Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 217, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXPLICIT LIMIT CYCLES OF A FAMILY OF POLYNOMIAL DIFFERENTIAL SYSTEMS

RACHID BOUKOUCHA

Abstract. We consider the family of polynomial differential systems

$$x' = x + (\alpha y - \beta x)(ax^2 - bxy + ay^2)^n,$$
$$y' = y - (\beta y + \alpha x)(ax^2 - bxy + ay^2)^n,$$

where a, b, α, β are real constants and n is positive integer. We prove that these systems are Liouville integrable. Moreover, we determine sufficient conditions for the existence of an explicit algebraic or non-algebraic limit cycle. Examples exhibiting the applicability of our result are introduced.

1. Introduction

An important problem in the qualitative theory of differential equations [9, 13, 20] is to determine the limit cycles of systems of the form

$$x' = \frac{dx}{dt} = P(x,y) \quad \text{and} \quad y' = \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are coprime polynomials. Here, the degree of system (1.1) is denoted by $n = \max\{\deg P, \deg Q\}$. A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1), and it is said to be algebraic if it is contained in the zero level set of a polynomial function [18]. In 1900 Hilbert [17] in the second part of his 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane $\mathbb{R}^2$. An even more difficult problem is to give an explicit expression of them [1, 15]. This has been one of the main problems in the qualitative theory of planar differential equations in the 20th century. We solve this last problem for a system of type (1.1). Until recently, the only limit cycles known in an explicit way were algebraic. In [3, 12, 15] examples of explicit limit cycles which are not algebraic are given. For instance, the limit cycle appearing in van der Pol's system is not algebraic as it is proved in [19].

Let Ω be a non-empty open and dense subset of $\mathbb{R}^2$. We say that a non-locally constant C^1 function $H : \Omega \to \mathbb{R}$ is a first integral of the differential system (1.1)

2010 *Mathematics Subject Classification.* 34A05, 34C05, 34CO7, 34C25.
Key words and phrases. Planar polynomial differential system; first integral;
Algebraic limit cycle; non-algebraic limit cycle.

Submitted October 4, 2016. Published September 13, 2017.

in Ω if H is constant on the trajectories of the system (1.1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x}P(x,y) + \frac{\partial H(x,y)}{\partial y}Q(x,y) \equiv 0 \quad \text{in } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait. There is a lot of literature on the existence of a first integral [2, 4].

A real or complex polynomial $U(x,y)$ is called algebraic solution of the polynomial differential system (1.1) if

$$\frac{\partial U(x,y)}{\partial x}P(x,y) + \frac{\partial U(x,y)}{\partial y}Q(x,y) = K(x,y)U(x,y),$$

for some polynomial $K(x,y)$, called the cofactor of $U(x,y)$. If $U(x,y)$ is non-algebraic the cofactor may not be algebraic [10, 11, 16]. If U is real, the curve $U(x,y) = 0$ is an invariant under the flow of differential system (1.1) and the set $\{(x,y) \in \mathbb{R}^2,\ U(x,y) = 0\}$ is formed by orbits of system (1.1). There are strong relationships between the integrability of system (1.1) and its number of invariant algebraic solutions. It is shown [8] that the existence of a certain number of algebraic solutions for a system implies the Darboux integrability of the system, that is the first integral is the product of the algebraic solutions with complex exponents [5, 6, 7, 14]. In [21], it is proved that, if a polynomial system (1.1) has a Liouvillian first integral, then it can be computed by using the invariant algebraic solutions and the exponential factors of the system (1.1).

In this paper, we consider the family of the polynomial differential system of the form

$$\begin{aligned} x' &= \frac{dx}{dt} = x + (\alpha y - \beta x)(ax^2 - bxy + ay^2)^n, \\ y' &= \frac{dy}{dt} = y - (\beta y + \alpha x)(ax^2 - bxy + ay^2)^n, \end{aligned} \tag{1.2}$$

where a, b, α, β are real constants and n is positive integer. We prove that these systems are Liouville integrable. Moreover, we determine sufficient conditions for a polynomial differential system to possess an explicit algebraic or non-algebraic limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

2. Main result

Our main result is contained in the following theorem.

Theorem 2.1. *Consider a multi-parameter polynomial differential system* (1.2). *Then the following statements hold.*

(1) The curve $U(x,y) = -\alpha(x^2+y^2)(ax^2 - bxy + ay^2)^n = 0$ *is an invariant algebraic of system* (1.2).

(2) If $\alpha > 0$ *and* $a > \frac{1}{2}|b|$, *then system* (1.2) *has the first integral*

$$H(x,y) = (x^2+y^2)^n \exp\Big(\frac{-2n\beta}{\alpha}\arctan\frac{y}{x}\Big) + 2n\int_0^{\arctan\frac{y}{x}} \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2}b\sin 2w)^n}\Big)dw.$$

(3) If $\alpha > 0$, $\beta > 0$ and $2a > |b|$ then system (1.2) has an explicit limit cycle, given in polar coordinates (r, θ) by

$$r(\theta, r_*) = \exp\big(\frac{\beta}{\alpha}\theta\big)\Big(r_*^{2n} - 2n\int_0^\theta \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2}b\sin 2w)^n}\Big)dw\Big)^{1/(2n)},$$

where

$$r_* = \exp\big(\frac{2\beta\pi}{\alpha}\big)\Big(\frac{2n\int_0^{2\pi}\big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a-\frac{1}{2}b\sin 2w)^n}\big)dw}{\exp(\frac{4n\beta\pi}{\alpha}) - 1}\Big)^{1/(2n)}.$$

Proof. (1) We prove that $U(x, y) = -\alpha(x^2 + y^2)(ax^2 - bxy + ay^2)^n = 0$ is an invariant algebraic curve of the differential system (1.2).

Indeed, we have

$$\frac{\partial U(x, y)}{\partial x}P(x, y) + \frac{\partial U(x, y)}{\partial y}Q(x, y) = U(x, y)K(x, y),$$

where

$$\begin{aligned} K(x, y) &= 2 + 2n + (2\beta - 2\beta n)(ax^2 - bxy + ay^2)^n \\ &\quad - n\alpha b((y^2 - x^2))(ax^2 - bxy + ay^2)^{n-1}. \end{aligned}$$

Therefore, $U(x, y) = -\alpha(x^2 + y^2)(ax^2 - bxy + ay^2)^n = 0$ is an invariant algebraic curve of the polynomial differential systems (1.2). Hence, statement (1) is proved.

(2) To prove our results (2) and (3) we write the polynomial differential system (1.2) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then the system becomes

$$\begin{aligned} r' &= r - \beta(a - \frac{1}{2}b\sin 2\theta)^n r^{2n+1}, \\ \theta' &= -\alpha(a - \frac{1}{2}b\sin 2\theta)^n r^{2n}, \end{aligned} \tag{2.1}$$

where $\theta' = \frac{d\theta}{dt}$, $r' = \frac{dr}{dt}$.

Taking as new independent variable the coordinate θ, this differential system reads

$$\frac{dr}{d\theta} = \frac{\beta}{\alpha}r + \frac{-1}{\alpha(a - \frac{1}{2}b\sin 2\theta)^n}r^{1-2n}, \tag{2.2}$$

which is a Bernoulli equation.

By introducing the standard change of variables $\rho = r^{2n}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \frac{2n\beta}{\alpha}\rho + \frac{-2n}{\alpha(a - \frac{1}{2}b\sin 2\theta)^n}. \tag{2.3}$$

The general solution of linear equation (2.3) is

$$r(\theta) = \exp\big(\frac{\beta}{\alpha}\theta\big)\Big(c - 2n\int_0^\theta \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2}b\sin 2w)^n}\Big)dw\Big)^{1/(2n)},$$

where $c \in \mathbb{R}$. From these solution we can obtain a first integral in the variables (x, y) of the form

$$H(x, y) = (x^2 + y^2)^n \exp\Big(\frac{-2n\beta}{\alpha}\arctan\frac{y}{x}\Big) + 2n\int_0^{\arctan\frac{y}{x}} \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2}b\sin 2w)^n}\Big)dw.$$

Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (1.2) is

Liouville integrable. The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.2), in Cartesian coordinates are written as

$$\begin{aligned} x^2 + y^2 = \Big(& h \exp\Big(\frac{2n\beta}{\alpha} \arctan \frac{y}{x}\Big) \\ &- 2n \exp\Big(\frac{2n\beta}{\alpha} \arctan \frac{y}{x}\Big) \int_0^{\arctan \frac{y}{x}} \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw\Big)^{1/n}, \end{aligned}$$

where $h \in \mathbb{R}$. Hence, statement (2) is proved.

(3) Since $\alpha > 0$ and $a > \frac{1}{2}|b|$, it follows that $-\alpha(a - \frac{1}{2} b \sin 2\theta)^n < 0$ for all $\theta \in \mathbb{R}$, then θ' is negative for all t, which means that each orbit of system (1.2) encircle the singularity at the origin.

Notice that system (1.2) has a periodic orbit if and only if equation (2.2) has a strictly positive 2π periodic solution. This, moreover, is equivalent to the existence of a solution of (2.2) that satisfies $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$.

The solution $r(\theta, r_0)$ of the differential equation (2.2) such that $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = \exp\Big(\frac{\beta}{\alpha}\theta\Big)\Big(r_0^{2n} - 2n \int_0^\theta \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw\Big)^{1/(2n)},$$

where $r_0 = r(0)$.

A periodic solution of system (1.2) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$, which leads to a unique value $r_0 = r_*$, given by

$$r_* = \exp\big(\frac{2\beta\pi}{\alpha}\big)\Big(\frac{2n \int_0^{2\pi} (\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}) dw}{\exp(\frac{4n\beta\pi}{\alpha}) - 1}\Big)^{1/(2n)}.$$

After the substitution of these value r_* into $r(\theta, r_0)$ we obtain

$$\begin{aligned} r(\theta, r_*) = \exp\big(\frac{\beta}{\alpha}\theta\big)\Big(& 2n \frac{\exp(\frac{4n\beta\pi}{\alpha})}{\exp(\frac{4n\beta\pi}{\alpha}) - 1} \int_0^{2\pi} \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw \\ &- 2n \int_0^\theta \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw\Big)^{1/(2n)}. \end{aligned}$$

Next we prove that $r(\theta, r_*) > 0$. Indeed

$$\begin{aligned} r(\theta, r_*) = & \sqrt[2n]{2n} \exp\big(\frac{\beta}{\alpha}\theta\big)\Big(\frac{\exp(\frac{4n\beta\pi}{\alpha})}{\exp(\frac{4n\beta\pi}{\alpha}) - 1} \int_0^{2\pi} (\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}) dw \\ &- \int_0^\theta \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw\Big)^{1/(2n)} \\ \geq & \sqrt[2n]{2n} \exp\big(\frac{\beta}{\alpha}\theta\big)\Big(\int_0^{2\pi} \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw \\ &- \int_0^\theta \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw\Big)^{1/(2n)} \\ = & \sqrt[2n]{2n} \exp\big(\frac{\beta}{\alpha}\theta\big)\Big(\int_\theta^{2\pi} \Big(\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a - \frac{1}{2} b \sin 2w)^n}\Big) dw\Big)^{1/(2n)} > 0, \end{aligned}$$

because

$$\frac{\exp(-\frac{2n\beta w}{\alpha})}{\alpha(a-\frac{1}{2}b\sin 2w)^n} > 0$$

for all $s \in \mathbb{R}$. Moreover, we compute

$$\frac{dr(2\pi, r_0)}{dr_0}\Big|_{r_0=r_*} = \exp\big(\frac{4\beta n\pi}{\alpha}\big) > 1.$$

This is a stable and hyperbolic limit cycle for the differential systems (1.2). This completes the proof of statement (3) of Theorem 2.1. □

3. Examples

The following examples illustrate our result.

Example 3.1. When $a = b = \alpha = \beta = n = 1$, system (1.2) reads

$$\begin{aligned} x' &= x + (y-x)(x^2 - xy + y^2), \\ y' &= y - (y+x)(x^2 - xy + y^2). \end{aligned} \tag{3.1}$$

This system is a cubic system that has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = e^{\theta}\Big(r_*^2 - 4\int_0^{\theta}\Big(\frac{e^{-2\omega}}{2-\sin 2\omega}\Big)d\omega\Big)^{1/2},$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point

$$r_* = \Big(\frac{2e^{4\pi}}{e^{4\pi}-1}\int_0^{2\pi}\Big(\frac{2}{2-\sin 2\omega}e^{-2\omega}\Big)d\omega\Big)^{1/2} \simeq 1.1912.$$

Moreover,

$$\frac{dr(2\pi, r_0)}{dr_0}\Big|_{r_0=r_*} = \exp(4\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle. This results presented was by Llibre and Rebiha [3].

Example 3.2. When $a = \alpha = \beta = n = 1$ and $b = 0$, system (1.2) reads

$$\begin{aligned} x' &= x - x^3 + x^2y - xy^2 + y^3, \\ y' &= y - x^3 - x^2y - xy^2 - y^3. \end{aligned} \tag{3.2}$$

This system is a cubic system that has a algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = 1,$$

where $\theta \in \mathbb{R}$. In Cartesian coordinates it is written

$$x^2 + y^2 = 1.$$

Moreover,

$$\frac{dr(2\pi, r_0)}{dr_0}\Big|_{r_0=r_*} = \exp(4\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

Example 3.3. When $\alpha = \beta = 1$ and $a = b = n = 2$, system (1.2) reads

$$\begin{aligned} x' &= x - 4x^5 + 12x^4y - 20x^3y^2 + 20x^2y^3 - 12xy^4 + 4y^5, \\ y' &= y - 4x^5 + 4x^4y - 4x^3y^2 - 4x^2y^3 + 4xy^4 - 4y^5, \end{aligned} \tag{3.3}$$

This system is a quintic system that has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = e^{\theta}\Big(r_*^4 - 4\int_0^{\theta}\Big(\frac{\exp(-4w)}{(2 - \sin 2w)^2}\Big)dw\Big)^{1/4},$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point

$$r_* = \exp(2\pi)\Big(\frac{4\int_0^{2\pi}\big(\frac{\exp(-4w)}{(2-\sin 2w)^2}\big)dw}{\exp(8\pi) - 1}\Big)^{1/4} \simeq 0.816\,28.$$

Moreover,

$$\frac{dr(2\pi, r_0)}{dr_0}\Big|_{r_0 = r_*} = \exp(8\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

Example 3.4. When $a = b = \alpha = \beta = 1$ and $n = 3$, system (1.2) reads

$$\begin{aligned} x' &= x - x^7 + 4x^6y - 9x^5y^2 + 13x^4y^3 - 13x^3y^4 + 9x^2y^5 - 4xy^6 + y^7, \\ y' &= y - x^7 + 2x^6y - 3x^5y^2 + x^4y^3 + x^3y^4 - 3x^2y^5 + 2xy^6 - y^7. \end{aligned} \tag{3.4}$$

This system has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = e^{\theta}\Big(r_*^6 - 6\int_0^{\theta}\Big(\frac{8\exp(-6w)}{(1 - \frac{1}{2}\sin 2w)^3}\Big)dw\Big)^{1/6},$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is

$$r_* = \Big(\frac{\exp(12\pi)}{-1 + \exp(12\pi)}\int_0^{2\pi}\Big(\frac{48\exp(-6w)}{(2 - \sin 2w)^3}\Big)dw\Big)^{1/6} \simeq 1.1189.$$

Moreover,

$$\frac{dr(2\pi, r_0)}{dr_0}\Big|_{r_0 = r_*} = \exp(12\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

References

[1] A. Bendjeddou, R. Boukoucha; *Explicit limit cycles of a cubic polynomial differential systems,* Stud. Univ. Babes-Bolyai Math., **61** (2016), no. 1, 77-85.

[2] A. Bendjeddou, J. Llibre, T. Salhi; *Dynamics of the differential systems with homogenous nonlinearity and a star node,* J. Differential Equations **254** (2013), 3530-3537.

[3] R. Benterki, J. Llibre; *Polynomial differential systems with explicit non-algebraic limit cycles,* Elect. J. of Diff. Equ., **2012**, no. 78 (2012), 1-6.

[4] R. Boukoucha, A. Bendjeddou; *On the dynamics of a class of rational Kolmogorov systems,* J. of Nonlinear Math. Phys., **23** (2016), no. 1, 21-27.

[5] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre; *Darboux integrability and the inverse integrating factor,* J. Differential Equations, **194** (2003), no. 1, 116-139.

[6] J. Chavarriga, M. Grau; *A family of non Darboux integrable quadratic polynomial differential systems with algebraic solution of arbitrarily high degree,* Appl. Math. Lett. **16** (2003), 833-837.

[7] C. Christopher, J. Llibre; *Integrability via invariant algebraic curves for planar polynomial differential systems,* Ann. Differential Equations **16** (2000), 5-19.

[8] G. Darboux; *Mémoire sur les équations diffé rentielles algébriques du premier ordre et du premier degré (Melanges),* Bull. Sci. Math., **32** (1978), 60-96, 123-144, 151-200.

[9] F. Dumortier, J. Llibre, J. Artés; *Qualitative Theory of Planar Differential Systems,* (Universitex) Berlin, Springer (2006).

[10] I. A. Garcia, H. Giacomini, J. Giné; *Generalized nonlinear superposition principles for polynomial planar vector fields,* J. Lie Theory, **15** (2005), no. 1, 89-104.

[11] I. A. Garcia, J. Giné; *Non-algebraic invariant curves for polynomial planar vector fields,* Discrete Contin. Dyn. Syst., **10** (2004), no. 3, 755-768.

[12] A. Gasull, H. Giacomini, J. Torregrosa; *Explicit non-algebraic limit cycles for polynomial systems,* J. Comput. Appl. Math. **200** (2007), 448-457.

[13] H. Giacominiy, J. Llibre, M. Viano; *On the nonexistence existence and uniqueness of limit cycles,* Nonlinearity **9** (1996), 501–516. Printed in the UK.

[14] J. Giné; *Reduction of integrable planar polynomial differential systems,* Appl. Math. Lett., **25** (2012), no. 11, 1862-1865.

[15] J. Giné, M. Grau; *Coexistence of algebraic and non-algebraic limit cycles, explicitly given, using Riccati equations,* Nonlinearity **19** (2006), 1939-1950.

[16] J. Giné, M. Grau, J. Llibre; *On the extensions of the Darboux theory of integrability,* Nonlinearity **26** (2013), no. 8, 2221-2229.

[17] D. Hilbert; *Mathematische Probleme,* Lecture, Second Internat. Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. Gttingen Math. Phys. Kl. (1900) 253-297, English transl, Bull. Amer. Math. Soc., **8** (1902), 437-479.

[18] J. Llibre, Y. Zhao; *Algebraic Limit Cycles in Polynomial Systems of Differential Equations,* J. Phys. A: Math. Theor., **40** (2007), 14207-14222.

[19] K. Odani; *The limit cycle of the van der Pol equation is not algebraic,* J. of Diff. Equ. **115** (1995), 146-152.

[20] L. Perko; *Differential Equations and Dynamical Systems,* Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.

[21] M. F. Singer; *Liouvillian first integrals of differential equations,* Trans. Amer. Math. Soc. **333** (1990), 673-688.

Rachid Boukoucha

Department of Technology, Faculty of Technology, University of Bejaia, 06000 Bejaia, Algeria

E-mail address: rachid_boukecha@yahoo.fr