Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 240, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POSITIVE SOLUTIONS FOR SUPERLINEAR RIEMANN-LIOUVILLE FRACTIONAL BOUNDARY-VALUE PROBLEMS

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Communicated by Marco Squassina

Abstract. Using a perturbation argument, we establish the existence and uniqueness of a positive continuous solution for the following superlinear Riemann-Liouville fractional boundary-value problem

$$D^\alpha u(x) - u(x)\varphi(x,u(x)) = 0, \quad 0 < x < 1,$$
$$u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha} u''(x) = 0, \quad u''(1) = a > 0,$$

where $3 < \alpha \le 4$ and $\varphi(x,t)$ satisfies a suitable integrability condition.

1. Introduction

Fractional differential equations have been of great interest recently. They can be used to model many phenomena in control theory of dynamic systems, fluid flow, electrochemistry of corrosion, rheology etc. For more applications, we refer to [5, 6, 7, 9, 11, 12, 13, 17, 18, 20, 23, 25, 26, 27, 28] and references therein.

By means of the lower and upper solution method and fixed-point theorems, Liang and Zhang established in [14] the existence of positive solutions for the following Riemann-Liouville fractional problem

$$\begin{gathered} D^\alpha u(x) + f(x,u(x)) = 0, \quad 0 < x < 1, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{gathered} \tag{1.1}$$

where $3 < \alpha \le 4$ and f is a nonnegative continuous function satisfying some adequate conditions.

Recently, Zhai et al. [29], studied problem (1.1) with $f(x,u(x)) = g(x,u(x)) + h(x,u(x))$. They proved the existence and uniqueness of positive solutions by using a fixed point theorem for a sum of operators.

For further existence results related to (1.1), we refer to [1, 2, 3, 4, 14, 21, 22, 24, 29] and the references therein.

2010 *Mathematics Subject Classification.* 34A08, 34B15, 34B18, 34B27.
Key words and phrases. Fractional differential equation; positive solution; Green's function; perturbation; Schäuder fixed point theorem.

Submitted August 18, 2017. Published October 4, 2017.

In this paper, we are concerned with the following superlinear Riemann-Liouville fractional boundary value problem

$$\begin{gathered} D^{\alpha}u(x) - u(x)\varphi(x, u(x)) = 0, \quad 0 < x < 1, \\ u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha}u''(x) = 0, \quad u''(1) = a > 0, \end{gathered} \tag{1.2}$$

where $3 < \alpha \le 4$ and $\varphi(x,t)$ is a nonnegative continuous function in $(0,1)\times[0,\infty)$ that is required to satisfy some appropriate integrability condition.

We emphasize that the condition $a > 0$ on the boundary is crucial to obtain a positive solution. Our approach is based on a perturbation argument.

Notation:

(i) $C(X)$ (resp.$C^+(X)$) is the set of continuous (resp. nonnegative continuous) functions in a metric space X;

(ii) $\mathcal{B}((0,1))$ (resp. $\mathcal{B}^+((0,1))$) is the set of Borel (resp., nonnegative Borel) measurable functions in $(0,1)$;

(iii) $L^1((0,1)) := \{q \in \mathcal{B}((0,1)),\ \int_0^1 |q(r)|dr < \infty\}$;

(iv) $C^1_+([0,\infty))$ is the set of nonnegative continuously differentiable functions on $[0,\infty)$;

(v) for $3 < \alpha \le 4$,

$$\mathcal{K}_\alpha := \{q \in \mathcal{B}^+((0,1)), \quad \int_0^1 r^{\alpha-1}(1-r)^{\alpha-3}q(r)dr < \infty\}; \tag{1.3}$$

(vi) for $3 < \alpha \le 4$ and $a > 0$, we let

$$\omega(x) = \frac{a}{(\alpha-1)(\alpha-2)}x^{\alpha-1}, \ 0 \le x \le 1. \tag{1.4}$$

Observe that $\omega(x)$ is the unique solution of the problem

$$\begin{gathered} D^{\alpha}u(x) = 0, \quad 0 < x < 1, \\ u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha}u''(x) = 0, \quad u''(1) = a > 0. \end{gathered} \tag{1.5}$$

(vii) For $3 < \alpha \le 4$, we denote by $G(x,t)$ the Green's function of the operator $u \to D^\alpha u$, with boundary conditions $u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha}u''(x) = u''(1) = 0$. We have (see Lemma 2.4)

$$G(x,t) = \frac{1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-1}(1-t)^{\alpha-3} - (x-t)^{\alpha-1}, & 0 \le t \le x \le 1; \\ x^{\alpha-1}(1-t)^{\alpha-3}, & 0 \le x \le t \le 1. \end{cases} \tag{1.6}$$

(viii) For each $q \in \mathcal{K}_\alpha$, we let

$$\alpha_q := \sup_{x,t\in(0,1)} \int_0^1 \frac{G(x,r)G(r,t)}{G(x,t)} q(r)dr. \tag{1.7}$$

It will be showed that if $q \in \mathcal{K}_\alpha$, then $\alpha_q < \infty$.

To state our main results, we need a combination of the following assumptions.

(H1) $\varphi \in C^+((0,1)\times[0,\infty))$.

(H2) There exists a function $q \in \mathcal{K}_\alpha \cap C((0,1))$ with $\alpha_q \le 1/2$ such that for each $x \in (0,1)$, the map $t \mapsto t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$.

(H3) For each $x \in (0,1)$, the function $t \mapsto t\varphi(x,t)$ is nondecreasing on $[0,\infty)$.

We first prove that if q belongs to $\mathcal{K}_\alpha \cap C((0,1))$ with $\alpha_q \leq 1/2$ and f belongs to $\mathcal{B}^+((0,1))$, then the fractional problem

$$\begin{gathered} D^\alpha u(x) - q(x)u(x) = -f(x), \quad 0 < x < 1,\ 3 < \alpha \leq 4, \\ u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha} u''(x) = u''(1) = 0, \end{gathered} \tag{1.8}$$

admits a positive Green's function $\mathcal{G}(x,t)$.

Exploiting properties of the Green's function $\mathcal{G}(x,t)$ and by using a perturbation argument, we establish the following result.

Theorem 1.1. *Under assumptions* (H1), (H2)*, problem* (1.2) *admits a positive solution u in $C([0,1])$ satisfying*

$$c_0\omega(x) \leq u(x) \leq \omega(x), \quad x \in [0,1], \tag{1.9}$$

where c_0 is a constant in $(0,1)$.

Moreover, the uniqueness of such solution is proved if, further, hypothesis (H3) *is satisfied.*

From Theorem 1.1, we deduce the following property.

Corollary 1.2. *Let $f \in C^1_+([0,\infty))$ be such that the map $r \mapsto \theta(r) = rf(r)$ is nondecreasing on $[0,\infty)$. Let $p \in C^+((0,1))$ be such that the function $x \mapsto \widetilde{p}(x) := p(x) \max_{0\leq\xi\leq\omega(x)} \theta'(\xi) \in \mathcal{K}_\alpha$. Then for $\lambda \in [0, \frac{1}{2\alpha_{\widetilde{p}}})$, the problem*

$$\begin{gathered} D^\alpha u(x) - \lambda p(x)u(x)f(u(x)) = 0, \quad x \in (0,1), \\ u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha} u''(x) = 0, \quad u''(1) = a > 0, \end{gathered} \tag{1.10}$$

has a unique positive solution u in $C([0,1])$ satisfying

$$(1 - \lambda\alpha_{\widetilde{p}})\omega(x) \leq u(x) \leq \omega(x), \quad x \in [0,1].$$

Note that hypotheses (H1)–(H3), are satisfied with $\varphi(x,t) = \lambda p(x)t^\sigma$, for $\sigma \geq 0$, $p \in C^+((0,1))$ such that

$$\int_0^1 r^{(\alpha-1)(1+\sigma)}(1-r)^{\alpha-3}p(r)dr < \infty.$$

This article is organized as follows. In section 2, some estimates on the Green's function $G(x,t)$ are obtained. In particular, we prove that for all $x,r,t \in (0,1)$,

$$\frac{G(x,r)G(r,t)}{G(x,t)} \leq \frac{4(\alpha-1)^2}{(\Gamma(\alpha))} r^{\alpha-1}(1-r)^{\alpha-3}.$$

This implies that for each $q \in \mathcal{K}_\alpha$, we have $\alpha_q < \infty$. In section 3, we start by deriving the Green's function $\mathcal{G}(x,t)$ associated to the boundary value problem (1.8). We also establish some basic estimates of this function. In particular, we show that for $(x,t) \in [0,1]\times[0,1]$, we have

$$(1-\alpha_q)G(x,t) \leq \mathcal{G}(x,t) \leq G(x,t).$$

We also prove the resolvent equation

$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f), \quad \text{for } f \in \mathcal{B}^+((0,1)),$$

where the kernels V and V_q are defined on $\mathcal{B}^+((0,1))$ by

$$Vf(x) := \int_0^1 G(x,t)f(t)dt \quad \text{and} \quad V_q f(x) := \int_0^1 \mathcal{G}(x,t)f(t)dt, x \in [0,1].$$

By using the above results and a perturbation argument, we prove Theorem 1.1.

2. Fractional calculus and estimates on the Green's function

2.1. **Fractional calculus.** We recall the following basic definitions and properties on fractional calculus (see [11, 23, 25]).

Definition 2.1. Let $\beta > 0$ and $\Gamma(\beta)$ be the Euler Gamma function. For a measurable function $f : (0, \infty) \to \mathbb{R}$, the integral (provided that it exists)

$$I^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t) dt, \quad x > 0,$$

is called the Riemann-Liouville fractional integral of order β.

Definition 2.2. Let $\beta > 0$ and $[\beta]$ be its integer part. For a measurable function $f : (0, \infty) \to \mathbb{R}$, the expression (provided that it exists)

$$D^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)} (\frac{d}{dx})^n \int_0^x (x-t)^{n-\beta-1} f(t) dt = (\frac{d}{dx})^n I^{n-\beta} f(x),$$

where $n = [\beta] + 1$, is called the Riemann-Liouville fractional derivative of order β.

Lemma 2.3. *Let $\beta > 0$ and $u \in C((0,1)) \cap L^1((0,1))$. Then we have*

(i) *For $0 < \gamma < \beta$, $D^{\gamma} I^{\beta} u = I^{\beta-\gamma} u$ and $D^{\beta} I^{\beta} u = u$.*
(ii) *$D^{\beta} u(x) = 0$ if and only if $u(x) = c_1 x^{\beta-1} + c_2 x^{\beta-2} + \dots + c_m x^{\beta-m}$, $c_i \in \mathbb{R}$, $i = 1, \dots, m$, where m is the smallest integer greater than or equal to β.*
(iii) *Assume that $D^{\beta} u \in C((0,1)) \cap L^1((0,1))$. Then*

$$I^{\beta} D^{\beta} u(x) = u(x) + c_1 x^{\beta-1} + c_2 x^{\beta-2} + \dots + c_m x^{\beta-m},$$

$c_i \in \mathbb{R}$, $i = 1, \dots, m$, where m is the smallest integer greater than or equal to β.

2.2. **Estimates on the Green's function.** In the next lemma, we give the explicit expression of the Green's function $G(x,t)$.

Lemma 2.4. *If $f \in C^+([0,1])$, then the fractional boundary value problem*

$$\begin{gathered} D^{\alpha} u(x) = -f(x), \quad 0 < x < 1, \ 3 < \alpha \leq 4, \\ u(0) = u'(0) = \lim_{x \to 0^+} x^{4-\alpha} u''(x) = u''(1) = 0, \end{gathered} \tag{2.1}$$

has a unique nonnegative solution

$$u(x) = \int_0^1 G(x,t) f(t) dt, \tag{2.2}$$

where for $x, t \in [0,1]$,

$$G(x,t) = \frac{1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-1}(1-t)^{\alpha-3} - (x-t)^{\alpha-1}, & 0 \leq t \leq x \leq 1; \\ x^{\alpha-1}(1-t)^{\alpha-3}, & 0 \leq x \leq t \leq 1. \end{cases}$$

Proof. Since $f \in C([0,1])$, by Lemma 2.3 and Definition 2.1, we have

$$u(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + c_3 x^{\alpha-3} + c_4 x^{\alpha-4} - I^{\alpha} f(x).$$

Using the fact that $u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha}u''(x) = u''(1) = 0$, we obtain $c_2 = c_3 = c_4 = 0$ and $c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-3}f(t)dt$. Then the unique solution of problem (2.1) is

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{\alpha-1}(1-t)^{\alpha-3}f(t)dt - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt \\ &= \int_0^1 G(x,t)f(t)dt. \end{aligned}$$

This completes the proof. □

Next, we establish sharp estimates on the Green's function $G(x,t)$.

Proposition 2.5. *The following properties hold:*

(i) *For* $x, t \in [0,1]$,
$$\frac{1}{\Gamma(\alpha)}H_0(x,t) \le G(x,t) \le \frac{2(\alpha-1)}{\Gamma(\alpha)}H_0(x,t),$$
where $H_0(x,t) = x^{\alpha-2}(1-t)^{\alpha-3}\min(x,t)$.

(ii) *For* $x, t \in [0,1]$,
$$\frac{1}{\Gamma(\alpha)}x^{\alpha-1}t(1-t)^{\alpha-3} \le G(x,t) \le \frac{2(\alpha-1)}{\Gamma(\alpha)}x^{\alpha-2}t(1-t)^{\alpha-3}.$$

(iii) *For* $x \in (0,1]$ *and* $t \in [0,1)$,
$$\frac{(\alpha-1)}{\Gamma(\alpha)}H(x,t) \le \frac{\partial}{\partial x}G(x,t) \le \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}H(x,t),$$
where $H(x,t) = x^{\alpha-3}(1-t)^{\alpha-3}\min(x,t)$.

(iv) *For* $x \in (0,1]$ *and* $t \in [0,1)$,
$$\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{\Gamma(\alpha)}\widetilde{H}(x,t) \le \frac{\partial^2}{\partial x^2}G(x,t) \le \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\widetilde{H}(x,t),$$
where $\widetilde{H}(x,t) = x^{\alpha-4}(1-t)^{\alpha-4}\min(x,t)(1-\max(x,t))$.

Proof. We first observe that for $\lambda, \mu \in (0,\infty)$ and $c, t \in [0,1]$, we have
$$\min(1,\frac{\mu}{\lambda})(1-ct^\lambda) \le 1-ct^\mu \le \max(1,\frac{\mu}{\lambda})(1-ct^\lambda). \tag{2.3}$$

(i) By Lemma 2.4, for $x, t \in [0,1]$, we have
$$\begin{aligned} G(x,t) &= \frac{1}{\Gamma(\alpha)}\begin{cases} x^{\alpha-1}(1-t)^{\alpha-3} - (x-t)^{\alpha-1}, & 0 \le t \le x \le 1; \\ x^{\alpha-1}(1-t)^{\alpha-3}, & 0 \le x \le t \le 1, \end{cases} \\ &= \frac{1}{\Gamma(\alpha)}x^{\alpha-1}(1-t)^{\alpha-3} - (\max(x-t,0))^{\alpha-1}, \\ &= \frac{1}{\Gamma(\alpha)}x^{\alpha-1}(1-t)^{\alpha-3}\Big[1-(1-t)^2\Big(\frac{\max(x-t,0)}{x(1-t)}\Big)^{\alpha-1}\Big]. \end{aligned}$$

Since $\frac{\max(x-t,0)}{x(1-t)} \in [0,1]$, for $x \in (0,1]$ and $t \in [0,1)$, the required result follows from (2.3) with $\mu = \alpha-1$, $\lambda = 1$ and $c = (1-t)^2$.

(ii) The assertion follows from (i) and the elementary inequalities
$$xt \le \min(x,t) \le t, \quad \text{for } x, t \in [0,1].$$

(iii) For all $x, t \in [0,1]$ we have

$$\begin{aligned}\frac{\partial}{\partial x}G(x,t) &= \frac{\alpha-1}{\Gamma(\alpha)}\begin{cases} x^{\alpha-2}(1-t)^{\alpha-3}-(x-t)^{\alpha-2}, & 0\le t\le x\le 1;\\ x^{\alpha-2}(1-t)^{\alpha-3}, & 0\le x\le t\le 1,\end{cases}\\ &= \frac{\alpha-1}{\Gamma(\alpha)}x^{\alpha-2}(1-t)^{\alpha-3}\Big[1-(1-t)\Big(\frac{\max(x-t,0)}{x(1-t)}\Big)^{\alpha-2}\Big].\end{aligned}$$

Then the required result follows from (2.3) with $\mu=\alpha-2$, $\lambda=1$ and $c=(1-t)$.

(iv) For all $x\in(0,1]$ and $t\in[0,1)$ we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}G(x,t) &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\begin{cases} x^{\alpha-3}(1-t)^{\alpha-3}-(x-t)^{\alpha-3}, & 0\le t\le x\le 1;\\ x^{\alpha-3}(1-t)^{\alpha-3}, & 0\le x\le t\le 1,\end{cases}\\ &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}x^{\alpha-3}(1-t)^{\alpha-3}\Big[1-\Big(\frac{\max(x-t,0)}{x(1-t)}\Big)^{\alpha-3}\Big].\end{aligned}$$

Then the required result follows again from (2.3) with $\mu=\alpha-3$, $\lambda=1$ and $c=1$. This completes the proof. ∎

From Proposition 2.5 (ii), we deduce the following characterization property.

Corollary 2.6. *Let $f\in\mathcal{B}^+((0,1))$. Then*

$$x\mapsto Vf(x)\in C([0,1]) \Leftrightarrow \int_0^1 t(1-t)^{\alpha-3}f(t)dt<\infty.$$

Proposition 2.7. *Let $3<\alpha<4$ and assume that the map $t\mapsto t(1-t)^{\alpha-3}f(t)\in C((0,1))\cap L^1((0,1))$. Then Vf is the unique solution in $C([0,1])$ of the problem*

$$\begin{gathered} D^\alpha u(x)=-f(x),\quad 0<x<1,\\ u(0)=u'(0)=\lim_{x\to0^+}x^{4-\alpha}u''(x)=u''(1)=0.\end{gathered}\tag{2.4}$$

Proof. Using Corollary 2.6 we deduce that $Vf\in C([0,1])$. This implies that $I^{4-\alpha}(V|f|)$ is finite on $[0,1]$. So, by Fubini's theorem,

$$\begin{aligned} I^{4-\alpha}(Vf)(x) &= \frac{1}{\Gamma(4-\alpha)}\int_0^x (x-t)^{3-\alpha}Vf(t)dt\\ &= \frac{1}{\Gamma(4-\alpha)}\int_0^1\Big(\int_0^x (x-t)^{3-\alpha}G(t,s)dt\Big)f(s)ds\\ &= \int_0^1 K(x,s)f(s)ds,\end{aligned}$$

where $K(x,s):=\frac{1}{\Gamma(4-\alpha)}\int_0^x(x-t)^{3-\alpha}G(t,s)dt$.

Next, we aim to give an explicit expression of the kernel $K(x,s)$. To this end, observe that by making the substitution $t=s+(x-s)\theta$, we obtain for $\gamma,\nu>-1$,

$$\int_s^x (x-t)^\gamma(t-s)^\nu dt=\frac{\Gamma(\gamma+1)\Gamma(\nu+1)}{\Gamma(\gamma+\nu+2)}(x-s)^{\gamma+\nu+1}.\tag{2.5}$$

Using this fact and (1.6), we deduce that

$$\begin{aligned} K(x,s) &= \frac{(1-s)^{\alpha-3}}{\Gamma(4-\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{3-\alpha}t^{\alpha-1}dt\\ &\quad -\frac{1}{\Gamma(4-\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{3-\alpha}(\max(t-s,0))^{\alpha-1}dt\end{aligned}$$

$$= \frac{1}{6}x^3(1-s)^{\alpha-3} - \frac{1}{\Gamma(4-\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{3-\alpha}(\max(t-s,0))^{\alpha-1}dt$$

Now, assume that $s \leq x$. Then by (2.5) we have

$$\begin{aligned}\int_0^x (x-t)^{3-\alpha}(\max(t-s,0))^{\alpha-1}dt &= \int_s^x (x-t)^{3-\alpha}(t-s)^{\alpha-1}dt| \\ &= \frac{\Gamma(\alpha)\Gamma(4-\alpha)}{6}(x-s)^3.\end{aligned} \tag{2.6}$$

On the other hand, if $x \leq s$ and $t \in (0,x)$, we have

$$\int_0^x (x-t)^{3-\alpha}(\max(t-s,0))^{\alpha-1}dt = 0. \tag{2.7}$$

So, combining (2.6) and (2.7), we obtain

$$K(x,s) = \frac{1}{6}x^3(1-s)^{\alpha-3} - \frac{1}{6}(\max(x-s,0))^3.$$

Hence for $x \in (0,1)$, we have

$$\begin{aligned}6I^{4-\alpha}(Vf)(x) &= 6\int_0^1 K(x,s)f(s)ds \\ &= x^3\int_0^x [(1-s)^{\alpha-3}-1]f(s)ds + 3x^2\int_0^x sf(s)ds \\ &\quad - 3x\int_0^x s^2 f(s)ds + \int_0^x s^3 f(s)ds + x^3\int_x^1 (1-s)^{\alpha-3}f(s)ds \\ &=: J_1(x) + J_2(x) + J_3(x) + J_4(x) + J_5(x).\end{aligned}$$

We claim that

$$D^\alpha(Vf)(x) := \frac{d^4}{dx^4}(I^{4-\alpha}(Vf))(x) = -f(x), \quad \text{for } x \in (0,1).$$

Since the function $s \mapsto sf(s)$ is continuous and integrable near 0 and the function $s \mapsto (1-s)^{\alpha-3}f(s)$ is continuous and integrable near 1, then the functions $J_2(x), J_3(x), J_4(x)$ and $J_5(x)$ are differentiable.

On the other hand, since $(1-s)^{\alpha-3} - 1 = O(s)$ near 0, it follows that $J_1(x)$ is differentiable. By simple computation, we obtain

$$\begin{aligned}\frac{d}{dx}(6I^{4-\alpha}(Vf))(x) &= 3x^2\int_0^x [(1-s)^{\alpha-3}-1]f(s)ds + 6x\int_0^x sf(s)ds \\ &\quad - 3x\int_0^x s^2 f(s)ds + 3x^2\int_x^1 (1-s)^{\alpha-3}f(s)ds.\end{aligned}$$

Using similar arguments as above, we obtain

$$\frac{d^4}{dx^4}(I^{4-\alpha}(Vf))(x) = -f(x), \quad \text{for } x \in (0,1).$$

Next, we need to verify the boundary conditions. Since $G(0,t) = 0$ and $Vf(x) \in C([0,1])$, then it follows that $Vf(0) = 0$.

On the other hand, by Proposition 2.5 (iii), there exists a constant $c > 0$ such that for all $x, t \in [0,1]$, we have

$$|\frac{\partial}{\partial x}G(x,t)| \leq c\min(x,t)(1-t)^{\alpha-3} \leq ct(1-t)^{\alpha-3}.$$

This implies, by Lebesgue's theorem, that $(Vf)'(0) = 0$.

By Proposition 2.5 (iv), there exists $c > 0$ such that each $x, t \in [0,1]$, we have

$$x^{4-\alpha}\Big|\frac{\partial^2}{\partial x^2}G(x,t)\Big| \le c\min(x,t)(1-t)^{\alpha-3} \le ct(1-t)^{\alpha-3}.$$

Hence, by Lebesgue's theorem, we deduce that $\lim_{x\to 0^+} x^{4-\alpha}(Vf)''(x) = 0$.

Let $\eta \in (0,1)$. Again by Proposition 2.5 (iv), there exists a constant $c > 0$, such that for $x \in [\eta, 1]$ and $t \in (0,1)$, we have

$$\Big|\frac{\partial^2}{\partial x^2}G(x,t)\Big| \le c\eta^{\alpha-4}t(1-t)^{\alpha-4}(1-\max(x,t)) \le c\eta^{\alpha-4}t(1-t)^{\alpha-3}.$$

So by the Lebesgue's theorem, we deduce that $(Vf)''(1) = 0$.

Finally, we need to prove the uniqueness. Let $u, v \in C([0,1])$ be two solutions of (2.4) and put $z = u - v$. Then $z \in C([0,1])$ and $D^\alpha z = 0$. Hence, by Lemma 2.3 (iii), we deduce that $z(x) = c_1x^{\alpha-1} + c_2x^{\alpha-2} + c_3x^{\alpha-3} + c_4x^{\alpha-4}$. Using the fact that $z(0) = z'(0) = \lim_{x\to 0^+} x^{4-\alpha}z''(x) = z''(1) = 0$, we deduce that $z = 0$ and therefore $u = v$. This completes the proof. $\square$

Remark 2.8. Note that the conclusion of Proposition 2.7 is also valid for $\alpha = 4$.

Proposition 2.9. *For each $x, r, t \in (0,1)$, we have*

$$\frac{G(x,r)G(r,t)}{G(x,t)} \le \frac{4(\alpha-1)^2}{\Gamma(\alpha)}r^{\alpha-1}(1-r)^{\alpha-3}. \tag{2.8}$$

Proof. Using Proposition 2.5 (i), for each $x, r, t \in (0,1)$, we have

$$\frac{G(x,r)G(r,t)}{G(x,t)} \le \frac{4(\alpha-1)^2}{\Gamma(\alpha)}r^{\alpha-2}(1-r)^{\alpha-3}\frac{\min(x,r)\min(r,t)}{\min(x,t)}.$$

So the result follows from this fact and that

$$\frac{\min(x,r)\min(r,t)}{\min(x,t)} \le r.$$

This completes the proof. $\square$

Proposition 2.10. *Let $q \in \mathcal{K}_\alpha$. Then*

(i)
$$\alpha_q \le \frac{4(\alpha-1)^2}{\Gamma(\alpha)}\int_0^1 r^{\alpha-1}(1-r)^{\alpha-3}q(r)dr < \infty. \tag{2.9}$$

(ii) *For $x \in [0,1]$, we have*
$$\int_0^1 G(x,t)\omega(t)q(t)dt \le \alpha_q\omega(x), \tag{2.10}$$
where $\omega(x) = \frac{a}{(\alpha-1)(\alpha-2)}x^{\alpha-1}$.

Proof. Let $q \in \mathcal{K}_\alpha$.

(i) Using (1.7) and (2.8), we obtain inequality (2.9).

(ii) For all $x, t \in (0,1]$, we have $\lim_{r\to 1}\frac{G(t,r)}{G(x,r)} = \frac{t^{\alpha-1}}{x^{\alpha-1}}$. Thus, by Fatou's lemma and (1.7), we deduce that

$$\int_0^1 G(x,t)\frac{\omega(t)}{\omega(x)}q(t)dt \le \liminf_{r\to 1}\int_0^1 G(x,t)\frac{G(t,r)}{G(x,r)}q(t)dt \le \alpha_q.$$

This implies that for $x \in [0,1]$,

$$\int_0^1 G(x,t)\omega(t)q(t)dt \le \alpha_q \omega(x),$$

which completes the proof. □

3. Proof of main results

Let $q \in \mathcal{K}_\alpha$ with $\alpha_q < 1$. Define the function $\mathcal{G}(x,t)$ on $[0,1]\times[0,1]$ by

$$\mathcal{G}(x,t) = \sum_{n=0}^{\infty} (-1)^n G_n(x,t), \tag{3.1}$$

where $G_0(x,t) = G(x,t)$ and

$$G_n(x,t) = \int_0^1 G(x,r)G_{n-1}(r,t)q(r)dr, \quad n \ge 1. \tag{3.2}$$

Lemma 3.1. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q < 1$. For all $n \ge 0$ and $(x,t) \in [0,1]\times[0,1]$, we have*

(i) $G_n(x,t) \le \alpha_q^n G(x,t)$. *In particular, $\mathcal{G}(x,t)$ is well defined in $[0,1]\times[0,1]$.*

(ii) *The following inequalities hold:*

$$L_n x^{\alpha-1} t(1-t)^{\alpha-3} \le G_n(x,t) \le R_n x^{\alpha-2} t(1-t)^{\alpha-3}, \tag{3.3}$$

where

$$L_n = \frac{1}{(\Gamma(\alpha))^{n+1}}\Big(\int_0^1 r^\alpha (1-r)^{\alpha-3} q(r)dr\Big)^n,$$
$$R_n = \Big(\frac{2\alpha-2}{\Gamma(\alpha)}\Big)^{n+1}\Big(\int_0^1 r^{\alpha-1}(1-r)^{\alpha-3}q(r)dr\Big)^n.$$

(iii)

$$G_{n+1}(x,t) = \int_0^1 G_n(x,r)G(r,t)q(r)dr.$$

(iv)

$$\int_0^1 \mathcal{G}(x,r)G(r,t)q(r)dr = \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)dr.$$

Proof. (i) Clearly, inequality in (i) holds for $n = 0$. Assume that inequality in (i) is valid for some $n \ge 0$, then by using (3.2) and (1.7), we obtain

$$G_{n+1}(x,t) \le \alpha_q^n \int_0^1 G(x,r)G(r,t)q(r)dr \le \alpha_q^{n+1} G(x,t).$$

Since $G_n(x,t) \le \alpha_q^n G(x,t)$, it follows that $\mathcal{G}(x,t)$ is well defined in $[0,1]\times[0,1]$.

(ii) We can prove (3.3) by using Proposition 2.5 (ii), (3.2) and using a standard induction argument.

(iii) We proceed by induction. The equality is true for $n = 0$. Assume that for a given integer $n \ge 1$ and $(x,t) \in [0,1]\times[0,1]$, we have

$$G_n(x,t) = \int_0^1 G_{n-1}(x,r)G(r,t)q(r)dr. \tag{3.4}$$

Using (3.2) and the Fubini-Tonelli theorem, we obtain

$$\begin{aligned} G_{n+1}(x,t) &= \int_0^1 G(x,r)(\int_0^1 G_{n-1}(r,\xi)G(\xi,t)q(\xi)d\xi)q(r)dr \\ &= \int_0^1 (\int_0^1 G(x,r)G_{n-1}(r,\xi)q(r)dr)G(\xi,t)q(\xi)d\xi \\ &= \int_0^1 G_n(x,\xi)G(\xi,t)q(\xi)d\xi. \end{aligned}$$

(iv) Let $n \geq 0$ and $x, r, t \in [0,1]$. By Lemma 3.1 (i) we have

$$0 \leq G_n(x,r)G(r,t)q(r) \leq \alpha_q^n G(x,r)G(r,t)q(r).$$

Hence the series $\sum_{n\geq 0} \int_0^1 G_n(x,r)G(r,t)q(r)dr$ converges. By the dominated convergence theorem and Lemma 3.1 (iii), we deduce that

$$\begin{aligned} \int_0^1 \mathcal{G}(x,r)G(r,t)q(r)dr &= \sum_{n=0}^{\infty} \int_0^1 (-1)^n G_n(x,r)G(r,t)q(r)dr \\ &= \sum_{n=0}^{\infty} \int_0^1 (-1)^n G(x,r)G_n(r,t)q(r)dr \\ &= \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)dr. \end{aligned}$$

This completes the proof. □

Proposition 3.2. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q < 1$. Then the function $(x,t) \mapsto \mathcal{G}(x,t)$ is continuous on $[0,1] \times [0,1]$.*

Proof. We claim that for $n \geq 0$, the function $(x,t) \mapsto G_n(x,t)$ is continuous on $[0,1] \times [0,1]$.

Clearly, $G_0(x,t)$ is continuous on $[0,1]\times[0,1]$. Assume that the function $(x,t) \mapsto G_{n-1}(x,t)$ is continuous on $[0,1] \times [0,1]$. So, for each $r \in [0,1]$, the function $(x,t) \mapsto G(x,r)G_{n-1}(r,t)$ is continuous on $[0,1] \times [0,1]$. By using Lemma 3.1 (i) and Proposition 2.5 (ii), we have for each $(x,t,r) \in [0,1] \times [0,1] \times [0,1]$,

$$\begin{aligned} G(x,r)G_{n-1}(r,t)q(r) &\leq \alpha_q^{n-1} G(x,r)G(r,t)q(r) \\ &\leq \big(\frac{2(\alpha-1)}{\Gamma(\alpha)}\big)^2 r^{\alpha-1}(1-r)^{\alpha-3}q(r). \end{aligned}$$

We deduce by (3.2) and the dominated convergence theorem, that the function $(x,t) \mapsto G_n(x,t)$ is continuous on $[0,1] \times [0,1]$. This proves our claim.

Using again Lemma 3.1 (i) and Proposition 2.5 (ii), we have for all $x, t \in [0,1]$,

$$G_n(x,t) \leq \alpha_q^n G(x,t) \leq \frac{2(\alpha-1)}{\Gamma(\alpha)} \alpha_q^n.$$

Therefore the series $\sum_{n\geq 0} (-1)^n G_n(x,t)$ is uniformly convergent on $[0,1] \times [0,1]$ and hence the function $(x,t) \mapsto \mathcal{G}(x,t)$ is continuous on $[0,1] \times [0,1]$. This completes the proof. □

Lemma 3.3. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q \leq 1/2$. Then for all $(x,t) \in [0,1]\times[0,1]$, we have*

$$(1-\alpha_q)G(x,t) \leq \mathcal{G}(x,t) \leq G(x,t). \tag{3.5}$$

Proof. Since $\alpha_q \leq 1/2$, we deduce by Lemma 3.1 (i) that
$$|\mathcal{G}(x,t)| \leq \sum_{n=0}^{\infty}(\alpha_q)^n G(x,t) = \frac{1}{1-\alpha_q}G(x,t). \tag{3.6}$$
Note that
$$\mathcal{G}(x,t) = G(x,t) - \sum_{n=0}^{\infty}(-1)^n G_{n+1}(x,t). \tag{3.7}$$
Since the series $\sum\limits_{n\geq 0}\int_0^1 G(x,r)G_n(r,t)q(r)dr$ is convergent, we deduce by (3.7) and (3.2) that
$$\begin{aligned}
\mathcal{G}(x,t) &= G(x,t) - \sum_{n=0}^{\infty}(-1)^n\int_0^1 G(x,r)G_n(r,t)q(r)dr \\
&= G(x,t) - \int_0^1 G(x,r)\Big(\sum_{n=0}^{\infty}(-1)^n G_n(r,t)\Big)q(r)dr.
\end{aligned}$$
Therefore,
$$\mathcal{G}(x,t) = G(x,t) - V(q\mathcal{G}(\cdot,t))(x). \tag{3.8}$$
Using (3.6) and Lemma 3.1 (i) (with $n=1$), we deduce that
$$V(q\mathcal{G}(.,t))(x) \leq \frac{1}{1-\alpha_q}V(qG(.,t))(x) = \frac{1}{1-\alpha_q}G_1(x,t) \leq \frac{\alpha_q}{1-\alpha_q}G(x,t).$$
This implies by (3.8) that
$$\mathcal{G}(x,t) \geq G(x,t) - \frac{\alpha_q}{1-\alpha_q}G(x,t) = \frac{1-2\alpha_q}{1-\alpha_q}G(x,t) \geq 0.$$
So $\mathcal{G}(x,t) \leq G(x,t)$ and by (3.8) and Lemma 3.1 (i) (with $n=1$), we have
$$\mathcal{G}(x,t) \geq G(x,t) - V(qG(\cdot,t))(x) \geq (1-\alpha_q)G(x,t).$$
The proof is now complete. □

Using Proposition 3.2, (3.5) and Proposition 2.5 (ii), we obtain the following property.

Corollary 3.4. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q \leq \frac{1}{2}$ and let $f \in \mathcal{B}^+((0,1))$. Then the following characterization property holds:*
$$x \mapsto V_q f(x) \in C([0,1]) \Leftrightarrow \int_0^1 t(1-t)^{\alpha-3}f(t)dt < \infty.$$

Lemma 3.5. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q \leq \frac{1}{2}$ and $f \in \mathcal{B}^+((0,1))$. Then we have*
$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f). \tag{3.9}$$
In particular, if $V(qf) < \infty$, we have
$$(I - V_q(q.))(I + V(q.))f = (I + V(q.))(I - V_q(q.))f = f. \tag{3.10}$$
Here, $V(q.)(f) := V(qf)$.

Proof. Let $(x,t)\in[0,1]\times[0,1]$. Then by (3.8), we have

$$G(x,t)=\mathcal{G}(x,t)+V(q\mathcal{G}(\cdot,t))(x),$$

which implies by the Fubini-Tonelli theorem that for all $f\in\mathcal{B}^+((0,1))$,

$$\begin{aligned}Vf(x)&=\int_0^1(\mathcal{G}(x,t)+V(q\mathcal{G}(\cdot,t))(x))f(t)dt\\&=V_qf(x)+V(qV_qf)(x).\end{aligned}$$

Using Lemma 3.1 (iii) and the Fubini-Tonelli theorem, we obtain that for all $f\in\mathcal{B}^+((0,1))$ and $x\in[0,1]$,

$$\int_0^1\int_0^1\mathcal{G}(x,r)G(r,t)q(r)f(t)drdt=\int_0^1\int_0^1G(x,r)\mathcal{G}(r,t)q(r)f(t)drdt.$$

It follows that

$$V_q(qVf)(x)=V(qV_qf)(x).$$

We deduce that

$$Vf=V_qf+V(qV_qf)=V_qf+V_q(qVf)(x).$$

This completes the proof. □

Proposition 3.6. *Let $q\in\mathcal{K}_\alpha\cap C((0,1))$ such that $\alpha_q\le 1/2$ and $f\in\mathcal{B}^+((0,1))$ such that $t\mapsto t(1-t)^{\alpha-3}f(t)\in C((0,1))\cap L^1((0,1))$. Then $V_qf\in C^+([0,1])$ and it is the unique solution of problem* (1.8) *satisfying*

$$(1-\alpha_q)Vf\le V_qf\le Vf. \tag{3.11}$$

Proof. By Corollary 3.4, we deduce that $x\mapsto V_qf(x)\in C^+([0,1])$. Therefore, the function $x\mapsto q(x)V_qf(x)\in C((0,1))$. Using (3.9) and Proposition 2.5 (ii), there exists a constant $c\ge 0$ such that

$$V_qf(x)\le Vf(x)\le\frac{2(\alpha-1)}{\Gamma(\alpha)}\int_0^1x^{\alpha-2}t(1-t)^{\alpha-3}f(t)dt=cx^{\alpha-2}. \tag{3.12}$$

So we deduce that

$$\int_0^1t(1-t)^{\alpha-3}q(t)V_qf(t)dt\le c\int_0^1t^{\alpha-1}(1-t)^{\alpha-3}q(t)dt<\infty.$$

Hence by Proposition 2.7, the function $u=V_qf=Vf-V(qV_qf)$ satisfies the equation

$$\begin{gathered}D^\alpha u(x)=-f(x)+q(x)u(x),\quad x\in(0,1),\\u(0)=u'(0)=\lim_{x\to0^+}x^{4-\alpha}u''(x)=u''(1)=0.\end{gathered}$$

By integrating inequalities (3.5), we obtain (3.11).

For the uniqueness, assume that v is another nonnegative solution in $C([0,1])$ of problem (1.8) satisfying (3.11). Since the function $t\mapsto q(t)v(t)$ is of class $C((0,1))$ and by (3.11), (3.12), the function $t\mapsto t(1-t)^{\alpha-3}q(t)v(t)$ is in $L^1((0,1))$, it follows by Proposition 2.7 that the function $\widetilde{v}:=v+V(qv)$ satisfies

$$\begin{gathered}D^\alpha\widetilde{v}(x)+f(x)=0,\quad x\in(0,1),\\\widetilde{v}(0)=\widetilde{v}'(0)=\lim_{x\to0^+}x^{4-\alpha}\widetilde{v}''(x)=\widetilde{v}''(1)=0.\end{gathered}$$

Using Proposition 2.7, we deduce that

$$\widetilde{v} := v + V(qv) = Vf,$$

hence

$$(I + V(q.))(v - u) = 0.$$

Using (3.11), (3.12) and Proposition 2.5 (i) we have

$$\begin{aligned} V(q|v-u|)(x) &\leq \frac{4c(\alpha-1)}{\Gamma(\alpha)} \int_0^1 t^{\alpha-2}(1-t)^{\alpha-3} \min(x,t) q(t) dt \\ &\leq \frac{4c(\alpha-1)}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\alpha-3} q(t) dt < \infty. \end{aligned}$$

So by (3.10), we deduce that $u = v$. This completes the proof. □

3.1. **Proof of Theorem 1.1.** Let $a > 0$ and recall that

$$\omega(x) = \frac{a}{(\alpha-1)(\alpha-2)} x^{\alpha-1}, \quad \text{for } 0 \leq x \leq 1.$$

Since φ satisfies (H_2), there exists a function q in $\mathcal{K}_\alpha \cap C((0,1))$ such that $\alpha_q \leq 1/2$ and for each $x \in (0,1)$, the map $t \mapsto t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$. Let

$$\Lambda := \{u \in \mathcal{B}^+((0,1)) : (1-\alpha_q)\omega \leq u \leq \omega\}.$$

Define the operator T on Λ by

$$Tu = \omega - V_q(q\omega) + V_q((q - \varphi(\cdot, u))u).$$

By (3.9) and (2.10), we have

$$V_q(q\omega) \leq V(q\omega) \leq \alpha_q \omega \leq \omega. \tag{3.13}$$

By (H2), we obtain

$$0 \leq \varphi(., u) \leq q, \quad \text{for all } u \in \Lambda. \tag{3.14}$$

Using (3.14) and (3.13), we have that for all $u \in \Lambda$,

$$\begin{aligned} Tu &\leq \omega - V_q(q\omega) + V_q(qu) \leq \omega, \\ Tu &\geq \omega - V_q(q\omega) \geq (1-\alpha_q)\omega. \end{aligned}$$

Therefore. $T(\Lambda) \subset \Lambda$.

Next, we prove that the operator T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ be such that $u \leq v$. Since the map $t \mapsto t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$, we obtain that for all $x \in (0,1)$,

$$Tv - Tu = V_q([v(q - \varphi(\cdot, v)) - u(q - \varphi(., u))]) \geq 0.$$

Now, we consider the sequence $\{u_n\}$ defined by $u_0 = (1-\alpha_q)\omega$ and $u_{n+1} = Tu_n$, for $n \in \mathbb{N}$. Since Λ is invariant under T, we have $u_1 = Tu_0 \geq u_0$ and by the monotonicity of T, we deduce that

$$(1-\alpha_q)\omega = u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq \omega.$$

Hence by dominated convergence theorem, (H1), and (H2), we conclude that the sequence $\{u_n\}$ converges to a function $u \in \Lambda$ satisfying

$$u = (I - V_q(q.))\omega + V_q((q - \varphi(\cdot, u))u).$$

It follows that

$$(I - V_q(q.))u = (I - V_q(q.))\omega - V_q(u\varphi(., u)).$$

On the other hand, by (3.13), we have $V(qu) \le V(q\omega) \le \omega < \infty$. Then by applying the operator $(I + V(q.))$ on both sides of the above equality and using (3.9) and (3.10), we conclude that u satisfies

$$u = \omega - V(u\varphi(\cdot, u)). \tag{3.15}$$

We claim that u is the required solution. From (3.14), we have

$$u(t)\varphi(t, u(t)) \le q(t)\omega(t) = \frac{a}{(\alpha-1)(\alpha-2)} t^{\alpha-1} q(t). \tag{3.16}$$

So $\int_0^1 t(1-t)^{\alpha-3} u(t)\varphi(t, u(t))dt < \infty$. Therefore by Corollary 2.6, the function $x \mapsto V(u\varphi(., u))(x) \in C([0, 1])$ and from (3.15), we conclude that $u \in C([0, 1])$.

Since by (H1) and (3.16), the function $t \mapsto t(1-t)^{\alpha-3} u(t)\varphi(t, u(t))$ belongs to $C((0, 1)) \cap L^1((0, 1))$, we deduce by Proposition 2.7 that u is the required solution.

To prove uniqueness, assume (H3) and let v be another nonnegative solution in $C([0, 1])$ of problem (1.2) satisfying (1.9). Since v satisfies (1.9), we deduce by (3.14)) and (3.16) that

$$0 \le v(t)\varphi(t, v(t)) \le q(t)\omega(t) = \frac{a}{(\alpha-1)(\alpha-2)} t^{\alpha-1} q(t).$$

So the function $t \mapsto t(1-t)^{\alpha-3} v(t)\varphi(t, v(t))$ belongs to $C((0, 1)) \cap L^1((0, 1))$, and by Proposition 2.7, we deduce that the function $\widetilde{v} := v + V(v\varphi(\cdot, v))$ satisfies

$$\begin{gathered} D^\alpha \widetilde{v}(x) = 0, \quad 0 < x < 1, \\ \widetilde{v}(0) = \widetilde{v}'(0) = \lim_{x \to 0^+} x^{4-\alpha} \widetilde{v}''(x) = 0, \quad \widetilde{v}''(1) = a > 0. \end{gathered}$$

Hence

$$\widetilde{v} := v + V(v\varphi(\cdot, v)) = \omega.$$

We deduce that

$$v = \omega - V(v\varphi(\cdot, v)). \tag{3.17}$$

Let h be the function defined on $(0, 1)$ by

$$h(x) = \begin{cases} \frac{v(x)\varphi(x,v(x)) - u(x)\varphi(x,u(x))}{v(x)-u(x)}, & \text{if } v(x) \ne u(x), \\ 0, & \text{if } v(x) = u(x). \end{cases}$$

Then by (H3), $h \in \mathcal{B}^+((0, 1))$ and by (3.15) and (3.17), we have

$$(I + V(h.))(v - u) = 0.$$

On the other hand, by (H2), we remark that $h \le q$ and by (2.10) we deduce that

$$V(h|v - u|) \le 2V(q\omega) \le 2\alpha_q \omega < \infty.$$

Hence by (3.10), we conclude that $u = v$. This completes the proof.

Example 3.7. Let $3 < \alpha \le 4$ and $a > 0$. Let $\sigma \ge 0$, and $p \in C^+((0, 1))$ such that

$$\int_0^1 r^{(\alpha-1)(1+\sigma)} (1-r)^{\alpha-3} p(r)dr < \infty.$$

Let $\widetilde{p}(x) := (\sigma+1)p(x)(\omega(x))^\sigma$. Since $\widetilde{p} \in \mathcal{K}_\alpha$, then for $\lambda \in [0, \frac{1}{2\alpha_{\widetilde{p}}})$, the problem

$$\begin{gathered} D^\alpha u(x) - \lambda p(x) u^{\sigma+1}(x) = 0, \quad 0 < x < 1, \\ u(0) = u'(0) = \lim_{x \to 0^+} x^{4-\alpha} u''(x) = 0, \quad u''(1) = a, \end{gathered}$$

has a unique positive solution u in $C([0,1])$ satisfying

$$(1-\lambda\alpha_{\widetilde{p}})\omega(x) \leq u(x) \leq \omega(x),\ x \in [0,1].$$

Example 3.8. Let $3 < \alpha \leq 4$ and $a > 0$. Let $\sigma \geq 0$, $\gamma > 0$ and $p \in C^+((0,1))$ such that

$$\int_0^1 r^{(\alpha-1)+(\alpha-1)(\sigma+\gamma)}(1-r)^{\alpha-3}p(r)dr < \infty.$$

Let $\theta(s) = s^{\sigma+1}\log(1+s^\gamma)$ and $\widetilde{p}(t) := p(t)\max_{0\leq\xi\leq\omega(t)}\theta'(\xi)$. Since $\widetilde{p} \in \mathcal{K}_\alpha$, then for $\lambda \in [0, \frac{1}{2\alpha_{\widetilde{p}}})$, the problem

$$\begin{gathered} D^\alpha u(x) - \lambda p(x)u^{\sigma+1}(x)\log(1+u^\gamma(x)) = 0, \quad 0 < x < 1, \\ u(0) = u'(0) = \lim_{x\to 0^+} x^{4-\alpha}u''(x) = 0, \quad u''(1) = a, \end{gathered}$$

has a unique positive solution u in $C([0,1])$ satisfying

$$(1-\lambda\alpha_{\widetilde{p}})\omega(x) \leq u(x) \leq \omega(x), \quad x \in [0,1].$$

Acknowledgements. The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding this Research group NO (RG-1435-043).

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