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LINEARIZATION OF HYPERBOLIC RESONANT FIXED POINTS OF DIFFEOMORPHISMS WITH RELATED GEVREY ESTIMATES IN THE PLANAR CASE

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Abstract. We show that any germ of smooth hyperbolic diffeomophism at a fixed point is conjugate to its linear part, using a transformation with a Mourtada type functions, which (roughly) means that it may contain terms like $x \log|x|$. Such a conjugacy admits a Mourtada type expansion. In the planar case, when the fixed point is a $p : -q$ resonant saddle, and if we assume that the diffeomorphism is of Gevrey class, we give an upper bound on the Gevrey estimates for this expansion.

1. Introduction

In a vicinity of a hyperbolic fixed point of a diffeomorphism, there exists a homeomorphism that conjugates the map with its linear part; this is called a 'linearization'. In the general context, this conjugacy is only topological and the presence of resonances is typically an obstruction to C^k linearization [23, 24, 25, 26] (in the plane, this homeomorphism can be taken C^1 and close to the identity). Instead one could try a change of variables that is perhaps not C^2 but is smooth in other 'elementary' functions. In order to fix the ideas, consider for example the C^1 planar transformation $H(x, y) = (x + x^2 \log|x|y, y)$, which is analytic in the variables $(x, x\log|x|, y)$, although it is obviously not C^2 in (x, y). This approach has already been used for singularities of vector fields, see e.g. [21, 9, 8, 4]. The goal of this paper is to show that such linearization can be written as the composition of a smooth function and logarithmic functions, i.e., to give suitable results for diffeomorphisms compared with [8]. We first introduce the following definition. In what follow $\mathbb{R}$ designates the set of real number.

Definition 1.1. Let $\mathbf{U} \subset \mathbb{R}^m$ be a neighbourhood of 0 and $f : \mathbf{U} \to \mathbb{R}$ be a continuous function. We say that f is a Mourtada type function if there exists a positive integer k, a neighbourhood $V_k \subset \mathbb{R}^{mk}$ of 0 and a C^∞ function $\mathbf{F} : V_k \to \mathbb{R}$ such that

$$f(z) = \mathbf{F}(z, Tz, \dots, T^{k-1}z), \quad \text{where } T = \log \sum_{i=1}^{m} a_i (z_i \bar{z}_i)^{n_i},\ a_i \geq 0,\ n_i \in \mathbb{N}.$$

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A homeomorphism $\Phi : \mathbf{U} \to \mathbb{R}^n$ is said to be of Mourtada type if each component of Φ is a Mourtada type function.

Theorem 1.2. *Let $\mathbf{U} \subset \mathbb{R}^m$ be a neighbourhood of 0 and*

$$\boldsymbol{\varphi} : \mathbf{U} \to \mathbb{R}^m, \quad \mathbf{x} \mapsto \boldsymbol{\varphi}(\mathbf{x})$$

be a smooth hyperbolic diffeomorphism at the origin. Assume that $\mathbf{L} = d\boldsymbol{\varphi}(0)$ is semi-simple. Then there exists a Mourtada type homeomorphism

$$\Phi : \tilde{\mathbf{U}} \subset \mathbf{U} \to \mathbb{R}^m$$

such that $\boldsymbol{\varphi} \circ \Phi = \Phi \circ \mathbf{L}$.

From Definition 1.1 above, this result means Φ reads $\Phi = (\Phi_1, \dots \Phi_m)$, where

$$\Phi_i = F_i(z, Tz, \dots, T^\ell z), \quad T = T(z) = \log \sum_{j=1}^{m} c_j |z_j|^{2n_i}, \quad i = 1, \dots, m$$

and where F_i is a smooth function. A natural question concerns the growth of the coefficients of the Taylor expansion (at the origin) of each F_i. In section 3 of this article we address this question in the planar saddle case. Recall that any planar C^2 diffeomorphism can always be C^1 linearized near a hyperbolic fixed point [13]. Before stating the second result of this article, we recall the following concept.

Definition 1.3. Let s be a non negative integer. We say that a formal power series

$$P(x) = \sum_{\mathbf{n} \in \mathbf{N}^m}^{\infty} a_{\mathbf{n}} x^{\mathbf{n}}$$

is of Gevrey class (or order) s if, there exist $C > 0$, $r > 0$ such that

$$|a_{\mathbf{n}}| \le C r^{|\mathbf{n}|} (|\mathbf{n}|!)^s.$$

Clearly, for $s = 0$ one has analyticity; there seems to be no consensus in the literature about the terminology: some authors call a series as in our definition 1.3 of class Gevrey class $s + 1$.

Now we consider a typical planar resonant saddle.

Definition 1.4. Let

$$\boldsymbol{\varphi} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0), (x, y) \mapsto (\varphi_1(x, y), \varphi_2(x, y))$$

be a germ of diffeomorphism at the origin. We say that this germ is $p : -q$ resonant if

$$\alpha^p \beta^q = 1,$$

where α and β are the eigenvalues of $d\boldsymbol{\varphi}(0)$ and where $\gcd(p, q) = 1$.

It is well known that if a germ $\boldsymbol{\varphi}$ is $p : -q$ resonant, the associated Poincaré Dulac normal form is

$$\varphi_1(x, y) = \alpha x + x H_1(u), \quad \varphi_2(x, y) = \beta y + y H_2(u)\,, \tag{1.1}$$

where $u = x^p y^q$. In the real case with $0 < \alpha < 1 < \beta$ this normal form can be obtained by a C^∞ conjugacy, if $\boldsymbol{\varphi}$ is C^∞. We shall often work on the level of formal power series, and may take x, y, α and β to be complex as well, with $0 < |\alpha| < 1 < |\beta|$. We expand

$$H_1(u) = \sum_{n \ge 1} a_n u^n, \quad H_2(u) = \sum_{n \ge 1} b_n u^n. \tag{1.2}$$

When starting from an analytic diffeomorphism, we doubt that there is, in general, an analytic change of coordinates into normal form: for vector fields it is known that 'divergence of the change of coordinates to the formal normal form is the rule' [14]. For the diffeomorphism case we have no knowledge of references, but, a bad sign could be the fact that the formal infinitesimal generator in general diverges, even for an analytic diffeomorphism [20].

On the other hand, let us now motivate the assumptions about (H_1, H_2) that we shall make in the statement of Theorem 1.5.

If the given diffeomorphism φ in Definition 1.4 is analytic (or even of Gevrey class 1) in the variables (x, y), then following the methods and estimates used in, for example, [5] and [6], one obtains a Poincaré-Dulac normal form (1.1) that is of Gevrey class 1. In that case, in the expansion of (H_1, H_2) above, a term in $u^n = x^{pn}y^{qn}$ is of order $(p+q)n$ in the variables (x, y), and hence we have estimates of the form

$$|a_n| \le Cr^n(n!)^{p+q}, \quad |b_n| \le Cr^n(n!)^{p+q} \tag{1.3}$$

for some constants $C > 0$ and $r > 0$. Therefore this will be the starting point for the second statement in Theorem 1.5 concerning this planar case. Furthermore, the notion of 'Mourtada type homeomorphism' will have a simpler expression in the plane, and we can be more specific.

Theorem 1.5. *Let $\boldsymbol{\varphi}$ have the normal form* (1.1). *There is a linearizing change of variables $(x, y) = \Phi(\tilde{x}, \tilde{y})$ of the formal form*

$$\begin{aligned} x &= \tilde{x}(1 + \psi_1(\tilde{u}, \tilde{u}\log|\tilde{x}|, \tilde{u}\log|\tilde{y}|)) \\ y &= \tilde{y}(1 + \psi_2(\tilde{u}, \tilde{u}\log|\tilde{x}|, \tilde{u}\log|\tilde{y}|)), \end{aligned} \tag{1.4}$$

where $\tilde{u} = \tilde{x}^p\tilde{y}^q$. Moreover, if the expansion in (1.2) *satisfies the Gevrey estimates in* (1.3)*, then (ψ_1, ψ_2) is of Gevrey class at most $4(p+q)$ in its variables $(\tilde{u}, \tilde{u}\log|\tilde{x}|, \tilde{u}\log|\tilde{y}|)$.*

1.1. **Background and motivation.** From the topological point of view, a map near a hyperbolic fixed point is, thanks to [13] well understood. However, if we want to improve the smoothness of the conjugacy, the map should satisfy additional properties, namely, to avoid resonances. More precisely, if the map admits no resonance at the fixed point, then the map is locally formally linearizable. Observe that in the non-hyperbolic case, a one dimensional germs is C^∞ conjugate with a polynomial vector field [27]. Since Chen's Theorem [11] guarantees the equivalence between C^∞ and formal, in this article we only need to consider formal conjugacies, because of the hyperbolicity. However, even if the map admits some resonance, which is an obstruction for a C^∞ linearization, there are several results [23, 24, 10] about optimal criteria for finitely smooth (C^k) linearization. On the other hand one can try to express a linearization in terms of Mourtada type functions. This notion, see [18, 21], has already turned out to be useful when dealing with resonant singularities of vector fields, and a version of Theorem 1.2 was obtained in [7, 8] in this case, and in [22] in the presence of a parameter.

1.2. **Initial set-up.** Let us fix some notation. We can assume that the fixed point of $\boldsymbol{\varphi}$ is located at the origin. Write

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{L}(\mathbf{x}) + \mathbf{h}(\mathbf{x}), \tag{1.5}$$

where $\mathbf{x} = (x_1, \dots, x_m)$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)$. The linear part $\mathbf{L} = d\boldsymbol{\varphi}(0)$ has eigenvalues with modulus different from 1. For simplicity we will assume that the eigenvalues are real. The general case where some eigenvalues are complex can be treated exactly the same way by considering $\boldsymbol{\varphi}$ in the m-dimensional complex space. The higher order terms are represented by $\mathbf{h} = (h_1, \dots, h_m)$. Assuming $\mathbf{L}$ to be semi-simple amounts to saying that $\mathbf{L}$ is represented by a diagonal matrix i.e.,

$$\mathbf{L}(\mathbf{x}) = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \operatorname{diag}\{\alpha_1, \dots, \alpha_m\}\,.$$

This condition is fulfilled in the case where the eigenvalues α_i's are different and therefore it is a generic condition. The eigenvalues are ordered in such a way that

$$0 < |\alpha_1| < |\alpha_2| < \dots < |\alpha_p| < 1 < |\alpha_{p+1}| < \dots < |\alpha_{p+q}|, \tag{1.6}$$

where $m = p + q$. We write $\boldsymbol{\varphi}$ in (1.5) as

$$\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{y}, \mathbf{z}) = (\mathbf{Y}, \mathbf{Z}) = \mathbf{X}\,,$$

where

$$\begin{gathered} y_1 = x_1,\ y_2 = x_2,\ \dots,\ y_p = x_p, \quad z_1 = x_{p+1}, \dots, z_q = x_{p+q}, \\ \mathbf{y} = (y_1, \dots, y_p), \quad \mathbf{z} = (z_1, \dots, z_q), \quad \mathbf{Y} = (Y_1, \dots, Y_p), \\ \mathbf{Z} = (Z_1, \dots, Z_q), \quad X = (X_1, \dots, X_{p+q}) \end{gathered}$$

and where

$$\begin{gathered} Y_i = \alpha_i y_i + \mathrm{h}_i(\mathbf{y}, \mathbf{z}), \quad i = 1, \dots, p \\ Z_j = \alpha_{p+j} z_j + \mathrm{h}_{p+j}(\mathbf{y}, \mathbf{z}), \quad j = 1, \dots, q. \end{gathered} \tag{1.7}$$

or, to avoid unnecessary heavy notation

$$X_i = \alpha_i x_i + \mathrm{h}_i(\mathbf{x}). \quad i = 1, \dots, p + q\,. \tag{1.8}$$

We also assume that the map (1.8) is already written into its Poincaré Dulac Normal Form [1, 3, 7, 10]. This implies that where for each integer $1 \le i \le p + q$, h_i is α_i-resonant or resonant associated to α_i. This means that each h_i admits the following expansion

$$\mathrm{h}_i(\mathbf{y}, \mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{N}^{p+q}} \Gamma_{i,\mathbf{n}} x_1^{n_1} \dots x_m^{n_{p+q}} = \sum_{\mathbf{n} \in \mathbb{N}^{p+q}} \Gamma_{i,\mathbf{n}} \mathbf{x}^{\mathbf{n}}\,, \tag{1.9}$$

where $\Gamma_{i,\mathbf{n}}$ are real coefficients, and each $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \dots x_m^{n_{p+q}}$ satisfies

$$\alpha_1^{n_1} \dots \alpha_m^{n_{p+q}} = \alpha_i. \tag{1.10}$$

The stable, unstable and total orders of such monomial are respectively

$$d_s(\mathbf{x}^{\mathbf{n}}) = n_1 + \dots + n_p, \quad d_u(\mathbf{x}^{\mathbf{n}}) = n_{p+1} + \dots + n_{p+q}, \quad d(\mathbf{x}^{\mathbf{n}}) = d_s(\mathbf{x}^{\mathbf{n}}) + d_u(\mathbf{x}^{\mathbf{n}}).$$

As a consequence of (1.10), a α_i-resonant function h_i taking the form (1.9) commutes with the linear map $\mathbf{L}$, more precisely we have

$$\mathrm{h}_i(\mathbf{L}(\mathbf{x})) = \alpha_i \mathrm{h}_i(\mathbf{x}). \tag{1.11}$$

We denote

$$\begin{gathered} d_s(\mathrm{h}_i) = \min_{\mathbf{n} \in \mathbb{N}^{p+q}} \{d_s(\mathbf{x}^{\mathbf{n}}), \quad \Gamma_{i,\mathbf{n}} \neq 0\}, \quad d_u(\mathrm{h}_i) = \min_{\mathbf{n} \in \mathbb{N}^{p+q}} \{d_u(\mathbf{x}^{\mathbf{n}}), \\ \Gamma_{i,\mathbf{n}} \neq 0\}, \quad d(\mathrm{h}_i) = \min_{\mathbf{n} \in \mathbb{N}^{p+q}} \{d(\mathbf{x}^{\mathbf{n}}), \quad \Gamma_{i,\mathbf{n}} \neq 0\}, \end{gathered}$$

respectively the stable, unstable and total order of the resonant function. We state the following proposition.

Proposition 1.6. *Let $i = 1, \dots, p+q$ be an integer, and let $\mathbf{N} = \mathbf{N}(\mathbf{x})$ be a resonant monomial (with respect to the $i-th$ coordinate) , i.e.,*

$$\mathbf{N}(\mathbf{x}) = \Gamma x_1^{a_1} \dots x_n^{a_n} \,,$$

where $\Gamma \in \mathbb{R}$ and $\alpha_1^{a_1} \dots \alpha_n^{a_n} = \alpha_i$. Let $d = a_1 + \dots a_{p+q}$ be the total order of $\mathbf{N}$. Then

$$\mathbf{N}(\boldsymbol{\varphi}(\mathbf{x})) = \alpha_i \mathbf{N}(\mathbf{x}) + \mathbf{f}_1(\mathbf{x}) \,,$$

where $\mathbf{f}_1$ is a countable sum of resonant monomials with respect to the $i-th$ coordinate of total order higher than d.

Proof. Let $i = 1, \dots, p+q$ be integers and let

$$\mathbf{N} = \mathbf{N}(\mathbf{x}) = \Gamma x_1^{a_1} \dots x_n^{a_n} \,,$$

where $\Gamma \in \mathbb{R}$, be a resonant monomial with respect to the i-th coordinates. Recall that

$$\mathbf{N}(\mathbf{L}(\mathbf{x})) = \alpha_i \mathbf{N}(\mathbf{x}),$$

which implies that

$$\alpha_1^{a_1} \dots \alpha_n^{a_n} = \alpha_i.$$

We also recall that

$$\boldsymbol{\varphi}(\mathbf{x}) = \Big(\alpha_1 x_1 + h_1(\mathbf{x}), \dots, \alpha_{p+q} x_{p+q} + h_{p+q}(\mathbf{x})\Big) \,,$$

where each h_j, $j = 1, \dots, p+q$ is resonant with respect to the j-th coordinates, i.e.,

$$h_j(\mathbf{L}(\mathbf{x})) = \alpha_j h_j(\mathbf{x}). \tag{1.12}$$

We also write $h_j(\mathbf{x}) = x_j \hat{h}_j(\mathbf{x})$. Observe that $\hat{h}_j$ is not well defined when $x_j = 0$. However, for each integer $j = 1, \dots, p+q$, $\hat{h}_j$ is a countable sum of (rational) monomial of the form

$$x_1^{n_1} \dots x_j^{n_j} \dots x_{p+q}^{n_{p+q}}, \quad \text{where} \quad n_i \geq 0,\ i \neq j,\ n_j \geq -1, \text{and} \ \sum_{j=1}^{p+q} n_j \geq 1 \,. \tag{1.13}$$

Moreover, from (1.12), we deduce that $\hat{h}_j$ is invariant under $\mathbf{L}$, i.e.,

$$\hat{h}_j(\mathbf{L}(\mathbf{x})) = \hat{h}_j(\mathbf{x}). \tag{1.14}$$

We now write

$$\begin{aligned}
\mathbf{N}(\boldsymbol{\varphi}(\mathbf{x}) &= \prod_{k=1}^{p+q} \Big(\alpha_k x_k + h_k(\mathbf{x})\Big)^{a_k} \\
&= \prod_{k=1}^{p+q} \Big(x_k(\alpha_k + \hat{h}_k(\mathbf{x}))\Big)^{a_k} \\
&= \prod_{k=1}^{p+q} x_k^{a_k} \Big(\alpha_k + \hat{h}_k(\mathbf{x})\Big)^{a_k} \,.
\end{aligned}$$

For each integer k, using binomial formula, we write

$$\Big(\alpha_k + \hat{h}_k(\mathbf{x})\Big)^{a_k} = \alpha_k^{a_k} + \tilde{h}_k(\mathbf{x}) \,,$$

where $\tilde{h}_k(\mathbf{x})$ satisfies (1.14) i.e., $\tilde{h}_k(\mathbf{L}(\mathbf{x})) = \tilde{h}_k(\mathbf{x})$ and $\tilde{h}_k(\mathbf{x})$ is a countable sum of (rational) monomial of the form

$$x_1^{m_1} \dots x_j^{m_j} \dots x_{p+q}^{m_{p+q}}, \quad \text{where } m_i \geq 0,\ i \neq k,\ m_k \geq -a_k,\ \text{and } \sum_{j=1}^{p+q} m_j \geq 1.$$

Then it follows that

$$\begin{aligned}
\mathbf{N}(\boldsymbol{\varphi}(\mathbf{x})) &= \prod_{k=1}^{p+q} x_k^{a_k} \Big(\prod_{k=1}^{p+q} \alpha_k^{a_k} + \tilde{h}_k(\mathbf{x}) \Big) \\
&= \prod_{k=1}^{p+q} x_k^{a_k} \Big(\alpha_i + \tilde{h}_k(\mathbf{x}) \Big) \\
&= \alpha_i \mathbf{N} + \prod_{k=1}^{p+q} x_k^{a_k} \tilde{h}_k(\mathbf{x}) \\
&= \alpha_i \mathbf{N} + \mathbf{f}_1(\mathbf{x}),
\end{aligned}$$

where $\mathbf{f}_1(\mathbf{x})$ is the sum of monomials of the form

$$x_1^{m_1} \dots x_{p+q}^{m_{p+q}}, \quad \text{where } m_i \geq 0 \text{ and } \sum_{j=1}^{p+q} m_j \geq 1 + \sum_{j=1}^{p+q} a_j = d + 1$$

and satisfies

$$f_1(\mathbf{L}(\mathbf{x})) = \alpha_i \mathbf{f}_1(\mathbf{x}),$$

completing the proof of the proposition. □

By linearity, we deduce that the above proposition extends to resonant functions. More precisely we have the following: for each integer $i = 1, \dots, p+q$, there exists a sequence of resonant functions $\{h_i^{\{k\}}\}_{k \geq 1}$ such that

$$\mathrm{h}_i^{\{1\}} = \mathrm{h}_i, \quad d(\mathrm{h}_i^{\{k\}}(\mathbf{x})) \geq k+1, \quad \mathrm{h}_i^{\{k\}}(\boldsymbol{\varphi}(\mathbf{x})) = \alpha_i \mathrm{h}_i^{\{k\}}(\mathbf{x}) + \mathrm{h}_i^{\{k+1\}}(\mathbf{x}). \tag{1.15}$$

Furthermore, we say that an α_i-resonant term of the form given in (1.10) is a 'good' resonant term if $n_i > 0$. Otherwise, if $n_i = 0$, the resonant term is said to be 'bad'. Observe that since the eigenvalues are ordered according to (1.6), for each integer $i = 1, \dots, p-1$, there is only a finite number of bad α_i-resonant monomials of the form given in (1.10) of stable order 1, (i.e., when $n_1 + \dots + n_p = 1$), and each of those resonant terms may only concern variables $x_{i+1}, \dots, x_p$. Similarly, for each integer $j = 1, \dots, q$ there is only a finite number of bad resonant α_{j+p}-resonant monomial in of unstable order one, (i.e., such as (1.10) with $n_{p+1} + \dots + n_{p+q} = 1$), and each of those resonant terms may only concern variables $x_{j+p+1}, \dots, x_{p+q}$.

1.3. **Strategy.** To avoid fuzziness and complications in the notation, we briefly explain the mains steps of the strategy to follow in the planar case. The strategy in higher dimension is essentially the same. Actually section 3 is completely devoted to the planar case, and for completeness, the procedure is presented again. In what follows, the notations used in this section are independent from those used before and those used in section 2.

In the planar case, the initial diffeomorphism $\boldsymbol{\varphi}$ takes the form

$$\boldsymbol{\varphi} : \mathbf{U} \to \mathbb{R}^2, \quad (x,y) \mapsto \boldsymbol{\varphi}(x,y) = (\alpha x + x H_1(x^p y^q), \beta y + y H_2(x^p y^q)),$$

where H_1 and H_2 are smooth functions, p and q are positive integer such that $\alpha^p\beta^q = 1$ and therefore each resonant monomial is of the form $x(x^py^q)^k$, $k \geq 1$ for the first component and $y(x^py^q)^k$, $k \geq 1$ for the second component. Define

$$\begin{gathered}\hat{\boldsymbol{\varphi}} : \mathbf{U} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ (x, y, s, t) \mapsto (\boldsymbol{\varphi}(x, y), s + 1/\alpha, t + 1/\beta).\end{gathered} \tag{1.16}$$

The new variables t and s are called *tag-functions*. The role played by s and t is essentially the same as the tagged function in [8] in the context of hyperbolic vector fields. At this stage of the presentation, one can see these variables as independent variables. However, later on, we will express both t and s as functions depending on the new variables in which the dynamic is linear.

We shall perform (in Section 2 in the general case, and in section 3 in the planar case) a normal form procedure to eliminate resonant terms for the map $\hat{\boldsymbol{\varphi}}$. Eventually we will find a semi-local formal transformation (and change of coordinates) of the form

$$(x_\infty, y_\infty, s, t) = \hat{\mathbf{g}}(x, y, s, t) = \Big(\mathbf{g}(x, y, s, t), s, t\Big), \tag{1.17}$$

semi local in the sense that (x, y) is near 0 in $\mathbb{R}^2$ but both t and s may take large values. More precisely $\mathbf{g} = (g_1(x, y, s, t), g_2(x, y, s, t))$ admits the following asymptotics

$$\begin{aligned} g_1(x, y, s, t) = x_\infty = x + x \sum_{k=0}^{\infty} g_{1,k}(x, y) s^{k+1} \\ g_2(x, y, s, t) = y_\infty = y + y \sum_{k=0}^{\infty} g_{2,k}(x, y) t^{k+1}, \end{aligned} \tag{1.18}$$

where for each integer k, both $g_{1,k}$ and $g_{2,k}$ admits the following asymptotics

$$\begin{aligned} g_{1,k}(x, y) \asymp (x^py^q)^{k+1} \sum_{j=0}^{\infty} A_j (x^py^q)^j, \\ g_{2,k}(x, y) \asymp (x^py^q)^{k+1} \sum_{j=0}^{\infty} B_j (x^py^q)^j. \end{aligned} \tag{1.19}$$

Thanks to this transformation, we show that $\hat{\mathbf{L}} \circ \hat{\mathbf{g}} = \hat{\mathbf{g}} \circ \hat{\boldsymbol{\varphi}}$ where $\hat{\mathbf{L}}$ is the 'tag-extension' of $\mathbf{L}$ i.e.,

$$\begin{gathered}\hat{\mathbf{L}} : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}, \\ (x, y, s, t) \mapsto (\alpha x, \beta y, s + 1/\alpha, t + 1/\beta).\end{gathered}$$

We then need to invert the expression given in (1.17), that is to express the old variable (x, y) by means of the new variable (x_∞, z_∞). To do so, we construct the following extension. More precisely, we define

$$\xi = xs, \quad \eta = yt, \quad \xi_1 = x_\infty s, \quad \eta_1 = y_\infty t,$$

and therefore, (1.18) induces the following formal transformation

$$\begin{aligned} x_\infty &= x + x\sum_{k=0}^{\infty} \hat{g}_{1,k}(x,y,\xi) \\ y_\infty &= y + y\sum_{k=0}^{\infty} \hat{g}_{2,k}(x,y,\eta) \\ \xi_1 &= \xi + \xi\sum_{k=0}^{\infty} \hat{g}_{1,k}(x,y,\xi) \\ \eta_1 &= \eta + \eta\sum_{k=0}^{\infty} \hat{g}_{2,k}(x,y,\eta)\,, \end{aligned} \tag{1.20}$$

where thanks to (1.19), both $\hat{g}_{1,k}$ and $\hat{g}_{2,k}$ read

$$\begin{aligned} \hat{g}_{1,k}(x,y,\xi) &\asymp \sum_{j=k+1}^{\infty} A_j(x^py^q)^{j-(k+1)}(x^{p-1}y^q)^{k+1}\xi^{k+1}, \\ \hat{g}_{1,k}(x,\xi,y) &\asymp \sum_{j=k+1}^{\infty} B_j(x^py^q)^{j-(k+1)}(x^py^{q-1})^{k+1}\eta^{k+1}. \end{aligned}$$

Using a theorem of Borel and the Inverse Function Theorem, we will invert the expression (1.20) and get

$$\begin{aligned} x &= x_\infty + x_\infty J_1(x_\infty, y_\infty, \xi_1, \eta_1) = x_\infty + x_\infty J_1(x_\infty, y_\infty, x_\infty s, y_\infty t) \\ y &= y_\infty + y_\infty J_2(x_\infty, y_\infty, \xi_1, \eta_1) = y_\infty + y_\infty J_2(x_\infty, y_\infty, x_\infty s, y_\infty t) \\ \xi &= \xi_1 + \xi_1 J_3(x_\infty, y_\infty, \xi_1, \eta_1) = \xi_1 + \xi_1 J_3(x_\infty, y_\infty, x_\infty s, y_\infty t) \\ \eta &= \eta_1 + \eta_1 J_4(x_\infty, y_\infty, \xi_1, \eta_1) = \eta_1 + \eta_1 J_4(x_\infty, y_\infty, x_\infty s, y_\infty t)\,, \end{aligned} \tag{1.21}$$

where J_1, J_2, J_3 and J_4 are smooth functions. Define the map

$$\hat{\mathbf{\Psi}}(x_\infty, y_\infty, s, t) = (\Psi_1(x_\infty, y_\infty, s, t), \Psi_2(x_\infty, y_\infty, s, t), s, t)\,,$$

where

$$\begin{aligned} \Psi_1(x_\infty, y_\infty, s, t) &= x_\infty + x_\infty J_1(x_\infty, y_\infty, x_\infty s, y_\infty t), \\ \Psi_2(x_\infty, y_\infty, s, t) &= y_\infty + y_\infty J_2(x_\infty, y_\infty, x_\infty s, y_\infty t). \end{aligned}$$

Since $\hat{\mathbf{\Psi}}$ is the inverse of $\mathbf{g}$, it follows that

$$\hat{\mathbf{\Psi}} \circ \hat{\mathbf{L}} = \hat{\boldsymbol{\varphi}} \circ \hat{\mathbf{\Psi}}. \tag{1.22}$$

Define now the map

$$h(x_\infty, y_\infty) = (\alpha^{-1}\log_{|\alpha|}|x_\infty|, \beta^{-1}\log_{|\beta|}|y_\infty|) = (h_1(x_\infty, y_\infty), h_2(x_\infty, y_\infty)).$$

Recall that $\mathbf{L}(x_\infty, y_\infty) = (\alpha x_\infty, \beta y_\infty)$. We have

$$h \circ \mathbf{L}(x_\infty, y_\infty) = (\alpha^{-1}, \beta^{-1}) + h(x_\infty, y_\infty). \tag{1.23}$$

We finally define the map

$$\begin{aligned} \mathbf{\Phi}(x_\infty, y_\infty) = \big(&\Psi_1(x_\infty, y_\infty, h_1(x_\infty, y_\infty), h_2(x_\infty, y_\infty)), \\ &\Psi_2(x_\infty, y_\infty, h_1(x_\infty, y_\infty), h_2(x_\infty, y_\infty))\big). \end{aligned}$$

Thanks to (1.22) and (1.23) we show that $\mathbf{\Phi} \circ \mathbf{L} = \boldsymbol{\varphi} \circ \mathbf{\Phi}$.

As we said earlier, the strategy in higher dimension is essentially the same. However, since the set of resonant monomials is, in general, more complicated, the tasks presented above are more delicate and require an extension in higher dimensions.

2. A singular normal form

For each integer $i = 1, \dots, p+q$, we define the sequence of functions $(\mathrm{w}_i^{\{n\}})$ such that it satisfies the equation

$$
\begin{gathered}
\mathrm{w}_i^{\{0\}}(\xi) = 1, \quad \mathrm{w}_i^{\{1\}}(\xi) = \xi, \\
\mathrm{w}_i^{\{n\}}(\xi + 1/\alpha_i) = \alpha_i \mathrm{w}_i^{\{n+1\}}(\xi + 1/\alpha_i) - \alpha_i \mathrm{w}_i^{\{n+1\}}(\xi).
\end{gathered}
$$

A straightforward computation shows that

$$
\mathrm{w}_i^{\{n\}}(\xi) = \frac{1}{n!}\xi(\xi + 1/\alpha_i)\dots(\xi + (n-1)/\alpha_i) = \frac{1}{n!}\prod_{j=0}^{n-1}(\xi + j/\alpha_i)\,. \tag{2.1}
$$

2.1. **A preliminary step.** We first add two variables (s and t) and extend the map (1.8) in the following way

$$
\hat{\boldsymbol{\varphi}}(\mathbf{x}, t, s) = (\mathbf{X}, t+1, s+1)\,.
$$

We define the change of coordinates

$$
g_1(\mathbf{x}, t, s) = (\mathbf{x}^{\{1\}}, t, s)\,,
$$

where

$$
\begin{gathered}
\mathbf{x}^{\{1\}} = (\mathbf{y}^{\{1\}}, \quad \mathbf{z}^{\{1\}}) = (x_1^{\{1\}}, \dots, x_{p+q}^{\{1\}}), \\
\mathbf{y}^{\{1\}} = (y_1^{\{1\}}, \dots, y_p^{\{1\}}), \quad \mathbf{z}^{\{1\}} = (z_1^{\{1\}}, \dots, z_q^{\{1\}}),
\end{gathered}
$$

and

$$
x_i^{\{1\}} = x_i - t_i h_i(\mathbf{x}) = x_i - \mathrm{w}_i^{\{1\}}(t_i) h_i(\mathbf{x}), \quad i = 1, \dots, p+q\,,
$$

where $t_i = t/\alpha_i$ $i = 1, \dots, p$, $t_{p+j} = s/\alpha_{p+j}$ $j = 1, \dots, q$. Consider

$$
\hat{\varphi}^{\{1\}}(\mathbf{x}, t, s) = g_1 \circ \hat{\varphi} \circ g_1^{-1}(\mathbf{x}, t, s) = (\mathbf{X}^{\{1\}}, t+1, s+1).
$$

We obtain

$$
\begin{aligned}
{X_i}^{\{1\}} &= X_i - \mathrm{w}_i^{\{1\}}\Big(\frac{t+1}{\alpha_i}\Big)\mathrm{h}_i(\mathbf{X}) \\
&= X_i - \mathrm{w}_i^{\{1\}}\Big(t_i + \frac{1}{\alpha_i}\Big)\mathrm{h}_i(\mathbf{X}) \\
&= X_i - (t_i + 1/\alpha_i)\mathrm{h}_i(\mathbf{X}), \quad i = 1\dots, p+q\,.
\end{aligned}
$$

However, since $\mathbf{h} = (\mathrm{h}_1, \dots, \mathrm{h}_{p+q})$ is resonant, we should have

$$
\mathrm{h}_i(\mathbf{X}) = \alpha_i \mathrm{h}_i(\mathbf{x}) + \mathrm{h}_i^{\{2\}}(\mathbf{x})\,,
$$

where $\mathbf{h}^{\{2\}}(x) = (\mathrm{h}_1^{\{2\}}, \dots, \mathrm{h}_{p+q}^{\{2\}})$ is again a resonant (with higher order). The above equation reads

$$
\begin{aligned}
{X_i}^{\{1\}} &= \alpha_i x_i + \mathrm{h}_i(\mathbf{x}) - (t_i + 1/\alpha_i)(\alpha_i \mathrm{h}_i(\mathbf{x}) + \mathrm{h}_i^{\{2\}}(\mathbf{x})) \\
&= \alpha_i x_i - \alpha_i t_i \mathrm{h}_i(\mathbf{x}) - (t_i + 1/\alpha_i)\mathrm{h}_i^{\{2\}}(\mathbf{x}) \\
&= \alpha_i x_i^{\{1\}} - \mathrm{w}_i^{\{1\}}\Big(t_i + \frac{1}{\alpha_i}\Big)\mathrm{h}_i^{\{2\}}(\mathbf{x}), \quad i = 1, \dots, p+q.
\end{aligned}
$$

Observe that the nonlinear term has been replaced by another nonlinear term of higher order (along with a tag function).

2.2. **Induction step.** Assume that we have found a finite sequence of transformations $\{g_1,\dots,g_n\}$ such that

$$\varphi^{\{n\}}(\mathbf{x},t,s)=g_n\circ\cdots\circ g_1\circ\varphi\circ g_1^{-1}\circ\cdots\circ g_n^{-1}(\mathbf{x},t,s)=(\mathbf{X}^{\{n\}},t+1,s+1)\,,$$

where $\mathbf{x}^{\{n\}}=(\mathbf{y}^{\{n\}},\mathbf{z}^{\{n\}})$, $\mathbf{X}^{\{n\}}=(\mathbf{Y}^{\{n\}},\mathbf{Z}^{\{n\}})$ and

$$X_i^{\{n\}}=\alpha_i x_i^{\{n\}}+(-1)^n\mathrm{w}_i^{\{n\}}\Big(t_i+1/\alpha_i\Big)\mathrm{h}_i^{\{n+1\}}(\mathbf{x}),\quad i=1,\dots,p+q.$$

We now verify that this property is inherited at the next step. Write

$$x_i^{\{n+1\}}=x_i^{\{n\}}-(-1)^n\mathrm{w}_i^{\{n+1\}}(t_i)\mathrm{h}_i^{\{n+1\}}(\mathbf{x})\quad i=1,\dots,p+q. \tag{2.2}$$

Therefore,

$$X_i^{\{n+1\}}=X_i^{\{n\}}-(-1)^n\mathrm{w}_i^{\{n+1\}}(t_i+1/\alpha_i)\mathrm{h}_i{}^{\{n+1\}}(\mathbf{X}),\quad i=1,\dots,p+q.$$

Since $\mathbf{h}^{\{n+1\}}$ is resonant we have

$$\mathrm{h}_i^{\{n+1\}}(\mathbf{X})=\alpha_i\mathrm{h}_i{}^{\{n+1\}}(\mathbf{x})+\mathrm{h}_i^{\{n+2\}}(\mathbf{x})\,,$$

where $\mathbf{h}^{\{n+2\}}$ is again resonant. We then get

$$\begin{aligned}X_i{}^{\{n+1\}}&=\alpha_i x_i{}^{\{n\}}+(-1)^n\mathrm{w}_i^{\{n\}}(t_i+1/\alpha_i)\mathrm{h}_i^{\{n+1\}}(\mathbf{x})\\&\quad-(-1)^n\mathrm{w}_i^{\{n+1\}}(t_i+1/\alpha_i)(\alpha_i\mathrm{h}_i^{\{n+1\}}(\mathbf{x})+\mathrm{h}_i^{\{n+2\}}(\mathbf{x}))\\&=\alpha_i x_i{}^{\{n\}}-(-1)^n\alpha_i\mathrm{w}_i^{\{n+1\}}(t_i)\mathrm{h}_i{}^{\{n+1\}}(\mathbf{x})\\&\quad+(-1)^n\mathbb{R}_{i,n}(t_i)h_i{}^{\{n+1\}}(\mathbf{x})\\&\quad+(-1)^{n+1}\mathrm{w}_i^{\{n+1\}}(t_i+1/\alpha_i)h_i{}^{\{n+2\}}(\mathbf{x}),\quad i=1,\dots,p+q\,,\end{aligned}$$

where

$$\mathbb{R}_{i,n}(t_i)=\mathrm{w}_i^{\{n\}}(t_i+1/\alpha_i)+\alpha_i\mathrm{w}_i^{\{n+1\}}(t_i)-\alpha_i\mathrm{w}_i^{\{n+1\}}(t_i+1/\alpha_i),$$

and according to the definition given in (2.1) we have $\mathbb{R}_{i,n}\equiv 0$. This implies that our initial diffeomorphism is conjugate with the map

$$X_i{}^{\{n+1\}}=\alpha_i x_i{}^{\{n+1\}}+(-1)^{n+1}\mathrm{w}_i^{\{n+1\}}(t_i+1/\alpha_i)\mathrm{h}_i^{\{n+2\}}(\mathbf{x}),\quad i=1,\dots,p. \tag{2.3}$$

In other words, using these new coordinates, $\hat{\boldsymbol{\varphi}}$ is conjugated with a linear map up to terms of arbitrarily high order.

2.3. **Analyzing the transformation.** From (2.2) one defines the formal transformation

$$\begin{gathered}\hat{\mathbf{g}}:(\mathbb{R}^{p+q},0)\times\times\mathbb{R}\times\mathbb{R}\to(\mathbb{R}^{p+q},0)\times\mathbb{R}\times\mathbb{R},\\(\mathbf{x},t,s)\mapsto(\mathbf{x}^{\{\infty\}},t,s)\,,\end{gathered}$$

where

$$x_i^{\{\infty\}}=x_i+\sum_{\ell=1}^{\infty}(-1)^\ell\mathrm{w}_i^{\{\ell\}}(t_i)\mathrm{h}_i{}^{\{\ell\}}(\mathbf{x}),\quad i=1,\dots,p+q\,.$$

Thanks to the previous subsection, we formally have

$$\hat{\mathbf{L}}\circ\hat{\mathbf{g}}\asymp\hat{\mathbf{g}}\circ\hat{\boldsymbol{\varphi}}.$$

By formally, we mean that the expansion $\hat{\mathbf{g}} \circ \hat{\boldsymbol{\varphi}}$ in terms of $\mathbf{x}$, t and s coincide with that of $\hat{\mathbf{L}} \circ \hat{\mathbf{g}}$. In the remaining part of this section we show how this formal transformation leads to a smooth conjugacy between $\hat{\boldsymbol{\varphi}}$ and $\hat{\mathbf{L}}$ leading itself to a conjugacy between $\boldsymbol{\varphi}$ and $\mathbf{L}$.

2.4. **Rewriting the transformation.** The above transformation can be written 'line by line' as follows, now distinguishing the stable and the unstable directions.

$$
\begin{gathered}
y_{i,\infty} = y_i + \sum_{n\geq 1}\Big(\mathrm{V}_{i,n}(\mathbf{x})t^n + \mathrm{B}_{i,n}(\mathbf{x})t^n + y_i\mathrm{G}_{i,n}(\mathbf{x})t^n\Big) \quad i = 1,\dots,p \\
z_{j,\infty} = z_j + \sum_{n\geq 1}\Big(\mathrm{V}_{p+j,n}(\mathbf{x})s^n + \mathrm{B}_{p+j,n}(\mathbf{x})s^n + z_j\mathrm{G}_{p+j,n}(\mathbf{x})s^n\Big) \\
j = 1,\dots,q\,,
\end{gathered}
\tag{2.4}
$$

where for each integer $i = 1,\dots,p+q$ and $n > 0$,

- $\mathrm{V}_{i,n}(\mathbf{x})$ is a sum of bad resonant monomial of stable order 1, if $i \leq p$ and a sum of bad resonant monomial of unstable order 1 if $p < i \leq p+q$;
- $\mathrm{B}_{i,n}(\mathbf{x})$ is a sum of bad resonant monomial of stable order at least 2 if $i \leq p$, and unstable order at least 2 if $p < i \leq p+q$;
- $y_i\mathrm{G}_{i,n}(\mathbf{x})$ and $z_j\mathrm{G}_{j+p,n}(\mathbf{x})$ are sums of good resonant monomial.

We introduce the following notation. For integer $i = 1,\dots,p$, $j = 1,\dots,q$, $0 < \ell$, we introduce

$$
u_{i,\ell} = y_i t^\ell, \quad v_{j,\ell} = z_j s^\ell, \quad u_{i,\ell,\infty} = y_i^{\{\infty\}} t^\ell, \quad v_{j,\ell,\infty} = z_j^{\{\infty\}} s^\ell, \tag{2.5}
$$

$$
\begin{gathered}
\mathbf{u} = (u_{i,\ell}, \quad \ell = 1,\dots,L_i,\ i = 1,\dots,p), \\
\mathbf{v} = (v_{j,\ell}, \quad \ell = 1,\dots,L_{p+j},\ j = 1,\dots,q), \\
\mathbf{u}_\infty = (u_{i,\ell,\infty}, \quad \ell = 1,\dots,L_i,\ i = 1,\dots,p), \\
\mathbf{v}_\infty = (v_{j,\ell,\infty}, \quad \ell = 1,\dots,L_{p+j},\ j = 1,\dots,q)\,,
\end{gathered}
$$

where each L_i and L_{p+j} will be determined later. For each $1 \leq \ell \leq L_i$, $1 \leq k \leq p$, write

$$
\boldsymbol{\omega}^s_{\ell,k} = (y_i t^l,\ 0 \leq l \leq \ell,\ k \leq i \leq p),
$$

and for each $1 \leq \ell \leq L_{j+q}$, $1 \leq k \leq q$, write

$$
\boldsymbol{\omega}^u_{\ell,k} = (z_j s^l,\ 1 \leq l \leq \ell,\ k \leq j \leq q).
$$

Thanks to this notation, we rewrite (2.4) as follows

$$
\begin{gathered}
y_{i,\infty} = y_i + \sum_{n\geq 1}\Big(V_{i,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{\ell_i,i+1}) + B_{i,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{\ell_i,1}) \\
+ y_i G_{i,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{\ell_i,1})\Big), \quad i = 1,\dots,p; \\
z_{j,\infty} = z_j + \sum_{n\geq 1}\Big(V_{p+j,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{\ell_j,j+1}) + B_{p+j,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{\ell_j,1}) \\
+ z_j G_{p+j,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{\ell_j,1})\Big), \quad j = 1,\dots,q\,,
\end{gathered}
\tag{2.6}
$$

where $V_{i,n}$, $B_{i,n}$, $G_{i,n}$ are monomial in the variable they depend upon. Furthermore, since the set of resonant monomials of (un)stable order 1 is finite, we have that

$$
\max\{\ell_i, \ell_j,\ i = 1,\dots,p,\ j = 1\dots,q\} = L < \infty.
$$

We consider now the first line of (2.6) and multiply by t^k, $k = 0, \dots, L_1 = 2^{p-1}L$. Since $\mathrm{B}_{1,n}$ consists in resonant term of stable order at least 2, we write

$$\begin{aligned} u_{1,k,\infty} = u_{1,k} + \sum_{n\geq 1}\Big(& V_{1,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L+k,2}) + B_{1,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L+\delta(k),1}) \\ &+ u_{1,k}G_{1,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L,1})\Big) \quad k = 0,\dots,2^{p-1}L\,, \end{aligned} \tag{2.7}$$

where

$$\delta(\ell) = \begin{cases} \frac{\ell}{2} & \text{if } \ell \text{ is even,} \\ \frac{\ell}{2} + \frac{1}{2} & \text{otherwise.} \end{cases}$$

Since the above equation involves $u_{i,k}$ for $k = 0, \dots, L + 2^{p-1}L$, $i = 2, \dots, p$ we multiply the second line of (2.6) by t^k when $k = 0, \dots, l_2 = L + 2^{p-1}L$. We get

$$\begin{aligned} u_{2,k,\infty} = u_{2,k} + \sum_{n\geq 1}\Big(& V_{2,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L+k,3}) + \mathrm{B}_{2,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L+\delta(k),1}) \\ &+ u_{2,k}\mathrm{G}_{2,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L,1})\Big) \quad k = 0,\dots,L+2^{p-1}L\,. \end{aligned} \tag{2.8}$$

Proceeding line by line, we multiply the r-th line of (2.6) by t^k for $k = 0, \dots, L_r = (r-1)L + 2^{p-1}L$. For each $2 < r \leq p-1$, we obtain

$$\begin{aligned} u_{r,k,\infty} = u_{r,k} + \sum_{n\geq 1}\Big(& V_{r,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L+k,r+1}) \\ &+ B_{r,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L+\delta(k),1}) + u_{r,k}G_{r,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L,1})\Big) \\ & k = 0,\dots,(r-1)L + 2^{p-1}L\,. \end{aligned} \tag{2.9}$$

Finally, for the case $r = p$, we multiply the p-th line of (2.6) by t^k for $k = 0, \dots, L_p = 2^pL$. We get

$$u_{p,k,\infty} = u_{p,k} + \sum_{n\geq 1}\Big(B_{p,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{\delta(k),1}) + u_{p,k}G_{p,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^s_{L,1})\Big) \tag{2.10}$$

for $k = 0, \dots, 2^pL$. We proceed in a complete similar way on the unstable manifold, i.e., for the z-variable and get

$$\begin{aligned} v_{1,k,\infty} = v_{1,k} + \sum_{n\geq 1}\Big(& V_{p+1,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L+k,2}) + B_{p+1,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L+\delta(k),1}) \\ &+ v_{1,k}G_{p+1,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L,1})\Big) \quad k = 0,\dots,L_{p+1} = 2^{q-1}L\,. \end{aligned} \tag{2.11}$$

For $r = 1, \dots, q-1$,

$$\begin{aligned} v_{r,k,\infty} = v_{r,k} + \sum_{n\geq 1}\Big(& V_{p+r,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L+k,r+1}) \\ &+ B_{p+r,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L+\delta(k),1}) + v_{r,k}G_{p+r,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L,1})\Big) \\ & k = 0,\dots,L_{r+p} = (r-1)L + 2^{q-1}L, \end{aligned} \tag{2.12}$$

and finally

$$v_{p,k,\infty} = v_{p,k} + \sum_{n\geq 1}\Big(B_{p+q,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{\delta(k),1}) + v_{p,k}\mathrm{G}_{p+q,n}(\mathbf{y},\mathbf{z},\boldsymbol{\omega}^u_{L,1})\Big) \tag{2.13}$$

for $k = 0, \dots, L_{p+q} = 2^q L$. Collecting equations (2.7) up to (2.13), defines a formal map

$$\mathbf{F} : \mathbf{M} \to \mathbf{M}, \quad (\mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}) \mapsto (\mathbf{y}_\infty, \mathbf{z}_\infty, \mathbf{u}_\infty, \mathbf{v}_\infty)\,,$$

where

$$\mathbf{M} = \mathbb{R}^p \times \mathbb{R}^q \times \prod_{i=1}^{p} \mathbb{R}^{L_i} \times \prod_{j=1}^{q} \mathbb{R}^{L_{p+j}}.$$

Thanks to the Inverse Function Theorem, we write

$$\begin{gathered} y_i = y_{i,\infty} + \sum_{n\geq 1} \mathcal{H}_{i,n}(\mathbf{y}_\infty, \mathbf{z}_\infty, \mathbf{u}_\infty) \quad i = 1, \dots, p; \\ z_j = z_{j,\infty} + \sum_{n\geq 1} \mathcal{H}_{p+j,n}(\mathbf{y}_\infty, \mathbf{z}_\infty, \mathbf{v}_\infty) \quad j = 1, \dots, q\,, \end{gathered}$$

where each $\mathcal{H}_{i,n}$ and $\mathcal{H}_{p+j,n}$ are sum of monomials in the variables they depend upon. Thanks to a theorem of Borel, the formal map written above corresponds to the asymptotic of a smooth map

$$\hat{\Phi} : \mathbf{M} \to \mathbb{R}^p \times \mathbb{R}^q,\ (\mathbf{y}_\infty, \mathbf{z}_\infty, \mathbf{u}_\infty, \mathbf{v}_\infty) \mapsto (\mathbf{y}, \mathbf{z}) = \hat{\Phi}(\mathbf{x}_\infty, \mathbf{y}_\infty, \mathbf{u}_\infty, \mathbf{v}_\infty).$$

Define

$$\begin{gathered} t = t(\mathbf{y}_\infty) = \frac{1}{\log(|\alpha_1|^{A_1})} \log(\sum_{\ell=1}^{p} |y_\ell^{\{\infty\}}|^{A_\ell}), \quad i = 1 \dots p; \\ s = s(\mathbf{z}_\infty) = \frac{1}{\log(|\alpha_1|^{A_1})} \log(\sum_{\ell=1}^{q} |z_\ell^{\{\infty\}}|^{B_\ell}), \quad j = 1, \dots, q\,, \end{gathered} \tag{2.14}$$

where $A_1, \dots, A_p$, $B_1, \dots, B_q$ are positive real number such that

$$|\alpha_1|^{A_1} = |\alpha_2|^{A_2} = \dots = |\alpha_p|^{A_p} = |\alpha_{p+1}|^{B_1} = |\alpha_{p+2}|^{B_2} = \dots = |\alpha_{p+q}|^{B_q}.$$

Therefore

$$t_i(\mathbf{y}_\infty) = \frac{t(\mathbf{y}_\infty)}{\alpha_i}, \quad s_j(\mathbf{z}_\infty) = \frac{s(\mathbf{z}_\infty)}{\alpha_{i+p}}.$$

Observe that for each integer $i = 1, \dots, p+q$ we have

$$t_i\big(\mathbf{L}(\mathbf{y})\big) = t_i(\mathbf{y}) + \frac{1}{\alpha_i}, \quad s_j\big(\mathbf{L}(\mathbf{z})\big) = s_j(\mathbf{z}) + \frac{1}{\alpha_{p+j}}\,.$$

Replacing $\mathbf{u}_\infty$ and $\mathbf{v}_\infty$ by their initial definition (2.5), we define

$$\begin{gathered} \Phi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q, \\ (\mathbf{y}_\infty, \mathbf{z}_\infty) \mapsto \hat{\Phi}(\mathbf{x}_\infty, \mathbf{y}_\infty, \mathbf{u}_\infty, \mathbf{v}_\infty)\big|_{t=t(\mathbf{y}_\infty),\, s=s(\mathbf{z}_\infty)}\,, \end{gathered}$$

completing the proof of Theorem 1.2.

3. Gevrey class in the planar case

In two dimensions the singular normal form procedure from section 2 becomes simpler, and we will be able to perform formal computations and estimates in order to prove Theorem 1.5. To understand the nature of the expansion and more precisely the Gevrey class of it, we need to revisit some steps of the proof of Theorem 1 in this specific case. We shall hence shorten some similar straightforward computations in this section.

We recall from the introduction that

$$f:(x,y)\mapsto(\alpha x(1+H_1(u)),\beta y(1+H_2(u))) \tag{3.1}$$

with $\alpha^p\beta^q=1$ and where $u=x^py^q$. The process will yield, in subsection 3.3, a formal change of variables $(x,y)=\hat{H}(\tilde{x},\tilde{y})$, which will turn out to be of the form $\hat{H}(\tilde{x},\tilde{y})=\hat{\mathcal{G}}(\tilde{x},\tilde{y},s(\tilde{x})\tilde{x},t(\tilde{y})\tilde{y})$, where $\hat{\mathcal{G}}$ is a formal power series. Moreover it will turn out that $\hat{H}$ is of the form

$$(x,y)=(\tilde{x}(1+\Psi_1(\tilde{u},s(\tilde{x})\tilde{u},t(\tilde{y})\tilde{u})),\tilde{y}(1+\Psi_2(\tilde{u},s(\tilde{x})\tilde{u},t(\tilde{y})\tilde{u})))\,,$$

where $\tilde{u}=\tilde{x}^p\tilde{y}^q$ and where (Ψ_1,Ψ_2) is a formal power series.

To have the notation consistent, we will write $(F^0,G^0)=(\alpha H_1,\beta H_2)$. Just like in section 2 we extend f in (3.1) to the four dimensional diffeomorphism f^0 as follows:

$$f^0(x,y,s,t)=(\alpha x+xF^0(u),\beta y+yG^0(u),s+\alpha^{-1},t+\beta^{-1})=:(X,Y,S,T). \tag{3.2}$$

We write

$$\begin{aligned} U:=X^pY^q&=\alpha^px^p(1+\alpha^{-1}F^0(u))^p\beta^qy^q(1+\beta^{-1}G^0(u))^q\\ &=u(1+\alpha^{-1}F^0(u))^p(1+\beta^{-1}G^0(u))^q\\ &=u(1+H^0(u)) \end{aligned} \tag{3.3}$$

for some $H^0(u)=O(u)$.

3.1. **First step.** We introduce the first change of variables

$$g_1(x,y,s,t)=(x-xsF^0(u)),y-ytG^0(u),s,t)=(x_1,y_1,s,t) \tag{3.4}$$

eliminating terms in (3.2) that are not of order $O(u^2)$. Let us denote $f^1=\varphi^1f^0(\varphi^1)^{-1}$. Then

$$g_1f^0g_1^{-1}(x_1,y_1,s,t)=g_1(X,Y,S,T). \tag{3.5}$$

A straightforward computation shows that the first component of (3.5) writes

$$\begin{aligned} f_1^1(x_1,y_1,s,t)&=\alpha x_1+xF^0(u).(\alpha s+1-\alpha S)\\ &\quad-\alpha xS.(F^0(U)-F^0(u))-xF^0(u)SF^0(U). \end{aligned} \tag{3.6}$$

Notice that in line (3.6) we have

$$\alpha s+1-\alpha S=\alpha s+1-\alpha(s+\alpha^{-1})=0. \tag{3.7}$$

Hence

$$f_1^1(x_1,y_1,s,t)=\alpha x_1-\alpha xS.(F^0(U)-F^0(u))-xF^0(u)SF^0(U). \tag{3.8}$$

To be consistent with the sequel we denote $\omega_0(s)=1$ and $\omega_1(s)=s$ and remark that

$$\alpha\omega_1(s)+\omega_0(S)-\alpha\omega_1(S)=0. \tag{3.9}$$

We also denote

$$F^1(u)=\alpha(F^0(U)-F^0(u))+F^0(u)F^0(U). \tag{3.10}$$

Then we can write

$$f_1^1(x_1,y_1,s,t)=\alpha x_1+x\omega_1(S)F^1(u). \tag{3.11}$$

Remark that $F^1(u)=O(u^2)$.

In a similar way we denote $\tau_0(t) = 1$, $\tau_1(t) = t$, and define

$$G^1(u) = \beta(G^0(U) - G^0(u)) + G^0(u)G^0(U) \tag{3.12}$$

and obtain for the second component of f^1:

$$f_2^1(x_1, y_1, s, t) = \beta y_1 - y\tau_1(T)G^1(u). \tag{3.13}$$

We conclude that

$$\begin{aligned} f^1(x_1, y_1, s, t) &= \Big(\alpha x_1 - x\omega_1(s + \alpha^{-1})F^1(u), \alpha^{-1}y_1 \\ &\quad - y\tau_1(t + \beta^{-1})G^1(u), s + \alpha^{-1}, t + \beta^{-1}\Big) \\ &=: (X_1, Y_1, S, T). \end{aligned} \tag{3.14}$$

3.2. **Induction step.** Suppose that we have a sequence of conjugating transformations $g_1, \dots, g_n$ leading to a diffeomorphism $f^n = (g_1 \circ \dots \circ g_n)_* f$ that is of the form

$$\begin{aligned} f^n(x_n, y_n, s, t) &= \Big(\alpha x_n + (-1)^n x\omega_n(s + \alpha^{-1})F^n(u), \beta y_n \\ &\quad + (-1)^n y\tau_n(t + \beta^{-1})G^n(u), s + \alpha^{-1}, t + \beta^{-1}\Big) \\ &=: (X_n, Y_n, S, T). \end{aligned}$$

So we have the following diagram:

$$\begin{array}{ccccccc} (x, y, s, t) & \xrightarrow{g_1} & (x_1, y_1, s, t) & \xrightarrow{g_2} & \dots & \xrightarrow{g_n} & (x_n, y_n, s, t) \\ \downarrow f^0 & & \downarrow f^1 & & & & \downarrow f^n \\ (X, Y, S, T) & \xrightarrow{g_1} & (X_1, Y_1, S, T) & \xrightarrow{g_2} & \dots & \xrightarrow{g_n} & (X_n, Y_n, S, T)\,. \end{array} \tag{3.15}$$

We introduce the following new change of variables

$$\begin{aligned} &g_{n+1}(x_n, y_n, s, t) \\ &= (x_n + (-1)^{n+1}x\omega_{n+1}(s)F^n(u), y_n + (-1)^{n+1}y\tau_{n+1}(t)G^n(u), s, t) \\ &=: (x_{n+1}, y_{n+1}, s, t)\,, \end{aligned} \tag{3.16}$$

where ω_{n+1}, τ_{n+1} are yet to be determined. Here x and y are the first and second component of $(g_1 \circ \dots \circ g_n)^{-1}(x_n, y_n, s, t)$. Denote $f^{n+1} = g_{n+1} \circ f^n \circ g_{n+1}^{-1}$. The first component of f^{n+1} is equal to

$$f^{n+1}(x_{n+1}, y_{n+1}, s, t) = g_{n+1}(X_n, Y_n, S, T). \tag{3.17}$$

A straightforward computation shows that the first component of (3.17) reads

$$\begin{aligned} &f_1^{n+1}(x_{n+1}, y_{n+1}, s, t) \\ &= \alpha x_{n+1} + (-1)^n xF^n(u).(\alpha\omega_{n+1}(s) + \omega_n(S) - \alpha\omega_{n+1}(S)) \\ &\quad + (-1)^{n+1}x\omega_{n+1}(S) \cdot (\alpha(F^n(U) - F^n(u)) + F^0(u)F^n(U)). \end{aligned} \tag{3.18}$$

To eliminate the terms in line (3.18) we take ω_{n+1} to be a solution of the equation

$$\alpha\omega_{n+1}(s) + \omega_n(S) - \alpha\omega_{n+1}(S) = 0. \tag{3.19}$$

For $n = 1$ we already had $\omega_1(s) = s$. A straightforward calculation, by induction, shows that

$$\omega_n(s) = \frac{1}{n!}\prod_{j=0}^{n-1}(s + j\alpha^{-1}) \tag{3.20}$$

solves (3.19). So using equation (3.19) we obtain for (3.18):

$$\begin{aligned} &f_1^{n+1}(x_{n+1}, y_{n+1}, s, t) \\ &= \alpha x_{n+1} + (-1)^{n+1} x\omega_{n+1}(S) \cdot (\alpha(F^n(U) - F^n(u)) + F^0(u)F^n(U)). \end{aligned} \tag{3.21}$$

We denote

$$F^{n+1}(u) = \alpha(F^n(U) - F^n(u)) + F^0(u)F^n(U) \tag{3.22}$$

and thus

$$f_1^{n+1}(x_{n+1}, y_{n+1}, s, t) = \alpha x_{n+1} + (-1)^{n+1} x\omega_{n+1}(S)F^{n+1}(u). \tag{3.23}$$

In a completely similar way we define for all $n \geq 1$:

$$\tau_n(t) = \frac{1}{n!} \prod_{j=0}^{n-1} (t + j\beta^{-1}), \tag{3.24}$$

$$G^{n+1}(u) = \beta(G^n(U) - G^n(u)) + G^0(u)G^n(U)) \tag{3.25}$$

and obtain with φ^{n+1} a conjugacy of f^n with

$$\begin{aligned} &f^{n+1}(x_{n+1}, y_{n+1}, s, t) \\ &= \Big(\alpha x_{n+1} + (-1)^{n+1}\omega_{n+1}(s + \alpha^{-1})F^{n+1}(u), \\ &\quad \beta y_{n+1}\tau_{n+1}(t + \beta^{-1})G^{n+1}(u), s + \alpha^{-1}, t + \beta^{-1}\Big). \end{aligned} \tag{3.26}$$

Note that $(F^n(u), G^n(u)) = O(u^{n+1})$.

3.3. **Formal limit of the induction.** From subsection 3.2 we see that for each $n \geq 0$, using (3.16), we have

$$x_{n+1} = x + x \sum_{k=0}^{n} (-1)^{k+1} \omega_{k+1}(s) F^k(u) \tag{3.27}$$

$$y_{n+1} = y + y \sum_{k=0}^{n} (-1)^{k+1} \tau_{k+1}(t) G^k(u)\,. \tag{3.28}$$

We consider the formal limit $n \to \infty$ of this induction and write

$$g_\infty = \lim_{n\to\infty} g_n \circ \dots \circ g_1,$$

i.e.,

$$(x_\infty, y_\infty, s, t) = g_\infty(x, y, s, t)\,, \tag{3.29}$$

where

$$\begin{aligned} x_\infty &= x + x \sum_{k=0}^{\infty} (-1)^{k+1} \omega_{k+1}(s) F^k(u) \\ y_\infty &= y + y \sum_{k=0}^{\infty} (-1)^{k+1} \tau_{k+1}(t) G^k(u) \\ s &= s, \quad t = t. \end{aligned} \tag{3.30}$$

Then, writing $f^\infty = g_\infty \circ f^0 \circ g_\infty^{-1}$, we obtain

$$f^\infty(x_\infty, y_\infty, s, t) = (\alpha x_\infty, \beta y_\infty, s + \alpha^{-1}, t + \beta^{-1}) =: (X_\infty, Y_\infty, S, T), \tag{3.31}$$

which corresponds to the diagram

$$\begin{array}{ccc} (x,y,s,t) & \xrightarrow{g_\infty} & (x_\infty,y_\infty,s,t) \\ \downarrow f^0 & & \downarrow f^\infty \\ (X,Y,S,T) & \xrightarrow{g_\infty} & (X_\infty,Y_\infty,S,T). \end{array} \tag{3.32}$$

Let us explain how to obtain from the above a change of variables $(x,y) = H(x_\infty,y_\infty)$ conjugating f in (3.1) to its linearization $L:(x_\infty,y_\infty)\mapsto(\alpha x_\infty,\beta y_\infty)$. Let us define the map

$$h(x_\infty,y_\infty) = (\alpha^{-1}\log_{|\alpha|}|x_\infty|, \beta^{-1}\log_{|\beta|}|y_\infty|) =: (h_1(x_\infty),h_2(y_\infty))\,. \tag{3.33}$$

Then

$$\begin{aligned} h(X_\infty,Y_\infty) &= (\alpha^{-1}\log_{|\alpha|}|X_\infty|, \beta^{-1}\log_{|\beta|}|Y_\infty|) \\ &= (\alpha^{-1}(\log_{|\alpha|}|\alpha| + \log_{|\alpha|}|x_\infty|), \beta^{-1}(\log_{|\beta|}|\beta| + \log_{|\beta|}|y_\infty|)) \\ &= (\alpha^{-1},\beta^{-1}) + h(x_\infty,y_\infty). \end{aligned} \tag{3.34}$$

Hence, if we put

$$(s,t) = h(x_\infty,y_\infty) \tag{3.35}$$

in (3.31) we obtain

$$\begin{aligned} f^\infty(x_\infty,y_\infty,h(x_\infty,y_\infty)) &= (X_\infty,Y_\infty,h(x_\infty,y_\infty) + (\alpha^{-1},\beta^{-1})) \\ &= (X_\infty,Y_\infty,h(X_\infty,Y_\infty)). \end{aligned} \tag{3.36}$$

Hence from (3.32) we obtain the diagram

$$\begin{array}{ccc} (x,y,h(x_\infty,y_\infty)) & \xrightarrow{g_\infty} & (x_\infty,y_\infty,h(x_\infty,y_\infty)) \\ \downarrow f^0 & & \downarrow f^\infty \\ (X,Y,h(X_\infty,Y_\infty)) & \xrightarrow{g_\infty} & (X_\infty,Y_\infty,h(X_\infty,Y_\infty)). \end{array} \tag{3.37}$$

Let us consider the 'space direction' component function of g_∞ in (3.29), that is: we write $g_\infty(x,y,s,t) = (\Psi(x,y,s,t),s,t)$ where, thus, $(x_\infty,y_\infty) = \Psi(x,y,s,t)$. Also here we insert (3.35) and investigate the equation

$$(x_\infty,y_\infty) = \Psi(x,y,h(x_\infty,y_\infty)). \tag{3.38}$$

We will show below that we can solve this as an implicit function problem, that is: we try to write (x,y) as a function H of (x_∞,y_∞), i.e. equation (3.38) holds if and only if $(x,y) = H(x_\infty,y_\infty)$. From (3.37) it follows that also

$$(X_\infty,Y_\infty) = \Psi(X,Y,h(X_\infty,Y_\infty)) \tag{3.39}$$

and hence $(X,Y) = H(X_\infty,Y_\infty)$. We conclude here that the diagram

$$\begin{array}{ccc} (x,y) & \xleftarrow{H} & (x_\infty,y_\infty) \\ \downarrow f & & \downarrow L \\ (X,Y) & \xleftarrow{H} & (X_\infty,Y_\infty) \end{array} \tag{3.40}$$

holds; this means that H linearizes f.

Let us now go to the solution of the equation (3.38). We use the expression (3.33) for h and introduce extra variables:

$$\begin{aligned} \xi &= h_1(x_\infty).x, \quad \xi_1 = h_1(x_\infty).x_\infty \\ \eta &= h_2(y_\infty).y, \quad \eta_1 = h_2(y_\infty).y_\infty. \end{aligned} \tag{3.41}$$

In the formulas in (3.30), which define Ψ, we observe that $\omega_{k+1}(s), \tau_{k+1}(t)$ are polynomials of degree at most $k+1$ and that $F^k(u) = O(u^{k+1})$; hence we can write Ψ in the form

$$(x_\infty, y_\infty) = \Psi(x, y, s, t) = (x + x\psi_1(u, su), y + y\psi_2(u, tu)), \tag{3.42}$$

where ψ_1, ψ_2 are $O(1)$ formal power series. Herein we substitute $(s,t) = h(x_\infty, y_\infty)$ and obtain

$$\begin{aligned}\Psi(x, y, h(x_\infty, y_\infty)) &= (x + x\psi_1(u, h_1(x_\infty)u), y + y\psi_2(u, h_2(y_\infty)u)) \\ &= (x + x\psi_1(u, \xi x^{p-1}y^q), y + y\psi_2(u, \eta x^p y^{q-1})) \\ &= (x_\infty, y_\infty).\end{aligned} \tag{3.43}$$

Hence from (3.41):

$$\begin{aligned}\xi_1 &= h_1(x_\infty).(x + x\psi_1(u, \xi x^{p-1}y^q)) = \xi + \xi\psi_1(u, \xi x^{p-1}y^q) \\ \eta_1 &= h_2(y_\infty).(y + y\psi_2(u, \xi x^p y^{q-1})) = \eta + \eta\psi_2(u, \eta x^p y^{q-1})\end{aligned}. \tag{3.44}$$

So we can summarize:

$$\begin{aligned}x_\infty &= x + x\psi_1(u, \xi x^{p-1}y^q) \\ y_\infty &= y + y\psi_2(u, \eta x^p y^{q-1}) \\ \xi_1 &= \xi + \xi\psi_1(u, \xi x^{p-1}y^q) \\ \eta_1 &= \eta + \eta\psi_2(u, \eta x^p y^{q-1}).\end{aligned} \tag{3.45}$$

Let us therefore consider the map $\mathcal{F} : (x, y, \xi, \eta) \mapsto (x_\infty, y_\infty, \xi_1, \eta_1)$. The linear part of $\mathcal{F}$ at $(x, y, \xi, \eta) = (0, 0, 0, 0)$ is the identity. Hence we can apply the inverse function theorem, at least on the level of formal power series. Consequently, we can write $(x, y, \xi, \eta) = \mathcal{F}^{-1}(x_\infty, y_\infty, \xi_1, \eta_1)$. We use the first components of this inverse now. Let us denote $\pi_{(x,y)}(x, y, \xi, \eta) = (x, y)$; then we can write

$$(x, y) = (\pi_{(x,y)} \circ \mathcal{F}^{-1})(x_\infty, y_\infty, h_1(x_\infty).x_\infty, h_2(y_\infty).y_\infty) =: H(x_\infty, y_\infty). \tag{3.46}$$

Up to terms with a zero ∞-jet, we may take ψ_1, ψ_2 to be C^∞ functions. The linear map $(x, y) \mapsto (\alpha x, \beta y)$ lifts to $(x, y, \xi, \eta) = (x, y, h_1(x).x, h_2(y).y) \mapsto (\alpha x, \beta y, \alpha\xi + x, \beta\eta + y)$, which is also hyperbolic. With the methods from the classical Sternberg-Chen theorem, see [11], it follows that there exists a transformation that is C^∞ in the variables $(x_\infty, y_\infty, h_1(x_\infty).x_\infty, h_2(y_\infty).y_\infty))$, and infinitely tangent to the identity, which linearizes f.

In the planar case, additionally to Theorem 1.2, the following proposition holds.

Proposition 3.1. *Assume that $m = 2$ in the statement of Theorem 1.2 and the origin is of saddle type. Then the linearization Φ given by Theorem 1.2 has an inverse which is also of Mourtada type.*

Proof. Observe first that equation (3.33) implies

$$\begin{aligned}x_\infty &= x[1 + G_1(x, y, \xi)] \\ y_\infty &= y[1 + G_2(x, y, \eta)] \\ \xi_1 &= \xi[1 + G_1(x, y, \xi)] \\ \eta_1 &= \eta[1 + G_2(x, y, \eta)]\end{aligned} \tag{3.47}$$

for some smooth functions G_1 and G_2 defined near 0. From the first two lines above we have:

$$\log|x_\infty| = \log|x| + \log[1 + G_1(x, y, \xi)]$$

$$\log|y_\infty| = \log|y| + \log[1 + G_2(x, y, \eta)].$$

Since $z \mapsto \log(1+z)$ is analytic near $z = 0$ and using the fact that

$$\log(1+z) = z - z^2/2 + \dots,$$

from (3.21) we can write

$$\begin{aligned} h_1(x_\infty) &= h_1(x) + J_1(x, y, \xi) \\ h_2(y_\infty) &= h_2(y) + J_2(x, y, \eta)\,, \end{aligned} \tag{3.48}$$

where both J_1 and J_2 are smooth. This implies

$$\begin{aligned} xh_1(x_\infty) &= xh_1(x) + xJ_1(x, y, \xi) \\ yh_2(y_\infty) &= yh_2(y) + yJ_2(x, y, \eta). \end{aligned} \tag{3.49}$$

Denote $xh_1(x) = w_1$, $yh_1(y) = z_1$ then

$$\begin{aligned} \xi &= w_1 + xJ_1(x, y, \xi) \\ \eta &= z_1 + yJ_2(x, y, \eta). \end{aligned} \tag{3.50}$$

Thanks to the Implicit Function Theorem, we have

$$\begin{aligned} \xi &= w_1 + xK_1(x, y, w_1) \\ \eta &= z_1 + yK_2(x, y, z_1) \end{aligned} \tag{3.51}$$

for some smooth functions K_1 and K_2. Finally, we can write

$$G_1(x, y, \xi) = M_1(x, y, w_1), \quad G_2(x, y, \eta) = M_2(x, y, z_1),$$

for some smooth functions M_1 and M_2 and rewriting the first 2 lines of (3.47) we obtain

$$\begin{aligned} x_\infty &= x + xM_1(x, y, w_1) = x + xM_1(x, y, xh_1(x)) \\ y_\infty &= y + yM_2(x, y, z_1) = y + yM_2(x, y, yh_2(y)), \end{aligned} \tag{3.52}$$

completing the proof. $\square$

3.4. **About the use of Mourtada type functions in applications.** Recall that $\boldsymbol{\varphi} = \boldsymbol{\Phi} \circ \mathbf{L} \circ \boldsymbol{\Phi}^{-1}$, where $\boldsymbol{\Phi}$ is of Mourtada type. If a point $\mathbf{x}$ is such that the set

$$\{\boldsymbol{\varphi}^i(\mathbf{x}),\ 0 \le i \le n\}$$

is included in the neighbourhood of linearization, (for some $n > 1$) we have

$$\boldsymbol{\varphi}^i(\mathbf{x}) = \boldsymbol{\Phi} \circ \mathbf{L}^i \circ \boldsymbol{\Phi}^{-1}(\mathbf{x}), \quad 0 \le i \le n.$$

In the situation where the inverse $\boldsymbol{\Phi}^{-1}$ of the linearization is also of Mourtada type, (which is the case in two dimensions, thanks to Proposition 3.1), one can deduce the asymptotics of $\boldsymbol{\Phi}^i(x)$. Although this gives us only local information, this tools is of great help in the study of the global dynamics of a map with orbits approaching the hyperbolic fixed point arbitrary close. For instance when such a map admits a homoclinic tangency that both the stable and the unstable manifold of the saddle point intersect at a given point in a non-transversal manner. This case has been studied by [17]. The authors computed the return map near the tangency and using rescaling and renormalization method see [19] they show that the return map is C^3 close to the Hénon map. As a consequence, the authors deduce the existence of complex dynamics including strange attractors and the existence of infinitely many sinks. However, the author make the assumption that the map is locally linearizable near the hyperbolic point. This latter assumption is necessary, precisely, to compute the part of the orbit of a point passing close the saddle. In the case where resonances occur such a map is no longer locally linearizable, however, thanks to Theorem 1.2, the asymptotic of the return map can be computed.

The idea that the presence of resonances at a fixed point for a map (or a singularity for a vector field) may increase the complexity of the dynamic is not new. For instance in [16], the authors study the dynamics that appear in the unfolding of a degenerate homoclinic orbit of a three-dimensional vector field where a resonance occurs for the unperturbed system. It is shown that the presence of a resonant term increases the complexity of the bifurcating dynamics. More precisely the authors show the existence of a "cubic" strange attractor, that is a strange attractor with a topological entropy close to log(3), see [16] for more details.

3.5. **Proof of Theorem 1.5.** First we treat the formal part of Theorem 1.5. From (3.42) we obtain

$$(x,y) = (x_\infty(1+\psi_1(u,su))^{-1}, y_\infty(1+\psi_2(u,tu)^{-1}). \tag{3.53}$$

We denote $u_\infty = x_\infty^p y_\infty^q$ and have, from (3.42)

$$\begin{aligned} u_\infty &= x^p y^q (1+\psi_1(u,su))^p(1+\psi_2(u,tu))^q \\ &= u(1+\psi_1(u,su))^p(1+\psi_2(u,tu))^q. \end{aligned} \tag{3.54}$$

Let us introduce more variables: $(\sigma,\tau,\sigma_1,\tau_1) := (su, tu, su_\infty, tu_\infty)$. We have

$$\begin{aligned} u_\infty &= u(1+\psi_1(u,\sigma))^p(1+\psi_2(u,\tau))^q \\ \sigma_1 &= \sigma(1+\psi_1(u,\sigma))^p(1+\psi_2(u,\tau))^q \\ \tau_1 &= \tau(1+\psi_1(u,\sigma))^p(1+\psi_2(u,\tau))^q \end{aligned} \tag{3.55}$$

which we can be inverted as

$$\begin{aligned} u &= u_\infty(1+\varphi(u_\infty,\sigma_1,\tau_1)) \\ \sigma &= \sigma_1(1+\varphi(u_\infty,\sigma_1,\tau_1)) \\ \tau &= \tau_1(1+\varphi(u_\infty,\sigma_1,\tau_1)) \end{aligned} \tag{3.56}$$

for some formal power series φ. Consequently, from (3.53):

$$\begin{aligned} x &= x_\infty(1+\psi_1(u_\infty(1+\varphi(u_\infty,\sigma_1,\tau_1)), \sigma_1(1+\varphi(u_\infty,\sigma_1,\tau_1)))^{-1} \\ y &= y_\infty(1+\psi_2(u_\infty(1+\varphi(u_\infty,\sigma_1,\tau_1)), \tau_1(1+\varphi(u_\infty,\sigma_1,\tau_1)))^{-1}. \end{aligned} \tag{3.57}$$

Second, we give the necessary estimates for Theorem 1.5. We shall express the functions ω_n and τ_n, defined in (3.20) resp. (3.24), using Stirling numbers $\begin{bmatrix} n \\ j \end{bmatrix}$ of the first kind as follows. Let us remind that these numbers can be obtained using the generating function

$$x(x+1)\dots(x+n-1) = \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j, \tag{3.58}$$

see for instance [12] for more information. We will use the fact that

$$n! = \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \tag{3.59}$$

which is obtained by putting $x = 1$ in (3.58). We can write

$$\begin{aligned}\omega_n(s) &= \frac{\alpha^{-n}}{n!}\alpha s(\alpha s+1)\dots(\alpha s+n-1)\\ &= \frac{\alpha^{-n}}{n!}\sum_{j=1}^{n}\begin{bmatrix} n\\ j\end{bmatrix}(\alpha s)^j\end{aligned} \tag{3.60}$$

and a similar formula holds for τ_n.

Assume that we can write

$$F^0(u) = \sum_{j\geq 1} a_j u^j \tag{3.61}$$

with $u = x^p y^q$; let us abbreviate $d = p + q$. If F_0 is Gevrey-1 in (x, y), there exist $C > 0$ and $r > 0$ with $|a_j| \leq C r^j j!^d$ for all $j \geq 1$; similarly for G^0. In order to abridge the estimates below, we first perform a rescaling $(\tilde{x}, \tilde{y}) = R_\rho(x, y) := (\rho x, \rho y)$. The diffeomorphism then becomes, writing $\tilde{u} = \tilde{x}^p\tilde{y}^q$:

$$\tilde{f}(\tilde{x},\tilde{y}) = R_\rho\circ f\circ R_\rho^{-1}(\tilde{x},\tilde{y}) = (\alpha\tilde{x}+\tilde{x}F^0(\frac{1}{\rho^d}\tilde{u}), \beta\tilde{y}+\tilde{y}G^0(\frac{1}{\rho^d}\tilde{u})) := (\tilde{X},\tilde{Y}) \tag{3.62}$$

and we also check that

$$\tilde{U} := \tilde{X}^p\tilde{Y}^q = \tilde{u}(1+H^0(\frac{1}{\rho^d}\tilde{u}))\,, \tag{3.63}$$

where H^0 is as in (3.3). For $\rho > 0$ large enough we have

$$\frac{r}{\rho^d} \leq \min\{\frac{1}{C}, 1\} \tag{3.64}$$

and hence we can majorate the series for $F^0(\frac{1}{\rho^d}\tilde{u})$ as follows (the symbol $\preceq$ means: 'is majorated by'):

$$F^0(\frac{1}{\rho^d}\tilde{u}) = \sum_{j\geq 1} a_j\frac{1}{\rho^d}\tilde{u}^j \preceq \sum_{j\geq 1} C(\frac{r}{\rho^d})^j j!^d\tilde{u}^j \preceq \sum_{j\geq 1} j!^d\tilde{u}^j. \tag{3.65}$$

Similar estimates for the series for $\tilde{U}$ hold. We conclude here that, up to a rescaling, we may assume that F^0 and U have the majorant (we omit form here on all the tilde symbols) $\sum_{j=1}^{\infty} j!^d u^j$. We reuse the symbol '$U$' from here on for this majorating series.

We will also need the following majorations, given $n \in \mathbf{N}_0$. First of all,

$$\begin{aligned}U^n - u^n &= (U-u)(U^{n-1}+uU^{n-2}+\dots+u^{n-2}U+u^{n-1})\\ &\preceq (U-u)nU^{n-1}.\end{aligned} \tag{3.66}$$

Second:

$$\begin{aligned} U^n &= \sum_{k_1\geq 1} k_1!^d u^{k_1}\cdots \sum_{k_n\geq 1} k_n!^d u^{k_n} \\ &= \sum_{m\geq n}\sum_{\substack{k\in\mathbf{N}_0^n\\ |k|=m}} k_1!^d\dots k_n!^d u^m \\ &\preceq \sum_{m\geq n}\sum_{\substack{k\in\mathbf{N}_0^n\\ |k|=m}} \frac{|k|!^d}{n!^d}u^m \\ &= \frac{1}{n!^d}\sum_{m\geq n} m!^d u^m \sum_{\substack{k\in\mathbf{N}_0^n\\ |k|=m}} 1 \\ &= \frac{1}{n!^d}\sum_{m\geq n} m!^d u^m \binom{m-1}{n-1}. \end{aligned} \tag{3.67}$$

We now give, inductively, a majorant for $F^N(u)$, $N\geq 0$. Suppose by induction that there exist $A, r>0$ and $j\in\mathbf{N}$, depending on N, such that

$$F^N(u)\preceq A\sum_{n\geq N+1} r^n(n+j)!^d u^n. \tag{3.68}$$

We know that

$$F^{N+1}(u)=\alpha(F^N(U)-F^N(u))+F^0(u)F^N(U). \tag{3.69}$$

Majorant of $F^N(U)$. We have, using (3.67):

$$\begin{aligned} F^N(U) &\preceq A\frac{1}{U^j}\sum_{n\geq N+j+1} r^{n-j} n!^d U^n \\ &\preceq A\frac{1}{U^j}\sum_{n\geq N+j+1} r^{n-j}\sum_{m\geq n} m!^d u^m\binom{m-1}{n-1}. \end{aligned} \tag{3.70}$$

Since $\sum_{n\geq N+j+1}\sum_{m\geq n}=\sum_{m\geq N+j+1}\sum_{n=N+j+1}^m$ we can continue (3.70) as follows:

$$\begin{aligned} F^N(U) &\preceq A\frac{1}{U^j}\sum_{m\geq N+j+1}\sum_{n=N+j+1}^{m} r^{n-j} m!^d u^m\binom{m-1}{n-1} \\ &= A\frac{1}{U^j}\sum_{m\geq N+j+1} m!^d u^m\sum_{n=N+j}^{m-1} r^{n-j+1}\binom{m-1}{n} \\ &\preceq A\frac{r^{-j+1}}{U^j}\sum_{m\geq N+j+1} m!^d u^m\sum_{n=0}^{m-1} r^n\binom{m-1}{n} \\ &= A\frac{r^{-j+1}}{U^j}\sum_{m\geq N+j+1} m!^d u^m(r+1)^{m-1} \\ &\preceq A\frac{r^{-j+1}}{u^j}\sum_{m\geq N+j+1} m!^d u^m(r+1)^{m-1} \\ &= A(1+\frac{1}{r})^{j-1}\sum_{m\geq N+1}(r+1)^m(m+j)!^d u^m. \end{aligned} \tag{3.71}$$

Majorant of $F^N(U)-F^N(u)$. One has, using also (3.66):

$$\begin{aligned}
&F^N(U)-F^N(u)\\
&\preceq A\sum_{n\geq N+1} r^n(n+j)!^d(U^n-u^n)\\
&\preceq A(U-u).\sum_{n\geq N+1} r^n(n+j)!^d nU^{n-1}\\
&\preceq A(U-u).\sum_{n\geq N+1} r^n(n+j+1)!^d U^{n-1}\\
&= A(U-u)\frac{1}{U^{j+2}}\sum_{n\geq N+j+2} r^{n-j-1}n!^d U^n\\
&\preceq A(U-u)\frac{1}{U^{j+2}}\sum_{n\geq N+j+2} r^{n-j-1}\sum_{m\geq n} m!^d u^m\binom{m-1}{n-1} \qquad (3.72)\\
&= A(U-u)\frac{1}{U^{j+2}}\sum_{m\geq N+j+2} m!^d u^m\sum_{n=N+j+1}^{m-1} r^{n-j}\binom{m-1}{n}\\
&\preceq A(U-u)\frac{r^{-j}}{U^{j+2}}\sum_{m\geq N+j+2} m!^d u^m(r+1)^{m-1}\\
&\preceq A(U-u)\frac{r^{-j}}{u^{j+2}}\sum_{m\geq N+j+2} m!^d u^m(r+1)^{m-1}\\
&= A(U-u)(1+\frac{1}{r})^j\sum_{m\geq N}(m+j+2)!^d u^m(r+1)^{m+1}\,.
\end{aligned}$$

Since $U-u\preceq\sum_{k\geq 2}k!^d u^k$ we can continue (3.72) as follows:

$$\begin{aligned}
&F^N(U)-F^N(u)\\
&\preceq A(1+\frac{1}{r})^j\sum_{k\geq 2}k!^d u^k\sum_{m\geq N}(m+j+2)!^d u^m(r+1)^{m+1}\\
&\preceq \frac{1}{2^d}A(1+\frac{1}{r})^j\sum_{l\geq N+2}\sum_{\substack{k\geq 2,m\geq N\\ k+m=l}}(k+m+j+2)!^d u^l(r+1)^{m+1}\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^j\sum_{l\geq N+2}(l+j+2)!^d u^l\sum_{m=N}^{l-2}(r+1)^{m+1} \qquad (3.73)\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^j\sum_{l\geq N+2}(l+j+2)!^d u^l(r+1)^{N+1}\frac{(r+1)^{l-N-1}-1}{r}\\
&\preceq \frac{1}{2^d}A(1+\frac{1}{r})^j\frac{1}{r}\sum_{l\geq N+2}(l+j+2)!^d u^l(r+1)^l.
\end{aligned}$$

Majorant of $F^0(u).F^N(U)$. We have

$$\begin{aligned}
&F^0(u).F^N(U)\\
&\preceq\sum_{n\geq 1}n!^d u^n.A(1+\frac{1}{r})^{j-1}\sum_{m\geq N+1}(r+1)^m(m+j)!^d u^m
\end{aligned}$$

$$\begin{aligned}
&\preceq \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\sum_{n\geq 1}\sum_{m\geq N+1}(r+1)^m(n+m+j)!^d u^{n+m}\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\sum_{l\geq N+2}\sum_{\substack{n\geq 1, m\geq N+1\\ n+m=l}}(r+1)^m(l+j)!^d u^{l}\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\sum_{l\geq N+2}(l+j)!^d u^{l}\sum_{\substack{n\geq 1, m\geq N+1\\ n+m=l}}(r+1)^m\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\sum_{l\geq N+2}(l+j)!^d u^{l}\sum_{m=N+1}^{l-1}(r+1)^m\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\sum_{l\geq N+2}(l+j)!^d u^{l}(r+1)^{N+1}\frac{(r+1)^{l-N-1}-1}{r}\\
&\preceq \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\frac{1}{r}\sum_{l\geq N+2}(l+j)!^d u^{l}(r+1)^{l}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
F^{N+1}(u) &\preceq |\alpha|\frac{1}{2^d}A(1+\frac{1}{r})^{j}\frac{1}{r}\sum_{l\geq N+2}(l+j+2)!^d u^{l}(r+1)^{l}\\
&\quad + \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\frac{1}{r}\sum_{l\geq N+2}(l+j)!^d u^{l}(r+1)^{l}\\
&\preceq |\alpha|\frac{1}{2^d}A(1+\frac{1}{r})^{j}\frac{1}{r}\sum_{l\geq N+2}(l+j+2)!^d u^{l}(r+1)^{l}\\
&\quad + \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\frac{1}{r}\sum_{l\geq N+2}(l+j+2)!^d u^{l}(r+1)^{l}\\
&= \frac{1}{2^d}A(1+\frac{1}{r})^{j-1}\frac{1}{r}(|\alpha|(1+\frac{1}{r})+1)\sum_{l\geq N+2}(l+j+2)!^d u^{l}(r+1)^{l}.
\end{aligned} \tag{3.74}$$

The constants A, r and j in (3.68) depend on N so we write $A = A_N$, $r = r_N$ and $j = j_N$. From (3.74) we obtain the recursion

$$\begin{gathered}
A_{N+1} = \frac{1}{2^d}A_N(1+\frac{1}{r_N})^{j_N-1}\frac{1}{r_N}(|\alpha|(1+\frac{1}{r_N})+1)\\
r_{N+1} = r_N + 1\\
j_{N+1} = j_N + 2.
\end{gathered} \tag{3.75}$$

together with initial conditions:

$$A_0 = 1, \quad r_0 = 1, \quad j_0 = 0, \tag{3.76}$$

so $r_N = N+1$ and $j_N = 2N$. We obtain

$$A_{N+1} = \frac{1}{2^d}A_N(1+\frac{1}{N+1})^{2N-1}\frac{1}{N+1}(|\alpha|(1+\frac{1}{N+1})+1). \tag{3.77}$$

Note that $A_{N+1} \leq \frac{1}{2^d}A_N\frac{1}{N+1}e^2(2|\alpha|+1)$. Let us abbreviate $E := \frac{1}{2^d}e^2(2|\alpha|+1)$ and examine the equation

$$A_{N+1} = A_N\frac{1}{N+1}E \tag{3.78}$$

which has the obvious solution

$$A_N = \frac{E^N}{N!}. \tag{3.79}$$

We conclude here that we may take the majorating series

$$F^N(u) \preceq A_N \sum_{n\geq N+1} (N+1)^n (n+2N)!^d u^n\,. \tag{3.80}$$

Majorant of the series for g_∞. Let us consider the majorating series

$$S(s,u) := \sum_{k\geq 0} \omega_{k+1}(s) F^k(u) \tag{3.81}$$

with F_k as in the right hand side of (3.80) and where we replace α by $|\alpha|$ and denote it again α. Thus:

$$\begin{aligned} S(u,s) &= \sum_{k\geq 0} \omega_{k+1}(s) A_k \sum_{n\geq k+1} (k+1)^n (n+2k)!^d u^n \\ &= \sum_{n\geq 1} \sum_{k=0}^{n-1} \omega_{k+1}(s) A_k (k+1)^n (n+2k)!^d u^n \\ &= \sum_{n\geq 1} u^n \sum_{k=0}^{n-1} \omega_{k+1}(s) A_k (k+1)^n (n+2k)!^d. \end{aligned} \tag{3.82}$$

We insert the expression (3.60) for $\omega_{k+1}(s)$:

$$S(u,s) = \sum_{n\geq 1} u^n \sum_{k=0}^{n-1} \frac{\alpha^{-k-1}}{(k+1)!} \sum_{j=1}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (\alpha s)^j A_k (k+1)^n (n+2k)!^d \tag{3.83}$$

and since $\sum_{k=0}^{n-1}\sum_{j=1}^{k+1} = \sum_{j=1}^{n}\sum_{k=j-1}^{n-1}$ we can rewrite this as

$$\begin{aligned} S(u,s) &= \sum_{n\geq 1} u^n \sum_{j=1}^{n} \sum_{k=j-1}^{n-1} \frac{\alpha^{-k-1}}{(k+1)!} \begin{bmatrix} k+1 \\ j \end{bmatrix} (\alpha s)^j A_k (k+1)^n (n+2k)!^d \\ &= \sum_{n\geq 1} u^n \sum_{j=1}^{n} (\alpha s)^j \sum_{k=j-1}^{n-1} \frac{\alpha^{-k-1}}{(k+1)!} \begin{bmatrix} k+1 \\ j \end{bmatrix} A_k (k+1)^n (n+2k)!^d \\ &= \sum_{n\geq 1} u^n \sum_{j=1}^{n} (\alpha s)^j \sum_{k=j}^{n} \frac{\alpha^{-k}}{k!} \begin{bmatrix} k \\ j \end{bmatrix} A_{k-1} k^n (n+2k-2)!^d. \end{aligned} \tag{3.84}$$

Let us abbreviate the last factor in (3.84):

$$d_{nj} := \sum_{k=j}^{n} \frac{\alpha^{-k}}{k!} \begin{bmatrix} k \\ j \end{bmatrix} A_{k-1} k^n (n+2k-2)!^d \tag{3.85}$$

so

$$S(u,s) = \sum_{n\geq 1} u^n \sum_{j=1}^{n} (\alpha s)^j d_{nj}. \tag{3.86}$$

We want to express this as a series in the variables u and su:

$$\begin{aligned} S(u,s) &= \sum_{n\geq 1}\sum_{j=1}^{n} d_{nj}u^{n-j}(\alpha us)^j \\ &= \sum_{j\geq 1}\sum_{n\geq j} d_{nj}u^{n-j}(\alpha us)^j \\ &= \sum_{j\geq 1}\sum_{i\geq 0} d_{i+j,j}u^{i}(\alpha us)^j. \end{aligned} \tag{3.87}$$

Let us now examine the coefficients d_{nj}. For this we will use the fact that $\begin{bmatrix}k\\j\end{bmatrix} \leq k!$:

$$\begin{aligned} d_{nj} &= \sum_{k=j}^{n} \frac{\alpha^{-k}}{k!}\begin{bmatrix}k\\j\end{bmatrix}\frac{E^{k-1}}{(k-1)!}k^n(n+2k-2)!^d \\ &\leq \sum_{k=j}^{n} \frac{\alpha^{-k}}{k!}k!\frac{E^{k-1}}{(k-1)!}k^n(n+2k-2)!^d \\ &= \alpha^{-1}\sum_{k=j}^{n} \frac{(\alpha^{-1}E)^{k-1}}{(k-1)!}k^n(n+2k-2)!^d \\ &\leq \alpha^{-1}\sum_{k=j}^{n} \frac{(\alpha^{-1}E)^{k-1}}{(k-1)!}n^n(n+2n-2)!^d \\ &= \alpha^{-1}n^n(3n-2)!^d\sum_{k=j}^{n} \frac{(\alpha^{-1}E)^{k-1}}{(k-1)!} \\ &\leq \alpha^{-1}n^n(3n-2)!^d\exp(\alpha^{-1}E). \end{aligned} \tag{3.88}$$

From the fact that

$$n^n \leq \frac{1}{\sqrt{2\pi}}n^{-1/2}e^n n! \tag{3.89}$$

it follows that we can continue (3.88) by:

$$\begin{aligned} d_{nj} &\leq \alpha^{-1}\frac{1}{\sqrt{2\pi}}n^{-1/2}e^n n!(3n-2)!^d\exp(\alpha^{-1}E) \\ &\leq \alpha^{-1}\frac{1}{\sqrt{2\pi}}n^{-1/2}e^n \frac{1}{2^d}(4n-2)!^d\exp(\alpha^{-1}E). \end{aligned} \tag{3.90}$$

Let us abbreviate the constant

$$\alpha^{-1}\frac{1}{\sqrt{2\pi}}\frac{1}{2^d}\exp(\alpha^{-1}E) =: B \tag{3.91}$$

then we conclude that

$$d_{nj} \leq Bn^{-1/2}e^n(4n-2)!^d \tag{3.92}$$

for all $n\geq 1$ and $j\geq 1$. For (3.87) we need

$$d_{i+j,j} \leq B(i+j)^{-1/2}e^{i+j}(4i+4j-2)!^d. \tag{3.93}$$

Note that in general, for I,J positive integers we have $(I+J)! \leq 2^{I+J}I!J!$: this follows from the binomial formula $\sum_{I=0}^{M}\binom{M}{I} = \sum_{I=0}^{M}\frac{M}{I!(M-I)!} = 2^M$. Hence we can continue (3.93) as follows:

$$d_{i+j,j} \leq B(i+j)^{-1/2}e^{i+j}2^{d(4i+4j-2)}(4i)!^d(4j-2)!^d \tag{3.94}$$

and hence (3.87) becomes

$$S(u,s) \preceq B \sum_{j\geq 1}\sum_{i\geq 0}(i+j)^{-1/2}e^{i+j}2^{d(4i+4j-2)}(4i)!^d(4j-2)!^d u^i(\alpha us)^j\,, \tag{3.95}$$

where the right hand side is thus a majorant series of Gevrey order $4d$ in the variables (u, su). In the inversion (3.56) we can apply an inverse function theorem for Gevrey series [2] and infer that the series $(\psi_1, \psi_2, \varphi)$ appearing in the right hand side of (3.57) are of Gevrey order at most $4d$ in their variables. This completes the proof of Theorem 1.5.

3.6. **Final remarks and questions.**

3.6.1. *Smoothness.* A natural question is whether Theorem 1.2 provides the most efficient linearization in terms of degree of differentiability in the variable $\mathbf{x}$, although this is not the target of this paper. It is not hard to see that in general, when choosing the tag functions t and s introduced in (2.14), the degree of differentiability in $\mathbf{x}$ proposed by Samovol's in [23] is not always reached, cf. the counterexamples in the vector field case in [7, 8]. The reason comes from the fact that our technique does not distinguish between one resonant term and another, more precisely the tag function associated with each resonant term on a single line is the same. However, we believe that it is possible to construct a similar theory where each resonant term is treated independently taking Samovol's conditions into account. We expect the asymptotics to be much more complicated. Having a more complicated normal form procedure in the diffeomorphism case, using tag functions, to increase the degree of differentiability is not the main goal of this paper, but might be worth knowing.

3.6.2. *Unfolding of a resonance.* It would be interesting to elaborate on similar theorems for a family of diffeomorphisms. We believe that Theorem 1.2 can be generalized to family of maps using the idea of replacing a logarithmic function with a 'compensator' like for example in formula (3.96) below. The difficulty one encounters when dealing with a map depending on a parameter, is that a resonance is typically not stable. The eigenvalues at the singular point will vary, and the resonance conditions between them will change.

For example in the planar case, a $p : -q$ resonance is unfolded by one parameter, say ε close to 0. Let φ_ε be the corresponding family of diffeomorphisms like in Definition 1.4, with eigenvalues of $d\varphi_\varepsilon(0)$ equal to $\alpha(\varepsilon), \beta(\varepsilon)$. Up to a reparametrization we can arrange that we have the 'almost resonance' relation $\alpha(\varepsilon)^{p+\varepsilon}\beta(\varepsilon)^q = 1$. Since the rational numbers are dense in the real numbers, we will inevitably generate additional resonances of order $O(1/\varepsilon)$ if ε is small. For such a family there are normal form results in the finitely smooth category [15], but one will have to confine the domain for ε if the order of normalization increases, even on the formal level. Also, when trying to remove resonant monomials, this could presumably be done using unfoldings of the functions like $\log|x|$; these are called Écalle-Roussarie compensators and are of the form

$$\omega_\varepsilon(x) = \frac{1-|x|^{-\varepsilon}}{\varepsilon}; \tag{3.96}$$

remark that $\lim_{\varepsilon\to 0}\omega_\varepsilon(x) = \log|x|$. Some unanswered questions arise: what happens for an infinite number of steps? how fast will the domain for ε shrink? is there a kind of Gevrey asymptotics when starting form analyticity?

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