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BLOW-UP FOR NONLINEAR MAXWELL EQUATIONS

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Abstract. We construct classical solutions to the nonlinear Maxwell system with periodic boundary conditions which blow up in $H(\mathrm{curl})$. A similar result is shown on the full space. Our construction is based on an analysis of a shock wave in one space dimension.

1. Introduction and statement of main results

The Maxwell system

$$\partial_t D(t) = \operatorname{curl} H(t), \quad \partial_t H(t) = -\operatorname{curl} E(t), \tag{1.1}$$

$$\operatorname{div} D(t) = 0, \quad \operatorname{div} B(t) = 0 \tag{1.2}$$

is one of the fundamental equations of physics (which is still poorly understood analytically in the nonlinear case). One has to complement (1.1) and (1.2) by material laws that connect the electric fields E and D, as well as the magnetic ones B and H. We focus on materials without magnetic response (as appearing in optics) and look at the case $H = B$. Moreover, we only treat instantaneous material relations where D is given as a pointwise function of E. This class of problems fits to the theory of quasilinear hyperbolic systems, cf. [3, 6, 10, 14]. For the physical background we refer to, e.g., [5, 11]. In our analysis we concentrate on conservative systems such as (1.1) and (1.2) without conductivity and given currents or charges.

For rather general instantaneous nonlinear material laws, the Cauchy problem for (1.1) and (1.2) on $\mathbb{R}^3$ is locally well posed due to standard results on quasilinear hyperbolic systems if one works in $H^3(\mathbb{R}^3)$ (or $H^s(\mathbb{R}^3)$ for $s > 5/2$), see [2, 3, 10]. Nothing seems to be known in lower regularity. The situation on domains is more delicate since one typically has characteristic boundary conditions. Here the general theory only yields result in even higher regularity, see e.g. [8]. For the Maxwell equations itself partial results were established in [12]. Again on $\mathbb{R}^3$ one knows at least for certain nonlinearities of Kerr type (1.5) that small initial data in $H^k(\mathbb{R}^3)$ (for large $k \in \mathbb{N}$) lead to global smooth solutions, see [13].

On the other hand, shock solutions with a pointwise blow-up of the first derivative exist for a large class of nonlinearities, see [1, 4, 6, 9, 10] for the general theory. We note that these solutions do not belong to $L^2(\mathbb{R}^3)$. To rule out global existence

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in natural spaces like $H(\mathrm{curl}) \times \mathrm{H(curl)}$, our main aim is to find solutions whose curl blows up even in L^2.

In this paper, we construct a classical solution (E^0, B^0) whose curl blows up in L^2 for the Maxwell system on a cube with periodic or certain mixed boundary conditions. This solution is essentially given by a shock solution for a one-dimensional subproblem constructed by the methods in [10]. However, the compactly supported initial data have to be carefully chosen so that one can show the explosion of the first derivative of the solution in $L^2(\mathbb{R})$ in finite time. To transfer this result to $\mathbb{R}^3$, one has to localize (E^0, B^0) since these fields only depend on one space variable. The analysis of the localized solution requires a local uniqueness result for (1.1) and (1.2) which does not seem to be available in $H(\mathrm{curl}) \times \mathrm{H(curl)}$. As a result, we obtain blow-up on $\mathbb{R}^3$ only within a smaller class of solutions, see Theorem 1.4. Our results apply to a large class of nonlinearities described below, where we impose no, respectively very mild, assumptions on their behavior at infinity.

To use subproblems in one space dimension, we look at scalar type material laws $D = \Phi(E)E$ for a given function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ and thus on the nonlinear Maxwell system

$$\partial_t[\Phi(E(t))E(t)] = \operatorname{curl} B(t), \quad \partial_t B(t) = -\operatorname{curl} E(t), \tag{1.3}$$

$$\operatorname{div}[\Phi(E(t))E(t)] = 0, \quad \operatorname{div} B(t) = 0 \tag{1.4}$$

An important special case is the Kerr model with

$$\Phi(x) = 1 + a\,|x|^2 \tag{1.5}$$

for some $a > 0$, see [5] or [11]. In Examples 2.1 and 3.1 we treat more general versions of (1.5). The one-dimensional version of (1.3) reads as

$$\partial_t b(u) = \partial_x v, \quad \partial_t v = \partial_x u, \quad (u(0,x), v(0,x)) = (u_0(x), v_0(x)) \tag{1.6}$$

In (1.6) the function b has to satisfy the following hypotheses.

Assumption 1.1. The map b belongs to $C^2(\mathbb{R}, \mathbb{R})$, there are numbers $w_- < 0 < w_0 < w_+$ such that $b' > 0$ on $J := (w_-, w_+)$, the map $q \in C(J, \mathbb{R})$ given by

$$q(s) := \frac{b''(s)}{2b'(s)^{3/2}}, \quad s \in J,$$

has a global maximum at $s = w_0$, q is C^1 near w_0, and $q(s) > 0$ for $0 < s \le w_0$.

We point out that there are no assumptions on the behavior of b at infinity. In particular, b could become constant or linear. We now state our basic blow-up result about (1.6), which we prove in Section 2.

Theorem 1.2 (1D case)**.** *Let Assumption 1.1 be true. Then there exist compactly supported initial data* $(u_0, v_0) \in C^1(\mathbb{R}, \mathbb{R}^2)$ *and a* C^1 *solution* (u, v) *to Problem* (1.6) *on* $[0, t_*) \times \mathbb{R}$ *for some* $t_* \in (0, \infty)$ *which is compactly supported in* x *for each* t *and satisfies* $\|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})} \to +\infty$ *as* $t \to t_*^-$.

In the proof one actually sees that this blow-up occurs for a large class of initial functions. We mainly require that the slope of the 'electric part' attains its positive maximum at $x = 0$, see (2.10) and (2.12). The blow-up solutions in the following two results are modifications of those from Theorem 1.2.

In three space dimensions we first look at the cube $Q_M = [-M, M]^3$ for some $M > 0$ with outer unit normal ν. Besides periodic boundary condition we treat the mixed conditions (BC) given as

(i) $E \cdot \nu = 0$ and $B \times \nu = 0$ for $x_3 = \pm M$;
(ii) $E \times \nu = 0$ and $B \cdot \nu = 0$ on the rest of the boundary.

Here (ii) are the usual conditions for perfectly conducting boundaries, while (i) corresponds to a perfect magnetic conductor. Though there is no real material with the behavior in (i), it is used as a symmetry boundary condition in numerical computations.

Theorem 1.3 (3D boundary value problem). *Assume that $b(s) := \Phi(s,0,0)s$ for $s \in \mathbb{R}$ fulfills Assumption 1.1. Consider the Cauchy problem for* (1.3) *and* (1.4) *on the cube Q_M with the boundary conditions (BC) or with periodic boundary conditions, where $M > 0$ is sufficiently large. Then there exist initial data (E_0^0, B_0^0) in $C^1(Q_M, \mathbb{R}^6)$ satisfying* (1.4) *and the respective boundary conditions and a corresponding C^1 solution (E^0, B^0) on $[0, t_*) \times Q_M$ for some $t_* \in (0,\infty)$ such that $\|\operatorname{curl} E^0(t,\cdot)\|_{L^2(Q_M)} \to +\infty$ as $t \to t_*^-$.*

This and the following theorem are shown in Section 3. For our result on $\mathbb{R}^3$, we also assume that $\Phi(E) = \beta(|E|)$ for a function $\beta \in C^2([0,\infty),\mathbb{R})$ and let $b(s) := s\beta(|s|)$ for $s \in \mathbb{R}$. We define the symmetric matrix

$$A_0(y) = \beta(|y|)I_{3\times 3} + \beta'(|y|)|y|^{-1}yy^\top \quad \text{for } y \in \mathbb{R}^3 \setminus \{0\} \tag{1.7}$$

and $A_0(0) = \beta(0)I_{3\times 3}$. Our assumptions will imply that this matrix is positive definite with a uniform lower bound. For C^1 solutions the equation (1.3) can be equivalently rewritten as the symmetric hyperbolic system

$$A_0(E)\partial_t E = \operatorname{curl} B, \quad \partial_t B = -\operatorname{curl} E. \tag{1.8}$$

The space of functions $\varphi \in L^2(\mathbb{R}^3,\mathbb{R}^3)$ such that $\operatorname{curl}\varphi \in L^2(\mathbb{R}^3,\mathbb{R}^3)$ is called $H(\mathrm{curl})$. It is a Hilbert space when endowed with the natural norm.

Theorem 1.4 (3D problem on $\mathbb{R}^3$). *Let $\Phi(E) = \beta(|E|)$ for a function β in $C^2([0,\infty),\mathbb{R})$ such that $b(s) := \beta(|s|)s$ for $s \in \mathbb{R}$ fulfills Assumption 1.1, $b'(s) \geq \kappa$ and $s\beta'(s) \leq c\beta(s)$ for some $c, \kappa > 0$ and all $s \geq 0$. Consider the Cauchy problem for* (1.3) *and* (1.4) *on $\mathbb{R}^3$. There exist compactly supported initial data (E_0^c, B_0^c) in $C^1(\mathbb{R}^3,\mathbb{R}^6)$ satisfying* (1.4) *such that the corresponding local C^1 solution (E^c, B^c) can not be continued to a global solution of* (1.8) *in $C([0,\infty), H(\mathrm{curl}) \times H(\mathrm{curl}))$ such that* $\operatorname{curl} B$ *is bounded on $[0,T] \times \mathbb{R}^3$ for each $T > 0$.*

As we will see in Example 2.1 and 3.1 the Kerr model (1.5) fulfills the assumptions of the above theorem, even if we modify $\beta(s) = 1 + as^2$ to a constant function for large $s > 0$.

Straightforward modifications of our proofs yield the following generalizations. First, if we assume in addition that b belongs to $C^\infty(\mathbb{R},\mathbb{R})$, then one can replace in the above theorems C^1 by C^∞. Second, Theorem 1.4 is also true for functions Φ as in Theorem 1.3 such that the map $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$; $\Psi(x) = \Phi(x)x$, is an diffeomorphism and its derivative $D\Psi(x)$ is symmetric and uniformly positive definite for $x \in \mathbb{R}^3$. The extra conditions on b and β in Theorem 1.4 just ensure these properties of Ψ.

Remark 1.5. Let $\Phi(E) = \beta(|E|)E$ with a function $\beta \in C^2(\mathbb{R},\mathbb{R})$ as in Theorem 1.4. For $Q = \mathbb{R}^3$ or $Q = [-M,M]^3$ as in Theorem 1.3 we define $h(s) = \int_0^s \beta(\sqrt{r})\,\mathrm{d}r$ for $s \geq 0$ and the 'energy'

$$\mathcal{E}(E,B) = \int_Q \Big(\tfrac{1}{2}\,|B|^2 + \beta(|E|)|E|^2 - \tfrac{1}{2}\,h(|E|^2)\Big)\mathrm{d}x$$

for functions on Q such that each summand is integrable. Take maps (E,B) in $C^1([0,T]\times Q,\mathbb{R}^6)$ with sufficient decay at ∞ if $Q=\mathbb{R}^3$, say, which solve (1.3) and satisfy (BC) or periodic boundary conditions if $Q=[-M,M]^3$. It is then easy to check that $\mathcal{E}(E(t),B(t))$ is constant for $t\in[0,T]$. (See [14] for more general material laws.) For the Kerr model one has

$$\mathcal{E}(E,B)=\int_Q\Big(\tfrac12|E|^2+\tfrac34 a|E|^2+\tfrac12|B|^2\Big)\mathrm{d}x.$$

But, these conserved quantities are not strong enough to prevent the blow-up in $H(\mathrm{curl})$ stated in the theorems.

2. The one dimensional case, proof of Theorem 1.2

Let b satisfy Assumption 1.1. For C^1 solutions taking values in $J\times\mathbb{R}$, we can rewrite system (1.6) as

$$\partial_t\begin{pmatrix}u\\ v\end{pmatrix}+A(u,v)\partial_x\begin{pmatrix}u\\ v\end{pmatrix}=0\quad\text{with}\quad A(u,v)=\begin{pmatrix}0 & -b'(u)^{-1}\\ -1 & 0\end{pmatrix}\tag{2.1}$$

on $\mathbb{R}$. For $(u,v)\in J\times\mathbb{R}$, the matrix $A(u,v)$ has the eigenvalues and -vectors

$$\lambda_{1,2}(u,v)=\pm b'(u)^{-1/2},\quad \eta_{1,2}(u,v)=\begin{pmatrix}\mp1\\ b'(u)^{1/2}\end{pmatrix}.$$

These observations are a special case of the analysis in Section 3 of [1]. In the following we take $\lambda=\lambda_1$ and $\eta=\eta_1$ and drop the index 1. Using the construction in [10, Section 1.4], we first construct a bounded C^1 solution of (2.1) whose first derivative has a finite time blow-up in the sup-norm. The main step is to show that it even blows up in L^2 if one chooses the initial functions in the right way.

Fix $(\alpha,\beta)\in(w_0,w_+)\times\mathbb{R}$ such that

$$q(s)>0\quad\text{for } 0<s\le\alpha.$$

Observe that the interval $\alpha-J=(\alpha-w_+,\alpha-w_-)$ contains $[0,\alpha]$. The C^2 function $\phi:\alpha-J\to J\times\mathbb{R}$

$$\phi_1(s)=\alpha-s,\quad \phi_2(s)=\beta+\int_0^s b'(\alpha-\tau)^{1/2}\,\mathrm{d}\tau,$$

solves the ordinary differential equation

$$\phi'(s)=\eta(\phi(s)),\quad s\in\alpha-J,\quad \phi(0)=(\alpha,\beta).\tag{2.2}$$

For later use, we note the identities

$$\nabla\lambda(\phi(s))\cdot\phi'(s)=\nabla\lambda(\phi(s))\cdot\eta(\phi(s))=q(\alpha-s),\quad s\in\alpha-J.\tag{2.3}$$

Let $\sigma_0:\mathbb{R}\to[0,\alpha]$ be C^1 and equal to α outside a compact set. There is a unique C^1 solution σ of the scalar partial differential equation

$$\begin{gathered}\partial_t\sigma(t,x)+\lambda(\phi(\sigma(t,x)))\partial_x\sigma(t,x)=0,\quad t\ge0,\ x\in\mathbb{R},\\ \sigma(0,x)=\sigma_0(x),\quad x\in\mathbb{R},\end{gathered}\tag{2.4}$$

on a sufficiently small (bounded) time interval $[0,\bar t)$, where σ takes values in $\alpha-J$. See e.g. [10, Theorems 2.1 and 2.2]. We now define

$$\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix}=\phi(\sigma(t,x)).$$

It is easy to check that (u, v) is a C^1 solution of (2.1) on $[0, \bar{t}) \times \mathbb{R}$. We observe

$$\partial_x u = \phi_1'(\sigma)\partial_x \sigma = -\partial_x \sigma. \tag{2.5}$$

The methods of characteristics yields the implicit formula

$$\sigma(t, x) = \sigma_0(x - t\lambda(\phi(\sigma(t, x)))) = \sigma_0(y(t, x)), \tag{2.6}$$

where

$$y(t, x) := x - t\lambda(\phi(\sigma(t, x)) = x - t \cdot b'(\alpha - \sigma(t, x))^{-1/2}, \tag{2.7}$$

as long as

$$\begin{aligned} &1 + t\nabla\lambda(\phi(\sigma(t, x))) \cdot \eta(\phi(\sigma(t, x)))\sigma_0'(x - t\lambda(\phi(\sigma(t, x)))) \\ &= 1 + t\sigma_0'(x - t\lambda(\phi(\sigma(t, x))))q(\alpha - \sigma(t, x)) > 0, \end{aligned} \tag{2.8}$$

see e.g. [7, p.114], as well as (2.3). We now set

$$\gamma(t) := \inf_{x\in\mathbb{R}} \sigma_0'(y(t, x))q(\alpha - \sigma(t, x)) \quad \text{for} \quad t \in [0, \bar{t}).$$

Let $t_0 \in \mathbb{R}_+$ be the supremum of $t \in [0, \bar{t})$ such that $\tau\gamma(\tau) > -1$ for all $\tau \in [0, t]$. In the following, we take $t \in [0, t_0)$ so that the inequality (2.8) is valid for all $x \in \mathbb{R}$. Equations (2.6) and (2.3) then imply

$$\begin{aligned} \partial_x\sigma(t, x) &= \sigma_0'(x - t\lambda(\phi(\sigma(t, x))))\Big(1 - tq(\alpha - \sigma(t, x))\partial_x\sigma(t, x)\Big), \\ \partial_x\sigma(t, x) &= \frac{\sigma_0'(x - t\lambda(\phi(\sigma(t, x))))}{1 + tq(\alpha - \sigma(t, x))\sigma_0'(x - t\lambda(\phi(\sigma(t, x))))} \\ &= \frac{\sigma_0'(y(t, x))}{1 + tq(\alpha - \sigma(t, x))\sigma_0'(y(t, x))}. \end{aligned}$$

In particular, σ and $\partial_x\sigma$ are bounded on $[0, t_0 - \delta] \times \mathbb{R}$ for each $\delta \in (0, t_0]$. The blow-up condition in [10, Theorem 2.2 Annex] thus yields $\bar{t} = t_0$. From formula (2.6) we further deduce $\partial_x\sigma(t, x) = \sigma_0'(y(t, x))\partial_x y(t, x)$ and hence

$$\partial_x y(t, x) = \frac{1}{1 + tq(\alpha - \sigma(t, x))\sigma_0'(y(t, x))} > 0. \tag{2.9}$$

(In the case $\sigma_0'(y(t, x)) = 0$, we have $\partial_x\sigma(t, x) = 0$ and the identity $\partial_x y(t, x) = 1 > 0$ follows from (2.7).) Using also (2.7), we see that the map $x \mapsto y(t, x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$; let $y_t^{-1} : \mathbb{R} \to \mathbb{R}$ be its inverse. This fact and (2.6) lead to the equation

$$\gamma(t) = \inf_{z\in\mathbb{R}} \sigma_0'(z)q(\alpha - \sigma_0(z)) =: \gamma_0$$

for all $t < t_0$. We now fix a C^1 function $\sigma_0 : \mathbb{R} \to [0, \alpha]$ which is equal to α outside some compact set and satisfies

$$\sigma_0(0) = \alpha - w_0, \quad \sigma_0'(0) = \min_{z\in\mathbb{R}} \sigma_0'(z) < 0. \tag{2.10}$$

In view of Assumption 1.1, we can determine

$$\gamma_0 = \sigma_0'(0)q(w_0) \quad \text{and} \quad t_0 = -\frac{1}{\gamma_0}. \tag{2.11}$$

Substituting $z = y(t, x)$ and employing (2.9), we deduce from formula (2.6)

$$\begin{aligned} \|\partial_x\sigma(t, \cdot)\|_2^2 &= \int_{\mathbb{R}} |\partial_x\sigma(t, x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\sigma_0'(y(t, x))\partial_x y(t, x)|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \frac{|\sigma_0'(z)|^2}{1 + tq(\alpha - \sigma_0(z))\,\sigma_0'(z)} \,\mathrm{d}z. \end{aligned}$$

We shall employ the expansions

$$\sigma_0(z) = \alpha - w_0 + O(z), \quad \sigma_0'(z) = \sigma_0'(0) + o_+(z), \quad q(w) = q(w_0) - o_+(w - w_0)$$

where $o_+(z)$ denotes any nonnegative function with the property $o_+(z)/z \to 0$ as $z \to 0$. Here we used the assumptions that q has a global maximum at w_0 while σ_0' has a global minimum at 0. Hence, (2.11) yields

$$\begin{aligned} 1 + tq(\alpha - \sigma_0(z))\, \sigma_0'(z) &= 1 + t\gamma_0 + t[q(w_0)o_+(z) + o_+(z)\, |\sigma_0'(0)| - o_+(z^2)] \\ &= 1 + t\gamma_0 + to_+(z). \end{aligned}$$

By means of this equality, we arrive at

$$\|\partial_x \sigma(t, \cdot)\|_2^2 = \int_{\mathbb{R}} \frac{|\sigma_0'(z)|^2}{1 + t\gamma_0 + to_+(z)} \, \mathrm{d}z.$$

Fix a number $\delta_0 > 0$ such that

$$|\sigma_0'(z)|^2 \geq \frac{|\sigma_0'(0)|^2}{2} =: c_0 \quad \text{for all} \quad |z| \leq \delta_0.$$

For all $\epsilon > 0$ there exists a radius $\delta \in (0, \delta_0)$ such that

$$0 \leq o_+(z) \leq \epsilon\delta \quad \text{for all } |z| < \delta.$$

We can then estimate

$$\|\partial_x \sigma(t, \cdot)\|_2^2 \geq \int_{-\delta}^{\delta} \frac{c_0}{1 + t\gamma_0 + t\epsilon\delta} \, \mathrm{d}z = \frac{2c_0\delta}{1 + t\gamma_0 + t\epsilon\delta}.$$

Because of $t_0 = -1/\gamma_0$ in (2.11), it follows that

$$\liminf_{t \to t_0^-} \|\partial_x \sigma(t, \cdot)\|_2^2 \geq \frac{2c_0}{t_0\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, equation (2.5) finally implies that

$$\liminf_{t \to t_0^-} \|\partial_x u(t, \cdot)\|_2^2 = \liminf_{t \to t_0^-} \|\partial_x \sigma(t, \cdot)\|_2^2 = +\infty.$$

Note that

$$u_0(x) := u(0, x) = \alpha - \sigma_0(x) \tag{2.12}$$

is compactly supported by construction. On the other hand, we have

$$v_0(x) := v(0, x) = \beta + \int_0^\alpha b'(\tau)^{1/2} \, \mathrm{d}\tau$$

for all sufficiently large $|x|$. Thus, v_0 has compact support if we choose

$$\beta = -\int_0^\alpha b'(\tau)^{1/2} \, \mathrm{d}\tau.$$

This concludes the proof of Theorem 1.2.

Example 2.1. Let $\gamma > 2$ and $a > 0$. Set

$$w_0 = \Big(\frac{2(\gamma - 2)}{a\gamma(\gamma + 1)}\Big)^{\frac{1}{\gamma - 1}} > 0 \quad \text{and} \quad w_- = -\Big(\frac{1}{a\gamma}\Big)^{\frac{1}{\gamma - 1}} < 0.$$

Then each function $b \in C^2(\mathbb{R})$ which is equal to

$$b_0(s) = s + a\, |s|^\gamma \quad \text{for all} \quad s \in (w_-, w_+)$$

and some $w_+ > w_0$ satisfies Assumption 1.1. If we take $\gamma = m \in \mathbb{N}$, we can also replace b_0 by $b_1(s) = s + as^m$.

3. The three dimensional case, proofs of Theorems 1.3 and 1.4

We begin with the proof of Theorem 1.3. We use the solution (u,v) of the one dimensional problem (1.6) constructed in the previous section. Equations (2.6) and (2.8) imply that the supports of $u(t,\cdot)$ and $v(t,\cdot)$ are contained in an interval $(-M,M)$ for all times $0\le t<t_0$ with t_0 from (2.11), provided $M>0$ is chosen large enough. We then define

$$(E^0(t,x),B^0(t,x))=(u(t,x_2),0,0,0,0,v(t,x_2)). \tag{3.1}$$

It is easy to check that these functions solve (1.3) and (1.4) on $Q_M=[-M,M]^3$ and that they satisfy the boundary conditions (BC) as well as the periodic ones. We set $(E_0^0(x),B_0^0(x))=(u_0(x_2),0,0,0,0,v_0(x_2))$. Since we have $\|\operatorname{curl} E^0(t,\cdot)\|_{L^2(Q_M)}=2M\|\partial_{x_2}u(t,\cdot)\|_{L^2(-M,M)}$, the conclusion of Theorem 1.3 follows from Theorem 1.2.

We pass now to the proof of Theorem 1.4. The functions (E^0,B^0) defined in (3.1) solve (1.3) and (1.4) on the whole strip $[0,t_0)\times\mathbb{R}^3$; and the initial data have compact support in the variable x_2, but not in x_1 and x_3. We now modify the initial functions outside a compact set so that they become compactly supported and still satisfy the divergence conditions (1.4). To this end, let χ be a test functions which is equal to 1 on an interval $[-r,r]$, vanishes outside the interval $[-r_1,r_1]$ for some $r_1>r>0$ with $r_1>M$, and has the additional property that $\int_{-\infty}^{\infty}\chi(s)\,\mathrm{d}s=0$. The support of

$$X(s):=\int_{-r_1}^{s}\chi(\sigma)\,\mathrm{d}\sigma,\quad s\in\mathbb{R},$$

is thus contained in $[-r_1,r_1]$. We next define the new field

$$D(t,x)=\Phi(E(t,x))E(t,x)=\beta(|E(t,x)|)E(t,x) \tag{3.2}$$

for any given E so that (1.4) reduces to

$$\operatorname{div} D=0,\quad \operatorname{div} B=0.$$

Since $b(0)=0$ and $b'(s)\ge\kappa>0$ for $s\ge0$, we have $b(s)=s\beta(s)\ge\kappa s$ and so $\beta(s)\ge\kappa$ for all $s\ge0$. The formula (3.2) then yields $|E(t,x)|=b^{-1}(|D(t,x)|)$ for all $(t,x)\in[0,t_0)\times\mathbb{R}$. Hence, the transformation in (3.2) has the inverse

$$E(t,x)=\frac{1}{\beta(b^{-1}(|D(t,x)|))}\,D(t,x)=:\psi(|D(t,x)|)\,D(t,x),$$

where $\psi:[0,b(w_+)]\to[0,w_+]$ is C^2.

Let $D_0^0(x)=\Phi(E_0^0(x))E_0^0(x)=\beta(|u_0(x_2)|)(u_0(x_2),0,0)$. We now introduce the functions $D_0^c,B_0^c:\mathbb{R}^3\to\mathbb{R}^3$ by

$$D_0^c(x)=\begin{bmatrix}\beta(|u_0(x_2)|)u_0(x_2)\chi(x_1)\chi(x_3)\\ 0\\ -\beta(|u_0(x_2)|)u_0(x_2)\chi'(x_1)X(x_3)\end{bmatrix},\quad B_0^c(x)=\begin{bmatrix}-v_0(x_2)X(x_1)\chi'(x_3)\\ 0\\ v_0(x_2)\chi(x_1)\chi(x_3)\end{bmatrix}.$$

Note that $D_0^c(x)$ and $B_0^c(x)$ vanish if $x\notin[-r_1,r_1]^3$. Moreover, for $x\in[-r,r]^3$ the identities $D_0^c(x)=D_0^0(x)$ and $B_0^c(x)=B_0^0(x)$ follow from $\chi=1$ and $\chi'=0$ on $[-a,a]$. In addition, on $\mathbb{R}^3$ we can easily compute

$$\operatorname{div} D_0^c=\operatorname{div} B_0^c=0.$$

We then define the initial field $E_0^c:\mathbb{R}^3\to\mathbb{R}^3$

$$E_0^c(x)=\psi(|D_0^c(x)|)D_0^c(x).$$

also supported in $[-r_1,r_1]^3$, which coincides with E_0^0 on $[-r,r]^3$.

[10, Theorem 2.1] yields a local in time C^1 solution (E^c, B^c) of (1.8) having the initial values (E_0^c, B_0^c). It satisfies the divergence conditions (1.4) for all t because of (1.3) and $\operatorname{div} D_0^c = \operatorname{div} B_0^c = 0$. We suppose that Theorem 1.4 was wrong. Then (E^c, B^c) can actually be extended to a global solution of (1.8) such that the functions $E^c(t,\cdot)$ and $B^c(t,\cdot)$ belong to $H(\mathrm{curl})$ and $\operatorname{curl} B^c(t,\cdot)$ to $L^\infty(\mathbb{R}^3)$ for each $t \geq 0$, and they are locally bounded in the respective spaces. We introduce the maps

$$E' = E^0 - E^c \quad \text{and} \quad B' = B^0 - B^c \quad \text{on } [0,t_0) \times \mathbb{R}^3. \tag{3.3}$$

Our assumptions yield that $b'(s) = \beta(s) + s\beta'(s) \geq \kappa > 0$ and $\beta(s) \geq \kappa > 0$ for all $s \geq 0$. Therefore the matrix $A_0(y)$ from (1.7) has the inverse

$$A_0(y)^{-1} = \frac{1}{\beta(|y|)} I_{3\times 3} - \frac{\beta'(|y|)}{|y|\,\beta(|y|)^2 + |y|^2\,\beta(|y|)\beta'(|y|)}\, yy^\top \tag{3.4}$$

for each $y \in \mathbb{R}^3 \setminus \{0\}$, and $A_0(0)^{-1} = \beta(0)^{-1} I_{3\times 3}$. These inverses are uniformly bounded since $\beta(s) \geq \kappa$ and $s\beta'(s) \leq c\beta(s)$ for all $s \geq 0$. As a result, the positive definite symmetric matrices $a_0(t,x) := A_0(E^0(t,x))$ and $a_c(t,x) := A_0(E^c(t,x))$ have a uniform lower bound $\eta \in (0,1]$ for $(t,x) \in [0,t_0) \times \mathbb{R}^3$. Using further that $\partial_t E^c = a_c^{-1} \operatorname{curl} B^c$ by (1.8) and that $E^c(0,\cdot)$ belongs to $L^\infty(\mathbb{R}^3)$, we deduce the boundedness of the functions $\partial_t E^c$ and E^c on $[0,t_0] \times \mathbb{R}^3$. It follows that a_c and a_0 are bounded on $[0,t_0) \times \mathbb{R}^3$, and $\partial_t a_0(t,x)$ on $[0,t_0-\delta] \times \mathbb{R}^3$ for each $\delta \in (0,t_0)$.

In view of (1.8), the function (E', B') satisfies the equation

$$\begin{gathered} a_0 \partial_t E' = \operatorname{curl} B' + (a_c - a_0) a_c^{-1} \operatorname{curl} B^c = \operatorname{curl} B' + ZE', \\ \partial_t B' = -\operatorname{curl} E', \end{gathered} \tag{3.5}$$

for the zero order term

$$ZE' := \int_0^1 DA_0(E^0 + s(E^c - E^0))[E', a_c^{-1} \operatorname{curl} B^c]\,\mathrm{d}s.$$

Formula (1.7) implies that the derivative $DA_0(y)$, $y \neq 0$, is bounded on bounded subsets. This fact and the above observations show that M is a uniformly bounded matrix function on $[0,t_0) \times \mathbb{R}^3$.

Recall that $E'(0,\cdot) = B'(0,\cdot) = 0$ on $[-r,r]^3$. Take now $r \geq \sqrt{3}M + \sqrt{6}t_0/\eta$. By the properties of a_0 and Z, we can apply local uniqueness results for the linear system (3.5), cf. [2, Theorem 4.11] and its proof, deducing that $E' = B' = 0$ and hence $E^c = E^0$ on the truncated cone $C = \{(t,x) : 0 \leq t < t_0, |x| \leq r - \sqrt{6}t/\eta\}$. Since the cuboid $[0,t_0) \times [-M,M]^3$ is contained in C, we see that $\operatorname{curl} E^c(t,\cdot)$ blows up in $L^2(\mathbb{R}^3)$ as $t \to t_0$. This contradicts the assumption made before (3.3), so that Theorem 1.4 is true.

Example 3.1. As in Example 2.1, let $\gamma > 2$ and $a > 0$. Set $\beta_1(s) = 1 + as^{\gamma-1}$ for $s \geq 0$. Alternatively, we modify β_1 to a function $\beta_2 \in C^2([0,\infty),\mathbb{R})$ with $\beta_2' \geq 0$ which is constant on $[w_2,\infty)$ for some $w_2 > w_0$, where $w_0 > 0$ is given by Example 2.1. Define $b_j(s) = s\beta_j(s)$ for $s \geq 0$. These functions satisfy the assumptions of Theorem 1.4.

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