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TIME PERIODIC SOLUTIONS FOR THE NON-ISENTROPIC COMPRESSIBLE QUANTUM HYDRODYNAMIC EQUATIONS WITH VISCOSITY IN $\mathbb{R}^3$

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Abstract. This article concerns the existence and uniqueness of a time periodic solution for the non-isentropic quantum hydrodynamic equations with viscosity. By applying the Leray-Schauder theory, subtle energy estimates and a limiting method, we obtain the existence of time periodic solutions under some smallness assumptions on the time periodic external force in $\mathbb{R}^3$. The uniqueness can be proved by similar energy estimates. In particular, the quantum effects and the energy equation are taken into account in this paper which play a significant role in the uniform (in the domain R and the positive constant ϵ) estimates, especially in the selection of the norm.

1. Introduction

In this article, we consider the three-dimensional compressible quantum hydrodynamic equations effected by a time periodic external force, which can be used widely in fluid models of nucleus, superconductivity, superfluidity and ultra small electronic devices [1, 9, 16],

$$\frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0, \tag{1.1a}$$

$$m[\frac{\partial(nu)}{\partial t} + \operatorname{div}(nu \otimes u)] + \operatorname{div} P = \operatorname{div} S + n\nabla\psi + nf - \frac{mnu}{\tau_m}, \tag{1.1b}$$

$$\frac{\partial W}{\partial t} + \operatorname{div}(uW + uP) + \operatorname{div} q = \operatorname{div}(uS) + nu \cdot \nabla\psi - \frac{W - \frac{3}{2}n}{\tau_e}, \tag{1.1c}$$

$$\Delta\psi = n - b(x), \tag{1.1d}$$

where n, u, ψ, $P = (P_{ij})_{3\times 3}$, W denote the electron density, the electron velocity, the electric potential, the stress tensor and the energy density respectively. m, $b(x)$, τ_m, τ_e are the electron mass, the prescribed background ion density, the momentum and the energy relaxation coefficients. Moreover, f is the given time periodic external force with the period $T^* > 0$ and $q = -\kappa\nabla T - \frac{\hbar^2 n}{24m}(\Delta u + 2\nabla \operatorname{div} u)$ is the dispersive heat flux, where the vorticity R in [15] is assumed to be "small"

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i.e. $R = O(\hbar^2)$, T and κ are the temperature and the heat conductivity coefficient respectively. To close the moment expansion at the third order, we define the above quantities by

$$P_{ij} = nT\delta_{ij} - \frac{\hbar^2 n}{12m}\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}\log n + O(\hbar^4),$$
$$W = \frac{3}{2}nT + \frac{1}{2}nm|u|^2 - \frac{\hbar^2 n}{24m}\Delta\log n + O(\hbar^4),$$

respectively, where the quantum correction is involved and $\hbar > 0$ is the Planck constant divided by 2π, much smaller than the macro quantities [9, 14]. In addition, we denote the viscous stress tensor S as

$$S = \mu(\nabla u + (\nabla u)^\top) - \frac{2\mu}{3}(\operatorname{div} u)I,$$

where $\mu > 0$. It follows from [29, 2] that the quantum stress tensor is closely related to the quantum Bohm potential. Moreover, by direct calculation, we obtain the following relation

$$-\nabla Q(n) = \frac{\hbar^2}{4nm}\operatorname{div}(n(\nabla\otimes\nabla)\log n) = \frac{\hbar^2}{4m}\Big(\frac{\Delta\nabla n}{n} - \frac{\nabla n(\Delta n + \nabla^2 n)}{(n)^2} + \frac{|\nabla n|^2\nabla n}{(n)^3}\Big).$$

Recently, a great deal of research has been devoted to many topics of the time periodic solutions, specially on the existence, stability and convergence of solutions for isentropic or non-isentropic models in bounded domain or in the whole space. See [7, 17, 18, 20, 27, 28] for Navier-Stokes equations. See [5, 13, 25, 26] and the references therein for other studies of the compressible Navier-Stokes-Korteweg equations, and [3, 4, 21, 24] for the magnetohydrodynamic equations. In the direction of a bounded domain of $\mathbb{R}^3$, one can see [7] for the strong and weak periodic solutions under the inhomogeneous boundary data of Dirichlet type for example.

Without the quantum effects, the above system (1.1) is well known as the classical Navier-Stokes equation and has been extensively studied by [3, 5, 17, 20, 24, 26, 31]. However, the dimension of the time periodic solutions in [5, 20, 24] need to satisfy $n > 3$, since the convergence of the integral with respect to the time depends closely on the space dimension. It is worthy of noting that the whole space case in two or three dimension seems more physically meaningful and mathematically difficult, and thus comparatively less studies were obtained to the best of our knowledge. By means of the spectral properties of the time-T-map in a hybrid type function space, Tsuda [26] confined the case to $n = 3$ and estimated the nonlinear terms by the contraction mapping argument. As for the isentropic Naver-Stokes equation in $\mathbb{R}^3$, Jin and Yang [17] studied the time periodic solutions by the symmetry condition and the topological degree. In this work, with the aid of the conservation law of mass and the fact that u is an odd function for the space variables, they obtain the "integration"$= 0$ conditions, which are necessary to construct a closed solution operator. Motivated by this, Cai and Tan considered the existence and uniqueness of the time periodic strong solutions to the isentropic magnetohydrodynamic model in the whole space [4] and the periodic domain [3]. Almost all of the above mentioned results are the isentropic case, the full Navier-Stokes equation is more interesting which brings new difficulty to the energy estimates.

The aim of our paper is to present the existence and uniqueness of time periodic solutions for the three dimensional full quantum hydrodynamic equations with the viscosity and the damping near the constant stationary solution $(n, u, T) = (1, 0, 1)$.

The main differences from the previous work [17] are in the following sense: we need neither the oddness assumptions for the space variable on the external force nor the "integration equal zero" conditions here since we introduce the damping terms and the Poisson equation, from which we can obtain the L^2-norm of the variables by using such a structure of system (1.1) directly. Precisely, we utilize carefully a function of entropy to derive the closed L^2 energy estimates. In addition, we take the quantum effects into account, which needs more effort for our purpose than those known results of classical hydrodynamic equations due to the quantum higher order terms.

The main results of this paper are stated in Theorems 1.1 and 1.2. To this end, we use the delicate energy approach and the Leray-Schauder degree theory to obtain the existence of time periodic solutions for the approximated system (2.1) in a periodic domain Ω^R, where we can see clearly how the quantum corrections and the energy equation affect the energy estimates, and then by means of a limiting process, we derive the sequence of solutions in the bounded domain with periodic boundary converges to that in the whole space. Furthermore, we not only need to construct suitable energy norms in coming estimates, but also to control the electric potential due to the special structure of system (2.1). We solve the difficult by employing the continuity equation and the curl-div decomposition of the gradient. The Sobolev space we finally adopt is a set of $H^5 \times H^4 \times H^3$ defined in (2.5) which includes the quantum parameter $\hbar$. On account of the third order quantum terms in (1.2c), obtaining the uniform estimate requires some elaborated treatments thereof, and thus we can not obtain the same Sobolev space for (ρ, u, θ).

Below we briefly review some results for the quantum fluid dynamics equations, and we only mention some results related to our paper. The derivation of the full compressible quantum hydrodynamic model for semiconductor devices is from Gardner [9] by the Wigner-Boltzmann equation. Moreover, in [6], the author formally derived nonlocal quantum hydrodynamic models. For recent works, Jüngel and Miliŝić [16] introduced the full compressible quantum hydrodynamic model with viscosity which is just the system (1.1). Moreover, Pu and Guo [22] studied the global existence of smooth solutions with small initial data and established the semiclassical limit to the classical hydrodynamic system. The global existence of smooth solutions to the quantum hydrodynamic equations without viscosity in a three-dimensional domain with the insulating boundary conditions was investigated recently by [23]. The interested readers may also refer to [2, 8, 9, 10, 29] for more results. For related monograph, one can see [12].

In this article, we only put emphasis on the two quantum terms in the non-isothermal model of quantum compressible fluids system (1.1). Letting $(\rho, u, \theta) = (n-1, u, T-1)$, system (1.1) transforms into

$$\partial_t \rho + \operatorname{div} u = -\operatorname{div}(\rho u), \tag{1.2a}$$

$$\begin{aligned}
&(1+\rho)\partial_t u + (1+\rho)u - \mu\Delta u - \frac{\mu}{3}\nabla \operatorname{div} u + \nabla\theta + \nabla\rho - \frac{\hbar^2}{12}\nabla\Delta\rho - (1+\rho)\nabla\psi \\
&= -\rho\nabla\theta - \theta\nabla\rho - (1+\rho)u\cdot\nabla u - \frac{\hbar^2}{12}\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{1+\rho} \\
&\quad + \frac{\hbar^2}{12}\frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\rho)^2} + (1+\rho)f,
\end{aligned} \tag{1.2b}$$

$$
\begin{aligned}
&\partial_t\theta + \theta + \frac{2}{3}\operatorname{div} u - \frac{2\kappa}{3}\frac{\Delta\theta}{1+\rho} - \frac{\hbar^2}{18}\operatorname{div}\Delta u \\
&= -u\cdot\nabla\theta - \frac{2}{3}\theta\operatorname{div} u + \frac{\hbar^2}{18}\frac{\nabla\rho\cdot\nabla\operatorname{div} u}{1+\rho} \\
&\quad + \frac{2}{3(1+\rho)}\Big(\frac{\mu}{2}|\nabla u + \nabla u^\top|^2 - \frac{2\mu}{3}(\operatorname{div} u)^2\Big) + \frac{|u|^2}{3} + \frac{\hbar^2}{36}\Delta\ln n,
\end{aligned} \tag{1.2c}
$$

$$
\Delta\psi = \rho, \tag{1.2d}
$$

where we let $m, \tau_m, \tau_e, b(x) = 1$ for the sake of brevity.

The existence of the time periodic solutions in $\mathbb{R}^3$ is stated as follows.

Theorem 1.1. *Assume the time periodic function $f \in L^2(0, T^*; H^3(\mathbb{R}^3))$. Then there are some sufficiently small constants λ and d_0, independent of ϵ, R, such that if*

$$
\int_0^{T^*} \|(f, \hbar\nabla f)\|^2_{H^2(\mathbb{R}^3)} dt < \lambda,
$$

then problem (1.2) admits a time periodic solution $(\rho, u, \theta) \in X_{d_0}$, where X_{d_0} is first defined in Definition 2.2.

Theorem 1.2. *Under the assumption of Theorem 1.1, provided that d_0 sufficiently small, there exists a unique solution $(\rho, u, \theta) \in X_{d_0}$.*

Remark 1.3. Studies pertaining to the unbounded domain over two or three dimensions is also open for the case of time periodic problem of the full Naver-Stokes equations without the damping. More precisely, the symmetry condition, which has been applied successfully to get the closed solution space for the isentropic compressible Navier-Stokes system in [17], is not applicable for the reason that the temperature is an even function of the no-conserved quantity.

This article is arranged as follows. In Section 2, we introduce the approximated system (2.1) and prove the completely continuous operator (2.12) to the linear system (2.6) is well defined. Then we derive some basic lemmas which will be used several times in the coming estimates. In Section 3, to obtain the existence of time periodic solutions to (2.1), we give several uniform estimates of system (3.1) with respect to the domain R and the positive constant ϵ. Moreover, with the help of the unform estimates and the Leray-Schauder degree theory, we complete the proof of Theorem 3.1. Finally, the main theorems are proved rigorously in Sections 4 by passing the limit of corresponding sequence of solutions in a bounded domain Ω^R to the whole space $\mathbb{R}^3$ and the similar energy estimates as Section 3.

Throughout this article, we denote α by the multi-index, and $\partial^\alpha = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$ by the partial differential derivatives. We sometimes abuse the notation $\partial^{\alpha\pm1}$ to take the place of $\partial^{\alpha\pm\beta}, |\beta| = 1$ for some multi-index β. Here and in the subsequent, H^s denote the usual Sobolev space with norm $\|f\|_{H^s} = \sum_{|\alpha|\le s}\|\partial^\alpha_x f\|_{L^2}$, and write $W^{k,p}(\Omega^R)$ as $W^{k,p}$ to simplify the symbol, but still denote $W^{k,p}(\mathbb{R}^3)$ by the Sobolev norm in the whole space to avoid confusion. Moreover, we use $[A, B] = AB - BA$ to denote the commutator of A and B. The commutator estimate appears on Lemma 2.5.

2. Preliminaries

2.1. **Regularized system.** To study the time periodic solutions for the quantum hydrodynamic system (1.2), we introduce the following regularized problem in a bounded domain $\Omega^R = (-R, R)^3 \subseteq \mathbb{R}^3$ with periodic boundary conditions

$$\partial_t\rho + \operatorname{div} u - \epsilon\Delta\rho = -\operatorname{div}(\rho u), \tag{2.1a}$$

$$\begin{aligned}
&(1+\rho)\partial_t u + (1+\rho)u - \mu\Delta u - \frac{\mu}{3}\nabla\operatorname{div} u + \nabla\theta + \nabla\rho - \frac{\hbar^2}{12}\nabla\Delta\rho \\
&- (1+\rho)\nabla\psi \\
&= -\rho\nabla\theta - \theta\nabla\rho - (1+\rho)u\cdot\nabla u - \frac{\hbar^2}{12}\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{1+\rho} \\
&\quad + \frac{\hbar^2}{12}\frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\rho)^2} + (1+\rho)f^R,
\end{aligned} \tag{2.1b}$$

$$\begin{aligned}
&\partial_t\theta + \frac{2}{3}\operatorname{div} u - \frac{2\kappa}{3}\frac{\Delta\theta}{1+\tau\rho} + \theta - \frac{\hbar^2}{18}\operatorname{div}\Delta u \\
&= -u\cdot\nabla\theta - \frac{2}{3}\theta\operatorname{div} u + \frac{\hbar^2}{18}\frac{\nabla\rho\cdot\nabla\operatorname{div} u}{1+\rho} \\
&\quad + \frac{2}{3(1+\rho)}\Big(\frac{\mu}{2}|\nabla u + \nabla u^\top|^2 - \frac{2\mu}{3}(\operatorname{div} u)^2\Big) + \frac{1}{3}|u|^2 + \frac{\hbar^2}{36}\Delta\ln(1+\rho),
\end{aligned} \tag{2.1c}$$

$$\Delta\psi = \rho, \tag{2.1d}$$

where f^R is a smooth time periodic function with the period T^* that satisfies

$$f^R \to f \quad \text{in } L^2(0, T^*; H^3(\mathbb{R}^3)), \tag{2.2}$$

as R tends to ∞.

Definition 2.1. Let X^R and X be the solution spaces in the bounded domain Ω^R and in the whole space $\mathbb{R}^3$ respectively

$$\begin{aligned}
X^R = \Big\{(\rho, u, \theta) :&\rho \in L^\infty(0, T^*; H^5(\Omega^R)) \cap L^2(0, T^*; H^5(\Omega^R)) \\
&u \in L^\infty(0, T^*; H^4(\Omega^R)) \cap L^2(0, T^*; H^5(\Omega^R)) \\
&\theta \in L^\infty(0, T^*; H^3(\Omega^R)) \cap L^2(0, T^*; H^4(\Omega^R))\Big\}
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
X = \Big\{(\rho, u, \theta) :&\rho \in L^\infty(0, T^*; H^5(\mathbb{R}^3)) \cap L^2(0, T^*; H^5(\mathbb{R}^3)) \\
&u \in L^\infty(0, T^*; H^4(\mathbb{R}^3)) \cap L^2(0, T^*; H^5(\mathbb{R}^3)) \\
&\theta \in L^\infty(0, T^*; H^3(\mathbb{R}^3)) \cap L^2(0, T^*; H^4(\mathbb{R}^3))\Big\},
\end{aligned} \tag{2.4}$$

where (ρ, u, θ) is periodic in time with the same period as the external force f.

Definition 2.2. We say that the solution $(\rho, u, \theta) \in X_d$(or X_d^R), if $(\rho, u, \theta) \in X$(or X^R) and satisfies

$$\begin{aligned} &\|\!|(\rho,u,\theta)\|\!|^2 \\ &= \sup_{t\in[0,T^*]} \Big(\|(\rho,u,\theta)\|_{H^3}^2 + \|(\hbar\nabla\rho, \hbar\nabla u, \hbar^2\Delta\rho)\|_{H^3}^2 \Big) \\ &\quad + \int_0^{T^*} \Big(\|(\rho,u,\theta)\|_{H^3}^2 + \|(\nabla u, \nabla\theta, \hbar\Delta u)\|_{H^3}^2 + \|(\hbar\nabla\rho, \hbar^2\Delta\rho)\|_{H^3}^2 \Big) dt \\ &< d^2, \end{aligned} \tag{2.5}$$

with d being suitably small.

2.2. **Introduction of an operator** χ**.** For any given $(\tilde\rho, \tilde u, \tilde\theta) \in X_d^R$, we consider the linear system

$$\partial_t\rho + \operatorname{div} u - \epsilon\Delta\rho = -\tau \operatorname{div}(\tilde\rho\tilde u), \tag{2.6a}$$

$$\begin{aligned} &\partial_t u + u - \mu\Delta u - \frac{\mu}{3}\nabla \operatorname{div} u + \nabla\theta + \nabla\rho - \frac{\hbar^2}{12}\nabla\Delta\rho - \nabla\psi \\ &= -\tau\tilde\rho\tilde u - \tau\tilde\rho\partial_t\tilde u - \tau\tilde\rho\nabla\tilde\theta - \tau\tilde\theta\nabla\tilde\rho - \tau(1+\tau\tilde\rho)\tilde u\cdot\nabla\tilde u \\ &\quad - \frac{\hbar^2\tau}{12}\frac{\nabla\tilde\rho\Delta\tilde\rho + \nabla\tilde\rho\cdot\nabla^2\tilde\rho}{1+\tau\tilde\rho} + \frac{\hbar^2\tau}{12}\frac{(\nabla\tilde\rho\cdot\nabla\tilde\rho)\nabla\tilde\rho}{(1+\tau\tilde\rho)^2} + \tau(1+\tau\tilde\rho)f^R, \end{aligned} \tag{2.6b}$$

$$\begin{aligned} &\partial_t\theta + \frac{2}{3}\operatorname{div} u - \frac{2\kappa}{3}\frac{\Delta\theta}{1+\tau\tilde\rho} + \theta - \frac{\hbar^2}{18}\operatorname{div}\Delta u - \frac{\hbar^2}{36}\frac{\Delta\rho}{1+\tau\tilde\rho} \\ &= -\tau\tilde u\cdot\nabla\tilde\theta - \frac{2}{3}\tilde\theta\operatorname{div}\tilde u + \frac{\hbar^2}{18}\frac{\nabla\tilde\rho\cdot\nabla\operatorname{div}\tilde u}{1+\tau\tilde\rho} \\ &\quad + \frac{2}{3(1+\tau\tilde\rho)}\Big(\frac{\mu}{2}|\nabla\tilde u + \nabla\tilde u^\top|^2 - \frac{2\mu}{3}(\operatorname{div}\tilde u)^2\Big) + \frac{\tau}{3}|\tilde u|^2 + \frac{\hbar^2\tau}{36}\frac{|\nabla\tilde\rho|^2}{(1+\tau\tilde\rho)^2}, \end{aligned} \tag{2.6c}$$

$$\Delta\psi = \rho, \tag{2.6d}$$

for any $\tau \in [0, 1]$.

Provided that $(\rho, u, \theta) \in X_d^R$, $0 < d < 1/2$, we have

$$\|(\rho,\theta)\|_{L^\infty} \le C\|(\nabla\rho,\nabla\theta)\|_{H^1} \le Cd. \tag{2.7}$$

This implies

$$\frac{1}{2} \le 1+\tau\rho \le \frac{3}{2}, \quad \frac{1}{2} \le 1+\tau\theta \le \frac{3}{2}, \tag{2.8}$$

which can be also applied to $1+\tau\tilde\rho, 1+\tau\tilde\theta$ when $(\tilde\rho, \tilde\theta) \in X_d^R$. Letting $U = (\rho, u, \theta), \tilde U = (\tilde\rho, \tilde u, \tilde\theta)$, system (2.6) takes the form

$$U_t + AU + QU = W(\tilde U) + F, \tag{2.9}$$

where $W(\tilde U)$ only depends on the known functions $(\tilde\rho, \tilde u, \tilde\theta)$, and

$$A = \begin{pmatrix} -\epsilon\Delta & \operatorname{div} & 0 \\ \nabla & -\mu\Delta - \frac{\mu}{3}\nabla\operatorname{div} + \mathbb{I} & \nabla \\ 0 & \frac{2}{3}\operatorname{div} & -\frac{2\kappa}{3}\frac{\Delta}{1+\tau\tilde\rho} + 1 \end{pmatrix}, \tag{2.10}$$

where $\mathbb{I}$ is the unit operator,

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\hbar^2}{12}\nabla\Delta & 0 & 0 \\ 0 & -\frac{\hbar^2}{18}\Delta \operatorname{div} & 0 \end{pmatrix}, \quad F = \big(0, \tau(1+\tau\tilde{\rho})f^R, 0\big)^\top. \tag{2.11}$$

We define the operator χ for system (2.6) in Ω^R as $\chi : X_d^R \times [0,1] \to X^R$ by

$$((\tilde{\rho}, \tilde{u}, \tilde{\theta}), \tau) \to (\rho, u, \theta). \tag{2.12}$$

In what follows, we focus on the properties of the operator χ.

Lemma 2.3. *For any $\tau \in [0,1]$, the operator χ is well defined.*

Proof. First of all, we show that U is the time periodic solution to (2.9) with the period T^*. Consider the liner system

$$\partial_t \rho + \operatorname{div} u - \epsilon\Delta\rho = 0, \tag{2.13a}$$

$$\partial_t u + u - \mu\Delta u - \frac{\mu}{3}\nabla \operatorname{div} u + \nabla\theta + \nabla\rho - \frac{\hbar^2}{12}\nabla\Delta\rho - \nabla\psi = 0, \tag{2.13b}$$

$$\partial_t\theta + \frac{2}{3}\operatorname{div} u + \theta - \frac{2\kappa}{3}\frac{\Delta\theta}{1+\tau\tilde{\rho}} - \frac{\hbar^2}{18}\operatorname{div}\Delta u + \frac{\hbar^2}{36}\frac{\Delta\rho}{1+\tau\tilde{\rho}} = 0, \tag{2.13c}$$

$$\Delta\psi = \rho. \tag{2.13d}$$

Multiplying (2.13b) by u, we obtain

$$\begin{aligned} \langle \mathcal{T}u, u\rangle &= -\int_{\Omega^R} \nabla\theta\cdot u - \int_{\Omega^R}\nabla\rho\cdot u + \frac{\hbar^2}{12}\int_{\Omega^R}\nabla\Delta\rho\cdot u + \int_{\Omega^R}\nabla\psi\cdot u \\ &\triangleq \sum_{i=1}^4 R_{1,i}. \end{aligned} \tag{2.14}$$

Here, we define the abbreviated operator $\mathcal{T}$ as

$$\langle \mathcal{T}u, u\rangle = \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{L^2}^2 + C_\mu\|\nabla u\|_{L^2}^2,$$

where C_μ is a constant dependent on μ, and the strong elliptic operator $\mu\Delta u + \frac{\mu}{3}\nabla \operatorname{div} u$ is equivalent to $\nabla^2 u$ because of the assumption $\mu > 0$.

Now, we deal with the right-hand side of (2.14) one by one. Integration by parts and using (2.13c), we have

$$\begin{aligned} R_{1,1} &= -\frac{3}{2}\int_{\Omega^R}\theta(\partial_t\theta + \theta - \frac{2\kappa}{3}\frac{\Delta\theta}{1+\tau\tilde{\rho}} - \frac{\hbar^2}{18}\operatorname{div}\Delta u + \frac{\hbar^2}{36}\frac{\Delta\rho}{1+\tau\tilde{\rho}}) \\ &= -\frac{3}{4}\frac{d}{dt}\int_{\Omega^R}|\theta|^2 - \frac{3}{2}\int_{\Omega^R}|\theta|^2 - \kappa\int_{\Omega^R}\frac{|\nabla\theta|^2}{1+\tau\tilde{\rho}} - \frac{\hbar^2}{12}\int_{\Omega^R}\nabla\theta\cdot\Delta u \\ &\quad - \kappa\int_{\Omega^R}\nabla(\frac{1}{1+\tau\tilde{\rho}})\cdot\nabla\theta\theta + \frac{\hbar^2}{24}\int_{\Omega^R}\frac{\nabla\rho\cdot\nabla\theta}{1+\tau\tilde{\rho}} + \frac{\hbar^2}{24}\int_{\Omega^R}\nabla(\frac{1}{1+\tau\tilde{\rho}})\cdot\nabla\rho\theta \\ &\le -\frac{3}{4}\frac{d}{dt}\int_{\Omega^R}|\theta|^2 - \frac{3}{2}\int_{\Omega^R}|\theta|^2 - \kappa\int_{\Omega^R}\frac{|\nabla\theta|^2}{1+\tau\tilde{\rho}} - \frac{\hbar^2}{12}\int_{\Omega^R}\nabla\theta\cdot\Delta u + \delta\|\nabla\theta\|_{L^2}^2 \\ &\quad + C\tau\|\nabla\tilde{\rho}\|_{L^\infty}\|\theta\|_{H^1}^2 + C\tau\|\nabla\tilde{\rho}\|_{L^\infty}\|(\theta, \hbar\nabla\rho)\|_{L^2}^2 + C\hbar^4\|\nabla\rho\|_{L^2}^2, \end{aligned}$$

where δ is a sufficiently small positive constant. Again integrating by parts and using (2.13a), we derive

$$R_{1,2} = \int_{\Omega^R} \rho \operatorname{div} u = -\int_{\Omega^R} \rho(\partial_t \rho - \epsilon\Delta\rho) = -\frac{1}{2}\frac{d}{dt}\int_{\Omega^R} |\rho|^2 - \epsilon\int_{\Omega^R} |\nabla\rho|^2.$$

Similarly,

$$\begin{aligned} R_{1,3} &= -\frac{\hbar^2}{12}\int_{\Omega^R} \Delta\rho \operatorname{div} u = \frac{\hbar^2}{12}\int_{\Omega^R} \Delta\rho(\partial_t\rho - \epsilon\Delta\rho) \\ &= -\frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R} |\nabla\rho|^2 - \frac{\hbar^2\epsilon}{12}\int_{\Omega^R} |\Delta\rho|^2. \end{aligned}$$

Using (2.13d) and the expression of $\operatorname{div} u$ again, we have

$$\begin{aligned} R_{1,4} &= -\int_{\Omega^R} \psi \operatorname{div} u \\ &= \int_{\Omega^R} \psi(\partial_t\Delta\psi - \epsilon\Delta^2\psi) \\ &= -\frac{1}{2}\frac{d}{dt}\int_{\Omega^R} |\nabla\psi|^2 - \epsilon\int_{\Omega^R} |\rho|^2. \end{aligned}$$

Putting the above estimates together, and using the bounds (2.8), we conclude that

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega^R} |(u,\rho,\theta,\nabla\psi,\hbar\nabla\rho)|^2 + \|\theta\|_{H^1}^2 + \|u\|_{H^1}^2 + \epsilon\|\rho\|_{H^1}^2 + \hbar^2\epsilon\|\Delta\rho\|_{L^2}^2 \\ &+ \frac{\hbar^2}{12}\int_{\Omega^R} \nabla\theta\cdot\Delta u \\ &\le \delta\|\nabla\theta\|_{L^2}^2 + C\tau(d_0+\hbar^2)\|(\theta,\nabla\theta,\hbar\nabla\rho)\|_{L^2}^2. \end{aligned} \tag{2.15}$$

On the other hand, multiplying (2.13b) by $-\hbar^2\Delta u$, we obtain

$$\begin{aligned} &\frac{\hbar^2}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \hbar^2 C_\mu\|\nabla u\|_{H^1}^2 - \hbar^2\int_{\Omega^R} \nabla\theta\cdot\Delta u \\ &= \hbar^2\int_{\Omega^R} \nabla\rho\cdot\Delta u - \hbar^2\int_{\Omega^R} \nabla\psi\cdot\Delta u - \frac{\hbar^4}{12}\int_{\Omega^R} \nabla\Delta\rho\cdot\Delta u \\ &\triangleq \sum_{i=5}^{7} R_{1,i}, \end{aligned} \tag{2.16}$$

thanks to (2.8) and the fact $\mu\Delta u + \mu/3\nabla \operatorname{div} u$ is a strong elliptic operator. Almost exactly as the estimate for $R_{1,3}$, $R_{1,5}$ can be bounded by

$$R_{1,5} = -\hbar^2\int_{\Omega^R} \rho\Delta \operatorname{div} u = -\frac{\hbar^2}{2}\frac{d}{dt}\int_{\Omega^R} |\nabla\rho|^2 - \hbar^2\epsilon\int_{\Omega^R} |\Delta\rho|^2. \tag{2.17}$$

It follows from (2.13a) and (2.13d) that

$$
\begin{aligned}
R_{1,6} =&\hbar^2 \int_{\Omega^R} \psi\Delta \operatorname{div} u \\
=&-\hbar^2 \int_{\Omega^R} \psi(\partial_t \Delta^2\psi - \epsilon\Delta^3\psi) \\
=&-\hbar^2 \frac{d}{dt}\int_{\Omega^R} |\Delta\psi|^2 - \hbar^2\epsilon \int_{\Omega^R} |\nabla\Delta\psi|^2 \\
=&-\hbar^2 \frac{d}{dt}\int_{\Omega^R} |\rho|^2 - \hbar^2\epsilon \int_{\Omega^R} |\nabla\rho|^2.
\end{aligned} \tag{2.18}
$$

With the aid of integration by parts and (2.13a), the last term $R_{1,7}$ can be estimated by

$$
\begin{aligned}
R_{1,7} =&-\frac{\hbar^4}{12}\int_{\Omega^R} \Delta\rho(\partial_t\Delta\rho - \epsilon\Delta\Delta\rho) \\
=&-\frac{\hbar^4}{24}\frac{d}{dt}\int_{\Omega^R} |\Delta\rho|^2 - \frac{\hbar^4\epsilon}{12}\int_{\Omega^R} |\nabla\Delta\rho|^2.
\end{aligned}
$$

Multiplying (2.16) by 1/12 on both sides, taking $\delta, \hbar$ to be sufficiently small and combining (2.15), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^R} |(\rho, u, \theta, \nabla\psi, \hbar\nabla u, \hbar\nabla\rho, \hbar^2\Delta\rho)|^2 + \epsilon\|\rho\|_{H^1}^2 + \|\theta\|_{H^1}^2 + \|u\|_{H^1}^2 \\
&+ \hbar^2\|\nabla^2 u\|_{L^2}^2 + \hbar^2\epsilon\|\Delta\rho\|_{L^2}^2 + \hbar^4\epsilon\|\nabla\Delta\rho\|_{L^2}^2 \le 0.
\end{aligned} \tag{2.19}
$$

Therefore, $\rho \equiv u \equiv \theta \equiv 0$. From system (2.9), we derive

$$
\|U(t)\|_{L^2} = \|S_\hbar(t)U_0\|_{L^2} \le C\|U_0\|_{L^2} e^{-C_{\epsilon,R}t},
$$

where $S_\hbar(t) = e^{-t(A+Q)}$ is the solution operator to system (2.9) and $C_{\epsilon,R}$ is a constant dependent on ϵ, R. Thanks to Duhamel's principle and the time periodic property of $(\tilde\rho, \tilde u, \tilde\theta, f^R)$, we derive solutions to system (2.9) can be estimated as

$$
\begin{aligned}
&\|U(t)\|_{L^2} \\
&= \int_{-\infty}^{t} \|S_\hbar(t-s)(W(\tilde U(s)) + F(s))\|_{L^2} ds \\
&= \int_{-\infty}^{t} e^{-C_{\epsilon,R}(t-s)}\|W(\tilde U(s)) + F(s)\|_{L^2} ds \\
&= \sum_{i=0}^{\infty}\int_{t-(i+1)T^*}^{t-iT^*} e^{-C_{\epsilon,R}(t-s)}\|W(\tilde U(s)) + F(s)\|_{L^2} ds \\
&\le \sum_{i=0}^{\infty}\Big(\int_0^{T^*} e^{-2C_{\epsilon,R}((i+1)T^*-s)} ds\Big)^{1/2}\Big(\int_t^{t+T^*} \|W(\tilde U(s)) + F(s)\|_{L^2}^2 ds\Big)^{1/2} \\
&\le C_{\epsilon,R}\sup_{t\in\mathbb{R}} \|W(\tilde U(t)) + F(t)\|_{L^2}.
\end{aligned} \tag{2.20}
$$

Since f^R and $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ are periodic in time with period T^*, we obtain

$$\begin{aligned} U(t+T^*) &= \int_{-\infty}^{t+T^*} S_\hbar(t+T^*-s)(W(\tilde{U}(s)) + F(s))ds \\ &= \int_{-\infty}^{t+T^*} S_\hbar(t-(s-T^*))(W(\tilde{U}(s-T^*)) + F(s-T^*))ds \\ &= \int_{-\infty}^{t} S_\hbar(t-s')(W(\tilde{U}(s') + F(s'))ds' = U(t). \end{aligned} \tag{2.21}$$

Therefore, by (2.20) and (2.21), we have that $U(t) =\in L^\infty(0, T^*; L^2(\Omega_R))$ is the desired periodic solution of system (2.9) with the time period T^*. Moreover, following the energy estimates similar but much easier to those in Section 3, we derive, for any $(\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in X_d^R, \tau \in [0,1]$, there exists a time periodic solution $(\rho, u, \theta) \in X^R$. In fact, we do not need the uniform (in ϵ, R) estimates here.

Finally, we will prove the uniqueness of the time periodic solutions to system (2.6). Assume that for some $\tilde{U} \in X_d^R, \tau \in [0,1]$, there exists two different solutions U_1 and U_2 to system (2.9). Then we can deduce $U_1 - U_2$ is the solution to the homogeneous system $\partial_t(U_1 - U_2) + (A+Q)(U_1 - U_2) = 0$. Recalling (2.19) and integrating the resultant inequality from 0 to T^*, we obtain $U_1 = U_2$. Therefore, it follows from (2.12) that

$$\chi((\tilde{\rho}, \tilde{u}, \tilde{\theta}), 0) = 0,$$

which implies we need only to consider the condition $\tau \in (0,1]$ in the following section. This completes the proof. □

Lemma 2.4. *Assume that d is suitably small, then the operator χ is compact and continuous.*

A proof of the above can be found in [3, Lemmas 2.4 and 2.5].

2.3. **Basic lemmas.** The following lemmas will be used later.

Lemma 2.5. *Let α be any multi-index with $|\alpha| = k$, $k \geq 1$ and $p \in (1, \infty)$. Then*

$$\begin{gathered} \|\partial^\alpha(fg)\|_{L^p} \leq C\|f\|_{L^{p^1}}\|g\|_{\dot{W}^{k,p^2}} + C\|f\|_{\dot{W}^{k,p^3}}\|g\|_{L^{p^4}}, \\ \|[\partial^\alpha, f]g\|_{L^p} \leq C\|\nabla f\|_{L^{p^1}}\|g\|_{\dot{W}^{k-1,p^2}} + C\|f\|_{\dot{W}^{k,p^3}}\|g\|_{L^{p^4}}, \end{gathered}$$

where $f, g \in \mathbb{S}$, the Schwartz class, $\dot{W}$ is the homogeneous Sobolev space, and $p_2, p_3 \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{p^1} + \frac{1}{p^2} = \frac{1}{p^3} + \frac{1}{p^4}$.

Lemma 2.6. *Let p, q, r be integers with $p, q, r \in [1, \infty]$, and $0 \leq a, m \leq l$. Then*

$$\|\nabla^a f\|_{L^p} \leq C\|\nabla^l f\|_{L^q}^c \|\nabla^m f\|_{L^r}^{1-c}, \tag{2.22}$$

for some generic constant $c \in [0,1]$, where

$$\frac{a}{3} - \frac{1}{p} = \Big(\frac{l}{3} - \frac{1}{q}\Big)c + \Big(\frac{m}{3} - \frac{1}{r}\Big)(1-c);$$

when $c = 1$, it holds $l - a \neq 3/q$.

3. Existence of time periodic solution in a bounded domain

This section is devoted to the proof of the existence of time periodic solutions to system (2.1) in the bounded domain Ω^R by combining the topological degree theory with the delicate energy estimates. The main theorem is stated as follows.

Theorem 3.1. *Let the time periodic force $f^R \in L^2(0, T^*; H^3)$. If there exist some sufficiently small constants λ and d_0 such that $C\lambda^2 + Cd_0^3 \le d_0^2/2$, where C is the generic positive constant, and*

$$\int_0^{T^*} \|(f^R, \hbar\nabla f^R)\|_{H^2}^2 dt < \lambda,$$

then$(\rho^R, u^R, \theta^R) \in X_{d_0}^R$ is the strong time periodic solution of the approximate system (2.1) in the bounded domain Ω^R with periodic boundary.

To do this, we introduce the nonlinear approximated equations

$$\partial_t \rho + \operatorname{div} u - \epsilon\Delta\rho = -\tau \operatorname{div}(\rho u), \tag{3.1a}$$

$$\begin{aligned}
&(1+\tau\rho)\partial_t u + (1+\tau\rho)u - \mu\Delta u - \frac{\mu}{3}\nabla \operatorname{div} u + (1+\tau\rho)\nabla\theta \\
&+ (1+\tau\theta)\nabla\rho - \frac{\hbar^2}{12}\nabla\Delta\rho - (1+\tau\rho)\nabla\psi \\
&= -\tau(1+\tau\rho)u\cdot\nabla u - \frac{\hbar^2\tau}{12}\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{1+\tau\rho} + \frac{\hbar^2\tau}{12}\frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^2} \\
&\quad + \tau(1+\tau\rho)f^R,
\end{aligned} \tag{3.1b}$$

$$\begin{aligned}
&\partial_t\theta + \theta + \frac{2}{3}(1+\tau\theta)\operatorname{div} u - \frac{2\kappa}{3}\frac{\Delta\theta}{1+\tau\rho} - \frac{\hbar^2}{18}\operatorname{div}\Delta u - \frac{\hbar^2}{36}\frac{\Delta\rho}{1+\tau\rho} \\
&= -\tau u\cdot\nabla\theta + \frac{\hbar^2\tau}{18}\frac{\nabla\rho\cdot\nabla \operatorname{div} u}{1+\tau\rho} \\
&\quad + \frac{2\tau}{3(1+\tau\rho)}\Big(\frac{\mu}{2}|\nabla u + \nabla u^\top|^2 - \frac{2\mu}{3}(\operatorname{div} u)^2\Big) + \tau\frac{|u|^2}{3} - \frac{\hbar^2\tau}{36}\frac{|\nabla\rho|^2}{(1+\tau\rho)^2},
\end{aligned} \tag{3.1c}$$

$$\Delta\psi = \rho, \tag{3.1d}$$

where $\tau \in (0,1]$. Provided $\tau = 1$, we naturally derive the existence of time periodic solutions to system (2.1).

Letting $(\rho, u, \theta) \in X_d^R$, we will utilize the Kronecker's existence theorem (see [30, Theorem 11.1.6]) to show that the approximated system (2.1) admits a solution $(\rho, u, \theta) \in X^R$, which is equivalent to prove

$$\chi(U,1) = U, \quad U = (\rho, u, \theta) \in X_d^R.$$

To this end, we need to choose $d_0 > 0$, such that, for any $\tau \in (0,1]$,

$$(I - \chi(\cdot,\tau))(\partial \hat{B}_{d_0}(0)) \neq 0, \tag{3.2}$$

where $\hat{B}_{d_0}(0)$ is a ball of radius d_0 centered at the origin in X^R.

Next, our plan is to derive inequality (3.2) and some suitable energy estimates by the deep analysis for the special structure of the approximated system (3.1). For this, we first give the basic L^2 estimate.

3.1. **Basic entropy estimate.** Motivated by [19], we give the zero order estimates for the approximated system (3.1) with the aid of the transform

$$\tau s = (1+\tau\theta)/(1+\tau\rho)^{2/3} - 1, \tag{3.3}$$

and we define the energy function E by

$$\begin{aligned} E(\rho,u,s) =& \frac{3(1+\tau s)}{2\tau^2}\big((1+\tau\rho)^{5/3} - 1 - \frac{5\tau\rho}{3}\big) + \frac{(1+\tau\rho)}{2}|u|^2 \\ &+ s\rho + \frac{3(1+\tau\rho)s^2}{4}. \end{aligned} \tag{3.4}$$

Lemma 3.2. *There exists a constant $0 < \rho_2 \le 1/2$, such that E is positive definite, and satisfies $E \cong \rho^2 + |u|^2 + \theta^2$.*

The above lemma can be proved by modifying [19, Lemma 2.5]. Under this notation (ρ, u, s), system (3.1) transforms into

$$\partial_t \rho + \operatorname{div} u - \epsilon\Delta\rho = -\tau \operatorname{div}(\rho u), \tag{3.5a}$$

$$\begin{aligned} &(1+\tau\rho)\partial_t u + (1+\tau\rho)u - \mu\Delta u - \frac{\mu}{3}\nabla \operatorname{div} u + \frac{1}{\tau}\nabla\big((1+\tau\rho)^{5/3}(1+\tau s)\big) \\ &- \frac{\hbar^2}{12}\nabla\Delta\rho - (1+\tau\rho)\nabla\psi \\ &= -\tau(1+\tau\rho)u\cdot\nabla u - \frac{\hbar^2\tau}{12}\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{1+\tau\rho} + \frac{\hbar^2\tau}{12}\frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^2} \\ &\quad + \tau(1+\tau\rho)f^R, \end{aligned} \tag{3.5b}$$

$$\begin{aligned} &\partial_t s + \frac{s}{(1+\tau\rho)^{2/3}} + \frac{2}{3}\frac{(1+\tau s)\rho}{(1+\tau\rho)^{2/3}} - \frac{2\kappa}{3}\operatorname{div}\Big(\frac{\nabla s}{1+\tau\rho} + \frac{2(1+\tau s)\nabla\rho}{3(1+\tau\rho)^2}\Big) \\ &- \frac{10}{9}\kappa\tau\Big(\frac{\nabla s}{(1+\tau\rho)^2} + \frac{2(1+\tau s)\nabla\rho}{3(1+\tau\rho)^3}\Big)\nabla\rho - \frac{\hbar^2}{18}\frac{\operatorname{div}\Delta u}{(1+\tau\rho)^{2/3}} \\ &- \frac{2\epsilon}{3}\frac{(1+\tau s)\Delta\rho}{1+\tau\rho} - \frac{\hbar^2}{36}\frac{\Delta\rho}{(1+\tau\rho)^{5/3}} \\ &= -\tau u\cdot\nabla s + \frac{\hbar^2\tau}{18}\frac{\nabla\rho\cdot\nabla\operatorname{div} u}{(1+\tau\rho)^{5/3}} + \frac{\tau}{3}\frac{|u|^2}{(1+\tau\rho)^{2/3}} - \frac{\hbar^2\tau}{36}\frac{|\nabla\rho|^2}{(1+\tau\rho)^{8/3}} \\ &\quad + \frac{2\tau}{3(1+\tau\rho)^{5/3}}\Big(\frac{\mu}{2}|\nabla u + \nabla u^\top|^2 - \frac{2\mu}{3}(\operatorname{div} u)^2\Big), \end{aligned} \tag{3.5c}$$

$$\Delta\psi = \rho. \tag{3.5d}$$

Lemma 3.3. *Let $(\rho, u, \theta) \in \partial\hat{B}_{d_0}(0)$ be a solution to system (3.1). Then there exists a sufficiently small positive constant d_0, independent ϵ, R, such that*

$$\begin{aligned} &\frac{d}{dt}\|(\rho,u,\theta,\nabla\rho,\nabla\psi,\hbar\nabla\rho)\|_{L^2}^2 + \|u\|_{H^1}^2 + \|\theta\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \epsilon\|\rho\|_{H^1}^2 \\ &+ \epsilon\|\Delta\rho\|_{L^2}^2 + \hbar^2\|\Delta\rho\|_{L^2}^2 + \hbar^2\epsilon\|\Delta\rho\|_{L^2}^2 \\ &\le C\tau d_0\|(u,\nabla u,\nabla\theta,\nabla\rho,\hbar\Delta\rho)\|_{L^2}^2 + C\hbar^4\|\Delta u\|_{L^2}^2 + C\tau\|f^R\|_{L^2}^2. \end{aligned} \tag{3.6}$$

Proof. Recalling the definition of $E(\rho, u, s)$ in (3.4), we have

$$\frac{d}{dt}\int_{\Omega^R} E(\rho,u,s) = \int_{\Omega^R}(1+\tau\rho)uu_t + \frac{3}{2\tau}\int_{\Omega^R}(1+\tau\rho)\big((1+\tau\rho)^{2/3} - 1 + \tau s\big)s_t$$

$$+\int_{\Omega^R}\Big(\tau\frac{|u|^2}{2}+\frac{5}{2\tau}(1+\tau s)\big((1+\tau\rho)^{2/3}-1\big)+s+\frac{3\tau}{4}s^2\Big)\rho_t.$$

Integrating by parts, using (3.5a) and doing careful computations, we have

$$\begin{aligned}
&\frac{1}{\tau}\int_{\Omega^R}\nabla\big((1+\tau\rho)^{5/3}(1+\tau s)\big)\cdot u\\
&+\int_{\Omega^R}\Big(\frac{5}{2\tau}(1+\tau s)\big((1+\tau\rho)^{2/3}-1\big)+s+\frac{3\tau}{4}s^2\Big)\operatorname{div}((1+\tau\rho)u)\\
&+\frac{3}{2}\int_{\Omega^R}(1+\tau\rho)\big((1+\tau\rho)^{2/3}-1+\tau s\big)(u\cdot\nabla s)=0,
\end{aligned}$$

and

$$\begin{aligned}
&-\tau\int_{\Omega^R}(1+\tau\rho)u\cdot\nabla u\cdot u+\tau\int_{\Omega^R}\frac{|u|^2}{2}\rho_t\\
&=\frac{\tau}{2}\int_{\Omega^R}\operatorname{div}\big((1+\tau\rho)u\big)|u|^2+\frac{\tau}{2}\int_{\Omega^R}\rho_t|u|^2\\
&=\frac{\tau\epsilon}{2}\int_{\Omega^R}|u|^2\Delta\rho\\
&=-\frac{\tau\epsilon}{2}\int_{\Omega^R}\nabla(|u|^2)\cdot\nabla\rho\\
&\le C\tau\|u\|_{L^\infty}\|(\nabla u,\nabla\rho)\|_{L^2}^2.
\end{aligned}$$

Combining the above estimates with Young's inequality and Lemma 2.6, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^R}E(\rho,u,s)\\
&\le\frac{\hbar^2}{12}\int_{\Omega^R}\nabla\Delta\rho\cdot u+\int_{\Omega^R}(1+\tau\rho)\nabla\psi\cdot u+\frac{\hbar^2}{24\tau}\int_{\Omega^R}\frac{\big((1+\tau\rho)^{2/3}-1+\tau s\big)\Delta\rho}{(1+\tau\rho)^{2/3}}\\
&\quad+\frac{\epsilon}{\tau}\int_{\Omega^R}(1+\tau s)\big((1+\tau\rho)^{2/3}-1+\tau s\big)\Delta\rho\\
&\quad-\frac{\hbar^2\tau}{12}\int_{\Omega^R}\Big(\frac{\nabla\rho\Delta\rho+\nabla\rho\cdot\nabla^2\rho}{1+\tau\rho}-\frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^2}\Big)\cdot u\\
&\quad+\frac{\hbar^2}{12\tau}\int_{\Omega^R}(1+\tau\rho)^{1/3}\big((1+\tau\rho)^{2/3}-1+\tau s\big)\operatorname{div}\Delta u-\frac{4\kappa}{3}\int_{\Omega^R}\nabla s\cdot\nabla\rho\\
&\quad-2\int_{\Omega^R}\rho s-\frac{4\kappa}{9}\|\nabla\rho\|_{L^2}^2-\|u\|_{L^2}^2-\frac{2}{3}\|\rho\|_{L^2}^2-\frac{3}{2}\|s\|_{L^2}^2-\frac{5\epsilon}{3}\|\nabla\rho\|_{L^2}^2\\
&\quad-\mu\|\nabla u\|_{L^2}^2-\kappa\|\nabla s\|_{L^2}^2-\frac{\mu}{3}\|\operatorname{div}u\|_{L^2}^2+\delta\|u\|_{L^2}^2\\
&\quad+C\tau\big(\|(u,\rho,s)\|_{L^\infty}+\|(\rho,s,\nabla\rho)\|_{L^\infty}^2\big)\|(u,\nabla u,\nabla s,\nabla\rho)\|_{L^2}^2\\
&\quad+C\tau\|f^R\|_{L^2}^2+C\epsilon\|\nabla s\|_{L^2}^2+C\hbar^4\|\nabla\operatorname{div}u\|_{L^2}^2\\
&\le\sum_{i=1}^{6}R_{2,i}-\frac{4\kappa}{3}\int_{\Omega^R}\nabla s\cdot\nabla\rho-2\int_{\Omega^R}\rho s-\frac{4\kappa}{9}\|\nabla\rho\|_{L^2}^2-\|u\|_{L^2}^2-\frac{2}{3}\|\rho\|_{L^2}^2\\
&\quad-\frac{3}{2}\|s\|_{L^2}^2-\frac{5\epsilon}{3}\|\nabla\rho\|_{L^2}^2-\mu\|\nabla u\|_{L^2}^2-\kappa\|\nabla s\|_{L^2}^2-\frac{\mu}{3}\|\operatorname{div}u\|_{L^2}^2\\
&\quad+C\tau d_0\|(u,\nabla u,\nabla s,\nabla\rho)\|_{L^2}^2+\delta\|u\|_{L^2}^2+C\tau\|f^R\|_{L^2}^2+C\epsilon\|\nabla s\|_{L^2}^2
\end{aligned}$$

$$+ C\hbar^4\|\nabla \operatorname{div} u\|_{L^2}^2, \tag{3.7}$$

where δ is a sufficiently small positive constant.

Estimates for the right-hand side of (3.7). For the first term $R_{2,1}$, by the equation (3.5a) and integration by parts, we obtain

$$\begin{aligned}
R_{2,1} =& \frac{\hbar^2}{12}\int_{\Omega^R} \frac{\Delta\rho(\partial_t\rho - \epsilon\Delta\rho + \tau u\cdot\nabla\rho)}{1+\tau\rho} \\
=& -\frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\rho|^2}{1+\tau\rho} - \frac{\hbar^2\epsilon}{12}\int_{\Omega^R}\frac{|\Delta\rho|^2}{1+\tau\rho} + \frac{\hbar^2\tau}{24}\int_{\Omega^R}\frac{\partial_t\rho|\nabla\rho|^2}{(1+\tau\rho)^2} \\
&+ \frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{\Delta\rho u\cdot\nabla\rho}{1+\tau\rho} \\
\le& -\frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\rho|^2}{1+\tau\rho} - \frac{\hbar^2\epsilon}{12}\int_{\Omega^R}\frac{|\Delta\rho|^2}{1+\tau\rho} - \frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{\nabla\rho^\top\nabla u\nabla\rho}{1+\tau\rho} \\
&+ \frac{\hbar^2\tau}{24}\int_{\Omega^R}\operatorname{div}(\frac{u}{1+\tau\rho})|\nabla\rho|^2 + C\tau\|\nabla\rho\|_{L^\infty}\|\hbar\nabla\rho\|_{L^2}\|\hbar\partial_t\rho\|_{L^2} \\
\le& -\frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\rho|^2}{1+\tau\rho} - \frac{\hbar^2\epsilon}{12}\int_{\Omega^R}\frac{|\Delta\rho|^2}{1+\tau\rho} + \delta\|\hbar\epsilon^{1/2}\Delta\rho\|_{L^2}^2 \\
&+ C\tau d_0\|(\hbar\nabla u, \hbar\nabla\rho)\|_{L^2}^2,
\end{aligned}$$

where δ is a sufficiently small positive constant. For the second term $R_{2,2}$, by (3.5a), (3.5d) and integration by parts, we obtain

$$\begin{aligned}
R_{2,2} =& -\int_{\Omega^R}\psi\operatorname{div}\big((1+\tau\rho)u\big) \\
=& \int_{\Omega^R}\psi(\partial_t\Delta\psi - \epsilon\Delta^2\psi) \\
=& -\frac{1}{2}\frac{d}{dt}\int_{\Omega^R}|\nabla\psi|^2 - \epsilon\int_{\Omega^R}|\rho|^2.
\end{aligned}$$

For the third term $R_{2,3}$, by integration by parts again, we obtain

$$\begin{aligned}
R_{2,3} =& -\frac{\hbar^2}{24\tau}\int_{\Omega^R}\nabla\Big(\frac{(1+\tau\rho)^{2/3} - 1 + \tau s}{(1+\tau\rho)^{2/3}}\Big)\cdot\nabla\rho \\
\le& -\frac{\hbar^2}{36}\|\nabla\rho\|_{L^2}^2 + C\tau\|(\rho,s)\|_{L^\infty}\|\nabla\rho\|_{L^2}^2 + C\hbar^2\|(\nabla s, \nabla\rho)\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$R_{2,4} \le -\frac{2\epsilon}{3}\|\nabla\rho\|_{L^2}^2 + C\tau\|(\rho,s)\|_{L^\infty}\|(\nabla s,\nabla\rho)\|_{L^2}^2 + C\epsilon\|(\nabla s,\nabla\rho)\|_{L^2}^2.$$

For the fifth term $R_{2,5}$, from integration by parts and Hölder's inequality, we have

$$\begin{aligned}
R_{2,5} =& -\frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{\nabla\rho\Delta\rho}{1+\tau\rho}\cdot u + \frac{\hbar^2\tau}{24}\int_{\Omega^R}\operatorname{div}\Big(\frac{u}{1+\tau\rho}\Big)|\nabla\rho|^2 + \frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^2}\cdot u \\
=& -\frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{\operatorname{div}(\nabla\rho\otimes\nabla\rho) - \frac{1}{2}\nabla(|\nabla\rho|^2)}{1+\tau\rho}\cdot u + \frac{\hbar^2\tau}{24}\int_{\Omega^R}\operatorname{div}\Big(\frac{u}{1+\tau\rho}\Big)|\nabla\rho|^2 \\
&+ \frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^2}\cdot u
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\hbar^2\tau}{12}\int_{\Omega^R}(\nabla\rho)^\top \operatorname{div}\Big(\frac{u}{1+\tau\rho}\Big)\nabla\rho+\frac{\hbar^2\tau}{12}\int_{\Omega^R}\frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^2}\cdot u\\
\le& C\tau(\|\nabla\rho\|_{L^\infty}+\|\nabla\rho\|_{L^\infty}^2)\|\hbar\nabla\rho\|_{L^2}^2,
\end{aligned}
$$

where we have used the vector analysis formulation

$$
\operatorname{div} ff=\operatorname{div}(f\otimes f)-\frac{1}{2}\nabla(|f|^2)-(\nabla\times f)\times f, \tag{3.8}
$$

for any vector function f. Again, by integration by parts, we obtain

$$
\begin{aligned}
R_{2,6}=&-\frac{\hbar^2}{36}\int_{\Omega^R}\frac{\nabla\rho\big((1+\tau\rho)^{2/3}-1+\tau s\big)\Delta u}{(1+\tau\rho)^{2/3}}\\
&-\frac{\hbar^2}{12\tau}\int_{\Omega^R}(1+\tau\rho)^{1/3}\nabla\big((1+\tau\rho)^{2/3}-1+\tau s\big)\Delta u\\
\le& C\hbar^4\|\Delta u\|_{L^2}^2+\delta\|(\nabla\rho,\nabla s)\|_{L^2}^2+C\tau\|(s,\rho)\|_{L^\infty}^2\|\nabla\rho\|_{L^2}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^R}E(\rho,u,s)+\frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\rho|^2}{1+\tau\rho}+\frac{1}{2}\frac{d}{dt}\int_{\Omega^R}|\nabla\psi|^2+\frac{7\epsilon}{3}\|\nabla\rho\|_{L^2}^2\\
&+\frac{\hbar^2}{36}\|\nabla\rho\|_{L^2}^2+\epsilon\|\rho\|_{L^2}^2+\|u\|_{L^2}^2+\mu\|\nabla u\|_{L^2}^2+\frac{3}{2}\|s\|_{L^2}^2+\kappa\|\nabla s\|_{L^2}^2\\
&+\frac{2}{3}\|\rho\|_{L^2}^2+\frac{4\kappa}{9}\|\nabla\rho\|_{L^2}^2+\frac{\mu}{3}\|\operatorname{div} u\|_{L^2}^2+\frac{\hbar^2\epsilon}{12}\|\Delta\rho\|_{L^2}^2\\
&\le-\frac{4\kappa}{3}\int_{\Omega^R}\nabla s\cdot\nabla\rho-2\int_{\Omega^R}s\rho+C\tau\|f^R\|_{L^2}^2+C\epsilon\|\nabla s\|_{L^2}^2\\
&\quad+\delta\|(u,\nabla\rho,\nabla s,\hbar\epsilon^{1/2}\Delta\rho)\|_{L^2}^2+C\hbar^2\|(\nabla\rho,\nabla s)\|_{L^2}^2+C\hbar^4\|\Delta u\|_{L^2}^2\\
&\quad+C\tau d_0\|(u,\nabla u,\nabla s,\nabla\rho)\|_{L^2}^2,
\end{aligned} \tag{3.9}
$$

where δ is a sufficiently small positive constant. Here, we have used the the Riesz operator R_j, $\widehat{(R_jf)}=\frac{i\xi_j}{|\xi|}\widehat{f}$ that

$$
\|\nabla^2 f\|_{L^2}=\|\nabla\Delta^{-1}\nabla\Delta f\|_{L^2}=\|R_iR_j\Delta f\|_{L^2}\le C\|\Delta f\|_{L^2}, \tag{3.10}
$$

where R_iR_j is bounded from L^p to L^p with $1<p<\infty$.

On the other hand, we observe that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^R}\Big(\frac{1}{2}|\nabla\rho|^2+\frac{3}{4\mu}(1+\tau\rho)^2\nabla\rho\cdot u\Big)\\
&=\int_{\Omega^R}\Big(\nabla\rho+\frac{3}{4\mu}(1+\tau\rho)^2u\Big)\cdot\nabla\rho_t+\frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)^2\nabla\rho\cdot u_t\\
&\quad+\frac{3\tau}{2\mu}\int_{\Omega^R}(1+\tau\rho)u\cdot\nabla\rho\rho_t\\
&=\int_{\Omega^R}\Big(\Delta\rho+\frac{3}{4\mu}(1+\tau\rho)^2\operatorname{div} u\Big)\Big((1+\tau\rho)\operatorname{div} u-\epsilon\Delta\rho+\tau u\cdot\nabla\rho\Big)\\
&\quad+\frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)^2\nabla\rho\cdot u_t\\
&\le-\frac{3}{8\mu}\frac{d}{dt}\int_{\Omega^R}(1+\tau\rho)|\rho|^2+\frac{3}{4\mu}\|\operatorname{div} u\|_{L^2}^2-\frac{3}{4\mu}\int_{\Omega^R}\nabla\rho\cdot\nabla s
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{5}{4\mu}\|\nabla\rho\|_{L^2}^2 - \epsilon\|\Delta\rho\|_{L^2}^2 - \frac{\hbar^2}{16\mu}\|\Delta\rho\|_{L^2}^2 - \frac{3}{4\mu}\|\rho\|_{L^2}^2 - \frac{3\epsilon}{4\mu}\|\nabla\rho\|_{L^2}^2 \\
&+ C\epsilon^{1/2}\|(\epsilon^{1/2}\Delta\rho, \operatorname{div} u)\|_{L^2}^2 + \delta\|(\nabla\rho, \epsilon^{1/2}\Delta\rho)\|_{L^2}^2 + C\tau\|f^R\|_{L^2}^2 \\
&+ C\tau(\|(u,\rho,\nabla\rho,\nabla u)\|_{L^\infty} + \|\nabla\rho\|_{L^\infty}^2)\|(\rho,\nabla\rho,\nabla s,\nabla u,\hbar\Delta\rho)\|_{L^2}^2, \qquad (3.11)
\end{aligned}
$$

where we have used

$$
\begin{aligned}
&\int_{\Omega^R} \Delta\rho(1+\tau\rho)\operatorname{div} u + \frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)\nabla\rho(\mu\Delta u + \frac{\mu}{3}\nabla\operatorname{div} u) \\
&= \int_{\Omega^R} \Delta\rho(1+\tau\rho)\operatorname{div} u - \int_{\Omega^R} \Delta\rho(1+\tau\rho)\operatorname{div} u - \tau\int_{\Omega^R}\operatorname{div} u|\nabla\rho|^2 \\
&\le C\tau\|\operatorname{div} u\|_{L^\infty}\|\nabla\rho\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)^2\nabla\rho\cdot u \\
&= \frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)\operatorname{div}\big((1+\tau\rho)u\big)\rho + \frac{3\tau}{4\mu}\int_{\Omega^R}\nabla\rho(1+\tau\rho)u\rho \\
&= -\frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)(\partial_t\rho - \epsilon\Delta\rho)\rho + \frac{3\tau}{4\mu}\int_{\Omega^R}\nabla\rho(1+\tau\rho)u\rho \\
&= -\frac{3}{8\mu}\frac{d}{dt}\int_{\Omega^R}(1+\tau\rho)|\rho|^2 + \frac{3\tau}{8\mu}\int_{\Omega^R}\rho_t|\rho|^2 - \frac{3\epsilon}{4\mu}\int_{\Omega^R}(1+\tau\rho)|\nabla\rho|^2 \\
&\quad - \frac{3\tau\epsilon}{4\mu}\int_{\Omega^R}\rho|\nabla\rho|^2 + \frac{3\tau}{4\mu}\int_{\Omega^R}\nabla\rho(1+\tau\rho)u\rho \\
&\le -\frac{3}{8\mu}\frac{d}{dt}\int_{\Omega^R}(1+\tau\rho)|\rho|^2 - \frac{3\epsilon}{4\mu}\int_{\Omega^R}(1+\tau\rho)|\nabla\rho|^2 + C\tau\|\rho_t\|_{L^2}\|\rho\|_{L^3}\|\rho\|_{L^6} \\
&\quad + C\tau\|\rho\|_{L^\infty}\|\nabla\rho\|_{L^2}^2 + C\tau\|\nabla\rho\|_{L^2}\|u\|_{L^3}\|\rho\|_{L^6} \\
&\le -\frac{3}{8\mu}\frac{d}{dt}\int_{\Omega^R}(1+\tau\rho)|\rho|^2 - \frac{3\epsilon}{4\mu}\int_{\Omega^R}(1+\tau\rho)|\nabla\rho|^2 + \delta\epsilon\|\Delta\rho\|_{L^2}^2 + C\tau d_0\|\rho\|_{H^1}^2,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)^2\nabla\rho\cdot\nabla\psi \\
&= -\frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)^2\rho\Delta\psi - \frac{3\tau}{2\mu}\int_{\Omega^R}(1+\tau\rho)\rho\nabla\rho\cdot\nabla\psi \\
&= -\frac{3}{4\mu}\int_{\Omega^R}(1+\tau\rho)^2|\rho|^2 - \frac{3\tau}{2\mu}\int_{\Omega^R}(1+\tau\rho)\rho\nabla\rho\cdot\nabla\psi \\
&\le -\frac{3}{4\mu}\|\rho\|_{L^2}^2 + C\tau\|\nabla\psi\|_{L^6}\|\rho\|_{L^3}\|\nabla\rho\|_{L^2} + C\tau\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2 \\
&\le -\frac{3}{4\mu}\|\rho\|_{L^2}^2 + C\tau\|\nabla^2\psi\|_{L^2}\|\rho\|_{H^1}\|\nabla\rho\|_{L^2} + C\tau\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2 \\
&\le -\frac{3}{4\mu}\|\rho\|_{L^2}^2 + C\tau d_0\|\rho\|_{H^1}^2,
\end{aligned}
$$

and

$$
\frac{\hbar^2}{16\mu}\int_{\Omega^R}(1+\tau\rho)\nabla\Delta\rho\cdot\nabla\rho
$$

$$
\begin{aligned}
&= -\frac{\hbar^2}{16\mu}\int_{\Omega^R}(1+\tau\rho)|\Delta\rho|^2 - \frac{\hbar^2\tau}{16\mu}\int_{\Omega^R}|\nabla\rho|^2\Delta\rho \\
&\le -\frac{\hbar^2}{16\mu}\int_{\Omega^R}(1+\tau\rho)|\Delta\rho|^2 + C\tau\|\nabla\rho\|_{L^\infty}\|(\nabla\rho,\hbar\Delta\rho)\|_{L^2}^2 \\
&\le -\frac{\hbar^2}{16\mu}\int_{\Omega^R}(1+\tau\rho)|\Delta\rho|^2 + C\tau d_0\|(\nabla\rho,\hbar\Delta\rho)\|_{L^2}^2,
\end{aligned}
$$

thanks to (3.1a), (3.1d), (3.10), integration by parts and Lemma 2.6.

Multiplying (3.11) by a positive constant m, and taking δ and ϵ sufficiently small, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^R}E(\rho,u,s) + \frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\nabla\rho|^2}{(1+\tau\rho)^2} + \frac{1}{2}\frac{d}{dt}\int_{\Omega^R}|\nabla\psi|^2 \\
&+ m\frac{d}{dt}\int_{\Omega^R}\Big(\frac{1}{2}|\nabla\rho|^2 + \frac{3}{4\mu}(1+\tau\rho)^2\nabla\rho\cdot u + \frac{3}{8\mu}(1+\tau\rho)|\rho|^2\Big) \\
&+ (\frac{3m}{4\mu}+\frac{2}{3})\|\rho\|_{L^2}^2 + \|u\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2 + (\frac{\mu}{3}-\frac{3m}{4\mu})\|\operatorname{div} u\|_{L^2}^2 + \frac{3}{2}\|s\|_{L^2}^2 \\
&\quad + \kappa\|\nabla s\|_{L^2}^2 + \frac{\hbar^2\epsilon}{12}\|\Delta\rho\|_{L^2}^2 + \frac{\hbar^2 m}{16\mu}\|\Delta\rho\|_{L^2}^2 + (\frac{7}{3}+\frac{3m}{4\mu})\epsilon\|\nabla\rho\|_{L^2}^2 \\
&\quad + \epsilon\|\rho\|_{L^2}^2 + m\epsilon\|\Delta\rho\|_{L^2}^2 + (\frac{4\kappa}{9}+\frac{5m}{4\mu})\|\nabla\rho\|_{L^2}^2 + \frac{\hbar^2}{36}\|\nabla\rho\|_{L^2}^2 \\
&\quad + (\frac{4\kappa}{3}+\frac{3m}{4\mu})\int_{\Omega^R}\nabla s\cdot\nabla\rho + 2\int_{\Omega^R}\rho s \\
&\le C\tau d_0\|(u,\rho,\nabla u,\nabla s,\nabla\rho,\hbar\Delta\rho)\|_{L^2}^2 + C\tau\|f^R\|_{L^2}^2 + C\hbar^4\|\Delta u\|_{L^2}^2.
\end{aligned}
$$

Moreover, we assume that

$$
\frac{2\mu(3-4a)}{9a} < m < \min\{\frac{2\mu^2}{9(1+\bar\rho)^4}, \frac{4\mu^2}{9}, \frac{16\kappa\mu}{3}\},
$$

where $0<\bar\rho\le 1/2,\ 0<a<3/4$, then we obtain

$$
\begin{gathered}
\int_{\Omega^R}E(\rho,u,s) + m\int_{\Omega^R}\Big(\frac{1}{2}|\nabla\rho|^2 + \frac{3}{4\mu}(1+\tau\rho)^2\nabla\rho\cdot u\Big) \\
\ge \frac{1}{8}(\|\rho\|_{L^2}^2 + \frac{7}{9}\|s\|_{L^2}^2 + \|u\|_{L^2}^2) + \frac{m}{4}\|\nabla\rho\|_{L^2}^2, \\
(\frac{3m}{4\mu}+\frac{2}{3})\|\rho\|_{L^2}^2 + \frac{3}{2}\|s\|_{L^2}^2 + 2\int_{\Omega^R}\rho s \\
\ge \big(\frac{3m}{4\mu}+\frac{4a-3}{6a}\big)\|\rho\|_{L^2}^2 + \big(\frac{3}{2}-2a\big)\|s\|_{L^2}^2,
\end{gathered}
$$

and

$$
\begin{aligned}
&\kappa\|\nabla s\|_{L^2}^2 + \big(\frac{\mu}{3}-\frac{3m}{4\mu}\big)\|\operatorname{div} u\|_{L^2}^2 + \big(\frac{4\kappa}{3}+\frac{3m}{4\mu}\big)\int_{\Omega^R}\nabla s\cdot\nabla\rho \\
&+ \big(\frac{4\kappa}{9}+\frac{5m}{4\mu}\big)\int_{\Omega^R}|\nabla\rho|^2 \\
&\ge \frac{5\kappa}{27}\|\nabla s\|_{L^2}^2 + \frac{5\kappa}{3}\|\nabla\rho\|_{L^2}^2.
\end{aligned}
$$

Combining the above estimates, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(\rho, u, s, \nabla\rho, \nabla\psi, \hbar\nabla\rho)\|_{L^2}^2 + \|u\|_{H^1}^2 + \|s\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \epsilon\|\rho\|_{H^1}^2 + \hbar^2\epsilon\|\Delta\rho\|_{L^2}^2 \\
&+ \epsilon\|\Delta\rho\|_{L^2}^2 + \hbar^2\|\Delta\rho\|_{L^2}^2 \\
&\le C\tau d_0\|(\nabla u, \nabla s, \nabla\rho, \hbar\Delta\rho)\|_{L^2}^2 + C\hbar^4\|\Delta u\|_{L^2}^2 + C\tau\|f^R\|_{L^2}^2.
\end{aligned}$$

Equivalently, we have

$$\begin{aligned}
&\frac{d}{dt}\|(\rho, u, \theta, \nabla\rho, \nabla\psi, \hbar\nabla\rho)\|_{L^2}^2 + \|u\|_{H^1}^2 + \|\theta\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \epsilon\|\rho\|_{H^1}^2 + \hbar^2\epsilon\|\Delta\rho\|_{L^2}^2 \\
&+ \epsilon\|\Delta\rho\|_{L^2}^2 + \hbar^2\|\Delta\rho\|_{L^2}^2 \\
&\le C\tau d_0\|(\nabla u, \nabla\theta, \nabla\rho, \hbar\Delta\rho)\|_{L^2}^2 + C\hbar^4\|\Delta u\|_{L^2}^2 + C\tau\|f^R\|_{L^2}^2,
\end{aligned}$$

thanks to Lemma 3.2 and (3.3). The proof is complete. □

It is worth mentioning that the derivative of the velocity u on the right of (3.6) is higher than that on the left which leads the following estimate.

Lemma 3.4. *Under the assumptions in Lemma 3.3, we arrive at*

$$\begin{aligned}
&\frac{d}{dt}\|(\hbar\nabla u, \hbar\nabla\rho, \hbar^2\Delta\rho)\|_{L^2}^2 + \hbar^2\|\Delta u\|_{L^2}^2 + \epsilon\|(\hbar\Delta\rho, \hbar^2\nabla\Delta\rho)\|_{H^3}^2 \\
&\le \delta\|(u, \epsilon^{1/2}\nabla\rho, \nabla\theta, \hbar\Delta u)\|_{H^3}^2 + C\tau d_0^2\|(\nabla u, \nabla\theta, \nabla\rho, \hbar\Delta\rho, \hbar^2\Delta\rho)\|_{L^2} \\
&\quad + C\tau\|f^R\|_{L^2}^2.
\end{aligned} \tag{3.12}$$

Proof. Multiplying (3.1b) by $1/(1+\tau\rho)$, we obtain

$$\begin{aligned}
&\partial_t u + \frac{u}{1+\tau\rho} - \mu\frac{\Delta u}{1+\tau\rho} - \frac{\mu}{3}\frac{\nabla \operatorname{div} u}{1+\tau\rho} + \nabla\theta + \frac{1+\tau\theta}{1+\tau\rho}\nabla\rho - \frac{\hbar^2}{12}\frac{\nabla\Delta\rho}{1+\tau\rho} - \nabla\psi \\
&= -\tau u\cdot\nabla u - \frac{\hbar^2\tau}{12}\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} + \frac{\hbar^2\tau}{12}\frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^3} + \tau f^R.
\end{aligned} \tag{3.13}$$

Taking inner product of (3.13) with $-\hbar^2\Delta u$, we obtain

$$\begin{aligned}
&\frac{\hbar^2}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \hbar^2 C_\mu\|\Delta u\|_{L^2}^2 \\
&= -\frac{\hbar^4}{12}\int_{\Omega^R}\frac{\nabla\Delta\rho\cdot\Delta u}{1+\tau\rho} + \hbar^2\int_{\Omega^R}\frac{1+\tau\theta}{1+\tau\rho}\nabla\rho\cdot\Delta u + \hbar^2\int_{\Omega^R}\rho \operatorname{div} u \\
&\quad + \frac{\hbar^4\tau}{12}\int_{\Omega^R}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^3}\Big)\Delta u + \hbar^2\int_{\Omega^R}\nabla\theta\cdot\Delta u \\
&\quad - \frac{\mu\hbar^2\tau}{3}\int_{\Omega^R}\frac{\nabla\rho \operatorname{div} u(\Delta u - \nabla \operatorname{div} u)}{(1+\tau\rho)^2} + \frac{\hbar^2\tau}{2}\int_{\Omega^R}\operatorname{div} u|\nabla u|^2 \\
&\quad + \hbar^2\tau\int_{\Omega^R}\partial_k u^i\partial_i u^j\partial_k u^j + \hbar^2\int\frac{u\Delta u}{1+\tau\rho} - \tau\hbar^2\int_{\Omega^R}f^R\Delta u,
\end{aligned} \tag{3.14}$$

where we have used the estimate

$$\begin{aligned}
&\frac{\mu\hbar^2}{3}\int_{\Omega^R}\frac{\nabla \operatorname{div} u\Delta u}{1+\tau\rho} \\
&= -\frac{\mu\hbar^2}{3}\int_{\Omega^R}\frac{\operatorname{div} u\Delta \operatorname{div} u}{1+\tau\rho} + \frac{\mu\hbar^2\tau}{3}\int_{\Omega^R}\frac{\nabla\rho \operatorname{div} u\Delta u}{(1+\tau\rho)^2}
\end{aligned}$$

$$
= \frac{\mu\hbar^2}{3}\int_{\Omega^R}\frac{|\nabla \operatorname{div} u|^2}{1+\tau\rho} - \frac{\mu\hbar^2\tau}{3}\int_{\Omega^R}\frac{\nabla\rho \operatorname{div} u\nabla \operatorname{div} u}{(1+\tau\rho)^2} + \frac{\mu\hbar^2\tau}{3}\int_{\Omega^R}\frac{\nabla\rho \operatorname{div} u\Delta u}{(1+\tau\rho)^2},
$$

$$
\begin{aligned}
\hbar^2\int_{\Omega^R}\nabla\psi\cdot\Delta u &= -\hbar^2\int_{\Omega^R}\psi\Delta \operatorname{div} u\\
&= -\hbar^2\int_{\Omega^R}\Delta\psi \operatorname{div} u = -\hbar^2\int_{\Omega^R}\rho \operatorname{div} u,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&\tau\hbar^2\int_{\Omega^R}u\cdot\nabla u\Delta u\\
&= -\tau\hbar^2\int_{\Omega^R}\partial_k u^i\partial_i u^j\partial_k u^j - \hbar^2\tau\int_{\Omega^R}u^i\partial_{ik}u^j\partial_k u^j\\
&= \hbar^2\tau\int_{\Omega^R}\partial_k u^i\partial_i u^j\partial_k u^j + \frac{\hbar^2\tau}{2}\int_{\Omega^R}\operatorname{div} u|\nabla u|^2.
\end{aligned}
\tag{3.16}
$$

We proceed to estimate the terms on the right-hand side of (3.14). For the first term, by integration by parts, (3.1a), Lemmas 2.5 and 2.6, we have

$$
\begin{aligned}
&-\frac{\hbar^4}{12}\int_{\Omega^R}\frac{\nabla\Delta\rho\cdot\Delta u}{1+\tau\rho}\\
&= \frac{\hbar^4}{12}\int_{\Omega^R}\frac{\Delta\rho\Delta \operatorname{div} u}{1+\tau\rho} + \frac{\hbar^4}{12}\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\cdot\Delta u\Delta\rho\\
&= -\frac{\hbar^4}{12}\int_{\Omega^R}\frac{\Delta\rho(\partial_t\Delta\rho - \epsilon\Delta^2\rho + \tau u\cdot\nabla\Delta\rho)}{(1+\tau\rho)^2}\\
&\quad - \frac{\hbar^4\tau}{12}\int_{\Omega^R}\frac{\Delta\rho([\Delta,u]\cdot\nabla\rho + [\Delta,\rho]\operatorname{div} u)}{(1+\tau\rho)^2} + \frac{\hbar^4}{12}\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\cdot\Delta u\Delta\rho\\
&= -\frac{\hbar^4}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\Delta\rho|^2}{(1+\tau\rho)^2} + \frac{\hbar^4}{24}\int_{\Omega^R}\partial_t\Big(\frac{1}{1+\tau\rho}\Big)^2|\Delta\rho|^2 - \frac{\hbar^4\epsilon}{12}\int_{\Omega^R}\frac{|\nabla\Delta\rho|^2}{(1+\tau\rho)^2}\\
&\quad + \frac{\hbar^4\epsilon\tau}{6}\int_{\Omega^R}\frac{\nabla\rho\cdot\nabla\Delta\rho\Delta\rho}{(1+\tau\rho)^3} + \frac{\hbar^4\tau}{12}\int_{\Omega^R}\operatorname{div}\Big(\frac{u}{(1+\tau\rho)^2}\Big)|\Delta\rho|^2\\
&\quad - \frac{\hbar^4\tau}{12}\int_{\Omega^R}\frac{\Delta\rho[\Delta,u]\cdot\nabla\rho}{(1+\tau\rho)^2} - \frac{\hbar^4\tau}{12}\int_{\Omega^R}\frac{\Delta\rho[\Delta,\rho]\operatorname{div} u}{(1+\tau\rho)^2}\\
&\quad + \frac{\hbar^4}{12}\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\cdot\Delta u\Delta\rho\\
&\le -\frac{\hbar^4}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\Delta\rho|^2}{(1+\tau\rho)^2} - \frac{\hbar^4\epsilon}{12}\int_{\Omega^R}\frac{|\nabla\Delta\rho|^2}{(1+\tau\rho)^2}\\
&\quad + \delta\|(\epsilon^{1/2}\nabla\rho, \hbar\Delta u, \hbar^2\epsilon^{1/2}\nabla\Delta\rho)\|_{H^3}^2 + C\tau d_0^2\|\hbar^2\Delta\rho\|_{L^2}.
\end{aligned}
$$

By integration by parts and (3.10), we have

$$
\begin{aligned}
&-\hbar^2\int_{\Omega^R}\frac{1+\tau\theta}{1+\tau\rho}\nabla\rho\cdot\Delta u\\
&= \hbar^2\int_{\Omega^R}\rho\nabla\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)\cdot\Delta u - \hbar^2\int_{\Omega^R}\rho\nabla\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)\cdot\nabla \operatorname{div} u\\
&\quad - \hbar^2\int_{\Omega^R}\frac{1+\tau\theta}{1+\tau\rho}\nabla\rho\cdot\nabla \operatorname{div} u
\end{aligned}
$$

$$
\begin{aligned}
&\le -\frac{\hbar^2}{2}\frac{d}{dt}\int_{\Omega^R}\frac{(1+\tau\theta)|\nabla\rho|^2}{(1+\tau\rho)^2} - \hbar^2\epsilon\int_{\Omega^R}\frac{(1+\tau\theta)|\Delta\rho|^2}{(1+\tau\rho)^2}\\
&\quad + \delta\|(\epsilon^{1/2}\nabla\rho,\nabla\theta,\hbar\epsilon^{1/2}\Delta\rho)\|_{H^3}^2 + C\tau d_0^2\|(\nabla\theta,\hbar\nabla\rho)\|_{L^2} + C\hbar^4\|\Delta u\|_{L^2}^2, \quad (3.17)
\end{aligned}
$$

where δ is a sufficiently small positive constant. Here, we have used the following estimates, for any $1\le p\le\infty$,

$$
\|\rho_t\|_{L^p} \le C\|(\operatorname{div} u,\epsilon\Delta\rho)\|_{L^p} + C\|u\|_{L^\infty}\|\nabla\rho\|_{L^p}, \quad (3.18)
$$

and

$$
\begin{aligned}
\|\theta_t\|_{L^p} \le& C\|(\theta,\Delta\theta,\operatorname{div} u,\hbar^2\Delta\rho,\hbar^2\operatorname{div}\Delta u)\|_{L^p} + C\tau\|u\|_{L^\infty}\|(u,\nabla\theta)\|_{L^p}\\
&+ C\tau\|\nabla\rho\|_{L^\infty}\|(\nabla\rho,\hbar^2\nabla\operatorname{div} u)\|_{L^p} + C\tau\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^p}\\
\le& C\|(\theta,\Delta\theta,\operatorname{div} u,\hbar^2\Delta\rho,\hbar^2\operatorname{div}\Delta u)\|_{L^p}\\
&+ C\tau\|(\nabla u,\nabla\rho)\|_{H^2}\|(u,\nabla u,\nabla\rho,\nabla\theta,\hbar\nabla\operatorname{div} u)\|_{L^p},
\end{aligned} \quad (3.19)
$$

thanks to (3.1a) and (3.1c). By Hölder's and Young's inequality, the other terms on the right-hand side of (3.14) can be bounded by

$$
\delta\|(u,\nabla\theta)\|_{L^2}^2 + C\hbar^4\|\Delta u\|_{L^2}^2 + C\tau d_0^2\|(\nabla u,\nabla\rho,\hbar^2\Delta\rho)\|_{L^2}.
$$

Putting the above estimates together, and taking $\delta,\hbar$ sufficiently small, we complete the proof. □

3.2. **First, second and third order estimates for (ρ,u,θ).**

Lemma 3.5. *Under the same assumptions in Lemma 3.3, it holds that*

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla\rho,\nabla u,\nabla\theta,\hbar\Delta\rho)\|_{H^2}^2 + \|\nabla u\|_{H^3}^2 + \|\nabla\theta\|_{H^3}^2 + \epsilon\|\nabla\rho\|_{H^3}^2 + \epsilon\hbar^2\|\Delta\rho\|_{H^3}^2\\
&\le C\tau d_0^2\|(\rho,\theta,\nabla u,\nabla\theta,\hbar^2\Delta\rho,\hbar^2\Delta u)\|_{H^3} + C\tau d_0\|(\rho,\theta,\nabla u,\nabla\theta)\|_{H^3}^2\\
&\quad + C\tau\|\nabla f^R\|_{H^1}^2 + \delta\|\hbar\Delta u\|_{H^3}^2 + C\hbar^4\|\Delta u\|_{H^3}^2 + C\hbar^2\|\hbar\nabla\rho\|_{H^3}^2.
\end{aligned} \quad (3.20)
$$

Proof. Next we define α by any multi-index with $|\alpha|=k$, $k=1,2,3$, and write $(\partial^\alpha\rho,\partial^\alpha u,\partial^\alpha\theta)=(\rho_\alpha,u_\alpha,\theta_\alpha)$ for simplicity.

Applying the operator ∂^α to (3.1a) and (3.1c), we derive

$$
\partial_t\rho_\alpha + (1+\tau\rho)\operatorname{div} u_\alpha - \epsilon\Delta\rho_\alpha = -\tau u\cdot\nabla\rho_\alpha + g_1, \quad (3.21a)
$$

$$
\begin{aligned}
&\partial_t\theta_\alpha + \theta_\alpha + \frac{2}{3}(1+\tau\theta)\operatorname{div} u_\alpha - \frac{2\kappa}{3}\frac{\Delta\theta_\alpha}{1+\tau\rho} - \frac{\hbar^2}{18}\operatorname{div}\Delta u_\alpha\\
&= -\tau u\cdot\nabla\theta_\alpha + \frac{\hbar^2}{36}\partial^\alpha\Big(\frac{\Delta\rho}{1+\tau\rho}\Big) + g_2,
\end{aligned} \quad (3.21b)
$$

where

$$
\begin{aligned}
g_1 &= -\tau[\partial^\alpha,\rho]\operatorname{div} u - \tau[\partial^\alpha,u]\cdot\nabla\rho,\\
g_2 &= \frac{2\kappa}{3}[\partial^\alpha,\frac{1}{1+\tau\rho}]\Delta\theta - \frac{2}{3}\tau[\partial^\alpha,\theta]\operatorname{div} u - \tau[\partial^\alpha,u]\cdot\nabla\theta\\
&\quad + \frac{\hbar^2\tau}{18}\partial^\alpha\Big(\frac{\nabla\rho\cdot\nabla\operatorname{div} u}{1+\tau\rho}\Big) + \frac{2\tau}{3}\partial^\alpha\Big(\frac{1}{1+\tau\rho}\big(\frac{\mu}{2}|\nabla u+\nabla u^\top|^2\\
&\quad - \frac{2\mu}{3}(\operatorname{div} u)^2\big)\Big) + \partial^\alpha\Big(\frac{\tau}{3}|u|^2 - \frac{\hbar^2\tau}{36}\frac{|\nabla\rho|^2}{(1+\tau\rho)^2}\Big).
\end{aligned} \quad (3.22)
$$

Multiplying system (3.21) by $\frac{1+\tau\theta}{1+\tau\rho}\rho_\alpha, \frac{3}{2}\frac{1+\tau\rho}{1+\tau\theta}\theta_\alpha$, integrating over the periodic domain Ω^R, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^R}\frac{(1+\tau\theta)|\rho_\alpha|^2}{1+\tau\rho}+\frac{3}{4}\frac{d}{dt}\int_{\Omega^R}\frac{(1+\tau\rho)|\theta_\alpha|^2}{1+\tau\theta}+\|(\theta_\alpha,\nabla\theta_\alpha)\|_{L^2}^2\\
&+\epsilon\|\nabla\rho_\alpha\|_{L^2}^2+\int_{\Omega^R}(1+\tau\theta)\operatorname{div}u_\alpha\rho_\alpha+\int_{\Omega^R}(1+\tau\rho)\operatorname{div}u_\alpha\theta_\alpha\\
&=\frac{1}{2}\int_{\Omega^R}\partial_t\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)|\rho_\alpha|^2+\frac{3}{4}\int_{\Omega^R}\partial_t\Big(\frac{1+\tau\rho}{1+\tau\theta}\Big)|\theta_\alpha|^2\\
&-\tau\int_{\Omega^R}\frac{1+\tau\theta}{1+\tau\rho}u\cdot\nabla\rho_\alpha\rho_\alpha-\frac{3\tau}{2}\int_{\Omega^R}\frac{1+\tau\rho}{1+\tau\theta}u\cdot\nabla\theta_\alpha\theta_\alpha\\
&+\int_{\Omega^R}\frac{1+\tau\theta}{1+\tau\rho}g_1\rho_\alpha+\frac{3}{2}\int_{\Omega^R}\frac{1+\tau\rho}{1+\tau\theta}g_2\theta_\alpha+\frac{\hbar^2}{24}\int_{\Omega^R}\frac{1+\tau\rho}{1+\tau\theta}\partial^\alpha\Big(\frac{\Delta\rho}{1+\tau\rho}\Big)\theta_\alpha\\
&-\frac{\hbar^2}{12}\int_{\Omega^R}\nabla\Big(\frac{1+\tau\rho}{1+\tau\theta}\Big)\cdot\Delta u_\alpha\theta_\alpha-\kappa\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\theta}\Big)\cdot\nabla\theta_\alpha\theta_\alpha\\
&-\frac{\hbar^2}{12}\int_{\Omega^R}\frac{(1+\tau\rho)\nabla\theta_\alpha\cdot\Delta u_\alpha}{1+\tau\theta}\triangleq\sum_{i=1}^{10}R_{3,i}.
\end{aligned}\tag{3.23}$$

Estimates for the right-hand side of (3.23). It follows from (3.18) and (3.19) that

$$\begin{aligned}
R_{3,1}+R_{3,2}\leq&C\|(\partial_t\rho,\partial_t\theta)\|_{L^\infty}\|(\rho,\theta)\|_{H^3}^2\\
\leq&\delta\epsilon\|\nabla\rho\|_{H^3}^2+\delta\|\nabla\theta\|_{H^3}^2+C\hbar^4\|\Delta u\|_{H^3}^2+C\tau\|(\rho,\theta)\|_{H^3}^4\\
&+C\tau\|(\nabla u,\nabla\rho,\nabla\theta,\hbar\Delta\rho)\|_{H^2}\|(\rho,\theta)\|_{H^3}^2\\
&+C\tau\|(\nabla u,\nabla\rho,\nabla\theta,\hbar\nabla\operatorname{div}u)\|_{H^2}^2\|(\rho,\theta)\|_{H^3}^2\\
\leq&\delta\epsilon\|\nabla\rho\|_{H^3}^2+\delta\|\nabla\theta\|_{H^3}^2+C\hbar^4\|\Delta u\|_{H^3}^2+C\tau d_0\|(\rho,\theta)\|_{H^3}^2.
\end{aligned}$$

By integration by parts and Hölder's inequality, we have

$$\begin{aligned}
R_{3,3}+R_{3,4}=&\frac{\tau}{2}\int_{\Omega^R}\operatorname{div}\Big(\frac{(1+\tau\theta)u}{1+\tau\rho}\Big)|\rho_\alpha|^2+\frac{3\tau}{4}\int_{\Omega^R}\operatorname{div}\Big(\frac{(1+\tau\rho)u}{1+\tau\theta}\Big)|\theta_\alpha|^2\\
&\leq C\tau(1+\|u\|_{L^\infty})\|(\nabla u,\nabla\rho,\nabla\theta)\|_{L^\infty}\|(\rho,\theta)\|_{H^3}^2\\
&\leq C\tau d_0\|(\rho,\theta)\|_{H^3}^2.
\end{aligned}$$

For the fifth term, by Lemma 2.5, we have

$$\begin{aligned}
R_{3,5}\leq&C\tau\big(\|(\nabla\rho,\nabla u)\|_{L^\infty}\|(\operatorname{div}u,\nabla\rho)\|_{H^2}+\|(\nabla\rho,\nabla u)\|_{H^2}\|(\operatorname{div}u,\nabla\rho)\|_{L^\infty}\big)\|\rho\|_{H^3}\\
\leq&C\tau d_0^2\|\rho\|_{H^3}.
\end{aligned}$$

By Lemmas 2.5 and 2.6, we have

$$\|\frac{1}{1+\tau\rho}\|_{\dot H^k}\leq C\tau(1+\|\nabla\rho\|_{H^2}^k),\tag{3.24}$$

where $k=2,3$, and $\dot H$ is the homogeneous Sobolev spaces. For the sixth term, using (3.24) and Lemma 2.5 again, we have

$$\begin{aligned}
R_{3,6}\leq&C\tau\|\nabla\Big(\frac{1}{1+\tau\rho}\Big)\|_{H^2}\|\Delta\theta\|_{H^2}\|\theta\|_{H^3}+C\tau\|(\nabla\theta,\nabla u)\|_{H^2}\|(\operatorname{div}u,\nabla\theta)\|_{H^2}\|\theta\|_{H^3}\\
&+C\tau\big(\|\nabla\rho\|_{L^\infty}\|\hbar^2\nabla\operatorname{div}u\|_{H^3}+\|\hbar\nabla\rho\|_{H^3}\|\hbar\nabla\operatorname{div}u\|_{L^\infty}\big)\|\theta\|_{H^3}\\
&+C\tau(\|\nabla\rho\|_{H^2}+\|\nabla\rho\|_{H^2}^4)\|\hbar\nabla\operatorname{div}u\|_{L^\infty}\|\theta\|_{H^3}
\end{aligned}$$

$$
\begin{aligned}
&+ C\tau\|(u,\nabla u)\|_{L^\infty}\|(\nabla u,\nabla^2 u)\|_{H^2}\|\theta\|_{H^3} + C\tau\|\hbar\nabla\rho\|_{H^3}^2\|\theta\|_{H^3}\\
&\le C\tau d_0^2\|(\theta,\nabla\theta,\hbar^2\Delta u)\|_{H^3}.
\end{aligned}
$$

For the seventh term, with the aid of integration by parts and (3.24), we have

$$
\begin{aligned}
R_{3,7} &= -\frac{\hbar^2}{24}\int_{\Omega^R}\partial\Big(\frac{1+\tau\rho}{1+\tau\theta}\Big)\partial^{\alpha-1}\Big(\frac{\Delta\rho}{1+\tau\rho}\Big)\theta_\alpha - \frac{\hbar^2}{24}\int_{\Omega^R}\frac{1+\tau\rho}{1+\tau\theta}\partial^{\alpha-1}\Big(\frac{\Delta\rho}{1+\tau\rho}\Big)\theta_{\alpha+1}\\
&\le C\tau\|(\nabla\rho,\nabla\theta)\|_{L^\infty}(1+\|\nabla\rho\|_{H^2}^2)\|\hbar\nabla\rho\|_{H^3}\|\theta\|_{H^3} + \delta\|\nabla\theta\|_{H^3}^2 + C\hbar^2\|\hbar\nabla\rho\|_{H^3}^2\\
&\le \delta\|\nabla\theta\|_{H^3}^2 + C\hbar^2\|\hbar\nabla\rho\|_{H^3}^2 + C\tau d_0^2\|\theta\|_{H^3},
\end{aligned}
$$

where δ is a sufficiently small positive constant. Due to Hölder's and Young's inequalities, the other terms on the right-hand side of (3.23) can be bounded by $\delta\|\hbar\Delta u\|_{H^3}^2 + C\hbar^2\|\nabla\theta\|_{H^3}^2 + C\hbar^4\|\Delta u\|_{H^3}^2 + C\tau d_0^2\|\theta\|_{H^4}$.

Adding the above estimates, and taking $\delta,\hbar,\epsilon$ sufficiently small, we have

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla\rho,\nabla\theta)\|_{H^2}^2 + \|\nabla\theta\|_{H^3}^2 + \epsilon\|\nabla\rho\|_{H^3}^2 + \int_{\Omega^R}(1+\tau\theta)\operatorname{div} u_\alpha\rho_\alpha\\
&+ \int_{\Omega^R}(1+\tau\rho)\operatorname{div} u_\alpha\theta_\alpha\\
&\le \delta\|\hbar\Delta u\|_{H^3}^2 + C\hbar^4\|\Delta u\|_{H^3}^2 + C\hbar^2\|\hbar\nabla\rho\|_{H^3}^2 + C\tau d_0^2\|(\rho,\theta,\nabla\theta,\hbar^2\Delta u)\|_{H^3}\\
&\quad + C\tau d_0\|(\rho,\theta,\nabla\theta)\|_{H^3}^2.
\end{aligned}
\tag{3.25}
$$

On the other hand, applying the operator div to (3.13), we have

$$
\begin{aligned}
&\partial_t\operatorname{div} u + \frac{\operatorname{div} u}{1+\tau\rho} - \frac{4\mu}{3}\frac{\Delta\operatorname{div} u}{1+\tau\rho} + \Delta\theta + \frac{1+\tau\theta}{1+\tau\rho}\Delta\rho - \frac{\hbar^2}{12}\frac{\Delta^2\rho}{1+\tau\rho}\\
&- \Delta\psi + \tau u\cdot\nabla\operatorname{div} u\\
&= \nabla\Big(\frac{1}{1+\tau\rho}\Big)(\mu\Delta u + \frac{\mu}{3}\nabla\Delta\operatorname{div} u) - \nabla\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)\cdot\nabla\rho\\
&\quad + \frac{\hbar^2}{12}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\cdot\nabla\Delta\rho - \tau\nabla u:\nabla u - \frac{\hbar^2\tau}{12}\operatorname{div}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2}\\
&\quad - \frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^3}\Big) + \tau\nabla\rho\cdot u + \tau\operatorname{div} f^R.
\end{aligned}
\tag{3.26}
$$

Applying the operator $\nabla\times$ to (3.13), we have

$$
\begin{aligned}
&\partial_t\nabla\times u + \frac{\nabla\times u}{1+\tau\rho} - \frac{\mu}{3}\frac{\Delta\nabla\times u}{1+\tau\rho} + \tau u\cdot\nabla\nabla\times u\\
&= \nabla\Big(\frac{1}{1+\tau\rho}\Big)(\mu\Delta u + \frac{\mu}{3}\nabla\Delta\operatorname{div} u) - \nabla\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)\cdot\nabla\rho\\
&\quad + \frac{\hbar^2}{12}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\cdot\nabla\Delta\rho - \tau(\nabla\times u\operatorname{div} u - \nabla\times u\cdot\nabla u)\\
&\quad - \frac{\hbar^2\tau}{12}\nabla\times\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^3}\Big) + \tau\nabla\rho\cdot u + \tau\nabla\times f^R,
\end{aligned}
\tag{3.27}
$$

where we have used the vector analysis formula

$$
\nabla\times(f\cdot\nabla f) = \nabla\times(\nabla\times f\otimes f), \tag{3.28}
$$

$$
\nabla\times(f\times g) = f\operatorname{div} g - g\operatorname{div} f + (g\cdot\nabla)f - (f\cdot\nabla)g, \tag{3.29}
$$

for any vector functions f,g.

Applying the operator $\partial^{\alpha-1}$ to systems (3.26) and (3.27), and multiplying the result by $(1+\tau\rho)(\nabla\times u_{\alpha-1}, \operatorname{div} u_{\alpha-1})$, integrating over the periodic domain Ω^R, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(1+\tau\rho)(\nabla\times u, \operatorname{div} u)\|_{H^2}^2 + \|(\nabla\times u, \operatorname{div} u)\|_{H^3}^2 \\
&-\frac{\tau}{2}\int_{\Omega^R}\partial_t\rho|\operatorname{div} u_{\alpha-1}|^2 \\
&+\tau\int_{\Omega^R}(1+\tau\rho)u\cdot\nabla\operatorname{div} u_{\alpha-1}\operatorname{div} u_{\alpha-1} - \frac{\tau}{2}\int_{\Omega^R}\partial_t\rho|\nabla\times u_{\alpha-1}|^2 \\
&+\tau\int_{\Omega^R}(1+\tau\rho)u\cdot\nabla\nabla\times u_{\alpha-1}\nabla\times u_{\alpha-1} + \int_{\Omega^R}(1+\tau\theta)\operatorname{div} u_{\alpha-1}\Delta\rho_{\alpha-1} \\
&+\int_{\Omega^R}(1+\tau\rho)\operatorname{div} u_{\alpha-1}\Delta\theta_{\alpha-1} - \int_{\Omega^R}(1+\tau\rho)\Delta\psi_{\alpha-1}\operatorname{div} u_{\alpha-1} \\
&-\frac{\hbar^2}{12}\int_{\Omega^R}\Delta^2\rho_{\alpha-1}\operatorname{div} u_{\alpha-1} \\
&=\int_{\Omega^R}(1+\tau\rho)\partial^{\alpha-1}g_3(\nabla\times u_{\alpha-1}, \operatorname{div} u_{\alpha-1}) \\
&\quad+\int_{\Omega^R}(1+\tau\rho)g_4(\nabla\times u_{\alpha-1}, \operatorname{div} u_{\alpha-1}) \triangleq R_{3,11}+R_{3,12},
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
g_3 =& 2\tau\nabla\rho\cdot u - \tau\nabla\times u\operatorname{div} u + \tau\nabla\times u\cdot\nabla u - \tau\nabla u:\nabla u \\
&+2\nabla\Big(\frac{1}{1+\tau\rho}\Big)(\mu\Delta u + \frac{\mu}{3}\nabla\Delta\operatorname{div} u) - 2\nabla\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)\cdot\nabla\rho \\
&+\frac{\hbar^2}{6}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\cdot\nabla\Delta\rho + \tau\nabla\times f^R + \tau\operatorname{div} f^R \\
&-\frac{\hbar^2\tau}{12}\nabla\times\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^3}\Big) \\
&-\frac{\hbar^2\tau}{12}\operatorname{div}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^3}\Big),
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
g_4 =& \frac{4\tau}{3}[\partial^{\alpha-1}, \frac{1}{1+\tau\rho}]\Delta\operatorname{div} u + \mu[\partial^{\alpha-1}, \frac{1}{1+\tau\rho}]\Delta\nabla\times u \\
&+\frac{\hbar^2}{12}[\partial^{\alpha-1}, \frac{1}{1+\tau\rho}]\Delta^2\rho - \tau[\partial^{\alpha-1}, u]\nabla\operatorname{div} u - \tau[\partial^{\alpha-1}, u]\nabla\nabla\times u \\
&-[\partial^{\alpha-1}, \frac{1+\tau\theta}{1+\tau\rho}]\Delta\rho - \tau[\partial^{\alpha-1}, \rho]\Delta\theta - [\partial^{\alpha-1}, \frac{1}{1+\tau\rho}]\operatorname{div} u \\
&-[\partial^{\alpha-1}, \frac{1}{1+\tau\rho}]\nabla\times u.
\end{aligned} \tag{3.32}$$

Estimate for the left-hand side of (3.30). By integration by parts and (3.1a), we obtain

$$\begin{aligned}
&\tau\int_{\Omega^R}(1+\tau\rho)u\cdot\nabla\operatorname{div} u_{\alpha-1}\operatorname{div} u_{\alpha-1} - \frac{\tau}{2}\int_{\Omega^R}\partial_t\rho|\operatorname{div} u_{\alpha-1}|^2 \\
&= -\frac{\tau}{2}\int_{\Omega^R}\big(\partial_t\rho + \operatorname{div}((1+\tau\rho)u)\big)|\operatorname{div} u_{\alpha-1}|^2
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\tau\epsilon}{2}\int_{\Omega^R} \Delta\rho|\operatorname{div} u_{\alpha-1}|^2 \\
&\geq -\epsilon^2\|\nabla\rho\|_{H^3}^2 - C\tau\|\nabla u\|_{H^2}^4.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&-\frac{\tau}{2}\int_{\Omega^R} \partial_t\rho|\nabla\times u_{\alpha-1}|^2 + \tau\int_{\Omega^R}(1+\tau\rho)u\cdot\nabla\nabla\times u_{\alpha-1}\nabla\times u_{\alpha-1} \\
&\geq -\epsilon^2\|\nabla\rho\|_{H^3}^2 - C\tau\|\nabla u\|_{H^2}^4.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
&\int_{\Omega^R}(1+\tau\theta)\Delta\rho_{\alpha-1}\operatorname{div} u_{\alpha-1} + \int_{\Omega^R}(1+\tau\rho)\operatorname{div} u_{\alpha-1}\Delta\theta_{\alpha-1} \\
&= -\tau\int_{\Omega^R}\nabla\theta\cdot\nabla\rho_{\alpha-1}\operatorname{div} u_{\alpha-1} - \int_{\Omega^R}(1+\tau\theta)\nabla\rho_{\alpha-1}\cdot\nabla\operatorname{div} u_{\alpha-1} \\
&\quad - \int_{\Omega^R}(1+\tau\rho)\nabla\operatorname{div} u_{\alpha-1}\cdot\nabla\theta_{\alpha-1} - \tau\int_{\Omega^R}\nabla\rho\cdot\nabla\theta_{\alpha-1}\operatorname{div} u_{\alpha-1} \\
&\geq -\int_{\Omega^R}(1+\tau\theta)\rho_\alpha\operatorname{div} u_\alpha - \int_{\Omega^R}(1+\tau\rho)\operatorname{div} u_\alpha\theta_\alpha - C\tau d_0\|(\nabla\theta,\nabla\rho,\nabla u)\|_{H^2}^2.
\end{aligned}
$$

For the last but one term on the left-hand side of (3.30), by integration by parts, (3.1a), (3.1d), Lemmas 2.5 and 2.6, we arrive at

$$
\begin{aligned}
&-\int_{\Omega^R}(1+\tau\rho)\Delta\psi_{\alpha-1}\operatorname{div} u_{\alpha-1} \\
&= -\int_{\Omega^R}(1+\tau\rho)\rho_{\alpha-1}\operatorname{div} u_{\alpha-1} \\
&= \int_{\Omega^R}\rho_{\alpha-1}(\partial_t\rho_{\alpha-1} - \epsilon\Delta\rho_{\alpha-1} + \tau u\cdot\nabla\rho_{\alpha-1} + \tau[\partial^{\alpha-1},\rho]\operatorname{div} u + \tau[\partial^{\alpha-1},u]\nabla\rho) \\
&= \frac{1}{2}\frac{d}{dt}\int_{\Omega^R}|\rho_{\alpha-1}|^2 + \epsilon\int_{\Omega^R}|\nabla\rho_{\alpha-1}|^2 - \frac{\tau}{2}\int_{\Omega^R}\operatorname{div} u|\rho_{\alpha-1}|^2 \\
&\quad + \tau\int_{\Omega^R}\rho_{\alpha-1}[\partial^{\alpha-1},\rho]\operatorname{div} u + \tau\int_{\Omega^R}\rho_{\alpha-1}[\partial^{\alpha-1},u]\cdot\nabla\rho \\
&\geq \frac{1}{2}\frac{d}{dt}\int_{\Omega^R}|\rho_{\alpha-1}|^2 + \epsilon\int_{\Omega^R}|\nabla\rho_{\alpha-1}|^2 - C\tau\|(\nabla u,\nabla\rho)\|_{L^\infty}\|(\rho,\nabla u)\|_{H^2}^2 \\
&\geq \frac{1}{2}\frac{d}{dt}\int_{\Omega^R}|\rho_{\alpha-1}|^2 + \epsilon\int_{\Omega^R}|\nabla\rho_{\alpha-1}|^2 - C\tau d_0\|(\rho,\nabla u)\|_{H^2}^2.
\end{aligned}
$$

For the last term on the left-hand side of (3.30), by (3.1a) and Lemma 2.5 again, we have

$$
\begin{aligned}
&-\frac{\hbar^2}{12}\int_{\Omega^R}\Delta^2\rho_{\alpha-1}\operatorname{div} u_{\alpha-1} \\
&= \frac{\hbar^2}{12}\int_{\Omega^R}\nabla\Delta\rho_{\alpha-1}\cdot\nabla\operatorname{div} u_{\alpha-1} \\
&= -\frac{\hbar^2}{12}\int_{\Omega^R}\Delta\rho_\alpha(\partial_t\rho_\alpha - \epsilon\Delta\rho_\alpha + \tau\partial^\alpha\operatorname{div}(\rho u)) \\
&= \frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R}|\nabla\rho_\alpha|^2 + \frac{\hbar^2\epsilon}{12}\int_{\Omega^R}|\Delta\rho_\alpha|^2 - \frac{\hbar^2\tau}{12}\int_{\Omega^R}\Delta\rho_\alpha\operatorname{div} u\rho_\alpha
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\hbar^2\tau}{24}\int_{\Omega^R} \operatorname{div} u|\nabla\rho_\alpha|^2 - \frac{\hbar^2\tau}{12}\int_{\Omega^R} \Delta\rho_\alpha[\partial^\alpha, \operatorname{div} u]\rho \\
&+ \frac{\hbar^2\tau}{12}\int_{\Omega^R} \nabla\rho_\alpha\cdot[\partial^\alpha\nabla, u]\cdot\nabla\rho \\
\geq\ & \frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R} |\nabla\rho_\alpha|^2 + \frac{\hbar^2\epsilon}{12}\int_{\Omega^R} |\Delta\rho_\alpha|^2 - C\tau\hbar^2\|\Delta\rho_\alpha\|_{L^2}(\|\nabla\operatorname{div} u\|_{L^3}\|\rho\|_{W^{2,6}} \\
&+ \|\nabla u\|_{H^3}\|\rho\|_{L^\infty}) - C\tau\hbar^2\|\nabla\rho_\alpha\|_{L^2}(\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{H^3} + \|\nabla u\|_{H^3}\|\nabla\rho\|_{L^\infty}) \\
&- C\tau\hbar^2(\|\Delta\rho\|_{H^3}\|\nabla\rho\|_{H^2} + \|\nabla\rho\|_{H^3}^2)\|\nabla u\|_{H^3} \\
\geq\ & \frac{\hbar^2}{24}\frac{d}{dt}\int_{\Omega^R} |\nabla\rho_\alpha|^2 + \frac{\hbar^2\epsilon}{12}\int_{\Omega^R} |\Delta\rho_\alpha|^2 - C\tau d_0^2\|\nabla u\|_{H^3}.
\end{aligned}
$$

Estimates for the right-hand side of (3.30). For the first term $R_{3,11}$, we consider here only the most difficult one, the other terms can be treated similarly. By the curl-div decomposition of the gradient, we have

$$
\|\nabla u\|_{H^k} \cong \|\nabla\times u\|_{H^k} + \|\operatorname{div} u\|_{H^k}, \tag{3.33}
$$

where $k = 0, 1, 2, 3$. Invoking (3.10), (3.24), (3.33), Lemmas 2.5 and 2.6, we obtain, for $\alpha = 2, 3$,

$$
\begin{aligned}
&- \frac{\hbar^2\tau}{12}\int_{\Omega^R} (1+\tau\rho)\partial^{\alpha-1}\operatorname{div}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^3}\Big)\operatorname{div} u_{\alpha-1} \\
=\ & \frac{\hbar^2\tau}{12}\int_{\Omega^R} (1+\tau\rho)\partial^{\alpha-2}\operatorname{div}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^3}\Big)\operatorname{div} u_{\alpha} \\
&+ \frac{\hbar^2\tau^2}{12}\int_{\Omega^R} \nabla\rho\partial^{\alpha-2}\operatorname{div}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^3}\Big)\operatorname{div} u_{\alpha-1} \\
\leq\ & C\tau\big(\|\nabla\rho\|_{L^\infty}\|\hbar\Delta\rho\|_{H^2} + \|\hbar\Delta\rho\|_{L^\infty}(\|\nabla\rho\|_{H^2} + \|\nabla\rho\|_{L^\infty}(1+\|\nabla\rho\|_{H^2}^2)) \\
&+ \|\nabla\rho\|_{H^2}^3(1+\|\nabla\rho\|_{H^2}^2)\big)\|\nabla u\|_{H^3} \\
\leq\ & C\tau\big(\|\nabla\rho\|_{H^2}\|\hbar\nabla\rho\|_{H^3} + \|\nabla\rho\|_{H^2}^3\|\hbar\nabla\rho\|_{H^3} + \|\nabla\rho\|_{H^2}^3 + \|\nabla\rho\|_{H^2}^5\big)\|\nabla u\|_{H^3} \\
\leq\ & C\tau d_0^2\|\nabla u\|_{H^3}.
\end{aligned}
$$

For $\alpha = 1$, we have

$$
-\frac{\hbar^2\tau}{12}\int_{\Omega^R}\operatorname{div}\Big(\frac{\nabla\rho\Delta\rho + \nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{(\nabla\rho\cdot\nabla\rho)\nabla\rho}{(1+\tau\rho)^3}\Big)\operatorname{div} u \leq C\tau d_0^2\|\nabla u\|_{L^2}.
$$

Therefore,

$$
\begin{aligned}
R_{3,11} \leq\ & C\tau\|\nabla u\|_{H^2}\|\nabla u\|_{H^2}^2 + C\tau\|\nabla u\|_{H^2}\|(\nabla\theta, \nabla\rho)\|_{H^2}^2 \\
&+ C\tau\|\nabla u\|_{H^2}\|\nabla\Big(\frac{1}{1+\tau\rho}\Big)\|_{H^2}(\|(\nabla u, \hbar^2\Delta\rho)\|_{H^3} + \|\nabla\rho\|_{H^2}) \\
&+ \delta\|\nabla u\|_{H^3}^2 + C\tau\|\nabla f^R\|_{H^1}^2 + C\tau d_0^2\|\nabla u\|_{H^3} \\
\leq\ & \delta\|\nabla u\|_{H^3}^2 + C\tau\|\nabla f^R\|_{H^1}^2 + C\tau d_0^2(\|(\nabla\theta, \nabla u, \hbar^2\Delta\rho)\|_{H^3} + \|\nabla\rho\|_{H^2}).
\end{aligned}
$$

For the second term $R_{3,12}$, with the aid of Hölder inequality, Lemmas 2.5 and 2.6, we have

$$
R_{3,12} \leq C\tau\|\nabla u\|_{H^2}(\|\nabla\frac{1}{1+\tau\rho}\|_{L^\infty}\|\nabla u\|_{H^1} + \|\nabla\frac{1}{1+\tau\rho}\|_{H^1}\|\nabla u\|_{L^\infty})
$$

$$
\begin{aligned}
&+ C\tau\|\nabla u\|_{H^2}(\|\nabla \frac{1}{1+\tau\rho}\|_{L^\infty}\|\nabla u\|_{H^3} + \|\nabla \frac{1}{1+\tau\rho}\|_{H^2}\|\nabla^3 u\|_{L^6})\\
&+ C\tau\|\nabla u\|_{H^2}(\|\nabla \frac{1}{1+\tau\rho}\|_{L^\infty}\|\hbar^2\Delta\rho\|_{H^3} + \|\nabla \frac{1}{1+\tau\rho}\|_{H^2}\|\hbar^2\Delta^2\rho\|_{L^6})\\
&+ C\tau\|\nabla u\|_{H^2}\|(\nabla\rho,\nabla\theta,\nabla u)\|_{L^\infty}\|(\nabla\rho,\nabla\theta,\nabla u)\|_{H^2}\\
&+ C\tau\|\nabla u\|_{H^2}\|(\nabla\rho,\nabla\theta,\nabla u)\|_{H^2}\|(\nabla^2 u,\nabla^2\rho,\nabla^2\theta)\|_{L^6}\\
\le& C\tau d_0^2\|\nabla u\|_{H^3}.
\end{aligned}
$$

Putting these above estimates together, and recalling $1 \le |\alpha| \le 3$, we complete the proof of Lemma 3.5 by the curl-div decomposition of the gradient. □

To close the uniform bound of solutions to system (3.1) with respect to ϵ, R, we must deal with the term $\|\Delta u\|_{H^3}$, which is caused by the quantum effect terms in the energy equation (3.21b).

3.3. **Higher order estimates for** (ρ, u, θ).

Lemma 3.6. *Under the assumptions in Lemma 3.3, we have*

$$
\begin{aligned}
&\frac{d}{dt}\|(\hbar\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2 + \hbar^2\|\nabla^2 u\|_{H^3}^2 + \epsilon\hbar^2\|\Delta\rho\|_{H^3}^2 + \hbar^4\epsilon\|\nabla\Delta\rho\|_{H^3}^2\\
&\le C\tau d_0^2\|(\rho,u,\theta,\nabla\theta,\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3} + C\hbar\|(\rho,\hbar\operatorname{div} u)\|_{H^3}^2\\
&\quad + C\hbar^2\|\nabla\theta\|_{H^3}^2 + \delta\epsilon\|\nabla\rho\|_{H^3}^2 + C\tau\|\hbar\nabla f^R\|_{H^2}^2.
\end{aligned}
\tag{3.34}
$$

Proof. Applying the operator ∂^α to (3.13), multiplying the result by $-\hbar^2\Delta u_\alpha$, we obtain

$$
\begin{aligned}
&\frac{\hbar^2}{2}\frac{d}{dt}\|\nabla u_\alpha\|_{L^2}^2 + \hbar^2 C_\mu\|\nabla u_\alpha\|_{H^1}^2\\
&= \tau\hbar^2\int_{\Omega^R} u\cdot\nabla u_\alpha\Delta u_\alpha + \hbar^2\int_{\Omega^R}\frac{1+\tau\theta}{1+\tau\rho}\nabla\rho_\alpha\cdot\Delta u_\alpha - \hbar^2\int_{\Omega^R}\nabla\psi_\alpha\cdot\Delta u_\alpha\\
&\quad - \frac{\hbar^4}{12}\int_{\Omega^R}\frac{\nabla\Delta\rho_\alpha\cdot\Delta u_\alpha}{1+\tau\rho} - \hbar^2\int_{\Omega^R} g_5\Delta u_\alpha\\
&\quad + \frac{\hbar^4\tau}{12}\int_{\Omega^R}\partial^\alpha\Big(\frac{\nabla\rho\Delta\rho+\nabla\rho\cdot\nabla^2\rho}{(1+\tau\rho)^2} - \frac{|\nabla\rho|^2\nabla\rho}{(1+\tau\rho)^3}\Big)\Delta u_\alpha\\
&\quad + \hbar^2\int_{\Omega^R}\partial^\alpha\Big(\frac{u}{1+\tau\rho}\Big)\Delta u_\alpha - \tau\hbar^2\int_{\Omega^R}\partial^\alpha f^R\Delta u_\alpha\\
&\quad - \frac{\mu\hbar^2\tau}{3}\int_{\Omega^R}\frac{\nabla\rho\operatorname{div} u_\alpha(\Delta u_\alpha - \nabla\operatorname{div} u_\alpha)}{(1+\tau\rho)^2} + \hbar^2\int_{\Omega^R}\nabla\theta_\alpha\cdot\Delta u_\alpha\\
&\triangleq \sum_{i=1}^{9} R_{4,i},
\end{aligned}
\tag{3.35}
$$

where

$$
\begin{aligned}
g_5 =& -[\partial^\alpha, u]\cdot\nabla u + [\partial^\alpha, \frac{1}{1+\tau\rho}](\mu\Delta u + \frac{\mu}{3}\nabla\operatorname{div} u) + [\partial^\alpha, \frac{1+\tau\theta}{1+\tau\rho}]\nabla\rho\\
&- \frac{\hbar^2}{12}[\partial^\alpha, \frac{1}{1+\tau\rho}]\nabla\Delta\rho.
\end{aligned}
\tag{3.36}
$$

Estimates for the right-hand side of (3.35). For the first three terms $R_{4,1} \sim R_{4,3}$, by a similar manner to the estimates (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned}
R_{4,1} =& -\hbar^2\tau\int_{\Omega^R}\partial_k u^i\partial_i u_\alpha^j\partial_k u_\alpha^j - \hbar^2\tau\int_{\Omega^R} u^i\partial_{ik}u_\alpha^j\partial_k u_\alpha^j \\
=& -\hbar^2\tau\int_{\Omega^R}\partial_k u^i\partial_i u_\alpha^j\partial_k u_\alpha^j + \frac{\hbar^2\tau}{2}\int_{\Omega^R}\operatorname{div} u|\nabla u_\alpha|^2 \\
\le& C\|\nabla u\|_{L^\infty}\|\hbar\nabla u\|_{H^3}^2,
\end{aligned}$$

$$\begin{aligned}
R_{4,2} \le& -\frac{\hbar^2}{2}\frac{d}{dt}\int_{\Omega^R}\frac{1+\tau\theta}{(1+\tau\rho)^2}|\nabla\rho_\alpha|^2 - \hbar^2\epsilon\int_{\Omega^R}\frac{1+\tau\theta}{(1+\tau\rho)^2}|\Delta\rho_\alpha|^2 \\
&+ \delta\|(\epsilon\nabla\rho, \hbar\epsilon^{1/2}\Delta\rho)\|_{H^3}^2 + C\tau d_0^2\|(\rho,\theta,\nabla u,\hbar\nabla\rho,\hbar\nabla u)\|_{H^3} + C\hbar^4\|\Delta u\|_{H^3}^2,
\end{aligned}$$

where δ is a sufficiently small positive constant, and

$$R_{4,3} = \hbar^2\int_{\Omega^R}\rho_\alpha\operatorname{div} u_\alpha \le C\hbar\|(\rho,\hbar\operatorname{div} u)\|_{H^3}^2.$$

For the fourth term, by the equation (3.1a), we can infer that

$$\begin{aligned}
R_{4,4} =& \frac{\hbar^4}{12}\int_{\Omega^R}\frac{\Delta\rho_\alpha\Delta\operatorname{div} u_\alpha}{1+\tau\rho} + \frac{\hbar^4}{12}\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\Delta u_\alpha\Delta\rho_\alpha \\
=& -\frac{\hbar^4}{12}\int_{\Omega^R}\frac{\Delta\rho_\alpha(\partial_t\Delta\rho_\alpha - \epsilon\Delta\Delta\rho_\alpha + \tau u\cdot\nabla\Delta\rho_\alpha)}{(1+\tau\rho)^2} \\
&- \frac{\hbar^4\tau}{12}\int_{\Omega^R}\frac{\Delta\rho_\alpha([\partial^\alpha\Delta, u]\cdot\nabla\rho + [\partial^\alpha\Delta,\rho]\operatorname{div} u)}{(1+\tau\rho)^2} \\
&+ \frac{\hbar^4}{12}\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\Delta u_\alpha\Delta\rho_\alpha \\
=& -\frac{\hbar^4}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\Delta\rho_\alpha|^2}{(1+\tau\rho)^2} + \frac{\hbar^4}{24}\int_{\Omega^R}\partial_t\Big(\frac{1}{(1+\tau\rho)^2}\Big)|\Delta\rho_\alpha|^2 \\
&- \frac{\hbar^4\epsilon}{12}\int_{\Omega^R}\frac{|\nabla\Delta\rho_\alpha|^2}{(1+\tau\rho)^2} + \frac{\hbar^4\epsilon\tau}{6}\int_{\Omega^R}\frac{\nabla\rho\cdot\nabla\Delta\rho_\alpha\Delta\rho_\alpha}{(1+\tau\rho)^3} \\
&+ \frac{\hbar^4\tau}{24}\int_{\Omega^R}\operatorname{div}\Big(\frac{u}{(1+\tau\rho)^2}\Big)|\Delta\rho_\alpha|^2 - \frac{\hbar^4\tau}{12}\int_{\Omega^R}\frac{\Delta\rho_\alpha[\partial^\alpha\Delta, u]\cdot\nabla\rho}{(1+\tau\rho)^2} \\
&- \frac{\hbar^4\tau}{12}\int_{\Omega^R}\frac{\Delta\rho_\alpha[\partial^\alpha\Delta,\rho]\operatorname{div} u}{(1+\tau\rho)^2} + \frac{\hbar^4}{12}\int_{\Omega^R}\nabla\Big(\frac{1}{1+\tau\rho}\Big)\Delta u_\alpha\Delta\rho_\alpha \\
\le& -\frac{\hbar^4}{24}\frac{d}{dt}\int_{\Omega^R}\frac{|\Delta\rho_\alpha|^2}{(1+\tau\rho)^2} - \frac{\hbar^4\epsilon}{12}\int_{\Omega^R}\frac{|\nabla\Delta\rho_\alpha|^2}{(1+\tau\rho)^2} + \delta\|(\epsilon^{1/2}\nabla\rho, \hbar^2\epsilon^{1/2}\nabla\Delta\rho)\|_{H^3}^2 \\
&+ C\tau d_0\|\hbar^2\Delta\rho\|_{H^3}^2 + C\tau d_0^2\|(\nabla u,\hbar\Delta u,\hbar^2\Delta\rho)\|_{H^3} + C\hbar^4\|\Delta u\|_{H^3}^2,
\end{aligned}$$

thanks to Lemmas 2.5 and 2.6, Hölder's and Young's inequalities. Invoking the definition of g_5, (3.24), Lemma 2.5 and Sobolev embedding, we arrive at

$$\begin{aligned}
R_{4,5} \le& C\hbar^2\|g_5\|_{L^2}\|\Delta u\|_{H^3} \\
\le& C\hbar^4\|\Delta u\|_{H^3}^2 + C\|g_5\|_{L^2}^2 \\
\le& C\hbar^4\|\Delta u\|_{H^3}^2 + C\|\nabla\Big(\frac{1}{1+\tau\rho}\Big)\|_{H^2}^2\|u\|_{H^3}^2 + C\tau\|\nabla u\|_{H^2}^2\|\nabla u\|_{H^3}^2
\end{aligned}$$

$$
\begin{aligned}
&+ C\|\nabla\Big(\frac{1}{1+\tau\rho}\Big)\|_{H^2}^2\|\nabla u\|_{H^3}^2 + C\|\nabla\Big(\frac{1+\tau\theta}{1+\tau\rho}\Big)\|_{H^2}^2\|\nabla\rho\|_{H^2}^2\\
&+ C\|\nabla\Big(\frac{1}{1+\tau\rho}\Big)\|_{H^2}^2\|\hbar^2\Delta\rho\|_{H^3}^2\\
&\le C\hbar^4\|\Delta u\|_{H^3}^2 + C\tau d_0^2\|(\rho,u,\nabla u,\hbar^2\Delta\rho)\|_{H^3}^2.
\end{aligned}
$$

For the sixth term $R_{4,6}$, using lemma 2.5 and (3.24) again, we have

$$
\begin{aligned}
R_{4,6} &\le C\tau\|\hbar^2\Delta u\|_{H^3}(\|\nabla\rho\|_{L^\infty}\|\hbar^2\Delta\rho\|_{H^3} + \|\hbar\frac{\nabla\rho}{(1+\tau\rho)^2}\|_{H^3}\|\hbar^2\Delta\rho\|_{L^\infty})\\
&\le C d_0^2\|(\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3} + C\hbar^4\|\Delta u\|_{H^2}^2.
\end{aligned}
$$

By Hölder's and Young's inequalities, the other terms on the right-hand side of (3.35) can be bounded by

$$
\begin{aligned}
\sum_{i=7}^{9} R_{4,i} &\le \delta\hbar^2\|\Delta u\|_{H^3}^2 + C\hbar^2\|\nabla\theta\|_{H^3}^2 + C d_0^2\|(\rho,u,\nabla u,\nabla\theta,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}\\
&\quad + C\tau\|\hbar\nabla f^R\|_{H^2}^2.
\end{aligned}
$$

Hence, choosing sufficiently small positive comstants $\delta,\hbar$, we complete the proof. □

3.4. **Estimates for $\nabla\rho$.**

Lemma 3.7. *Under the conditions in lemma 3.3, for any positive constant $m<1$, we have*

$$
\begin{aligned}
& m^2\|(\rho,\nabla\rho,\hbar\Delta\rho)\|_{\dot H^2}^2 + m^2\frac{d}{dt}\int_{\Omega^R}\partial^2 u\partial^2\nabla\rho\\
&\le Cm^2(\|\epsilon^{1/2}\nabla\rho\|_{H^3}^2 + \|(\nabla\theta,\nabla u)\|_{H^2}^2) + C\tau\|\nabla f^R\|_{H^1}^2 + C\tau\|\nabla f^R\|_{H^1}^4\\
&\quad + C\tau d_0(\|\hbar\nabla\rho\|_{H^3}^2 + \|(\nabla\theta,\nabla u,\nabla\rho)\|_{H^2}^2),
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
& m^2\|(\hbar\rho,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{\dot H^3}^2 + m^2\hbar^2\frac{d}{dt}\int_{\Omega^R}\partial^3 u\partial^3\nabla\rho\\
&\le Cm^2\|(\nabla\theta,\nabla u,\hbar\Delta u,\hbar\epsilon^{1/2}\Delta\rho)\|_{H^3}^2 + C\tau\|\nabla f^R\|_{H^1}^2\\
&\quad + C\tau d_0\|(\nabla\rho,\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2.
\end{aligned}
\tag{3.38}
$$

Proof. Applying the operator ∂^2 to (3.13), then taking inner product with $\partial^2\nabla\rho$, we can obtain (3.37). The process is similar but much easier than the proof of the estimate (3.38). More precisely, applying the operator ∂^3 to (3.13), multiplying the resultant equation by $\hbar^2\partial^3\nabla\rho$, and using integration by parts, Sobolev embedding, Young's inequality and Hölder's inequality, we obtain

$$
\begin{aligned}
&\|(\hbar\rho,\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{\dot H^3}^2\\
&\le -\hbar^2\int_{\Omega^R}\partial^3 u_t\partial^3\nabla\rho + C\|(\nabla\theta,\hbar\Delta u)\|_{H^3}^2 + \delta\|(\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2\\
&\quad + C\tau d_0\|(\nabla\rho,\nabla u,\hbar\nabla\rho,\hbar^2\Delta\rho\|_{H^3}^2 + C\tau\|\nabla f^R\|_{H^1}^2,
\end{aligned}
$$

where δ is a sufficiently small positive constant. By integration by parts again, we have

$$
\begin{aligned}
&-\hbar^2\int_{\Omega^R}\partial^3 u_t\partial^3\nabla\rho\\
&=-\hbar^2\frac{d}{dt}\int_{\Omega^R}\partial^3 u\partial^3\nabla\rho-\hbar^2\int_{\Omega^R}\partial^3\operatorname{div} u\partial^3\rho_t\\
&=-\hbar^2\frac{d}{dt}\int_{\Omega^R}\partial^3 u\partial^3\nabla\rho+\hbar^2\int_{\Omega^R}\partial^3\operatorname{div} u\partial^3\big((1+\tau\rho)\operatorname{div} u-\epsilon\Delta\rho+\tau u\cdot\nabla\rho\big)\\
&\le-\hbar^2\frac{d}{dt}\int_{\Omega^R}\partial^3 u\partial^3\nabla\rho+C\|\operatorname{div} u\|_{H^3}^2+C\|\hbar\epsilon^{1/2}\Delta\rho\|_{H^3}^2+C\tau d_0\|(\nabla u,\hbar\nabla\rho)\|_{H^3}^2).
\end{aligned}
$$

Therefore, multiplying a suitably small positive constant m^2, we derive (3.38). □

3.5. **Proof of Theorem 3.1.** In what follows, we will use the above uniform (in ϵ, R) estimates to show the condition (3.2) of the Leray-Schauder degree theory is satisfied. Then, we establish the existence of time periodic solutions to the approximated system (2.1) in the bounded domain Ω^R by the topological degree theory. It is worth noting that the uniform (in ϵ, R) bounds of solutions to system (3.1) are needed especially for the proof of Theorem 1.1.

Proof. Adding Lemmas 3.3-3.7 up, and integrating the resultant inequality from 0 to T^*, we obtain, for some suitably small positive constant m,

$$
\begin{aligned}
&\int_0^{T^*}\Big(\|(\rho,u,\theta)\|_{H^3}^2+\|(\nabla\theta,\nabla u,\hbar\Delta u)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar^2\Delta\rho)\|_{H^3}^2\Big)dt\\
&+\epsilon\int_0^{T^*}\|(\rho,\nabla\rho,\hbar\Delta\rho,\hbar^2\nabla\Delta\rho)\|_{H^3}^2dt\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt\\
&\quad+C\tau d_0^2\Big(\int_0^{T^*}\|(\rho,\theta,u,\nabla u,\nabla\theta,\hbar\nabla\rho,\hbar\Delta u,\hbar^2\Delta\rho)\|_{H^3}^2dt\Big)^{1/2}\\
&\quad+C\tau d_0\int_0^{T^*}\|(\rho,\theta,u,\nabla u,\nabla\theta,\hbar\nabla\rho,\hbar\Delta u,\hbar^2\Delta\rho)\|_{H^2}^2dt\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3,
\end{aligned}
\tag{3.39}
$$

thanks to the choice of some suitably small positive constants $m,\delta,\hbar$ such that those terms without τ ahead disappear. It follows from the above inequality (3.39) that there exists some time $t_0\in(0,T^*)$ such that

$$
\begin{aligned}
&T^*\|(\rho,u,\theta)(t_0)\|_{H^3}^2+T^*\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t_0)\|_{H^3}^2\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3.
\end{aligned}
\tag{3.40}
$$

Combining the results (3.6), (3.12), (3.20) with (3.34), we derive

$$
\begin{aligned}
&\frac{d}{dt}\big(\|(\rho,\theta,u)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)\|_{H^3}^2\big)\\
&\le C\tau\|f^R\|_{H^2}^2+C\tau\hbar^2\|\nabla f^R\|_{H^2}^2\\
&\quad+C\tau d_0\big(\|(\rho,u,\theta)\|_{H^3}^2+\|(\nabla u,\nabla\theta,\hbar\nabla\rho,\hbar\Delta u,\hbar^2\Delta\rho)\|_{H^3}^2\big)\\
&\quad+C\tau d_0^2\big(\|(\rho,\theta,u)\|_{H^3}+\|(\nabla\theta,\nabla u,\hbar\nabla\rho,\hbar\Delta u,\hbar^2\Delta\rho)\|_{H^3}\big).
\end{aligned}\tag{3.41}
$$

Integrating (3.41) from t_0 to t ($t_0<t\le T^*$), then (3.40) guarantees the inequality

$$
\begin{aligned}
&\|(\rho,\theta,u)(t)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2\\
&\le\|(\rho,\theta,u)(t_0)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t_0)\|_{H^3}^2\\
&\quad+C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3.
\end{aligned}\tag{3.42}
$$

For the reason that (ρ,u,θ) is periodic in T^*, we have

$$
\begin{aligned}
&\|(\rho,\theta,u)(0)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(0)\|_{H^3}^2\\
&=\|(\rho,\theta,u)(T^*)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(T^*)\|_{H^3}^2\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3.
\end{aligned}\tag{3.43}
$$

Therefore,

$$
\begin{aligned}
&\sup_{t\in[0,T^*]}\big(\|(\rho,\theta,u)(t)\|_{H^3}^2+\|(\hbar\nabla\rho,\hbar\nabla u,\hbar^2\Delta\rho)(t)\|_{H^3}^2\big)\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3.
\end{aligned}\tag{3.44}
$$

Combining (3.39) with (3.44), and recalling the definition (2.5), we obtain

$$
\begin{aligned}
&|||(\rho,u,\theta)|||^2+\epsilon\int_0^{T^*}\|(\rho,\nabla\rho,\hbar\Delta\rho,\hbar^2\nabla\Delta\rho)\|_{H^3}^2dt\\
&\le C\tau\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt+C\tau d_0^3\le\frac{d_0^2}{2},
\end{aligned}\tag{3.45}
$$

in which we have used the assumption that d_0 and $\int_0^{T^*}\|(f^R,\hbar\nabla f^R)\|_{H^2}^2dt$ are sufficiently small such that $C\lambda^2+Cd_0^3\le d_0^2/2$. This prove (3.2). Recalling $\chi((\tilde\rho,\tilde u,\tilde\theta),0)=0$ in the proof of Lemma 2.3, we have

$$
\deg(I-\chi(\cdot,1),\hat B_{d_0}(0),0)=deg(I-\chi(\cdot,0),\hat B_{d_0}(0),0)=deg(I,\hat B_{d_0}(0),0)=1,
$$

where we have used the normality and invariance of compact homotopy properties of Leray-Schauder degree $deg(I-\chi(\cdot,\tau),\hat B_{d_0}(0),0)$. Hence, system (2.1) admits a time periodic solution $(\rho,u,\theta)\in X_{d_0}^R$. □

4. Proof of the main results

4.1. **Proof of Theorem 1.1.** In the following, we will prove the existence of time periodic solution in $\mathbb{R}^3$ by a limiting process. Before starting the technical part of the proof, we list an available Lemma 4.1 which has been proved in [11].

Lemma 4.1. *Let $\{f_l\}_{l=1}^{\infty}$ be a sequence of functions defined on a set Y. If there exists an exhausting sequence of subsets of Y satisfying $\cup_{i=1}^{\infty} Y_i = Y$, and for $i \geqslant 1$, $\{f_l^i\}_{l=1}^{\infty}$ is a uniformly convergent subsequence of $\{f_l^{i-1}\}_{l=1}^{\infty}$ in Y_i in which we assume $f_l^0 = f_l (l = 1, 2, 3, \cdots,)$, then we can obtain a subsequence $\{f_{l'}\}$ of $\{f_l\}$ such that $\{f_{l'}\}$ converges uniformly in Y.*

In the following, we let $Y = \mathbb{R}^3$ and $Y_i = \Omega^R$.

Lemma 4.2. *Assume $g \in L^\infty(0, T^*, H^2)$, $\partial_t g \in L^2(0, T^*, L^2)$, we can derive g in $C^{\frac{1}{8}, \frac{1}{2}}\big((0, T^*) \times \Omega^R\big)$, where $g \in C^{\frac{1}{8}, \frac{1}{2}}\big((0, T^*) \times \Omega^R\big)$ means for any function $g \in L^\infty\big((0, T^*) \times \Omega^R\big)$, it holds*

$$\sup_{(x,t)\neq(y,s)} \frac{|g(x, t) - g(y, s)|}{|x - y|^{1/2} + |t - s|^{1/8}} \leq C.$$

Proof. By the condition $g(t) \in H^2$ and Sobolev embedding, we have $g(t) \in C^{1/2}$, for any $t \in (0, T^*)$. In the following, we assume $0 < t_1 \leq t_2 < T^*$. Without loss of generality, we denote B_r by a ball centered at $x \in \Omega^R$ with radius $r = |t_2 - t_1|^a$. Using the fact $\partial_t g \in L^2(0, T^*, L^2)$, we obtain

$$\begin{aligned}
\int_{B_r} |g(x, t_1) - g(x, t_2)|dx &\leq \int_{B_r} \Big| \int_{t_1}^{t_2} \partial_t g dt \Big| dx \\
&\leq \Big(\int_{t_1}^{t_2} \int_{B_r} |\partial_t g|^2 dt \Big)^{1/2} \Big(\int_{t_1}^{t_2} \int_{B_r} 1^2 dt \Big)^{1/2} dx \\
&\leq C|t_1 - t_2|^{\frac{1}{2} + \frac{an}{2}},
\end{aligned} \tag{4.1}$$

which implies there exists a point $\tilde{x} \in B_r$, such that

$$|g(\tilde{x}, t_1) - g(\tilde{x}, t_2)| \leq C|t_1 - t_2|^{\frac{1}{2} - \frac{an}{2}}.$$

Hence, we deduce that for any $x \in \Omega^R$,

$$\begin{aligned}
|g(x, t_1) - g(x, t_2)| \leq& |g(x, t_1) - g(\tilde{x}, t_1)| + |g(\tilde{x}, t_1) - g(\tilde{x}, t_2)| + |g(\tilde{x}, t_2) - g(x, t_2)| \\
\leq& 2C|t_1 - t_2|^{\frac{a}{2}} + C|t_1 - t_2|^{\frac{1}{2} - \frac{an}{2}}.
\end{aligned}$$

Then we can choose $a = \frac{1}{4}$ such that $\frac{1}{2} - \frac{an}{2} = \frac{a}{2}$. That is to say,

$$|g(x, t_1) - g(x, t_2)| \leq C|t_1 - t_2|^{\frac{1}{8}}.$$

Therefore, we prove that, for any $x, y \in \Omega^R$ and $t, s \in (0, T^*)$,

$$|g(x, t) - g(y, s)| \leq C|t - s|^{\frac{1}{8}} + C|x - y|^{1/2}.$$

This completes the proof of Lemma 4.2. □

Proof of Theorem 1.1. To avoid any confusion, for any fixed R, we denote the solutions to the approximated system (2.1) by $(\rho^{R,\epsilon}, u^{R,\epsilon}, \theta^{R,\epsilon})$. Using Lemma 4.2 and the result (3.45), we deduce

$$\|(\rho^{R,\epsilon}, u^{R,\epsilon}, \theta^{R,\epsilon})\|_{C^{\frac{1}{8}, \frac{1}{2}}\left((0,T^*)\times\Omega^R\right)} \leq Cd_0.$$

Then there exists a subsequence $\{(\rho_n^{R,\epsilon}, u_n^{R,\epsilon}, \theta_n^{R,\epsilon})\}_{n=1}^{\infty}$ such that

$$\begin{aligned} (\rho_n^{R,\epsilon}, u_n^{R,\epsilon}, \theta_n^{R,\epsilon}) &\to (\rho^R, u^R, \theta^R) \quad \text{uniformly in } \Omega^R \text{ as } n \to \infty, \\ (\rho_n^{R,\epsilon}, u_n^{R,\epsilon}, \theta_n^{R,\epsilon}) &\to (\rho^R, u^R, \theta^R) \quad \text{strongly in } L^2(0,T^*; L^2(\Omega^R), \end{aligned}$$

where $(\rho^R, u^R, \theta^R) \in X_{d_0}^R$.

Letting $\epsilon \to 0$ ($n \to \infty$), we can conclude that the limit function $(\rho^R, u^R, \theta^R) \in X_{d_0}^R$. This completes the proof of the time periodic solutions to system (1.2) in the periodic domain Ω^R.

In what follows, we choose a subsequence R_k such that $R_k \to \infty$ ($k \to \infty$). Let $(\rho_n^k, u_n^k, \theta_n^k)$ be the convergent function sequence in Ω^{R_k}, and $(\rho_n^{k+1}, u_n^{k+1}, \theta_n^{k+1})$ be convergent function sequence in $\Omega^{R_{k+1}}$, which is also the subsequence of $(\rho_n^k, u_n^k, \theta_n^k)$. Repeating the process and using the diagonal argument, one can show that there exists a Cantor diagonal subsequence $(\rho_n^n, u_n^n, \theta_n^n)$ such that $(\rho_n^n, u_n^n, \theta_n^n) \to (\rho, u, \theta)$ as $n \to \infty$ in the whole space $\mathbb{R}^3$ based on Lemma 4.1 and the bound (3.45) (uniform in R). Hence, we extend the time periodic classical solution (ρ, u, θ) to system (1.2) to the whole space $\mathbb{R}^3$. Then the proof is complete. □

4.2. **Proof of Theorem 1.2.** Assume that (ρ_1, u_1, θ_1) and (ρ_2, u_2, θ_2) are the periodic solutions from Theorem 1.1. Let $q = \rho_1 - \rho_2,\ v = u_1 - u_2,\ \vartheta = \theta_1 - \theta_2, \phi = \psi_1 - \psi_2$. Then system (1.2) can be rewritten as

$$\partial_t q + \operatorname{div} v = -q \operatorname{div} u_1 - \rho_2 \operatorname{div} v - v \cdot \nabla \rho_1 - u_2 \cdot \nabla q, \tag{4.2a}$$

$$\begin{aligned}
&\partial_t v + \frac{v}{1+\rho_2} + \nabla \vartheta + \frac{1+\theta_2}{1+\rho_2}\nabla q - \frac{\hbar^2}{12}\frac{\nabla\Delta q}{\rho_2+1} \\
&- \frac{\mu}{3}\frac{3\Delta v + \nabla \operatorname{div} v}{\rho_2+1} - \nabla\phi \\
&= -\frac{\hbar^2}{12}\frac{q\nabla\Delta\rho_1}{(\rho_1+1)(\rho_2+1)} - v\cdot\nabla u_1 - u_2\cdot\nabla v - \big(\frac{1+\theta_1}{1+\rho_1} - \frac{1+\theta_2}{1+\rho_2}\big)\nabla\rho_1 \\
&- \frac{\hbar^2}{12}\Big(\frac{\nabla\rho_1}{(1+\rho_1)^2} - \frac{\nabla\rho_2}{(1+\rho_2)^2}\Big)(\Delta\rho_1 + \nabla^2\rho_1) + \frac{\hbar^2}{12}\frac{\nabla\rho_2}{(1+\rho_2)^2}\Delta q \\
&- \frac{\mu}{3}\frac{q(3\Delta v + \nabla\operatorname{div} v)}{(\rho_1+1)(\rho_2+1)} - \frac{\hbar^2}{12}\frac{|\nabla\rho_2|^2}{(1+\rho_2)^3}\nabla q \\
&+ \frac{\hbar^2}{12}\Big(\frac{|\nabla\rho_1|^2}{(1+\rho_1)^3} - \frac{|\nabla\rho_2|^2}{(1+\rho_2)^3}\Big)\nabla\rho_1 + \frac{(\rho_2-\rho_1)u_1}{(1+\rho_1)(1+\rho_2)},
\end{aligned}$$

$$\begin{aligned}
&\partial_t\vartheta + \vartheta + \frac{2}{3}\operatorname{div} v - \frac{2\kappa}{3}\frac{\Delta\vartheta}{\rho_2+1} - \frac{\hbar^2}{18}\operatorname{div}\Delta v - \frac{\hbar^2}{36}\frac{\Delta q}{1+\rho_2} \\
&= -v\cdot\nabla\theta_1 - u_2\cdot\nabla\vartheta - \frac{2}{3}\vartheta\operatorname{div} u_1 - \frac{2}{3}\theta_2\operatorname{div} v \\
&+ \frac{\hbar^2}{18}\frac{\nabla\rho_2}{\rho_2+1}\nabla\operatorname{div} v + \frac{\hbar^2}{18}\big(\frac{\nabla\rho_1}{\rho_1+1} - \frac{\nabla\rho_2}{\rho_2+1}\big)\nabla\operatorname{div} u_1 - \frac{2\kappa}{3}\frac{q\Delta\theta_1}{(\rho_1+1)(\rho_2+1)} \\
&+ \frac{2}{3}(\tilde{R}(u_1,\rho_1) - \tilde{R}(u_2,\rho_2)) + \frac{\hbar^2}{36}\frac{(\rho_2-\rho_1)\Delta\rho_1}{(1+\rho_1)(1+\rho_2)},
\end{aligned}$$

$$\Delta\phi = q, \tag{4.2b}$$

where

$$\tilde{R}(u,\rho)=\frac{\frac{\mu}{2}|\nabla u+\nabla u^{\top}|^2-\frac{2\mu}{3}(\operatorname{div} u)^2}{\rho+1}+\frac{\hbar^2}{24}\frac{|\nabla\rho|^2}{(1+\rho)^2}+\frac{1}{2}|u|^2.$$

Note that the energy estimates share similar arguments as those of the proof for the existence of time periodic solutions, provided that we assume $f^R=0$ and $d_0>0$ is sufficiently small. Hence, we derive

$$\int_0^{T^*}\Big(\|(q,\vartheta,v)\|_{H^3}^2+\|(\nabla v,\nabla\vartheta,\hbar\nabla q,\hbar\nabla v,\hbar^2\Delta q)\|_{H^3}^2\Big)dt\le 0, \tag{4.3}$$

which implies that $(\rho_1,u_1,\theta_1)=(\rho_2,u_2,\theta_2)$, then the proof of uniqueness is thus complete.

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