Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 284, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

REGULARITY LIFTING RESULT FOR AN INTEGRAL SYSTEM INVOLVING RIESZ POTENTIALS

YAYUN LI, DEYUN XU

Abstract. In this article, we study the integral system involving the Riesz potentials

$$u(x) = \sqrt{p}\int_{\mathbb{R}^n} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-\alpha}}, \quad u > 0 \text{ in } \mathbb{R}^n,$$
$$v(x) = \sqrt{p}\int_{\mathbb{R}^n} \frac{u^{p}(y)dy}{|x-y|^{n-\alpha}} \quad v > 0 \text{ in } \mathbb{R}^n,$$

where $n \geq 1$, $0 < \alpha < n$ and $p > 1$. Such a system is related to the study of a static Hartree equation and the Hardy-Littlewood-Sobolev inequality. We investigate the regularity of positive solutions and prove that some integrable solutions belong to $C^1(\mathbb{R}^n)$. An essential regularity lifting lemma comes into play, which was established by Chen, Li and Ma [20].

1. Introduction

Recently, many authors have studied the stationary Hartree type equation

$$(-\Delta)^{\alpha/2}u = pu^{p-1}(|x|^{\alpha-n} * u^p), \quad u > 0 \text{ in } \mathbb{R}^n, \tag{1.1}$$

where $n \geq 1$, $\alpha \in (0, n)$ and $p > 1$.

When $\alpha = 2$, (1.1) is a simplified model of the Maxwell-Schrödinger system (cf. [1, 3, 10] and references therein). It is also [4, Example 3.2.8]. A more general form is the Choquard type equation in the papers [13, 21]. Paper [8] studied the existence and the regularity results of positive solutions of the static Schrödinger equation with the fractional Laplacian. Another interesting work related to (1.1) are paper [11] and the references therein. Equation (1.1) is also helpful in understanding the blowing up or the global existence and scattering of the solutions of the dynamic Hartree equation (cf. [16]), which arises in the study of boson stars and other physical phenomena, and also appears as a continuous-limit model for mesoscopic molecular structures in chemistry. Such an equation also arises in the Hartree-Fock theory of the nonlinear Schrödinger equations (cf. [18]). More related mathematical and physical background can be found in [9, 12, 22].

2010 *Mathematics Subject Classification.* 35J10, 35Q55, 45E10, 45G05.
Key words and phrases. Riesz potential; integral system; regularity lifting lemma; Hartree equation; Hardy-Littlewood-Sobolev inequality.

Submitted May 16, 2017. Published November 14, 2017.

Since (1.1) has a convolution term, it seems difficult to investigate the existence directly. Write

$$v(x) = \sqrt{p} \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}.$$

Then $v > 0$ in $\mathbb{R}^n$. As in [14, 15, 21], we introduce an integral system

$$\begin{gathered} u(x) = \sqrt{p} \int_{\mathbb{R}^n} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-\alpha}}, \quad u > 0 \text{ in } \mathbb{R}^n, \\ v(x) = \sqrt{p} \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}, \quad v > 0 \text{ in } \mathbb{R}^n. \end{gathered} \tag{1.2}$$

According to the results in [6], we can also see that the equivalence between (1.1) and (1.2) if omitting constants.

In addition, (1.2) is analogous to the system

$$\begin{gathered} u(x) = \int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}}, \quad u, v > 0 \text{ in } \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}, \quad p, q > 0. \end{gathered} \tag{1.3}$$

It is the Euler-Lagrange equations which the extremal functions of the following Hardy-Littlewood-Sobolev inequality satisfies

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta}\, dx\, dy \le C_{\alpha,\beta,s,\lambda,n} \|f\|_r \|g\|_s,$$

where $1 < s, r < \infty$, $0 < \lambda < n$, $\lambda \le \overline{\lambda} = \lambda + \alpha + \beta \le n$, $\frac{1}{r} + \frac{1}{s} + \frac{\overline{\lambda}}{n} = 2$, $\frac{\alpha}{n} < 1 - \frac{1}{r} < \frac{\lambda+\alpha}{n}$, $\frac{\beta}{n} < 1 - \frac{1}{s} < \frac{\lambda+\beta}{n}$. Some classical work can be found in [2, 5, 7, 17] and many other papers.

The main conclusions of this paper are stated as follows, which are proved in section 2.

Theorem 1.1. *Let $n \ge 1$ and $0 < \alpha < n$. If $1 < p \le \frac{n}{n-\alpha}$, (1.2) does not have any positive solution.*

Theorem 1.2. *Assume u is a positive solution of (1.2) and $1 < \alpha < n$. If $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n)$, then $u \in C^1(\mathbb{R}^n)$.*

To prove Theorem 1.2, we need a regularity lifting lemma in [5] which was established by Chen, Li and Ma [20]. This powerful technique was successfully applied to obtain the Lipschitz continuity of positive solutions of integral systems involving the Riesz potential, Bessel potential and the Wolff potential (cf. [13, 20, 25]). In particular, those regularity properties of (1.3) are helpful to understand well the shape of the extremal functions of the Hardy-Littlewood-Sobolev inequality.

Let V be a function space equipped with two norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Define

$$X = \{v \in V : \|v\|_X < \infty\}, \quad Y = \{v \in V : \|v\|_Y < \infty\}.$$

Assume that spaces X and Y are complete under the corresponding norms and the convergence in X or in Y implies the convergence in V.

From [5, Theorem 3.3.5 and Remark 3.3.5], we have the following regularity lifting lemma.

Lemma 1.3. *Let $X = L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ and $Y = C^{0,1}(\mathbb{R}^n) \times C^{0,1}(\mathbb{R}^n)$ with the norms*

$$\|(f,g)\|_X = \|f\|_\infty + \|g\|_\infty, \quad \text{and} \quad \|(f,g)\|_Y = \|f\|_{0,1} + \|g\|_{0,1}.$$

Define their closed subset

$$X_1 = \{(f,g) \in X; \|f\|_\infty + \|g\|_\infty \le C(\|u\|_\infty + \|v\|_\infty)\},$$
$$Y_1 = \{(f,g) \in Y; \|f\|_\infty + \|g\|_\infty \le C(\|u\|_\infty + \|v\|_\infty)\}.$$

Assume

(i) *T is a contraction map from $X_1 \to X$;*
(ii) *T is a shrinking map from $Y_1 \to Y$;*
(iii) *$(F,G) \in X_1 \cap Y_1$;*
(iv) *$T(\cdot,\cdot) + (F,G)$ is a map from $X_1 \cap Y_1$ to itself.*

If $(u,v) \in X$ is a pair of solutions of the operator equation $(f,g) = T(f,g) + (F,G)$, then $(u,v) \in Y$.

2. Proof of main results

Theorem 2.1. *If $1 < p \le n/(n-\alpha)$, then there is no positive solution of* (1.2).

Proof. If u, v are positive solutions, we can deduce a contradiction by the ideas in [2]. Clearly,

$$u(x) \ge c\int_{B_R(0)} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-\alpha}} \ge \frac{c}{(R+|x|)^{n-\alpha}} \int_{B_R(0)} u^{p-1}(y)v(y)dy. \tag{2.1}$$

Therefore,

$$\begin{aligned}\int_{B_R(0)} u^p(x)dx &\ge c\int_{B_R(0)} \frac{dx}{(R+|x|)^{p(n-\alpha)}} (\int_{B_R(0)} u^{p-1}(y)v(y)dy)^p \\ &\ge \frac{c}{R^{p(n-\alpha)-n}} (\int_{B_R(0)} u^{p-1}(y)v(y)dy)^p.\end{aligned} \tag{2.2}$$

Here c is independent of R. Similarly, from

$$v(x) \ge \frac{c}{(R+|x|)^{n-\alpha}} \int_{B_R(0)} u^p(y)dy, \tag{2.3}$$

and (2.1), (2.2), we deduce

$$\begin{aligned}\int_{B_R(0)} u^{p-1}(x)v(x)dx &\ge \int_{B_R(0)} \frac{cu^{p-1}(x)dx}{(R+|x|)^{n-\alpha}} \int_{B_R(0)} u^p(y)dy \\ &\ge \frac{c}{R^{2[p(n-\alpha)-n]}} (\int_{B_R(0)} u^{p-1}(y)v(y)dy)^p,\end{aligned}$$

which implies

$$\int_{B_R(0)} u^{p-1}(x)v(x)dx \le CR^{2[p(n-\alpha)-n]/(p-1)}. \tag{2.4}$$

If $1 < p < n/(n-\alpha)$, then (2.4) with $R \to \infty$ leads to $\|u^{p-1}v\|_{L^1(\mathbb{R}^n)} = 0$. This contradicts with $u^{p-1}v > 0$.

If $p = n/(n-\alpha)$, then (2.4) implies $u^{p-1}v \in L^1(\mathbb{R}^n)$ if we let $R \to \infty$. Multiplying (2.3) by u^{p-1} and integrating on $A_R := B_R(0) \setminus B_{R/2}(0)$, we still have

$$\int_{A_R} u^{p-1}(x)v(x)dx \geq c(\int_{B_R(0)} u^{p-1}(y)v(y)dy)^p.$$

Letting $R \to \infty$ and noting $u^{p-1}v \in L^1(\mathbb{R}^n)$, we obtain $\|u^{p-1}v\|_{L^1(\mathbb{R}^n)} = 0$. It is also impossible. □

Note that Theorem 2.1 implies

$$p > \frac{n}{n-\alpha} \tag{2.5}$$

which is the necessary condition of the existence of positive solutions for (1.2).

Theorem 2.2. *Assume u is a positive solution of (1.2) with $\alpha \in (1, n)$. If $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n)$, then $u \in C^1(\mathbb{R}^n)$.*

Proof. **Step 1.** By [24, Lemmas 2.3 and 2.4], we know that u, v are bounded.

Step 2. Moreover, we claim that $u, v \in C^{0,1}(\mathbb{R}^n)$. We use the regularity lifting lemma (Lemma 1.3) to prove this claim. Let $X = L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ and $Y = C^{0,1}(\mathbb{R}^n) \times C^{0,1}(\mathbb{R}^n)$ with the norms

$$\|(f,g)\|_X = \|f\|_\infty + \|g\|_\infty\,,$$
$$\|(f,g)\|_Y = \|f\|_{0,1} + \|g\|_{0,1}.$$

Define their closed subset

$$X_1 = \{(f,g) \in X; \|f\|_\infty + \|g\|_\infty \leq C(\|u\|_\infty + \|v\|_\infty)\},$$
$$Y_1 = \{(f,g) \in Y; \|f\|_\infty + \|g\|_\infty \leq C(\|u\|_\infty + \|v\|_\infty)\}.$$

Let $d > 0$. Set

$$T_1(f,g) = \sqrt{p}\int_{B_d(x)} \frac{f^{p-1}(y)g(y)dy}{|x-y|^{n-\alpha}},$$
$$T_2(f) = \sqrt{p}\int_{B_d(x)} \frac{f^p(y)dy}{|x-y|^{n-\alpha}},$$
$$F(x) = \sqrt{p}\int_{\mathbb{R}^n \setminus B_d(x)} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-\alpha}},$$
$$G(x) = \sqrt{p}\int_{\mathbb{R}^n \setminus B_d(x)} \frac{u^p(y)dy}{|x-y|^{n-\alpha}},$$

and $T(f,g) = (T_1(f,g), T_2(f))$. Then (u,v) solves the operator equation

$$(f,g) = T(f,g) + (F,G).$$

Claim 1. T is a contracting map from X_1 to X. In fact, for two functions $(f_1,g_1), (f_2,g_2) \in X_1$, we deduce that

$$\begin{aligned}
&\|T_1(f_1,g_1) - T_1(f_2,g_2)\|_\infty \\
&\leq C(\|\int_{B_d(x)} \frac{|g_1(f_1^{p-1} - f_2^{p-1})|}{|x-y|^{n-\alpha}} dy\|_\infty
\end{aligned}$$

$$+\|\int_{B_d(x)}\frac{|(g_1-g_2)f_2^{p-1}|}{|x-y|^{n-\alpha}}dy\|_\infty).$$

By the mean value theorem and noting the definition of X_1, we obtain

$$\begin{aligned}&\|T_1(f_1,g_1)-T_1(f_2,g_2)\|_\infty\\&\le Cd^\alpha(\|u\|_\infty+\|v\|_\infty)^{p-1}[\|g_1-g_2\|_\infty+\|f_1-f_2\|_\infty].\end{aligned}$$

Similarly, we obtain

$$\|T_2(f_1)-T_2(f_2)\|_\infty\le Cd^\alpha(\|u\|_\infty+\|v\|_\infty)^{p-1}\|f_1-f_2\|_\infty.$$

Choose d sufficiently small such that $C(\|u\|_\infty+\|v\|_\infty)^{p-1}d^\alpha<1$, then T is a contracting map.

Claim 2. T is a shrinking map from Y_1 to Y. In fact, for $(f,g)\in Y_1$ and for any $x_1,x_2\in\mathbb{R}^n$, we have

$$\begin{aligned}&|T_1(f,g)(x_1)-T_1(f,g)(x_2)|\\&\le C|\int_{B_d(0)}|y|^{\alpha-n}((gf^{p-1})(y+x_1)-(gf^{p-1})(y+x_2))dy|\\&\le Cd^\alpha(\|u\|_\infty+\|v\|_\infty)^{p-1}(\|f\|_{0,1}+\|g\|_{0,1})|x_1-x_2|.\end{aligned}\tag{2.6}$$

Choosing d sufficiently small, we have

$$\frac{|T_1(f,g)(x_1)-T_1(f,g)(x_2)|}{|x_1-x_2|}\le\frac{1}{3}(\|f\|_{0,1}+\|g\|_{0,1}).$$

Similarly, we deduce that

$$\frac{|T_2(f)(x_1)-T_2(f)(x_2)|}{|x_1-x_2|}\le Cd^\alpha(\|u\|_\infty+\|v\|_\infty)^{p-1}\|f\|_{0,1}\le\frac{1}{3}\|f\|_{0,1}.$$

Thus, T is a shrinking map.

Claim 3. $(F,G)\in X_1\cap Y_1$. First, (1.2) and the definitions of F and G imply $F\le u$ and $G\le v$. So $(F,G)\in X_1$.

Next, for any $x_1,x_2\in\mathbb{R}^n$ satisfying $|x_1-x_2|:=\delta<d/3$, we have

$$\begin{aligned}&|F(x_2)-F(x_1)|/\sqrt{p}\\&\le\int_{\mathbb{R}^n\setminus B_d(x_1)}||x_2-y|^{\alpha-n}-|x_1-y|^{\alpha-n}|u^{p-1}(y)v(y)dy\\&\quad+\int_{B_d(x_1)\setminus B_{d-\delta}(x_1)}|x_2-y|^{\alpha-n}u^{p-1}(y)v(y)dy\\&:=I_1+I_2.\end{aligned}$$

Using the mean value theorem and the integrability, we obtain

$$I_1\le C\|u\|_s^{p-1}\|v\|_\infty\Big(\int_d^\infty r^{n-t(n-\alpha+1)}\frac{dr}{r}\Big)^{1/t}|x_1-x_2|\le C\delta,$$

where $\frac{p-1}{s}+\frac{1}{t}=1$ with $s=\frac{n+\epsilon}{n-\alpha}$. Here $\epsilon>0$ is suitably small such that $n<t(n-\alpha+1)$. On the other hand,

$$I_2\le C\|u\|_\infty^{p-1}\|v\|_\infty\int_{B_d(x_1)\setminus B_{d-\delta}(x_1)}|x_2-y|^{\alpha-n}dy\le C\delta.$$

Combining the estimates of I_1 and I_2, we see $F\in C^{0,1}(\mathbb{R}^n)$.

Finally, we prove $G \in C^{0,1}(\mathbb{R}^n)$. Interchanging the order of integration, we obtain

$$\begin{aligned} G(x) &= \sqrt{p}(n-\alpha)\Big(\int_d^1 \frac{\int_{B_t(x)} u^p(y)dy}{t^{n-\alpha}}\frac{dt}{t} + \int_1^\infty \frac{\int_{B_t(x)} u^p(y)dy}{t^{n-\alpha}}\frac{dt}{t}\Big) \\ &:= \sqrt{p}(n-\alpha)[G_1(x)+G_2(x)]. \end{aligned}$$

For any $x_1, x_2 \in \mathbb{R}^n$ satisfying $|x_1 - x_2| := \delta < 1/3$, by scaling we obtain

$$G_2(x_2) \le \int_1^\infty \frac{\int_{B_{t+\delta}(x_1)} u^p(y)dy}{t^{n-\alpha}}\frac{dt}{t} \le G_2(x_1)(1+\delta)^{n-\alpha+1}.$$

Therefore, $|G_2(x_2) - G_2(x_1)| \le G_2(x_1)[(1+\delta)^{n-\alpha+1} - 1] \le C\delta$. In addition,

$$|G_1(x_2) - G_1(x_1)| \le C\int_d^1 \frac{\int_{B_{t+\delta}(x_1)\setminus B_t(x_1)} u^p(y)dy}{t^{n-\alpha}}\frac{dt}{t} \le C\|u\|_\infty^p \delta.$$

Thus, we deduce $G \in C^{0,1}(\mathbb{R}^n)$. Hence, $(F, G) \in Y$. Claim 3 is complete.

Claim 4. $T(\cdot,\cdot)+(F,G)$ is a map from $X_1 \cap Y_1$ to itself. In fact, for $(f,g) \in X_1 \cap Y_1$,

$$\begin{aligned} \|T(f,g)\|_\infty &= \|T_1(f,g)\|_\infty + \|T_2(f)\|_\infty \\ &\le C(\|u\|_\infty + \|v\|_\infty)^p d^\alpha. \end{aligned} \tag{2.7}$$

Similar to (2.6), we have

$$\|T(f,g)\|_{0,1} = \|T_1(f,g)\|_{0,1} + \|T_2(f)\|_{0,1} \le C.$$

Thus, $T(f,g) \in X \cap Y$.

In addition, (2.7) implies $\|T(f,g)\|_\infty \le \|u\|_\infty + \|v\|_\infty$ as long as d is chosen suitably small. Thus,

$$\|T(f,g) + (F,G)\|_\infty \le \|T(f,g)\|_\infty + \|(F,G)\|_\infty \le C(\|u\|_\infty + \|v\|_\infty).$$

Claim 4 is verified.

Since (u,v) solves $(f,g) = T(f,g) + (F,G)$, claims 1-4 lead to $u, v \in C^{0,1}(\mathbb{R}^n)$ by Lemma 1.3.

Step 3. We claim that $u \in C^1(\mathbb{R}^n)$. We use the classical potential estimation to verify $u \in C^1(\mathbb{R}^n)$ and ∇u can be expressed formally as

$$\nabla u(x) = (\alpha - n)\int_{\mathbb{R}^n} u^{p-1}(y)v(y)\frac{x-y}{|x-y|^{n-\alpha+2}}dy. \tag{2.8}$$

Write

$$J_1 = (\alpha - n)\int_{\mathbb{R}^n\setminus B_d(x)} u^{p-1}(y)v(y)\frac{x-y}{|x-y|^{n-\alpha+2}}dy$$

$$J_2 = \int_{B_d(x)\setminus B_\varepsilon(x)} u^{p-1}(y)v(y)\nabla(|x-y|^{\alpha-n})dy.$$

We claim that the improper integral J_1 converges uniformly about x. In fact,

$$\begin{aligned} |J_1| &\le C\int_{\mathbb{R}^n\setminus B_d(x)} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-\alpha+1}} \\ &\le C\|u\|_s^{p-1}\|v\|_\infty\Big(\int_d^\infty \rho^{n-(n-\alpha+1)t}\frac{d\rho}{\rho}\Big)^{1/t}, \end{aligned}$$

where $\frac{p-1}{s} + \frac{1}{t} = 1$. Let $s = \frac{n+\delta}{n-\alpha}$. Here $\delta > 0$ is sufficiently small such that $\frac{1}{t} < \frac{n-\alpha+1}{n}$. Thus, from the integrability it follows $J_1 < \infty$.

Clearly,

$$
\begin{aligned}
|J_2| \le & \int_{B_d(x)\setminus B_\varepsilon(x)} \frac{|u^{p-1}(y)v(y) - u^{p-1}(x)v(x)|}{|x-y|^{n-\alpha+1}} dy \\
& + u^{p-1}(x)v(x)| \int_{B_d(x)\setminus B_\varepsilon(x)} \nabla(|x-y|^{\alpha-n}) dy| \\
:= & J_{21} + J_{22}.
\end{aligned}
$$

In view of $u, v \in C^{0,1}(\mathbb{R}^n)$,

$$
J_{21} \le C(\|u^{p-1}\|_\infty \|v\|_{0,1} + \|u^{p-2}\|_\infty \|v\|_\infty \|u\|_{0,1}) \int_{B_d(x)\setminus B_\varepsilon(x)} \frac{|x-y| dy}{|x-y|^{n-\alpha+1}} < \infty.
$$

On the other hand, integration by parts yields

$$
J_{22} \le C\|u\|_\infty^{p-1} \|v\|_\infty | \int_{\partial(B_d(x)\setminus B_\varepsilon(x))} |x-y|^{\alpha-n} ds| < \infty
$$

as long as $\alpha > 1$. Hence, J_ε is convergent uniformly about x when $\varepsilon \to 0$.

Combining the estimates of J_1 and J_2, we know that (2.8) makes sense, and hence $u \in C^1(\mathbb{R}^n)$. □

Acknowledgements. This research was supported by the NSF (No. 11471164) of China.

References

[1] A. Ambrosetti; *On Schrödinger-Poisson systems*, Milan J. Math., **76** (2008), 257–274.
[2] G. Caristi, L. D'Ambrosio, E. Mitidieri; *Representation formulae for solutions to some classes of higher order systems and related Liouville theorems*, Milan J. Math., **76** (2008), 27–67.
[3] I. Catto, J. Dolbeault, O. Sanchez, J. Soler; *Existence of steady states for the Maxwell-Schrödinger-Poisson systems: exploring the applicability of the concentration-compactness principle*, Math. Models Methods Appl. Sci., **23** (2013), 1915–1938.
[4] T. Cazenave; *Semilinear schrödinger equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
[5] W. Chen, C. Li; *Methods on Nonlinear Elliptic Equations*, AIMS Book Series on Diff. Equa. Dyn. Sys., Vol. 4, 2010.
[6] W. Chen, C. Li; *Super polyharmonic property of solutions for PDE systems and its applications*, Commun. Pure Appl. Anal., **12** (2013), 2497–2514.
[7] W. Chen, C. Li, B. Ou; *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., **59** (2006), 330–343.
[8] P. Felmer, A. Quaas, J. Tan; *Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A., **142** (2012), 1237–1262.
[9] J. Ginibre, G. Velo; *Long range scattering and modified wave operators for some Hartree type equations*, Rev. Math. Phys., **12** (2000), 361–429.
[10] E. Hebey, J. Wei; *Schrödinger-Poisson systems in the 3-sphere*, Calc. Var. Partial Differential Equations, **47** (2013), 25–54.
[11] L. Jeanjean, T. Luo; *Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations*, Z. Angew. Math. Phys., **64** (2013), 937–954.
[12] V. I. Karpman, A. G. Shagalov; *Stabilization of soliton describe by nonlinear Schrödinger type equations with higher-order dispersion*, Phys. D., **144** (2000), 194–210.
[13] Y. Lei; *On the regularity of positive solutions of a class of Choquard type equations*, Math. Z., **273** (2013), 883–905.
[14] Y. Lei; *Qualitative analysis for the static Hartree-type equations*, SIAM J. Math. Anal., **45** (2013), 388–406.

[15] C. Li, L. Ma; *Uniqueness of positive bound states to Schrödinger systems with critical exponents*, SIAM J. Math. Anal., **40** (2008), 1049–1057.

[16] D. Li, C. Miao, X. Zhang; *The focusing energy-critical Hartree equation*, J. Differential Equations, **246** (2009), 1139–1163.

[17] E. Lieb; *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math., **118** (1983), 349–374.

[18] E. Lieb, B. Simon; *The Hartree-Fock theory for Coulomb systems*, Comm. Math. Phys., **53** (1977), 185–194.

[19] S. Liu; *Regularity, symmetry, and uniqueness of some integral type quasilinear equations*, Nonlinear Anal., **71** (2009), 1796–1806.

[20] C. Ma, W. Chen, C. Li; *Regularity of solutions for an integral system of Wolff type*, Adv. Math., **226** (2011), 2676–2699.

[21] L. Ma, L. Zhao; *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Rational Mech. Anal., 195 (2010), 455–467.

[22] K. Nakanishi; *Energy scattering for Hartree equations*, Math. Res. Lett., **6** (1999), 107–118.

[23] E. M. Stein; *Singular integrals and differentiability properties of function*, Princetion Math. Series, Vol. 30, Princetion University Press, Princetion, NJ, 1970.

[24] D. Xu, Y. Lei; *Classification of positive solutions for a static Schrödinger-Maxwell equation with fractional Laplacian*, Appl. Math. Lett., **43** (2015), 85–89.

[25] Y, Zhao; *Regularity and symmetry for solutions to a system of weighted integral equations*, J. Math. Anal. Appl., **391** (2012), 209–222.

Yayun Li

Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

E-mail address: liyayun.njnu@qq.com

Deyun Xu

Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

E-mail address: 954816700@qq.com