\int_{\mathbb{R}^2}\mu u^4dx\big)^{1/2}} \tag{2.5}
$$

and

$$
T_{\lambda,\mu}=\inf_{\mathcal{N}_0}\Big\{\frac{1}{2}\|u\|_\lambda^2-\frac{1}{4}\int_{\mathbb{R}^2}\mu u^4dx\Big\},
$$

where $\mathcal{N}_0=\{u\in H: u\not\equiv 0,\ \|u\|_\lambda^2=\int_{\mathbb{R}^2}\mu u^4dx\}$.

Proposition 2.2. *The function*

$$w_{\lambda,\mu} = \mu^{-1/2}\sqrt{\lambda}\, Q\left(\sqrt{\lambda}x\right)$$

is a minimizer for $T_{\lambda,\mu}$, and it is the unique positive solution of the equation

$$-\Delta w + \lambda w + \frac{w}{|x|^2} = \mu w^3 \quad \text{in } \mathbb{R}^2.$$

In addition,

$$T_{\lambda,\mu} = \frac{1}{4}S^2_{\lambda,\mu}, \quad S_{\lambda,\mu} = \mu^{-1/2}\lambda^{1/2}S_{1,1}.$$

The assertion of the above proposition follows by scaling arguments for Q since $Q(x)$ is the unique positive ground state solution of equation (1.10).

In the following, we set $w_\lambda(x) = w_{\lambda,1}(x) = \sqrt{\lambda}Q(\sqrt{\lambda}x)$, $T_\lambda = T_{\lambda,1}$ and $S_\lambda = S_{\lambda,1}$. We introduce a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$h(\lambda) := \frac{\int_{\mathbb{R}^2} Q^2(x)w_\lambda^2(x)dx}{\int_{\mathbb{R}^2} Q^4(x)dx}. \tag{2.6}$$

Proposition 2.3. *For every $\lambda \geq 1$, we have*

$$1 \leq h(\lambda) \leq \lambda^{1/2}. \tag{2.7}$$

Proof. Since $Q(x)$ is radial and strictly decreasing in $|x|$, we have

$$Q(x) \geq Q(\sqrt{\lambda}x) \quad \text{for } \lambda \geq 1 \text{ and } x \in \mathbb{R}^2.$$

Using this fact and a scaling argument, (2.7) readily follows. □

Next, we find a bound for $\mathcal{I}$.

Lemma 2.4. *Let $h(\lambda)$ be defined in (2.6) with $\lambda = \sqrt{\lambda_2/\lambda_1}$ for $\lambda_2 \geq \lambda_1 > 0$. If $k, l > 0$ satisfy the linear system*

$$\begin{aligned} \mu_1 k + \beta h(\lambda) l &= 1, \\ \beta h(\lambda) k + \mu_2 \lambda l &= \lambda. \end{aligned} \tag{2.8}$$

Then

(a) $(\sqrt{k}w_{\lambda_1}, \sqrt{l}w_{\lambda_2}) \in \mathcal{M}$;

(b) *there exists a $\rho_0 > 0$ such that*

$$0 < \rho_0 \leq \mathcal{I} \leq \frac{1}{4}\left(\lambda_1 k + \lambda_2 l\right) S_1^2. \tag{2.9}$$

Proof. To prove that $(\sqrt{k}w_{\lambda_1}, \sqrt{l}w_{\lambda_2}) \in \mathcal{M}$, it is suffices to show that $(u, v) = (\sqrt{k}w_{\lambda_1}, \sqrt{l}w_{\lambda_2})$ satisfies

$$\|u\|^2_{\lambda_1} = \int_{\mathbb{R}^2} \left(\mu_1 u^4 + \beta u^2 v^2\right) dx \quad \text{and} \quad \|v\|^2_{\lambda_2} = \int_{\mathbb{R}^2} \left(\mu_2 v^4 + \beta u^2 v^2\right) dx. \tag{2.10}$$

Apparently,

$$\|\sqrt{k}w_{\lambda_1}\|^2_{\lambda_1} = k\lambda_1\Big(\int_{\mathbb{R}^2} |\nabla Q|^2 + Q^2 + \frac{Q^2}{|x|^2}dx\Big) = k\lambda_1\int_{\mathbb{R}^2} Q^4 dx.$$

Substitution x by $\frac{x}{\sqrt{\lambda_1}}$, one can see that

$$\mu_1\int_{\mathbb{R}^2}(\sqrt{k}w_{\lambda_1})^4 dx + \beta\int_{\mathbb{R}^2}(\sqrt{k}w_{\lambda_1})^2(\sqrt{l}w_{\lambda_2})^2 dx$$

$$
\begin{aligned}
&= \mu_1 k^2 \lambda_1^2 \int_{\mathbb{R}^2} (Q(\sqrt{\lambda_1}x))^4 dx + \beta k l \lambda_1 \lambda_2 \int_{\mathbb{R}^2} (Q(\sqrt{\lambda_1}x))^2 (Q(\sqrt{\lambda_2}x))^2 dx \\
&= \mu_1 k^2 \lambda_1 \int_{\mathbb{R}^2} Q^4(x) dx + \beta k l \lambda_2 \int_{\mathbb{R}^2} Q^2(x) \Big(Q\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} x\Big)\Big)^2 dx \\
&= k\lambda_1 \left(\mu_1 k + \beta h(\lambda) l\right) \int_{\mathbb{R}^2} Q^4(x) dx,
\end{aligned}
$$

So the first equality in (2.10) holds if $\mu_1 k + \beta h(\lambda) l = 1$. Similarly, we can prove that the second equality in (2.10) also satisfied if $\beta h(\lambda) k + \mu_2 \lambda l = \lambda$.

Since $(\sqrt{k} w_{\lambda_1}, \sqrt{l} w_{\lambda_2}) \in \mathcal{M}$, by Proposition 2.2,

$$
\begin{aligned}
\mathcal{I} &\leq E(\sqrt{k} w_{\lambda_1}, \sqrt{l} w_{\lambda_2}) \\
&= \frac{k}{4} \|w_{\lambda_1}\|_{\lambda_1}^2 + \frac{l}{4} \|w_{\lambda_2}\|_{\lambda_2}^2 \\
&= k T_{\lambda_1} + l T_{\lambda_2} \\
&= \frac{k}{4} S_{\lambda_1}^2 + \frac{l}{4} S_{\lambda_2}^2 \\
&= \frac{1}{4} \left(k\lambda_1 + l\lambda_2\right) S_1^2.
\end{aligned} \tag{2.11}
$$

On the other hand, if $(u, v) \in \mathcal{M}$, we have

$$
\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 = \mu_1 \|u\|_{L^4}^4 + \mu_2 \|v\|_{L^4}^4 + 2\beta \|uv\|_{L^2}^2.
$$

By Sobolev inequality $H \hookrightarrow H_r^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and Hölder inequality, we have

$$
\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 \leq c_0 (\|u\|_{\lambda_1}^4 + \|v\|_{\lambda_2}^4 + 2\|u\|_{\lambda_1}^2 \|v\|_{\lambda_2}^2),
$$

where $c_0 = c_0(\mu_1, \mu_2, \beta)$ is a positive constant. Therefore, for $(u, v) \in \mathcal{M}$,

$$
E(u, v) = \frac{1}{4} (\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) \geq \frac{1}{4c_0}, \tag{2.12}
$$

which implies $\mathcal{I} \geq \rho_0 > 0$ for some ρ_0. Item (b) follows from (2.11) and (2.12). $\square$

Now, we solve (2.8). It follows from (2.8) that

$$
(\mu_1 \mu_2 \lambda - \beta^2 h^2(\lambda)) k = \mu_2 \lambda - \lambda h(\lambda) \beta. \tag{2.13}
$$

Hence, (2.13) is solvable for $k > 0$ and $l > 0$ if either

$$
\mu_1 \mu_2 \lambda - \beta^2 h^2(\lambda) > 0 \quad \text{and} \quad \beta h(\lambda) < \min\{\mu_2, \mu_1 \lambda\}, \tag{2.14}
$$

or

$$
\mu_1 \mu_2 \lambda - \beta^2 h^2(\lambda) < 0 \quad \text{and} \quad \beta h(\lambda) > \max\{\mu_2, \mu_1 \lambda\}. \tag{2.15}
$$

By Proposition 2.3, we know that (2.14) is satisfied if

$$
-\sqrt{\mu_1 \mu_2} < \beta < \lambda^{-1/2} \min\{\mu_2, \mu_1 \lambda\} = \min\{\nu_1, \nu_2\}, \tag{2.16}
$$

where ν_1 and ν_2 are defined in (1.14). Similarly, (2.15) is satisfied if

$$
\beta > \max\{\mu_2, \mu_1 \lambda\} = \lambda^{1/2} \max\{\nu_1, \nu_2\}. \tag{2.17}
$$

Hence,

$$
k = \frac{\lambda(\mu_2 - \beta h(\lambda))}{\mu_1 \mu_2 \lambda - \beta^2 h^2(\lambda)} \quad \text{and} \quad l = \frac{\mu_1 \lambda - \beta h(\lambda)}{\mu_1 \mu_2 \lambda - \beta^2 h^2(\lambda)} \tag{2.18}
$$

if either (2.16) or (2.17) holds. Define

$$
a(\lambda) = g(\lambda)(2 - g(\lambda)) \quad \text{where} \quad g(\lambda) = \lambda^{-1/2} h(\lambda). \tag{2.19}
$$

By Proposition 2.3,

$$\lambda^{-1/2} \le g(\lambda) \le 1 \quad \text{and} \quad \lambda^{-1/2}(2-\lambda^{-1/2}) \le a(\lambda) \le 1 \quad \text{for} \quad \lambda \ge 1. \tag{2.20}$$

We consider now the minimization problem $\mathcal{I}$.

Proposition 2.5. *Suppose that $\lambda = \lambda_2/\lambda_1 \ge 1$. Let χ_1 be the smaller root of the quadratic equation*

$$a(\lambda)x^2 - (\nu_1+\nu_2)x + \nu_1\nu_2 = 0.$$

Assume that

$$-\infty < \beta < \chi_1. \tag{2.21}$$

Let $\{(u_n, v_n)\} \subset \mathcal{M}$ be a sequence such that $E(u_n, v_n) \to \mathcal{I}$ as $n \to \infty$. Then there exists a constant $c_0 > 0$ such that $\|u_n\|_{L^4} \ge c_0$ and $\|v_n\|_{L^4} \ge c_0$ for all $n \in \mathbb{N}$.

Remark 2.6. Recall the constant χ_0 defined in Theorem 1.3 and (1.13), we see from (2.20) that $\chi_0 \le \chi_1 \le \min\{\nu_1, \nu_2\}$.

Proof of Proposition 2.5. Let $\{(u_n, v_n)\} \subset \mathcal{M}$ be a minimizing sequence for $\mathcal{I}$, that is,

$$E(u_n, v_n) = \frac{1}{4}\big(\|u_n\|_{\lambda_1}^2 + \|v_n\|_{\lambda_2}^2\big) = \frac{1}{4}\int_{\mathbb{R}^2} \mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4 dx \to \mathcal{I},$$

as $n \to \infty$. It follows that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. We recall that $u_n \not\equiv 0$ and $v_n \not\equiv 0$, then we define that

$$z_{1,n} = \Big(\int_{\mathbb{R}^2} u_n^4\,dx\Big)^{1/2}, \quad z_{2,n} = \Big(\int_{\mathbb{R}^2} v_n^4\,dx\Big)^{1/2}.$$

By the Sobolev and Hölder inequalities, the definition of S_λ and Proposition 2.2, we have

$$\lambda_1^{1/2} S_1 z_{1,n} \le \|u_n\|_{\lambda_1}^2 = \int_{\mathbb{R}^2} \mu_1 u_n^4 + \beta u_n^2 v_n^2 dx \le \mu_1 z_{1,n}^2 + \beta^+ z_{1,n} z_{2,n}\,, \tag{2.22}$$

$$\lambda_2^{1/2} S_1 z_{2,n} \le \|v_n\|_{\lambda_2}^2 = \int_{\mathbb{R}^2} \mu_2 v_n^4 + \beta u_n^2 v_n^2 dx \le \mu_2 z_{2,n}^2 + \beta^+ z_{1,n} z_{2,n}\,, \tag{2.23}$$

where $\beta^+ = \max\{\beta, 0\}$.

If $\beta \le 0$, the conclusion follows from (2.22)-(2.23). Therefore, we assume that $\beta > 0$. By (2.22)-(2.23), we have

$$S_1\big(\lambda_1^{1/2} z_{1,n} + \lambda_2^{1/2} z_{2,n}\big) \le \int_{\mathbb{R}^2} \mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4 dx = 4\mathcal{I} + o_n(1), \tag{2.24}$$

where $o_n(1) \to 0$ as $n \to \infty$.

Let $\tilde z_{i,n} = \lambda_1^{-1/2} S_1^{-1} z_{i,n}$ for $i = 1, 2$. By (2.9), (2.22) and (2.23), we obtain the following inequalities

$$\begin{gathered} \tilde z_{1,n} + \lambda^{1/2}\tilde z_{2,n} \le k + \lambda l + o_n(1),\\ \mu_1 \tilde z_{1,n} + \beta \tilde z_{2,n} \ge 1,\\ \beta \tilde z_{1,n} + \mu_2 \tilde z_{2,n} \ge \lambda^{1/2}, \end{gathered} \tag{2.25}$$

where $\lambda = \sqrt{\lambda_2/\lambda_1}$, k, l are given in Lemma 2.4.

To prove that the two sequences $\{\tilde z_{1,n}\}$, $\{\tilde z_{2,n}\}$ stay uniformly away zero, we need to show that each two of the following lines

$$l_1 = \big\{(z_1, z_2) \in \mathbb{R}^2 : z_1 + \lambda^{1/2} z_2 = k + \lambda l\big\},$$

$$l_2 = \big\{(z_1,\, z_2) \in \mathbb{R}^2 : \mu_1 z_1 + \beta z_2 = 1\big\},$$
$$l_3 = \big\{(z_1,\, z_2) \in \mathbb{R}^2 : \beta z_1 + \mu_2 z_2 = \lambda^{1/2}\big\}$$

meet, and their crossing points have strictly positive coordinates. This can be achieved if the following set of conditions are met:

$$\beta\lambda^{1/2} < \mu_2, \quad \beta < \mu_1\lambda^{1/2}, \tag{2.26}$$
$$\mu_1(k+\lambda l) > 1, \tag{2.27}$$
$$\mu_2(k+\lambda l) > \lambda, \tag{2.28}$$
$$\beta(k+\lambda l) < \lambda^{1/2}. \tag{2.29}$$

By Remark 2.6, $\beta < \chi_1 \le \min\{\nu_1,\, \nu_2\}$, then (2.26) holds. Next, we deduce from (2.14) and (2.18) that

$$\mu_1(k+\lambda l) - 1 = \frac{(\mu_1\lambda_1 - \beta h(\lambda))^2}{\mu_1\mu_2\lambda - \beta^2 h^2(\lambda)} > 0, \tag{2.30}$$

this implies (2.27) holds. Similarly, (2.28) holds. Finally, (2.29) is equivalent to

$$\Big[\frac{2h(\lambda)\lambda^{1/2} - h^2(\lambda)}{\lambda}\Big]\beta^2 - \Big(\lambda^{-1/2}\mu_2 + \lambda^{1/2}\mu_1\Big)\beta + \mu_1\mu_2 > 0,$$

that is,

$$a(\lambda)\beta^2 - (\nu_1+\nu_2)\beta + \nu_1\nu_2 > 0.$$

Therefore, by the definition of χ_1, one sees that (2.26)-(2.29) are satisfied if $0 < \beta < \chi_1$. This completes the proof. $\square$

3. Proof of Theorems 1.1 and 1.3

In this section we assume $\lambda_1 = \lambda_2 > 0$.

Proof of Theorem 1.1. If $\lambda = \lambda_2/\lambda_1 = 1$, then (2.25) becomes

$$\begin{gathered} \tilde z_{1,n} + \tilde z_{2,n} \le k + l + o_n(1), \\ \mu_1\tilde z_{1,n} + \beta\tilde z_{2,n} \ge 1, \\ \beta\tilde z_{1,n} + \mu_2\tilde z_{2,n} \ge 1. \end{gathered} \tag{3.1}$$

If either $0 < \beta < \min\{\mu_1, \mu_2\}$, or $\beta > \max\{\mu_1, \mu_2\}$ holds, equation (1.11) has a solution (k, l) satisfying $k > 0$ and $l > 0$. We set $w_{1,n} = \tilde z_{1,n} - k$ and $w_{2,n} = \tilde z_{2,n} - l$. By (1.11) and (3.1), we deduce that

$$\begin{gathered} w_{1,n} + w_{2,n} \le o_n(1), \\ \mu_1 w_{1,n} + \beta w_{2,n} \ge 0, \\ \beta w_{1,n} + \mu_2 w_{2,n} \ge 0. \end{gathered} \tag{3.2}$$

Therefore, $w_{1,n} \to 0$ and $w_{2,n} \to 0$ as $n \to +\infty$, that is, $\tilde z_{1,n} \to k$ and $\tilde z_{2,n} \to l$ as $n \to +\infty$.

Noting $\tilde z_{i,n} = \lambda_1^{-1/2} S_1^{-1} z_{i,n}$ for $i = 1, 2$ and passing to the limit in (2.24) with $\lambda_1 = \lambda_2$, we obtain

$$\mathcal{I} \ge \frac{1}{4}\lambda_1(k+l)S_1^2.$$

On the other hand, by Lemma 2.4, we have

$$\mathcal{I} \le E\big(\sqrt{k} w_{\lambda_1},\, \sqrt{l} w_{\lambda_1}\big) = \frac{1}{4}\lambda_1\,(k+l)\,S_1^2. \tag{3.3}$$

This implies

$$\mathcal{I} = E\big(\sqrt{k}w_{\lambda_1},\, \sqrt{l}w_{\lambda_1}\big) = \frac{1}{4}\lambda_1\,(k+l)\,S_1^2, \tag{3.4}$$

which proves part (1) in Theorem 1.1. For part (2), multiplying the u-equation in (1.7) by v, and the v-equation by u, subtracting and integrating over $\mathbb{R}^2$, we obtain

$$\int_{\mathbb{R}^2} uv[(\mu_1-\beta)u^2 + (\beta-\mu_2)v^2]dx = 0.$$

Thus, (1.7) does not have nontrivial nonnegative solutions if

$$\beta \in [\min\{\mu_1,\mu_2\}, \max\{\mu_1,\mu_2\}] \quad \text{and} \quad \mu_1 \neq \mu_2.$$

That is, (2) in Theorem 1.1 holds. This completes the proof. □

Proof of Theorem 1.3. Let $\{(u_n,v_n)\} \subset \mathcal{M}$ be a minimizing sequence for $\mathcal{I}$. Since $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$, we may assume that

$$\begin{gathered}(u_n,v_n) \rightharpoonup (u,v) \quad \text{in} \quad \mathcal{H},\\ (u_n,v_n) \to (u,v) \quad \text{in} \quad L^4(\mathbb{R}^2)\times L^4(\mathbb{R}^2),\end{gathered}$$

with $(u,v)\in\mathcal{H}$. The weak continuity of norms yields

$$\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 \le \liminf_{n\to+\infty}\big(\|u_n\|_{\lambda_1}^2 + \|v_n\|_{\lambda_2}^2\big) = 4\mathcal{I}. \tag{3.5}$$

By Proposition 2.5 and Remark 2.6, both $\{\|u_n\|_{L^4}\}$ and $\{\|v_n\|_{L^4}\}$ are bounded away from zero, so the limit (u,v) is nontrivial. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^2}\mu_1u^4 + 2\beta u^2v^2 + \mu_2v^4dx &= \lim_{n\to+\infty}\int_{\mathbb{R}^2}\mu_1u_n^4 + 2\beta u_n^2v_n^2 + \mu_2v_n^4dx \qquad (3.6)\\ &= 4\lim_{n\to+\infty}E(u_n,\,v_n) = 4\mathcal{I}. \qquad (3.7)\end{aligned}$$

Since (u,v) is nontrivial and the matrix A in (2.4) is positively definite, there exists a unique couple (t_1,t_2) satisfying

$$\begin{gathered}\Big(\int_{\mathbb{R}^2}\mu_1u^4dx\Big)t_1 + \Big(\int_{\mathbb{R}^2}\beta u^2v^2dx\Big)t_2 = \|u\|_{\lambda_1}^2,\\ \Big(\int_{\mathbb{R}^2}\beta u^2v^2dx\Big)t_1 + \Big(\int_{\mathbb{R}^2}\mu_2v^4dx\Big)t_2 = \|v\|_{\lambda_2}^2.\end{gathered} \tag{3.8}$$

We claim that $t_1>0$ and $t_2>0$. If $\beta\le 0$, the claim is obvious. Now we deal with the case $\beta>0$. Let us prove that $t_1>0$. The case $t_2>0$ can be proved in the same way. We deduce from (3.8) that

$$\begin{aligned}&\Big\{\Big(\int_{\mathbb{R}^2}\mu_1u^4dx\Big)\Big(\int_{\mathbb{R}^2}\mu_2v^4dx\Big) - \Big(\int_{\mathbb{R}^2}\beta u^2v^2dx\Big)^2\Big\}t_1\\ &= \|u\|_{\lambda_1}^2\Big(\int_{\mathbb{R}^2}\mu_2v^4dx\Big) - \|v\|_{\lambda_2}^2\Big(\int_{\mathbb{R}^2}\beta u^2v^2dx\Big).\end{aligned} \tag{3.9}$$

Since the matrix A in (2.4) is diagonally dominate, in order to prove $t_1>0$, it is sufficient to show that

$$\|u\|_{\lambda_1}^2\Big(\int_{\mathbb{R}^2}\mu_2v^4dx\Big) > \|v\|_{\lambda_2}^2\Big(\int_{\mathbb{R}^2}\beta u^2v^2dx\Big). \tag{3.10}$$

By the Hölder inequality and the definition of S_{λ_1} in (2.5), we have

$$\int_{\mathbb{R}^2}\beta u^2v^2dx \le \beta\Big(\int_{\mathbb{R}^2}u^4dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2}v^4dx\Big)^{1/2} \tag{3.11}$$

and

$$S_{\lambda_1} \le \frac{\|u\|_{\lambda_1}^2}{\big(\int_{\mathbb{R}^2} u^4 dx\big)^{1/2}}.$$

Therefore, (3.10) follows once we prove

$$\mu_2\Big(\int_{\mathbb{R}^2} v^4 dx\Big)^{1/2} > \frac{\beta}{S_{\lambda_1}}\|v\|_{\lambda_2}^2. \tag{3.12}$$

Now we show (3.12). Since $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathcal{H}$ and $(u_n, v_n) \to (u, v)$ in $L^4(\mathbb{R}^2) \times L^4(\mathbb{R}^2)$, we have

$$\begin{aligned} \|u\|_{\lambda_1}^2 &\le \liminf_{n\to+\infty} \|u_n\|_{\lambda_1}^2 = \liminf_{n\to+\infty} \int_{\mathbb{R}^2} \mu_1 u_n^4 + \beta u_n^2 v_n^2 dx \\ &= \int_{\mathbb{R}^2} \mu_1 u^4 + \beta u^2 v^2 dx. \end{aligned} \tag{3.13}$$

Similarly,

$$\|v\|_{\lambda_2}^2 \le \int_{\mathbb{R}^2} \mu_2 v^4 + \beta u^2 v^2 dx. \tag{3.14}$$

We deduce from (3.11) that

$$\|v\|_{\lambda_2}^2 \le \int_{\mathbb{R}^2} \mu_2 v^4 dx + \beta\Big(\int_{\mathbb{R}^2} u^4 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2} v^4 dx\Big)^{1/2}. \tag{3.15}$$

Hence, we see that (3.12) holds if

$$1 > \frac{\beta}{S_{\lambda_1}}\Big(\Big(\int_{\mathbb{R}^2} v^4 dx\Big)^{1/2} + \frac{\beta}{\mu_2}\Big(\int_{\mathbb{R}^2} u^4 dx\Big)^{1/2}\Big). \tag{3.16}$$

By Proposition 2.2, $S_{\lambda_1} = \lambda_1^{1/2} S_1$, equation (3.16) can be written as

$$\lim_{n\to+\infty} \beta\big(\frac{\beta}{\mu_2}\tilde{z}_{1,n} + \tilde{z}_{2,n}\big) < 1. \tag{3.17}$$

We claim that (3.17) is valid. Indeed, by the first inequality in (2.25) and (2.26), as well as (2.29) we see that

$$\lim_{n\to+\infty} \beta\Big(\frac{\beta}{\mu_2}\tilde{z}_{1,n} + \tilde{z}_{2,n}\Big) \le \lim_{n\to+\infty} \frac{\beta}{\lambda^{1/2}}\Big(\tilde{z}_{1,n} + \lambda^{1/2}\tilde{z}_{2,n}\Big) \le \frac{\beta}{\lambda^{1/2}}(k + \lambda l) < 1.$$

Similarly, by (3.13) and the second inequality in (2.26) we can prove that $t_2 > 0$.

Since $t_1 > 0$ and $t_2 > 0$, we know from (3.8) that $(\sqrt{t_1}u, \sqrt{t_2}v) \in \mathcal{M}$. So

$$\mathcal{I} \le E(\sqrt{t_1}u, \sqrt{t_2}v) = \frac{1}{4}\left(t_1\|u\|_{\lambda_1}^2 + t_2\|v\|_{\lambda_2}^2\right). \tag{3.18}$$

Equations (3.5) and (3.18) yield

$$\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_1}^2 \le t_1\|u\|_{\lambda_1}^2 + t_2\|v\|_{\lambda_2}^2. \tag{3.19}$$

Substituting (3.19) into (3.8), we obtain

$$t_1\Big(\|u\|_{\lambda_1}^2 - \int_{\mathbb{R}^2} (\mu_1 u^4 + \beta u^2 v^2) dx\Big) + t_2\Big(\|v\|_{\lambda_2}^2 - \int_{\mathbb{R}^2} (\mu_2 v^4 + \beta u^2 v^2) dx\Big) \ge 0.$$

By (3.13)-(3.15) and $t_1, t_2 > 0$, we have

$$\begin{gathered} \|u\|_{\lambda_1}^2 - \int_{\mathbb{R}^2} (\mu_1 u^4 + \beta u^2 v^2) dx = 0, \\ \|v\|_{\lambda_2}^2 - \int_{\mathbb{R}^2} (\mu_2 v^4 + \beta u^2 v^2) dx = 0, \end{gathered}$$

which means $(u, v) \in \mathcal{M}$. Thus, by (3.5)-(3.7), we know (u, v) is a minimizer of $\mathcal{I}$. Furthermore, by Remark 2.6,

$$\beta < \chi_0 \le \min\{\nu_1, \nu_2\} < \sqrt{\mu_1\mu_2},$$

Proposition 2.1 then implies that (u, v) is a nontrivial solution of (1.7). $\square$

4. Proof of Theorem 1.2

In this section, we prove the uniqueness of solution for problem (1.7) inspired by [9].

Proof of Theorem 1.2. There are two cases to be considered, the first one is $\mu_1 > 0$, $\mu_2 > 0$ and $0 < \beta < \min\{\mu_1, \mu_2\}$, another one is $\mu_1 > 0$, $\mu_2 > 0$ and $\beta > \max\{\mu_1, \mu_2\}$.

In the first case: $\mu_1 > 0$, $\mu_2 > 0$ and $0 < \beta < \min\{\mu_1, \mu_2\}$, suppose (u_0, v_0) is a positive least energy solution of (1.7). By Theorem 1.1, $(\sqrt{k}w_{\lambda_1}, \sqrt{l}w_{\lambda_1})$ is a least energy solution, we claim that

$$\int_{\mathbb{R}^2} u_0^4 \, dx = k^2 \int_{\mathbb{R}^2} w_{\lambda_1}^4 \, dx, \tag{4.1}$$

$$\int_{\mathbb{R}^2} v_0^4 \, dx = l^2 \int_{\mathbb{R}^2} w_{\lambda_1}^4 \, dx, \tag{4.2}$$

$$\int_{\mathbb{R}^2} \beta u_0^2 v_0^2 \, dx = kl \int_{\mathbb{R}^2} w_{\lambda_1}^4 \, dx. \tag{4.3}$$

To prove the claim, we perturb the parameter μ. In fact, there exists a $\delta > 0$, such that $0 < \beta < \min\{\mu, \mu_2\}$ for any $\mu \in (\mu_1 - \delta, \mu_1 + \delta)$. We can show as the proof of Theorem 1.1 that $\mathcal{I}$ is attained if we replace μ_1 by μ. Since E, $\mathcal{M}$ and $\mathcal{I}$ are all depend on μ, we denote them by E_μ, $\mathcal{M}_\mu$ and $\mathcal{I}(\mu)$. Hence, we infer from (1.11) and (3.4) that

$$\mathcal{I}(\mu) = \frac{\mu + \mu_2 - 2\beta}{4(\mu\mu_2 - \beta^2)}\lambda_1 S_1^2,$$

and so $\mathcal{I}'(\mu_1) := \frac{d}{d\mu}\mathcal{I}(\mu)|_{\mu=\mu_1}$ exists. Define

$$\begin{aligned} f(t, s, \mu) &:= t\mu \int_{\mathbb{R}^2} u_0^4 dx + s \int_{\mathbb{R}^2} \beta u_0^2 v_0^2 dx - \int_{\mathbb{R}^2} \Big(|\nabla u_0|^2 + \lambda_1 u_0^2 + \frac{u_0^2}{|x|^2}\Big) dx, \\ g(t, s, \mu) &:= s \int_{\mathbb{R}^2} \mu_2 v_0^4 dx + t \int_{\mathbb{R}^2} \beta u_0^2 v_0^2 dx - \int_{\mathbb{R}^2} \Big(|\nabla v_0|^2 + \lambda_2 v_0^2 + \frac{v_0^2}{|x|^2}\Big) dx. \end{aligned}$$

Then $f(1, 1, \mu_1) = g(1, 1, \mu_1) = 0$ and

$$\begin{aligned} \frac{\partial f}{\partial t}(1, 1, \mu_1) &= \mu_1 \int_{\mathbb{R}^2} u_0^4 dx, \quad \frac{\partial f}{\partial s}(1, 1, \mu_1) = \beta \int_{\mathbb{R}^2} u_0^2 v_0^2 dx, \\ \frac{\partial g}{\partial t}(1, 1, \mu_1) &= \beta \int_{\mathbb{R}^2} u_0^2 v_0^2 dx, \quad \frac{\partial g}{\partial s}(1, 1, \mu_1) = \mu_2 \int_{\mathbb{R}^2} v_0^4 dx. \end{aligned}$$

Set

$$B = \begin{pmatrix} \frac{\partial f}{\partial t}(1, 1, \mu_1) & \frac{\partial f}{\partial s}(1, 1, \mu_1) \\ \frac{\partial g}{\partial t}(1, 1, \mu_1) & \frac{\partial g}{\partial s}(1, 1, \mu_1) \end{pmatrix}.$$

Then $\det(B) > 0$. By the implicit function theorem, we know functions $t(\mu)$ and $s(\mu)$ are well defined and belongs to the class C^1 on $(\mu_1 - \delta_1, \mu_1 + \delta_1)$ for some

$\delta_1 \le \delta$. Moreover, $t(\mu_1) = s(\mu_1) = 1$, so we can assume that $t(\mu) > 0$ and $s(\mu) > 0$ for all $\mu \in (\mu_1 - \delta_1,\, \mu_1 + \delta_1)$ by choosing a small $\delta_1 > 0$. We also know

$$f(t(\mu), s(\mu), \mu) \equiv g(t(\mu), s(\mu), \mu) \equiv 0. \tag{4.4}$$

It can be verified that

$$t'(\mu_1) = -\frac{1}{B}\int_{\mathbb{R}^2} u_0^4 dx \int_{\mathbb{R}^2} \mu_2 v_0^4 dx, \quad s'(\mu_1) = \frac{1}{B}\int_{\mathbb{R}^2} u_0^4 dx \int_{\mathbb{R}^2} \beta u_0^2 v_0^2 dx.$$

By Taylor expansion, we see that

$$\begin{aligned} t(\mu) &= 1 + t'(\mu_1)(\mu - \mu_1) + O((\mu - \mu_1)^2), \\ s(\mu) &= 1 + s'(\mu_1)(\mu - \mu_1) + O((\mu - \mu_1)^2). \end{aligned}$$

By (4.4), $(\sqrt{t(\mu)}u_0,\, \sqrt{s(\mu)}v_0) \in \mathcal{M}_\mu$. We find that

$$\begin{aligned} \mathcal{I}(\mu) &\le E_\mu\Big(\sqrt{t(\mu)}u_0,\, \sqrt{s(\mu)}v_0\Big) \\ &= \frac{t(\mu)}{4}\int_{\mathbb{R}^2}\Big(|\nabla u_0|^2 + \lambda_1 u_0^2 + \frac{u_0^2}{|x|^2}\Big)dx + \frac{s(\mu)}{4}\int_{\mathbb{R}^2}\Big(|\nabla v_0|^2 + \lambda_2 v_0^2 + \frac{v_0^2}{|x|^2}\Big)dx \\ &= \mathcal{I}(\mu_1) + \frac{1}{4}\Theta \cdot (\mu - \mu_1) + O((\mu - \mu_1)^2), \end{aligned}$$

where

$$\begin{aligned} \Theta &:= t'(\mu_1)\int_{\mathbb{R}^2}\Big(|\nabla u_0|^2 + \lambda_1 u_0^2 + \frac{u_0^2}{|x|^2}\Big)dx \\ &\quad + s'(\mu_1)\int_{\mathbb{R}^2}\Big(|\nabla v_0|^2 + \lambda_2 v_0^2 + \frac{v_0^2}{|x|^2}\Big)dx \\ &= -\frac{1}{B}\int_{\mathbb{R}^2} u_0^4 dx \int_{\mathbb{R}^2} \mu_2 v_0^4 dx \int_{\mathbb{R}^2}(\mu_1 u_0^4 + \beta u_0^2 v_0^2)dx \\ &\quad + \frac{1}{B}\int_{\mathbb{R}^2} u_0^4 dx \int_{\mathbb{R}^2} \beta u_0^2 v_0^2 dx \int_{\mathbb{R}^2}(\mu_2 v_0^4 + \beta u_0^2 v_0^2)dx \\ &= -\int_{\mathbb{R}^2} u_0^4 dx. \end{aligned}$$

It follows that

$$\frac{\mathcal{I}(\mu) - \mathcal{I}(\mu_1)}{\mu - \mu_1} \ge \frac{\Theta}{4} + O(\mu - \mu_1), \quad \text{as } \mu \nearrow \mu_1.$$

As a result, $B'(\mu_1) \ge \frac{\Theta}{4}$. Similarly,

$$\frac{\mathcal{I}(\mu) - \mathcal{I}(\mu_1)}{\mu - \mu_1} \le \frac{\Theta}{4} + O(\mu - \mu_1), \quad \text{as } \mu \searrow \mu_1,$$

that is, $B'(\mu_1) \le \frac{\Theta}{4}$. Therefore,

$$\mathcal{I}'(\mu_1) = \frac{\Theta}{4} = -\frac{1}{4}\int_{\mathbb{R}^2} u_0^4 dx. \tag{4.5}$$

On the other hand, by Theorem 1.1, $(\sqrt{k}w_{\lambda_1},\, \sqrt{l}w_{\lambda_1})$ is also a positive least energy solution of (1.7). Hence,

$$\mathcal{I}'(\mu_1) = -\frac{k^2}{4}\int_{\mathbb{R}^2} w_{\lambda_1}^4 dx. \tag{4.6}$$

Consequently, (4.1) is true.

Similarly, using $\mathcal{I}'(\mu_2)$ and $\mathcal{I}'(\beta)$ respectively, we can prove that

$$\int_{\mathbb{R}^2} v_0^4 dx = l^2 \int_{\mathbb{R}^2} w_{\lambda_1}^4 dx, \quad \int_{\mathbb{R}^2} \beta u_0^2 v_0^2 dx = kl \int_{\mathbb{R}^2} w_{\lambda_1}^4 dx.$$

Therefore,

$$\int_{\mathbb{R}^2} \beta u_0^2 v_0^2 dx = \frac{l}{k} \int_{\mathbb{R}^2} u_0^4 dx = \frac{k}{l} \int_{\mathbb{R}^2} v_0^4 dx.$$

Let $(\tilde{u}, \tilde{v}) = (\frac{1}{\sqrt{k}} u_0, \frac{1}{\sqrt{l}} v_0)$. Since $(u_0, v_0) \in \mathcal{M}$, by (1.11), we can verify that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla \tilde{u}| + \lambda_1 \tilde{u}^2 + \frac{\tilde{u}^2}{|x|^2} dx &= \int_{\mathbb{R}^2} \tilde{u}^4 dx, \\ \int_{\mathbb{R}^2} |\nabla \tilde{v}| + \lambda_2 \tilde{v}^2 + \frac{\tilde{v}^2}{|x|^2} dx &= \int_{\mathbb{R}^2} \tilde{v}^4 dx. \end{aligned} \tag{4.7}$$

Noting $\tilde{u}, \tilde{v} \in \mathcal{N}_0$, by Proposition 2.2, we have

$$\int_{\mathbb{R}^2} |\nabla \tilde{u}| + \lambda_1 \tilde{u}^2 + \frac{\tilde{u}^2}{|x|^2} dx \geq \lambda_1 S_1^2, \quad \int_{\mathbb{R}^2} |\nabla \tilde{v}| + \lambda_2 \tilde{v}^2 + \frac{\tilde{v}^2}{|x|^2} dx \geq \lambda_1 S_1^2.$$

Therefore,

$$\begin{aligned} \mathcal{I} &= \frac{1}{4} \lambda_1 (k + l) S_1^2 \\ &= \frac{1}{4} \int_{\mathbb{R}^2} \Big(|\nabla u_0| + \lambda_1 u_0^2 + \frac{u_0^2}{|x|^2} + |\nabla v_0| + \lambda_2 v_0^2 + \frac{v_0^2}{|x|^2} \Big) dx \\ &= \frac{k}{4} \int_{\mathbb{R}^2} \Big(|\nabla \tilde{u}| + \lambda_1 \tilde{u}^2 + \frac{\tilde{u}^2}{|x|^2} \Big) dx + \frac{l}{4} \int_{\mathbb{R}^2} \Big(|\nabla \tilde{v}| + \lambda_2 \tilde{v}^2 + \frac{\tilde{v}^2}{|x|^2} \Big) dx \\ &\geq \frac{1}{4} \lambda_1 (k + l) S_1^2. \end{aligned}$$

This implies

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla \tilde{u}| + \lambda_1 \tilde{u}^2 + \frac{\tilde{u}^2}{|x|^2} dx &= \lambda_1 S_1^2 = S_{\lambda_1}, \\ \int_{\mathbb{R}^2} |\nabla \tilde{v}| + \lambda_2 \tilde{v}^2 + \frac{\tilde{v}^2}{|x|^2} dx &= \lambda_1 S_1^2 = S_{\lambda_2}. \end{aligned}$$

By (4.7), we know $\tilde{u}$ and $\tilde{v}$ are positive ground state solutions of

$$-\Delta w + \lambda_1 w + \frac{w}{|x|^2} = w^3 \quad \text{in } \mathbb{R}^2.$$

The uniqueness of positive ground solution of (1.10) implies

$$\tilde{u}(x) = \tilde{v}(x) = \sqrt{\lambda_1} Q\big(\sqrt{\lambda_1} x\big) = w_{\lambda_1}(x),$$

namely,

$$(u_0, v_0) = (\sqrt{k}\tilde{u}, \sqrt{l}\tilde{v}) = (\sqrt{k} w_{\lambda_1} \sqrt{l} w_{\lambda_1}).$$

Finally, the case $\mu_1 > 0$, $\mu_2 > 0$ and $\beta > \max\{\mu_1, \mu_2\}$ can be treated in the same way since $det(B) < 0$, the implicit function theorem can also be used. $\square$

5. Proof of Theorem 1.4

This section is devoted to prove Theorem 1.4. To highlight the dependence on β, we write $E_\beta, \mathcal{M}_\beta$ instead of $E, \mathcal{M}$. Let

$$\mathcal{I}_\beta := \inf_{(u,v)\in\mathcal{M}_\beta} E_\beta(u,v).$$

The energy functional Φ of the problem

$$-\Delta w + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)w^+ + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)w^- = \mu_1(w^+)^3 + \mu_2(w^-)^3 \quad \text{in } \mathbb{R}^2, \tag{5.1}$$

where $w^+ := \max\{w,0\}$ and $w^- := \min\{w,0\}$, is given by

$$\begin{aligned}\Phi(w) := &\frac{1}{2}\int_{\mathbb{R}^2}\Big[|\nabla w|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(w^+)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(w^-)^2\Big]dx\\ &- \frac{1}{4}\int_{\mathbb{R}^2}\big[\mu_1(w^+)^4 + \mu_2(w^-)^4\big]dx,\end{aligned}$$

and we define the corresponding Nehari manifold $\mathcal{N}$ by

$$\begin{aligned}\mathcal{N} &:= \{w \in H : w \neq 0,\ \Phi'(w)w = 0\}\\ &= \Big\{w \in H : w \neq 0,\\ &\int_{\mathbb{R}^2}\Big[|\nabla w|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(w^+)^2 + \big(\lambda_2 + \frac{1}{|x|^2}\big)(w^-)^2\Big]dx\\ &= \int_{\mathbb{R}^2}[\mu_1(w^+)^4 + \mu_2(w^-)^4]dx\Big\}.\end{aligned}$$

Sign-changing solutions of (5.1) belong to the set

$$\mathcal{E} := \{w \in H : w^+ \in \mathcal{N},\ w^- \in \mathcal{N}\}.$$

Observe that, if $u, v \in H\backslash\{0\}$, $u \geq 0$, $v \geq 0$, there exist unique numbers $s, t \in (0,\infty)$ such that $su \in \mathcal{N}$ and $-tv \in \mathcal{N}$, that is,

$$s^2 = \frac{\int_{\mathbb{R}^2}|\nabla u|^2 + \lambda_1 u^2 + \frac{u^2}{|x|^2}dx}{\int_{\mathbb{R}^2}\mu_1 u^4 dx} \quad \text{and} \quad t^2 = \frac{\int_{\mathbb{R}^2}|\nabla v|^2 + \lambda_2 v^2 + \frac{v^2}{|x|^2}dx}{\int_{\mathbb{R}^2}\mu_2 v^4 dx}. \tag{5.2}$$

If, moreover, $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$, then $su - tv \in \mathcal{E}$. Hence, $\mathcal{E} \neq \emptyset$. We define

$$\mathcal{I}_\infty := \inf_{w\in\mathcal{E}} \Phi(w).$$

Then $\mathcal{I}_\infty$ is finite.

Proposition 5.1. *For $\beta_n \to -\infty$, let $(u_n, v_n) \in \mathcal{M}_{\beta_n}$ satisfy $u_n \geq 0$, $v_n \geq 0$ and $E_{\beta_n}(u_n, v_n) = \mathcal{I}_{\beta_n}$. Then, after passing to a subsequence, we have $u_n \to u_\infty$ and $v_n \to v_\infty$ strongly in $\mathcal{H}$, and (u_∞, v_∞) satisfies*

- (a) *$u_\infty, v_\infty \in \mathcal{N}$, $u_\infty \geq 0$, $v_\infty \geq 0$, $u_\infty v_\infty = 0$. Then, $u_\infty - v_\infty \in \mathcal{E}$.*
- (b) *$\lim_{n\to+\infty} \mathcal{I}_{\beta_n} = \Phi(u_\infty - v_\infty) = \mathcal{I}_\infty$.*
- (c) *$u_\infty - v_\infty$ solves problem (5.1).*

Proof. If $w \in \mathcal{E}$, we have $w^+w^- = 0$, and then $(w^+,\, w^-) \in \mathcal{M}_\beta$,

$$\Phi(w) = E_\beta(w^+, w^-)$$

for every $\beta < 0$. Therefore, $\mathcal{I}_\beta \le \mathcal{I}_\infty$ for every $\beta < 0$. This implies, in particular, that

$$\frac{1}{4}(\|u_n\|^2_{\lambda_1} + \|v_n\|^2_{\lambda_1}) = E_{\beta_n}(u_n, v_n) \le \mathcal{I}_\infty \quad \text{for all } n \in \mathbb{N}.$$

So, after passing to a subsequence, there exist $u_\infty, v_\infty \in \mathcal{H}$ such that

$$\begin{gathered}(u_n, v_n) \rightharpoonup (u_\infty, v_\infty) \quad \text{weakly in } \mathcal{H},\\ (u_n, v_n) \to (u_\infty, v_\infty) \quad \text{strongly in } L^4(\mathbb{R}^2) \times L^4(\mathbb{R}^2),\\ (u_n, v_n) \to (u_\infty, v_\infty) \quad \text{a.e. in } \mathbb{R}^2 \times \mathbb{R}^2.\end{gathered}$$

Hence, $u_\infty \ge 0$ and $v_\infty \ge 0$. Since $(u_n, v_n) \in \mathcal{M}_{\beta_n}$, we see that

$$0 \le -2\beta_n \int_{\mathbb{R}^2} u_n^2 v_n^2 dx \le \mu_1 \int_{\mathbb{R}^2} u_n^4 dx + \mu_2 \int_{\mathbb{R}^2} v_n^4 dx \le C_0.$$

Using Fatou's lemma, we obtain

$$\int_{\mathbb{R}^2} u_\infty^2 v_\infty^2 dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^2} u_n^2 v_n^2 dx \le \frac{C_0}{2} \lim_{n\to+\infty} \frac{1}{(-\beta_n)} = 0.$$

Hence, $u_\infty v_\infty = 0$ a.e. in $\mathbb{R}^2$.

On the other hand, by Proposition 2.5, we know $u_\infty \ne 0$ and $v_\infty \ne 0$. Then, we may show as (5.2) that there exists $s, t \in (0, \infty)$ such that $su_\infty, tv_\infty \in \mathcal{N}$ and $su_\infty - tv_\infty \in \mathcal{E}$. By the fact that

$$E(u, v) = \max\{E(su, tv) : s > 0,\ t > 0\}$$

if $(u, v) \in \mathcal{M}$, seeing (d) of Proposition 2.1 in [11], we deduce that

$$\begin{aligned}
\mathcal{I}_\infty &\le \frac{1}{2}\int_{\mathbb{R}^2} \Big[|\nabla su_\infty|^2 + |\nabla tv_\infty|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(su_\infty)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(tv_\infty)^2\Big] dx\\
&\quad - \frac{1}{4}\int_{\mathbb{R}^2} [\mu_1 (su_\infty)^4 + \mu_2 (tv_\infty)^4] dx\\
&\le \frac{1}{2}\liminf_{n\to+\infty} \int_{\mathbb{R}^2} \Big[|\nabla su_n|^2 + |\nabla tv_n|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(su_n)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(tv_n)^2\Big] dx\\
&\quad - \frac{1}{4}\lim_{n\to+\infty}\int_{\mathbb{R}^2} [\mu_1 (su_n)^4 + \mu_2 (tv_n)^4] dx\\
&\le \frac{1}{2}\liminf_{n\to+\infty} \int_{\mathbb{R}^2} \Big[|\nabla su_n|^2 + |\nabla tv_n|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(su_n)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(tv_n)^2\Big] dx\\
&\quad - \frac{1}{4}\lim_{n\to+\infty}\int_{\mathbb{R}^2} [\mu_1 (su_n)^4 + \mu_2 (tv_n)^4] dx + \lim_{n\to+\infty}(-\beta_n)\int_{\mathbb{R}^2} (su_n)^2 (tv_n)^2 dx\\
&\le \liminf_{n\to+\infty} E_{\beta_n}(su_n, tv_n) \le \liminf_{n\to+\infty} E_{\beta_n}(u_n, v_n)\\
&= \liminf_{n\to+\infty} \mathcal{I}_{\beta_n} \le \limsup_{n\to+\infty} \mathcal{I}_{\beta_n} \le \mathcal{I}_\infty,
\end{aligned}$$

It follows that

$$\lim_{n\to+\infty} (-\beta_n) \int_{\mathbb{R}^2} u_n^2 v_n^2 dx = 0$$

and that

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{\mathbb{R}^2} \Big[|\nabla su_n|^2 + |\nabla tv_n|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(su_n)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(tv_n)^2\Big] dx\\
&= \int_{\mathbb{R}^2} \Big[|\nabla su_\infty|^2 + |\nabla tv_\infty|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(su_\infty)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(tv_\infty)^2\Big] dx.
\end{aligned}$$

Since $(su_n, tv_n) \to (su_\infty, tv_\infty)$ weakly in $\mathcal{H}$, we conclude that $(u_n, v_n) \to (u_\infty, v_\infty)$ strongly in $\mathcal{H}$. As a result,

$$\begin{aligned}\mathcal{I}_\infty &= \lim_{n\to+\infty} E_{\beta_n}(u_n, v_n)\\ &= \frac{1}{2}\int_{\mathbb{R}^2}\Big[|\nabla u_\infty|^2 + |\nabla v_\infty|^2 + \Big(\lambda_1 + \frac{1}{|x|^2}\Big)(u_\infty)^2 + \Big(\lambda_2 + \frac{1}{|x|^2}\Big)(v_\infty)^2\Big]dx\\ &\quad - \frac{1}{4}\int_{\mathbb{R}^2}[\mu_1 (u_\infty)^4 + \mu_2 (v_\infty)^4]dx\\ &= \Phi(u_\infty - v_\infty).\end{aligned}$$

The fact $(u_n, v_n) \in \mathcal{M}_{\beta_n}$ yields u_∞, $v_\infty \in \mathcal{N}$. This completes the proof of (a) and (b).

We have shown that $u_\infty - v_\infty$ is a minimizer for Φ on $\mathcal{E}$. By the Sobolev compact embedding, we know Φ satisfies the Palais-Smale condition on $\mathcal{N}$. The same argument of the proof of Lemma 2.6 in [8] leads to the conclusion that $u_\infty - v_\infty$ is a critical point of Φ. This proves (c). $\square$

Proof of Theorem 1.4. Let $\beta_n \to -\infty$. Correspondingly, we have $(u_n, v_n) \in \mathcal{M}_{\beta_n}$ satisfying $u_n \geq 0$ and $v_n \geq 0$ and $E_{\beta_n}(u_n, v_n) = \mathcal{I}_{\beta_n}$. By Proposition 5.1, after passing to a subsequence, we have that $(u_n, v_n) \to (u_\infty, v_\infty)$ strongly in $\mathcal{H}$, $u_\infty \geq 0$, $v_\infty \geq 0$, and $u_\infty - v_\infty$ is a nontrivial solution to the problem (5.1).

Observing that

$$-\Delta u_n \leq \mu_1 u_n^3, \quad -\Delta v_n \leq \mu_2 v_n^3 \quad \text{in } \mathbb{R}^2,$$

by a Brézis-Kato argument we can verify that the uniform boundedness of (u_n, v_n) in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ implies the uniform boundedness of (u_n, v_n) in $L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$, see [7]. By the interior $W^{2,2}$-regularity, see Theorem 1 in p.329 of [14], we obtain that $(u_n, v_n) \in W^{2,2}_{\rm loc}(\mathbb{R}^2) \times W^{2,2}_{\rm loc}(\mathbb{R}^2)$. It follows from the L^p-regularity , see Theorem B.2 in [27], and the fact that (u_n, v_n) is bounded in $\in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ that $(u_n, v_n) \in W^{2,p}_{\rm loc}(\mathbb{R}^2) \times W^{2,p}_{\rm loc}(\mathbb{R}^2)$ for all $p \geq 2$. Thanks to the Sobolev embedding theorem, we have $(u_n, v_n) \in C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2)$. It follows from Arsela-Ascoli theorem that $(u_n, v_n) \to (u_\infty, v_\infty)$ strongly in $C_{\rm loc}(\mathbb{R}^2) \times C_{\rm loc}(\mathbb{R}^2)$ as $n \to +\infty$ with $(u_\infty, v_\infty) \in C(\mathbb{R}^2) \times C(\mathbb{R}^2)$. Now, for $x, y \in \mathbb{R}^2$, we can see from the fact that ∇u_∞ is bounded in $L^\infty(\mathbb{R}^2)$ that

$$\begin{aligned}|u_\infty(x) - u_\infty(y)| &\leq |u_\infty(x) - u_n(x)| + |u_n(x) - u_n(y)| + |u_n(y) - u_\infty(y)|\\ &\leq \|\nabla u_n\|_{L^\infty(\mathbb{R}^2)}|x - y| + o_n(1)\\ &\leq M|x - y| + o_n(1).\end{aligned}$$

Lettin $n \to +\infty$, we obtain that u_∞ is locally Lipschitz in $\mathbb{R}^2$. Similarly, v_∞ is locally Lipschitz in $\mathbb{R}^2$. Therefore, $u_\infty - v_\infty$ is locally Lipschitz in $\mathbb{R}^2$.

As $u_\infty = (u_\infty - v_\infty)^+$ and $v_\infty = (u_\infty - v_\infty)^-$, these functions are continuous and the sets $\{u_\infty > 0\}$ and $\{v_\infty > 0\}$ are both open. Since $u_\infty - v_\infty$ is a minimizer of Φ in $\mathcal{N}$, these sets are connected. Moreover, we have $\overline{\{u_\infty > 0\} \cup \{v_\infty > 0\}} = \mathbb{R}^2$ because, otherwise, $u_\infty - v_\infty$ would vanish in an open set, contradicting with the unique continuation principle. Obviously, u_∞ solves the problem

$$-\Delta u + \lambda_1 u + \frac{u^2}{|x|^2} = \mu_1 u^3 \quad \text{in} \quad \{u_\infty > 0\},$$

and v_∞ solves the problem

$$-\Delta v+\lambda_2 v+\frac{v^2}{|x|^2}=\mu_2 v^3 \quad \text{in} \quad \{v_\infty>0\}.$$

This completes the proof. $\square$

Acknowledgments. A. Xia was supported by the Foundation of Jiangxi Provincial Education Department (NoĠJJ160335), and by the National Natural Science Foundation of China (Nos. 11701239 and 11871253). J. Yang was supported by the National Natural Science Foundation of China (Nos. 11671179 and 11771300).

References

[1] A. Ambrosetti, E. Colorado; Standing waves of some coupled nonlinear Schrödinger equations, *J. Lond. Math. Soc.*, **75** (2007), 67-82.

[2] N. Akhmediev, A. Ankiewicz; Partially coherent solitons on a finite background, *Phys. Rev. Lett.*, **82** (1999), 2661-2664.

[3] M. Badiale, V. Benci, S. Rolando; A nonlinear elliptic equation with singular potential and applications to nonlinear field equations, *J. Eur. Math. Soc.*, **9** (2007), no. 3, 355-381.

[4] M. Badiale, V. Benci, S. Rolando; Three dimensional vortices in the nonlinear wave equation, *Boll. Unione Mat. Ital.*, (9) **2** (2009), no. 1, 105-134.

[5] T. Bartsch, N. Dancer, Z.-Q. Wang; A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Calc. Var. Partial Differential Equations*, **37** (2010), 345-361.

[6] T. Bartsch, Z.-Q. Wang; Note on ground states of nonlinear Schrödinger systems, *J. Partial Differ. Equ.*, **19** (2006), 200-207.

[7] H. Brézis, T. Kato; Remarks on the Schrödinger operator with singular complex potentials, *J. Math. Pures Appl.*, (9) **58** (2) (1979), 137-151.

[8] A. Castro, J. Cossio, J. M. Neuberger; A sign-changing solution for a superlinear Dirichlet problem, *Rocky Mt. J. Math.*, (4) **27** (1997), 1041-1053.

[9] Z. Chen, W. Zou; Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent, *Arch. Ration. Mech. Anal.*, **205** (2012), no. 2, 515-551.

[10] Z. Chen, W. Zou; An optimal constant for the existence of least energy solutions of a coupled Schrödinger system, *Calc. Var. Partial Differential Equations*, **48** (2013), no. 3-4, 695-711.

[11] M. Clapp, A. Pistoia; Existence and phase separation of entire solutions to a pure critical competitive elliptic system, *Calc. Var. Partial Differential Equations*, **57** (2018), no. 1, Art. 23, 20 pp.

[12] N. Dancer, J. Wei, T. Weth; A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27** (2010), 953-969.

[13] B. Esry, C. Greene, J. Burke, J. Bohn; Hartree-Fock theory for double condensates, *Phys. Rev. Lett.*, **78** (1997), 3594-3597.

[14] L. C. Evans; Partial Differential Equations. Second edition, Graduate Studies in Mathematics, *American Mathematical Society, Providence, RI*, Vol. **19** (2010).

[15] G. Fibich, N. Gavish; Theory of singular vortex solutions of the nonlinear Schrödinger equation, *Phys. D*, **237** (2008), no. 21, 2696-2730.

[16] N. Gialelis; Inviscid limit of linearly damped and forced nonlinear Schrödinger equations, *Electron. J. Differential Equations*, **2020** (2020), No. 66, pp. 1-18.

[17] T. Lin, J. Wei; Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \geq 3$, *Commun. Math. Phys.*, **255** (2005), 629-653.

[18] T. Lin, J. Wei; Spikes in two coupled nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **22** (2005), 403-439.

[19] T. Lin, J. Wei; Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials, *J. Differ. Equ.*, **229** (2006), 538-569.

[20] Z. Liu, Z.-Q. Wang; Multiple bound states of nonlinear Schrödinger systems, *Commun. Math. Phys.*, **282** (2008), 721-731.

[21] L. Maia, E. Montefusco, B. Pellacci; Positive solutions for a weakly coupled nonlinear Schrödinger systems, *J. Differ. Equ.*, **229** (2006), 743-767.

[22] L. Maia, B. Pellacci, M. Squassina; Semiclassical states for weakly coupled nonlinear Schrödinger systems, *J. Eur. Math. Soc.*, **10** (2007), 47-71.

[23] B. Noris, M. Ramos; Existence and bounds of positive solutions for a nonlinear Schrödinger system, *Proc. Am. Math. Soc.*, **138** (2010), 1681-1692.

[24] B. Noris, H. Tavares, S. Terracini, G. Verzini; Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, *Comm. Pure Appl. Math.*, **63** (2010), no. 3, 267-302.

[25] A. Pomponio; Coupled nonlinear Schrödinger systems with potentials, *J. Differ. Equ.*, **227** (2006), 258-281.

[26] B. Sirakov; Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, *Commun. Math. Phys.*, **271** (2007), 199-221.

[27] M. Struwe; Variational Methods. Second edition, *Springer-Verlag*, Berlin-Heidelberg, 1996.

[28] E. Timmermans; Phase separation of Bose-Einstein condensates, *Phys. Rev. Lett.*, **81** (1998), 5718-5721.

[29] J. Wei, T. Weth; Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Ration. Mech. Anal.*, **190** (2008), 83-106.

Jin Deng
School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
Email address: Dengjin@jxnu.edu.cn

Aliang Xia
School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
Email address: aliang_xia@jxnu.edu.cn

Jianfu Yang
School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
Email address: jfyang_2000@yahoo.com