Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 213, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS TO SUPERCRITICAL NEUMANN PROBLEMS VIA A NEW VARIATIONAL PRINCIPLE

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Communicated by Claudianor O. Alves

Abstract. We use a new variational principle to obtain a positive solution of

$$-\Delta u + u = a(|x|)|u|^{p-2}u \quad \text{in } B_1,$$

with Neumann boundary conditions where B_1 is the unit ball in $\mathbb{R}^N$, a in nonnegative, radial and increasing and $p > 2$. Note that for $N \geq 3$ this includes supercritical values of p. We find critical points of the functional

$$I(u) := \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u + u|^q \,dx - \frac{1}{p}\int_{B_1} a(|x|)|u|^p \,dx,$$

over the set of $\{u \in H^1_{\rm rad}(B_1) : 0 \leq u, u \text{ is increasing}\}$, where q is the conjugate of p. We would like to emphasize the energy functional I is different from the standard Euler-Lagrange functional associated with the above equation, i.e.

$$E(u) := \int_{B_1} \frac{|\nabla u|^2 + u^2}{2}\,dx - \int_{B_1} \frac{a(|x|)|u|^p}{p}\,dx.$$

The novelty of using I instead of E is the hidden symmetry in I generated by $\frac{1}{p}\int_{B_1} a(|x|)|u|^p\,dx$ and its Fenchel dual. Additionally we are able to prove the existence of a positive nonconstant solution, in the case $a(|x|) = 1$, relatively easy and without needing to cut off the supercritical nonlinearity. Finally, we use this new approach to prove existence results for gradient systems with supercritical nonlinearities.

1. Introduction

In this paper we consider the existence of positive solutions of the Neumann problem

$$\begin{gathered}
-\Delta u + u = a(|x|)|u|^{p-2}u, \quad x \in B_1 \\
u > 0, \quad x \in B_1, \\
\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial B_1,
\end{gathered} \tag{1.1}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$ and $p > 2$ and where we assume a satisfies

(H1) $a \in L^1(0,1)$ is increasing, not constant and $a(r) > 0$ a.e. in $[0,1]$.

2010 *Mathematics Subject Classification.* 35J15, 58E30.
Key words and phrases. Variational principles, supercritical, Neumann boundary conditions.

Submitted April 12, 2017. Published september 13, 2017.

Before we outline our approach we mention prior works regarding (1.1). For $p < 2^*$ one can utilize the standard critical point theory, which relies on the compact embedding of $H^1(B_1)$ into $L^p(B_1)$, to obtain a positive solution of (1.1). With this in mind we are interested in the supercritical case $p > 2^*$ where one no longer has the needed compact embedding. We are also interested in the gradient elliptic system given by

$$\begin{gathered} -\Delta u + u = f_u(|x|, u, v), \quad x \in B_1 \\ -\Delta v + v = f_v(|x|, u, v), \quad x \in B_1 \\ \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial B_1, \end{gathered} \tag{1.2}$$

under suitable assumptions on f. Our assumption do allow some supercritical nonlinearities.

We begin by reviewing some known results for (1.1) in the supercritical case. In [18] they considered the variant of (1.1) where u^p is replaced with $f(u)$ where $f(u)$ is still a supercritical nonlinearity. They then considered the associated classical energy

$$E(u) := \int_{B_1} \frac{|\nabla u|^2 + u^2}{2} \, dx - \int_{B_1} a(|x|) F(u) \, dx,$$

where $F'(u) = f(u)$. Their goal was to find critical points of E over $H^1_{\rm rad}(B_1)$ (the $H^1(B_1)$ radial functions). Of course since f is supercritical the standard approach of finding critical points will present difficulties and hence their idea was to find critical points of E over the cone $\{u \in H^1_{\rm rad}(B_1) : 0 \leq u, u \text{ is increasing}\}$. Doing this is somewhat standard but now the issue is the critical points do not necessarily correspond to critical points over $H^1_{\rm rad}(B_1)$ and hence one can not conclude the critical points solve the equation. The majority of their work is to show that in fact the critical points of E on the cone are really critical points over the full space. In [12],

$$\begin{gathered} -\Delta u + V(|x|)u = |u|^{p-2}u, \quad x \in B_1 \\ u > 0, \quad x \in B_1, \end{gathered} \tag{1.3}$$

was examined under both homogeneous Neumann and Dirichlet boundary conditions. We will restrict our attention to their results regarding the Neumann boundary conditions. Consider $G(r,s)$ the Green function of the operator

$$\mathcal{L}(u) = -u'' - \frac{N-1}{r} u' + V(r)u, \quad u'(0) = 0,$$

with $u'(1) = 0$. Define now $H(r) := (G(r,r))^{-1} |\partial B_1| r^{N-1}$ for $0 < r \leq 1$. One of their results states that for $V \geq 0$ (not identically zero) if H has a local minimum at $\bar{r} \in (0,1]$ then for p large enough, (1.3) has a solution with Neumann boundary conditions and the solutions have a prescribed asymptotic behavior as $p \to \infty$. Additionally they can find as many solutions as H has local minimums. This work contains many results and we will list one more related result. For $V = \lambda > 0$, the problem (1.3) has a positive nonconstant solution with Neumann boundary conditions provided p is large enough. This methods used in [12] appear to be very different from the methods used in the all the other works. It appears the works of [18] and [12] were done completely independent of each other. The next work related to (1.1) was [2] where they considered

$$\begin{gathered} -\Delta u + b(|x|)x \cdot \nabla u + u = a(|x|)f(u), \quad x \in B_1 \\ u > 0, \quad x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial B_1, \end{gathered} \tag{1.4}$$

where again f was allowed to be supercritical and where various assumptions were imposed on b. Their approach was similar to [18] in the sense that they also worked on the cone $\{u \in H^1_{\rm rad}(B_1) : 0 \le u, u \text{ is increasing}\}$ but instead of using a variational approach they used a topological approach. They were able to weaken the assumptions needed on f. In the case of $a = 1$ one sees that the constant u_0 is a solution provided $f(u_0) = u_0$. In [2] they have showed that (1.4) has a positive nonconstant solution in the case of $b = 0$ provided there is some $u_0 > 0$ with $f(u_0) = u_0$ and $f'(u_0) > \lambda_2^{\rm rad}$ which is the second radial eigenvalue of $-\Delta + I$ in the unit ball with Neumann boundary conditions. Note that this result shows there is a positive nonconstant solution of (1.1) provided $p - 1 > \lambda_2^{\rm rad}$. In [3] they considered various elliptic systems of the form

$$\begin{gathered} -\Delta u + u = f(|x|, u, v), \quad x \in B_1 \\ -\Delta v + v = g(|x|, u, v), \quad x \in B_1 \\ \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial B_1. \end{gathered}$$

In particular they examined the gradient system when

$$f(|x|, u, v) = G_u(|x|, u, v), g(|x|, u, v) = G_v(|x|, u, v)$$

and they also considered the Hamiltonian system version where

$$f(|x|, u, v) = H_v(|x|, u, v), g(|x|, u, v) = H_u(|x|, u, v).$$

In both cases there obtain positive solutions under various assumptions (which allowed supercritical nonlinearities). They also obtain positive nonconstant solutions in the case of $f(|x|, u, v) = f(u, v)$, $g(|x|, u, v) = g(u, v)$; note in this case there is the added difficulty of avoiding the possible constant solutions.

These results were extended to p-Laplace versions in [19]. The methods of [12] were extended to prove results regarding multi-layer radials solutions in [1]. Finally we mention the work of [4] where problems on the annulus were considered.

One final point we mention is that there is another type of supercritical problem that one can examine on B_1. One can examine supercritical equations like (1.1) or the case of zero Dirichlet boundary conditions when a is radial and $a = 0$ at the origin; a well known case of this is the Hénon equation given by $-\Delta u = |x|^\alpha u^p$ in B_1 with $u = 0$ on ∂B_1 where $0 < \alpha$. In [17] it was shown the Hénon equation has a positive solution if and only if $p < \frac{N+2+2\alpha}{N-2}$, and note this includes a range of supercritical p. This increased range of p is coming from the fact that $a = 0$ at the origin. We mention this phenomena is very different than what is going on in the above works. Results regarding positive solutions of supercritical Hénon equations on general domains have also been obtained, see [5] and [11].

Remark 1.1. We would like to stress the fact that the results we obtain regarding (1.1) have already been obtained in [2, 3, 12, 18]. Our main contribution, we believe, is two-fold. The first is that in our approach we can apply a new variational principle, see Theorem 1.2, to obtain results. The second benefit of our approach is related to finding positive nonconstant solutions of (1.1) in the case of $a(r)$ a constant. We are able to use the mountain pass level directly to rule out that the solution is constant without needing to cut the nonlinearity off appropriately and make the problem subcritical. This seems to shorten and simplify the proof. Even though, we are stating our results for the nonlinearity $f(u) = |u|^{p-1}u$, one can easily consider other nonlinearities as long as f is an increasing function.

Our approach. Our plan is to prove existence for (1.1) by making use of a new variational principle established recently in [13] (see also [14, 15, 16]). To be more specific, let V be a reflexive Banach space, V^* its topological dual and K be a closed convex subset of V. Assume that $\Phi : V \to \mathbb{R}$ is convex, Gâteaux differentiable and lower semi-continuous and that $\Lambda : \mathrm{Dom}(\Lambda) \subset V \to V^*$ is a linear symmetric operator. Let Φ^* be the Fenchel dual of Φ, i.e.

$$\Phi^*(u^*) = \sup\{\langle u^*, u\rangle - \Phi(u); u \in V\}, \quad u^* \in V^*,$$

where the pairing between V and V^* is denoted by $\langle \cdot, \cdot\rangle$. Define the function $\Psi_K : V \to (-\infty, +\infty]$ by

$$\Psi_K(u) = \begin{cases} \Phi^*(\Lambda u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \tag{1.5}$$

Consider the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K(w) := \Psi_K(w) - \Phi(w).$$

A point $u \in \mathrm{Dom}(\Psi_K)$ is said to be a critical point of I_K if $D\Phi(u) \in \partial\Psi_K(u)$ or equivalently,

$$\Psi_K(v) - \Psi_K(u) \geq \langle D\Phi(u), v - u\rangle, \quad \forall v \in V.$$

We shall now recall the following variational principle established in [13].

Theorem 1.2. *Let V be a reflexive Banach space and K be a closed convex subset of V. Let $\Phi : V \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous function, and let the linear operator $\Lambda : \mathrm{Dom}(\Lambda) \subset V \to V^*$ be symmetric and positive. Assume that u is a critical point of $I_K(w) = \Psi_K(w) - \Phi(w)$, and that there exists $v \in K$ satisfying the linear equation*

$$\Lambda v = D\Phi(u).$$

Then $u \in K$ is a solution of the equation

$$\Lambda u = D\Phi(u).$$

To adapt Theorem 1.2 to our case, consider the Banach space $V = H^1_{\rm rad}(B_1) \cap L^p_a(B_1)$, where

$$L^p_a(B_1) := \Big\{u : \int_{B_1} a(|x|)|u|^p \, dx < \infty\Big\},$$

and V is equipped with the norm

$$\begin{aligned} \|u\| &:= \|u\|_{H^1} + \Big(\int_{B_1} a(|x|)|u|^p\Big)^{1/p} \\ &= \Big(\int_{B_1} (|\nabla u|^2 + |u|^2) \, dx\Big)^{1/2} + \Big(\int_{B_1} a(|x|)|u|^p \, dx\Big)^{1/p}. \end{aligned}$$

Let $W = L_a^p(B_1)$. It is easily seen that W^*, the topological dual of W, is of the form

$$W^* = \{g : \int a(|x|)^{1-q}|g(x)|^q\,dx < \infty\},$$

where $1/p + 1/q = 1$. Note that by using Lemma 3.2 and a density argument we have that the trace $\frac{\partial u}{\partial n}$ is well-defined for functions $u \in V$ with $-\Delta u + u \in W^*$. Thus, for each $u \in V$ we can define the operator $A : \mathrm{Dom}(A) \subset V \to W^*$ by $Au := -\Delta u + u$, where

$$\mathrm{Dom}(A) = \{u \in V; Au \in W^* \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_1\}.$$

Note that one can rewrite problem (1.1) as $Au = D\varphi(u)$, where

$$\varphi(u) = \frac{1}{p}\int_{B_1} a(|x|)|u|^p\,dx.$$

Our ambient set is that of radially increasing functions;

$$K_0 := \big\{u \in V : u(r) \geq 0, u(r) \leq u(s), \forall r, s \in [0,1], r \leq s\big\}.$$

We also define

$$K := K_0 \cap \mathrm{Dom}(A) = \big\{u \in K_0 : -\Delta u + u \in W^* \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B_1\big\}.$$

Recall that q is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and consider

$$\psi(u) = \begin{cases} \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u + u|^q\,dx, & u \in K \\ +\infty, & u \notin K, \end{cases} \tag{1.6}$$

with $\mathrm{Dom}(\psi) = \{u \in V;\, \psi(u) < \infty\}$. Here is a direct consequence of Theorem 1.2.

Corollary 1.3. *Assume that u is a critical point of*

$$I(w) := \psi(w) - \frac{1}{p}\int_{B_1} a(|x|)|w|^p\,dx. \tag{1.7}$$

If there exists $v \in \mathrm{Dom}(\psi)$ satisfying the linear equation

$$\begin{gathered} -\Delta v + v = a(|x|)|u|^{p-2}u, \quad x \in B_1 \\ \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial B_1, \end{gathered} \tag{1.8}$$

then u is a solution of the equation

$$\begin{gathered} -\Delta u + u = a(|x|)|u|^{p-2}u, \quad x \in B_1 \\ \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial B_1, \end{gathered}$$

Even though this corollary follows directly from Theorem 1.2, for the convenience of the reader we shall prove it in this paper. Here is our existence Theorem.

Theorem 1.4. *Assume that* (H1) *holds. Then problem* (1.1) *admits at least one radially increasing positive solution.*

Evidently, Corollary 1.3 maps out the plan for the prove of Theorem 1.4. Indeed, by using a non-smooth critical point theory we show that the functional I defined in (1.7) has a non-trivial critical point and then we shall prove that the linear equation (1.8) has a solution. We can also make use of the critical value of the functional I given in (1.7) to show that if $a(x) = 1$ then problem (1.1) may admit a non-constant solution. In fact, let λ_2 be the second radial eigenvalue of $-\Delta + I$ in the unit ball with Neumann boundary conditions. We have the following result.

Proposition 1.5. *If $\lambda_2 < p - 1$ then problem* (1.1) *admits at least one positive non-constant radially increasing solution.*

Even though the latter result is already contained in [2], our proof is much shorter and is based on the new proposed variational principle. In the next section we shall recall some preliminaries and then we proceed with the proofs regarding to problem (1.1) in Section 3. The last section is devoted to gradient systems.

2. Preliminaries

In this section we recall some important definitions and results from Convex Analysis and minimax principles for lower semi-continuous functions.

Let V be a real Banach space and V^* its topological dual and let $\langle \cdot, \cdot \rangle$ be the pairing between V and V^*. The weak topology on V induced by $\langle \cdot, \cdot \rangle$ is denoted by $\sigma(V, V^*)$. A function $\Psi : V \to \mathbb{R}$ is said to be weakly lower semi-continuous if

$$\Psi(u) \le \liminf_{n\to\infty} \Psi(u_n),$$

for each $u \in V$ and any sequence u_n approaching u in the weak topology $\sigma(V, V^*)$. Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential $\partial\Psi$ of Ψ is defined to be the following set-valued operator: if $u \in \mathrm{Dom}(\Psi) = \{v \in V;\ \Psi(v) < \infty\}$, set

$$\partial\Psi(u) = \{u^* \in V^*; \langle u^*, v - u\rangle + \Psi(u) \le \Psi(v) \text{ for all } v \in V\}$$

and if $u \notin \mathrm{Dom}(\Psi)$, set $\partial\Psi(u) = \emptyset$. If Ψ is Gâteaux differentiable at u, denote by $D\Psi(u)$ the derivative of Ψ at u. In this case $\partial\Psi(u) = \{D\Psi(u)\}$.

The Fenchel dual of an arbitrary function Ψ is denoted by Ψ^*, that is function on V^* and is defined by

$$\Psi^*(u^*) = \sup\{\langle u^*, u\rangle - \Psi(u); u \in V\}.$$

Clearly $\Psi^* : V^* \to \mathbb{R} \cup \{\infty\}$ is convex and weakly lower semi-continuous. The following standard result is crucial in the subsequent analysis (see [8, 7, 6] for a proof).

Proposition 2.1. *Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be an arbitrary function. The following statements hold:*

1. *$\Psi^{**}(u) \le \Psi(u)$ for all $u \in V$.*
2. *$\Psi(u) + \Psi^*(u^*) \ge \langle u^*, u\rangle$ for all $u \in V$ and $u^* \in V^*$.*
3. *If Ψ is convex and lower-semi continuous then $\Psi^{**} = \Psi$ and the following assertions are equivalent:*
   - *$\Psi(u) + \Psi^*(u^*) = \langle u, u^*\rangle$.*
   - *$u^* \in \partial\Psi(u)$.*
   - *$u \in \partial\Psi^*(u^*)$.*

We shall now recall some notation and results for the minimax principles of lower semi-continuous functions.

Definition 2.2. Let V be a real Banach space, $\varphi \in C^1(V,\mathbb{R})$ and $\psi : V \to (-\infty,+\infty]$ be proper (i.e. $\mathrm{Dom}(\psi) \neq \emptyset$), convex and lower semi-continuous. A point $u \in V$ is said to be a critical point of

$$I := \psi - \varphi \tag{2.1}$$

if $u \in \mathrm{Dom}(\psi)$ and if it satisfies the inequality

$$\langle D\varphi(u), u - v\rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in V. \tag{2.2}$$

Definition 2.3. We say that I satisfies the Palais-Smale compactness condition (PS) if every sequence $\{u_n\}$ such that

- $I[u_n] \to c \in \mathbb{R}$,
- $\langle D\varphi(u_n), u_n - v\rangle + \psi(v) - \psi(u_n) \geq -\epsilon_n \|v - u_n\|, \quad \forall v \in V$.

where $\epsilon_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

The following Mountain Pass Theorem is proved in [20].

Theorem 2.4. *Suppose that $I : V \to (-\infty,+\infty]$ is of the form (2.1) and satisfies the Palais-Smale condition and the Mountain Pass Geometry (MPG):*

1. *$I(0) = 0$.*
2. *there exists $e \in V$ such that $I(e) \leq 0$.*
3. *there exists some ρ such that $0 < \rho < \|e\|$ and for every $u \in V$ with $\|u\| = \rho$ one has $I(u) > 0$.*

Then I has a critical value $c \geq \rho$ which is characterized by

$$c = \inf_{g\in\Gamma} \sup_{t\in[0,1]} I[g(t)],$$

where $\Gamma = \{g \in C([0,1],V) : g(0) = 0, g(1) = e\}$.

3. Supercritical Neumann equations

We shall need some preliminary results before proving Theorem 1.4 and Corollary 1.3. Recall that

$$L^p_a(B_1) = \{u : \int a(|x|)|u|^p\,dx < \infty\}.$$

Let $W = L^p_a(B_1)$. It is easily seen that W^*, the topological dual of W, is of the form,

$$W^* = \{g : \int a(|x|)^{1-q}|g(x)|^q\,dx < \infty\},$$

where, as before, $1/p + 1/q = 1$.

Lemma 3.1. *For each $g \in W^*$ we have*

$$\varphi^*(g) = \frac{1}{q}\int a(x)^{1-q}|g(x)|^q\,dx,$$

where $\varphi : V \to \mathbb{R}$ is defined by $\varphi(v) = \frac{1}{p}\int a(|x|)|v|^p\,dx$.

Proof. Take $g \in W^*$. It follows from the density of V in W that

$$\begin{aligned}
\varphi^*(g) &= \sup_{v\in V}\{\langle v, g\rangle - \varphi(v)\} \\
&= \sup_{v\in V}\Big\{\int v(x)g(x)\,dx - \frac{1}{p}\int a(|x|)|v|^p\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{v\in W}\Big\{\int v(x)g(x)\,dx - \frac{1}{p}\int a(|x|)|v|^p\Big\}\\
&= \frac{1}{q}\int a(|x|)^{1-q}|g(x)|^q\,dx
\end{aligned}
$$

as desired. □

Lemma 3.2. *There exists a constant $C>0$ such that*

$$
C\Big|\int_{\partial B_1}\frac{\partial u}{\partial n}\,d\sigma\Big|^q \le \int_{B_1} a(|x|)^{1-q}|-\Delta u+u|^q\,dx + \Big|\int_{B_1} u\,dx\Big|^q \tag{3.1}
$$

for all $u\in C^2(\overline{B_1})$. In particular, if u is radial, i.e. $u(x)=u(|x|)$, then

$$
\gamma_n^q C\big|u'(1)|^q \le \int_{B_1} a(|x|)^{1-q}|-\Delta u+u|^q\,dx + \Big|\int_{B_1} u\,dx\Big|^q, \tag{3.2}
$$

where γ_n is the surface area of the unit ball in $\mathbb{R}^n$.

Proof. Define $h: B_1\setminus\{0\}\times\mathbb{R}\to\mathbb{R}$ by

$$
h(x,y)=a(x)^{1-q}\frac{|y|^q}{q}.
$$

Note that the function $y\to h(x,y)$ is convex for each $x\in B_1\setminus\{0\}$, and its Fenchel dual $h^*(x,\cdot)$ with respect to the second variable is given by

$$
h^*(x,z)=a(x)\frac{|z|^p}{p}.
$$

It then from $h(x,y)+h^*(x,z)\ge yz$ it follows that

$$
a(x)^{1-q}\frac{|y|^q}{q}\ge yz-a(x)\frac{|z|^p}{p},\quad \forall y,z\in\mathbb{R},\ x\in B_1\setminus\{0\}.
$$

Now substituting y by $-\Delta u+u$ in the latter inequality and integrating over B_1 we obtain that

$$
\frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u+u|^q\,dx \ge z\int_{B_1}(-\Delta u)\,dx + z\int_{B_1} u\,dx - \frac{|z|^p}{p}\int_{B_1} a(x)\,dx,
$$

for all $z\in\mathbb{R}$. Maximizing the latter inequality over all $z\in\mathbb{R}$ implies that

$$
\frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u+u|^q\,dx \ge \frac{1}{q}\Big|\int_{B_1}(-\Delta u)\,dx+\int_{B_1} u\,dx\Big|^q\Big(\int_{B_1} a(x)\,dx\Big)^{1-q}.
$$

It then follows that

$$
\Big|\int_{B_1}(-\Delta u)\,dx+\int_{B_1} u\,dx\Big|^q \le \Big(\int_{B_1} a(x)\,dx\Big)^{q-1}\int_{B_1} a(|x|)^{1-q}|-\Delta u+u|^q\,dx \tag{3.3}
$$

On the other hand by the Green's theorem

$$
\int_{B_1}\Delta u\,dx = \int_{\partial B_1}\frac{\partial u}{\partial n}\,d\sigma,
$$

from which together with (3.3) the inequality (3.1) follows. If now u is radial then inequality (3.1) simply yields (3.2). □

Note that by using Lemma (3.2) and a density argument we have that the trace $\frac{\partial u}{\partial n}$ is well-defined for functions $u \in V$ with $-\Delta u + u \in W^*$. Recall from the introduction that the operator $A : \mathrm{Dom}(A) \subset V \to W^*$ is defined by $Av := -\Delta v + v$, where

$$\mathrm{Dom}(A) = \big\{v \in V; Av \in W^* \text{ and } \frac{\partial v}{\partial n} = 0\big\},$$

and $\varphi : V \to \mathbb{R}$ is defined by

$$\varphi(u) = \frac{1}{p}\int_{B_1} a(|x|)|u|^p \, dx,$$

and finally $\psi : V \to [0, \infty]$ is defined by

$$\psi(u) = \begin{cases} \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u + u|^q \, dx, & u \in K \\ +\infty, & u \notin K, \end{cases}$$

where

$$K = \{u \in \mathrm{Dom}(A) : u(r) \geq 0, u(r) \leq u(s), \forall r, s \in [0, 1], r \leq s\}.$$

Proof of Corollary 1.3. Note first that the duality mapping $\langle \cdot, \cdot \rangle$ between V and V^* is defined by

$$\langle f, g\rangle = \int_{B_1} f(x)g(x) \, dx, \quad \forall f \in V, \forall g \in V^*.$$

Since u is a critical point of I, it follows from Definition 2.2 that

$$\psi(w) - \psi(u) \geq \langle D\varphi(u), w - u\rangle, \quad \forall w \in V. \tag{3.4}$$

Since $I(u)$ is finite we have that $u \in \mathrm{Dom}(\psi)$ and

$$\psi(u) = \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u + u|^q \, dx < \infty.$$

It then follows that $Au \in W^*$ and $\psi(u) = \varphi^*(Au)$ as shown in Lemma 3.1. By assumption, there exists $v \in \mathrm{Dom}(\psi)$ satisfying $Av = D\varphi(u)$. Substituting $w = v$ in (3.4) yields that

$$\varphi^*(Av) - \varphi^*(Au) = \psi(v) - \psi(u) \geq \langle D\varphi(u), v - u\rangle = \langle Av, v - u\rangle. \tag{3.5}$$

On the other hand it follows from $Av = D\varphi(u)$ and Proposition 2.1 that $u \in \partial\varphi^*(Av)$ from which we obtain

$$\varphi^*(Au) - \varphi^*(Av) \geq \langle u, Au - Av\rangle. \tag{3.6}$$

Adding (3.5) and (3.6) we obtain

$$\langle u, Au - Av\rangle + \langle Av, v - u\rangle \leq 0.$$

Since A is symmetric we obtain that $\langle u - v, Au - Av\rangle \leq 0$ from which we obtain

$$\int_{B_1} |\nabla u - \nabla v|^2 \, dx + \int_{B_1} |u - v|^2 \, dx \leq 0,$$

thereby giving that $u = v$. It then follows that $Au = Av = D\varphi(u)$ as claimed. $\square$

We shall need a few preliminary lemmas before proving our main theorem.

Lemma 3.3. *The functional $\psi : V \to (-\infty, \infty]$ defined by*

$$\psi(u) = \begin{cases} \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u + u|^q\,dx, & u \in K \\ +\infty, & u \notin K, \end{cases}$$

is weakly lower semi-continuous.

Proof. Let $\{u_n\}$ be a sequence in V that converges weakly to some $u \in V$. If $\alpha := \liminf_{n\to\infty}\psi(u_n) = \infty$ the there is nothing to prove. Let us assume that $\alpha < \infty$. Thus, up to a subsequence, $u_n \to u$ a.e., $\psi(u_n) < \infty$ and $\lim_{n\to\infty}\psi(u_n) = \alpha$. Since $u_n \to u$ a.e. we have that $u \in K_0$. We now show that $u \in K$.

Take $v \in C_c^2(\Omega)$. It follows that

$$\psi(u_n) = \varphi^*((-\Delta u_n + u_n) \geq \int_\Omega v(x)(-\Delta u_n + u_n)\,dx - \varphi(v),$$

from which we obtain

$$1 + \alpha + \varphi(v) \geq \int_\Omega u_n(x)(-\Delta v + v)\,dx,$$

for n large. Letting $n \to \infty$ we obtain

$$1 + \alpha + \varphi(v) \geq \int_\Omega u(x)(-\Delta v + v)\,dx, \quad \forall v \in C_c^2(\Omega).$$

This indeed implies that $-\Delta u + u \in L^1_{\rm loc}(B_1)$. Therefore,

$$1 + \alpha + \varphi(v) \geq \int_\Omega v(x)(-\Delta u + u)\,dx, \quad \forall v \in C_c^2(\Omega).$$

Since $C_c^2(\Omega)$ is dense in $W = L_a^p(\Omega)$, we obtain

$$1 + \alpha + \varphi(v) \geq \int_\Omega v(x)(-\Delta u + u)\,dx, \quad \forall v \in W.$$

This indeed implies that $-\Delta u + u \in W^*$. Take now $v \in W$ and note that

$$\psi(u_n) \geq \int_\Omega v(x)(-\Delta u_n + u_n)\,dx - \varphi(v),$$

from which we obtain

$$\liminf_{n\to\infty}\psi(u_n) \geq \int_\Omega v(x)(-\Delta u + u)\,dx - \varphi(v),$$

Taking supremum over all $v \in W$ implies that

$$\liminf_{n\to\infty}\psi(u_n) \geq \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|-\Delta u + u|^q\,dx = \psi(u),$$

from which the lower semi-continuity of ψ follows. □

Lemma 3.4. *There exists $C = C(R, N) > 0$ such that*

$$\|u\|_\infty \leq C\|u\|_{H^1}, \quad \forall u \in K.$$

Proof. Let $0 < r < 1$ and B_r be a ball centered at the origin with radius r. It follows from the continuous embedding of $H^1(B_1 \setminus B_r) \subseteq L^\infty(B_1 \setminus B_r)$ that there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(B_1)} = \|u\|_{L^\infty(B_1\setminus B_r)} \leq C\|u\|_{H^1(L^\infty(B_1\setminus B_r))}.$$

□

Lemma 3.5. *Let $V = H^1_{\rm rad}(B_1) \cap L^p_a(B_1)$ and consider the functional $I : V \to \mathbb{R}$ by*

$$I(u) := \psi(u) - \varphi(u),$$

with φ and ψ as in Corollary 1.3. Then I has a nontrivial critical point.

Proof. We make use Theorem 2.4 to prove this lemma. We shall do this in several steps. First note that

$$D\varphi(u) = a(|x|)|u|^{p-2}u,$$

and therefore φ is C^1 on the space V. Note also that ψ is proper, convex and lower semi-continuous as K is closed in V.

Step 1. We verify (MPG) for I. It is clear that $I(0) = 0$. Take $e \in K$ with $Ae \in W^*$. It follows that

$$\begin{aligned} I(te) &= \frac{1}{q}\int_{B_1} a(|x|)^{1-q}|tAe|^q\,dx - \frac{1}{p}\int_{B_1} a(|x|)|te|^p\,dx \\ &= \frac{t^q}{q}\int_{B_1} a(|x|)^{1-q}|Ae|^q\,dx - \frac{t^p}{p}\int_{B_1} a(|x|)|e|^p\,dx \end{aligned}$$

Now, since $p > 2$ one has that $q < 2$. Thus for t sufficiently large $I(te)$ is negative. We now prove condition 3) of (MPG). Take $u \in \mathrm{Dom}(\psi)$ with $\|u\|_V = \rho > 0$. We have

$$I(u) = \varphi^*(Au) - \varphi(u) \geq \langle Au, u\rangle - 2\varphi(u) = \|u\|^2_{H^1} - 2\varphi(u) \tag{3.7}$$

Note that from Lemma 3.4, for $u \in K$ one has $\|u\|_\infty \leq C_1\|u\|_{H^1}$. Therefore,

$$\|u\|_V = \|u\|_{H^1} + \Big(\int_{B_1} a(|x|)|u|^p\,dx\Big)^{1/p} \leq (1 + C_2)\|u\|_{H^1} \tag{3.8}$$

Also

$$\varphi(u) = \frac{1}{p}\int_{B_1} a(|x|)|u|^p\,dx \leq C_3\|u\|^p_{H^1} \leq C_3\rho^p \tag{3.9}$$

Therefore from (3.7), (3.8) and (3.9) we obtain

$$I[u] \geq \frac{\rho^2}{(1 + C_2)^2} - 2C_3\rho^p > 0$$

provided $\rho > 0$ is small enough as $p > 2$ and C_2 and C_3 are constants. If $u \notin \mathrm{Dom}(\psi)$, then clearly $I(u) > 0$. Therefore (MPG) holds for the functional I.

Step 2. We verify (PS) compactness condition. Suppose that $\{u_n\}$ is a sequence in K such that $I(u_n) \to c \in \mathbb{R}$, $\epsilon_n \to 0$ and

$$\langle D\varphi(u_n), u_n - v\rangle + \Psi(v) - \Psi(u_n) \geq -\epsilon_n\|v - u_n\|_V, \quad \forall v \in V. \tag{3.10}$$

We must show that $\{u_n\}$ has a convergent subsequence in V. First, note that $u_n \in \mathrm{Dom}(\psi)$ and therefore,

$$I(u_n) = \varphi^*(Au_n) - \varphi(u_n) \to c, \quad \text{as } n \to \infty.$$

Thus, for large values of n we have

$$\varphi^*(Au_n) - \varphi(u_n) \leq 1 + c. \tag{3.11}$$

In (3.10), set $v = ru_n$, where $r := p - 1 > 1$. Then

$$(1 - r)\langle D\varphi(u_n), u_n\rangle + (r^q - 1)\varphi^*(Au_n) \geq -\epsilon_n(r - 1)\|u_n\|_V. \tag{3.12}$$

On the other hand,

$$\langle D\varphi(u_n), u_n\rangle = \int_{B_1} a(|x|)u_n(x)^p\,dx = p\varphi(u_n) \tag{3.13}$$

It now follows from (3.12), (3.13) and (3.8) that

$$(r-1)p\varphi(u_n) - (r^q-1)\varphi^*(Au_n) \le \epsilon_n(r-1)\|u_n\|_V \le C\epsilon_n\|u_n\|_{H^1}, \tag{3.14}$$

Now observe that $r^q-1 < p(r-1)$. Take $\alpha > 0$ such that $r^q - 1 < \alpha < p(r-1)$. Multiply (3.11) by α and add it to (3.14) to get

$$[\alpha-(r^q-1)]\varphi^*(Au_n) + [(r-1)p-\alpha]\varphi(u_n) \le C_1(1+\|u_n\|_{H^1}),$$

and therefore

$$\varphi^*(Au_n) + \varphi(u_n) \le C_2(1+\|u_n\|_{H^1}), \tag{3.15}$$

for an appropriate constant $C_2 > 0$. On the other hand

$$\varphi^*(Au_n) + \varphi(u_n) \ge \langle Au_n, u_n\rangle = \|u_n\|^2_{H^1},$$

which according to (3.15) results in

$$\|u_n\|^2_{H^1} \le C_2(1+\|u_n\|_{H^1}).$$

Therefore $\{u_n\}$ is bounded in H^1. Using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $\bar u \in H^1$ such that $u_n \rightharpoonup \bar u$ weakly in H^1, $u_n \to \bar u$ strongly in L^2 and $u_n \to \bar u$ a.e.. Also according to Lemma 3.4 from boundedness of $\{u_n\}$ in H^1 one can deduce that $\{u_n\}$ is bounded in L^∞, thus $\|u_n\|_\infty \le C$ for a positive constant C. Note that every u_n is radial, so $\bar u$ is radial too and moreover $\bar u \in K$. It also follows from (3.15) that $\{\varphi^*(Au_n)\}$ is bounded and therefore,

$$\varphi^*(A\bar u) \le \liminf_{n\to\infty} \varphi^*(Au_n) < \infty,$$

from which we obtain $\bar u \in \mathrm{Dom}(\psi)$. Now in (3.10) set $v = \bar u$:

$$-\int a(|x|)|u_n|^{p-1}(\bar u - u_n)\,dx + \varphi^*(A\bar u) - \varphi^*(Au_n) \ge -\epsilon_n\|\bar u - u_n\|_V, \tag{3.16}$$

One has

$$\Big|\int a(|x|)|u_n|^{p-1}(\bar u - u_n)\,dx\Big| \le C\Big|\int a(|x|)(\bar u - u_n)\,dx\Big|$$

Note that $\bar u - u_n \to 0$ a.e., also

$$|a(|x|)(\bar u(x) - u_n(x))| \le a(|x|)(\|u_n\|_\infty + \|\bar u\|_\infty) \le Ca(|x|)$$

since $a \in L^1$, then from Dominated Convergence Theorem one can deduce that

$$\int a(|x|)|u_n|^{p-1}|\bar u - u_n|\,dx \to 0.$$

Therefore passing into limits in (3.16) results in

$$\limsup_{n\to\infty} \varphi^*(Au_n) \le \varphi^*(A\bar u). \tag{3.17}$$

The latter inequality together with the fact that $\varphi^*(A\bar u) \le \liminf_{n\to\infty}\varphi^*(Au_n)$ yield that

$$\varphi^*(A\bar u) = \lim_{n\to\infty} \varphi^*(Au_n).$$

Now observe that

$$\begin{aligned} \|u_n\|^2_{H^1} - \|\bar u\|^2_{H^1} &= \langle Au_n, u_n\rangle - \langle A\bar u, \bar u\rangle \\ &= \langle Au_n, u_n - \bar u\rangle + \langle Au_n - A\bar u, \bar u\rangle. \end{aligned} \tag{3.18}$$

But weakly convergence of u_n to $\bar{u}$ in H^1 means that $Au_n \rightharpoonup A\bar{u}$ weakly in H^{-1}, thus

$$\langle Au_n - A\bar{u}, \bar{u}\rangle \to 0, \quad \text{as } n \to \infty. \tag{3.19}$$

We also have

$$\begin{aligned} |\langle Au_n, u_n - \bar{u}\rangle| &\le \int_{B_1} a(x)^{\frac{1-q}{q}} |Au_n| a(x)^{\frac{q-1}{q}} |u_n - \bar{u}|\, dx \\ &\le \Big(\int_{B_1} a(x)^{1-q}|Au_n|^q\Big)^{1/q}\Big(\int_{B_1} a(x)|u_n - \bar{u}|^p\Big)^{1/p} \end{aligned} \tag{3.20}$$

Now since $\bar{u} - u_n \to 0$ a.e., and

$$|a(|x|)||\bar{u}(x) - u_n(x)|^p \le Ca(|x|),$$

it follows from the dominated convergence theorem that

$$\int_{B_1} a|u_n - \bar{u}|^p\, dx \to 0, \quad \text{as } n \to \infty. \tag{3.21}$$

It now follows from (3.20), (3.21) and the boundedness of $\int_{B_1} a^{1-q}|Au_n|^q\, dx$ that

$$\langle Au_n, u_n - \bar{u}\rangle \to 0, \quad \text{as } n \to \infty. \tag{3.22}$$

Therefore, from (3.18), (3.19) and (3.22) one has

$$u_n \to \bar{u} \quad \text{strongly in } H^1$$

and from (3.8) $u_n \to \bar{u}$ strongly in V as desired. □

Lemma 3.6. *Let $u \in \mathrm{Dom}(\psi)$. Then there exists $v \in \mathrm{Dom}(\psi)$ such that $Av = a(x)u(x)^{p-1}$.*

This result is essentially contained in a portion of [2]. We give a proof for the convenience of the reader.

Proof. Let $u \in \mathrm{Dom}(\psi)$ and so note that $0 \le u \in K \cap H^1_{\rm rad}(B_1) \cap L^\infty(B_1)$. We need to show the existence of $v \in \mathrm{Dom}(\psi)$ which satisfies (1.8). Instead we find a solution $v_m \in \mathrm{Dom}(\psi)$ of

$$\begin{gathered} -\Delta v_m + v_m = a_m(|x|)u^{p-1}, \quad x \in B_1 \\ \frac{\partial v_m}{\partial \nu} = 0, \quad x \in \partial B_1, \end{gathered} \tag{3.23}$$

where $0 \le a_m \le a$ is a smoothed version of a which is increasing and nonconstant on $(0,1)$ and such that $a_m \to a$ in $L^1(0,1)$; see below where we give an approach to construct these a_m. By standard methods we see there exists some $0 \le v_m \in H^1_{\rm rad}(B_1)$ which satisfies (3.23). By elliptic regularity one sees that $v_m \in H^3(B_1)$ after considering the fact that a_m is smooth and $u \in K \cap H^1_{\rm rad}(B_1) \cap L^\infty(B_1)$ along with the fact that $p > 2$. For $0 < r < 1$ note that $w_m(x) := (v_m)_r(|x|)$ satisfies

$$\begin{gathered} -\Delta w_m + \Big(\frac{N-1}{|x|^2} + 1\Big) w_m = g_m, \quad x \in B_1\backslash\{0\} \\ w_m = 0, \quad x \in \partial B_1, \end{gathered} \tag{3.24}$$

where $g_m(x) = a_m'(r)u(r)^{p-1} + a_m(r)(p-1)u(r)^{p-2}u'(r) \ge 0$ on $(0,1)$ where $r = |x|$. Note that $w_m \in H^1_{\rm rad}(B_1)$ and has enough regularity to extend the solution of (3.24) to the full ball B_1. Then one can apply a weak maximum principle to see that $w_m \ge 0$ in B_1. In particular we have $(v_m)_r \ge 0$ in $(0,1)$. We now multiply

(3.23) by v_m and integrate by parts to see that $\{v_m\}$ is bounded in $H^1(B_1)$. By passing to a subsequence we can assume there is some $0 \le v \in H^1_{\rm rad}(B_1)$ such that $v_m \rightharpoonup v$ in $H^1(B_1)$. Additionally one has that v is increasing on $(0,1)$ and so $v \in K$. We now show that v satisfies $Av = a(x)u(x)^{p-1}$. From (3.23) we see that

$$\int_{B_1} \nabla v_m \cdot \nabla \eta + v_m \eta \, dx = \int_{B_1} a_m u^{p-1} \eta \, dx \tag{3.25}$$

for all $\eta \in H^1(B_1) \cap L^\infty(B_1)$. Since $v_m \rightharpoonup v$ in $H^1(B_1)$ we can pass to the limit in (3.25) to see that v satisfies $Av = a(x)u(x)^{p-1}$ in the weak sense. Using (3.23) one sees that $\{v_m\}$ is bounded in $W^{2,2q}_{\rm loc}(B_1)$; note that the right hand side of (3.23) is bounded in $L^\infty(B_R)$ for $R<1$. By a diagonal argument in R and passing to another subsequence) one can assume that $v_m \rightharpoonup v$ in $W^{2,2q}_{\rm loc}(B_1)$. Fix $\frac12 < R < 1$ and then note by (3.23) we have

$$\int_{B_R} |-\Delta v_m + v_m|^q a_m^{1-q} \, dx = \int_{B_R} u^{(p-1)q} a_m \, dx \le \int_{B_1} u^{(p-1)q} a \, dx < \infty. \tag{3.26}$$

We now let $0 < \epsilon < \frac14$ be small and recall that a is bounded away from zero on any compact interval in $(0,1]$. Then note

$$\begin{aligned}
&\int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q a^{1-q} \, dx \\
&\le \int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q a_m^{1-q} \, dx + \int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q |a^{1-q} - a_m^{1-q}| \, dx, \\
&\le \int_{B_1} u^{(p-1)q} a \, dx + \int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q |a^{1-q} - a_m^{1-q}| \, dx,
\end{aligned}$$

where we have utilized (3.26). Then note that

$$\begin{aligned}
&\int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q |a^{1-q} - a_m^{1-q}| \, dx \\
&\le \| |-\Delta v_m + v_m|^q \|_{L^2(B_R)} \| a^{1-q} - a_m^{1-q} \|_{L^2(B_R \setminus B_\epsilon)},
\end{aligned}$$

and note that $\| |-\Delta v_m + v_m|^q \|_{L^2(B_R)}$ is bounded in m and $\| a^{1-q} - a_m^{1-q} \|_{L^2(B_R \setminus B_\epsilon)} \to 0$ as $m \to 0$. This gives

$$\limsup_m \int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q a^{1-q} \, dx \le \int_{B_1} u^{(p-1)q} a \, dx, \tag{3.27}$$

and hence we just need to pass to the limit in the left hand side. Since $v_m \rightharpoonup v$ in $W^{2,2q}(B_R)$ (and hence in $W^{2,q}(B_R)$) we have $-\Delta v_m + v_m \rightharpoonup -\Delta v + v$ in $L^q(B_R)$ and therefore we also have this weak convergence in $L^q(B_R \setminus B_\epsilon)$. Noting that the dual of $L^q(B_R \setminus B_\epsilon)$ and $L^q(B_R \setminus B_\epsilon, a^{1-q}\, dx)$ are equal we have that $-\Delta v_m + v_m \rightharpoonup -\Delta v + v$ in $L^q(B_R \setminus B_\epsilon, a^{1-q}\, dx)$. We now use the fact that a norm is weakly lower semi continuous to see that

$$\int_{B_R \setminus B_\epsilon} |-\Delta v + v|^q a^{1-q} \, dx \le \liminf_m \int_{B_R \setminus B_\epsilon} |-\Delta v_m + v_m|^q a^{1-q} \, dx.$$

Combining this with (3.27) shows that

$$\int_{B_R \setminus B_\epsilon} |-\Delta v + v|^q a^{1-q} \, dx \le \int_{B_1} u^{(p-1)q} \, dx.$$

Since $|-\Delta v + v|^q \in L^2(B_R)$ we can send $\epsilon \searrow 0$ to obtain

$$\int_{B_R} |-\Delta v + v|^q a^{1-q}\,dx \le \int_{B_1} u^{(p-1)q}\,dx,$$

and we can now send $R \nearrow 1$ to see that $v \in \mathrm{Dom}(\psi)$.

We now construct a_m; which will involve cutting a off and then using a mollifier to smooth the cut off. For large integers m we define b_m on $[0,\infty)$ via $b_m(r) = \min\{a(r), m\}$ and so note for each m that b_m is increasing on $(0,1)$. Now extend $b_m(r)$ to $b_m(1)$ for $r > 1$ and $b_m = 0$ for $r < 0$. Let $0 \le \eta$ be smooth with $\eta = 0$ on $(-\infty,-1) \cup (0,\infty)$ and $\eta > 0$ on $(-1,0)$. We also assume that $\int_{-1}^0 \eta(\tau)d\tau = 1$. For $\epsilon > 0$ define $\eta_\epsilon(r) := \frac{1}{\epsilon}\eta(\frac{r}{\epsilon})$ and

$$b_m^\epsilon(r) := \int_{-\epsilon}^0 \eta_\epsilon(\tau) b_m(r+\tau)d\tau,$$

note that this is just the usual mollification except the support of η is adjusted slightly. Since b_m is increasing we see that for each fixed $\epsilon > 0$ that b_m^ϵ is increasing in r. Then note that we have

$$0 \le b_m^\epsilon(r) = \int_{-\epsilon}^0 \eta_\epsilon(\tau) b_m(r+\tau)d\tau \le b_m(r)\int_{-\epsilon}^0 \eta_\epsilon(\tau)d\tau = b_m(r) \le a(r).$$

We now let $\epsilon_m \searrow 0$ and we set $a_m(r) := b_m^{\epsilon_m}$. So we have $0 \le a_m(r) \le a(r)$ for all m. Also $r \mapsto a_m(r)$ is increasing in r. One can now show that $a_m \to a$ in $L^1(0,1)$. $\square$

Proof of Theorem 1.4. It follows from Lemma 3.5 that the functional I has a nontrivial critical point u. It also follows from Lemma 3.6 that there exists $v \in \mathrm{Dom}(\psi)$ satisfying the linear equation $Av = D\varphi(u)$. It now follows from Corollary 1.3 that u must be a nontrivial nonnegative solution of (1.1). Setting $C(x) := 1 - a(|x|)u(x)^{p-2}$ one sees that $-\Delta u + C(x)u = 0$ in B_1. We now show that $u > 0$ in B_1. Assuming not one must have $u(0) = 0$ after considering the fact that u is radial and increasing. We can now apply the strong maximum principle to see that u is identically zero in B_1, giving us the needed contradiction. $\square$

To prove Proposition 1.5 we first recall the following result from [2, Lemma 4.8].

Lemma 3.7. *Let w be an eigenfunction associated to λ_2, the second radial eigenvalue of $-\Delta + I$ in the unit ball, that is*

$$\begin{gathered} -\Delta w + w = \lambda_2 w, \quad x \in B_1 \\ w \text{ radial}, \\ \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial B_1, \end{gathered} \tag{3.28}$$

Then w is unique up to a multiplicative factor and we can choose it increasing. Moreover, $\int_{B_1} w\,dx = 0$.

Proof of Proposition 1.5. It follows from Theorem 1.4 that the problem (1.1) has a positive solution u with $I(u) = c > 0$ where the critical value c is characterized by

$$c = \inf_{g\in\Gamma} \sup_{t\in[0,1]} I[g(t)],$$

where $\Gamma = \{g \in C([0,1],V) : g(0) = 0 \neq g(1), I\big(g(1)\big) \le 0\}$. Note that the constant function $u_0 = 1$ is the only positive constant solution of (1.1). We shall show that $I(u) = c < I(1)$ from which one can easily deduce that u is not a constant solution. Let w be as in Lemma 3.7 and $s \in \mathbb{R}^+$ with $|s| < 1/\|w\|_\infty$. It follows that $1 + sw \in K$. Take now $r \in R^+$ such that $I\big((1+sw)r\big) = 0$. Define $g : [0,1] \to V$ by $g(t) = t(1+sw)r$ and note that $I\big(g(0)\big) = I\big(g(1)\big) = 0$. It follows that $c \le max_{t\in[0,1]} I\big(g(t)\big)$ where

$$I\big(g(t)\big) = \frac{t^q}{q}\int_{B_1} |r(1+s\lambda_2 w)|^q\,dx - \frac{t^p}{p}\int_{B_1} |r(1+sw)|^p\,dx.$$

An easy computation shows that

$$\max_{t\in[0,1]} I\big(g(t)\big) = (\frac{1}{q} - \frac{1}{p})\frac{\big(\int |1+\lambda_2 sw|^q\,dx\big)^{\frac{p}{p-q}}}{\big(\int |1+sw|^p\,dx\big)^{\frac{q}{p-q}}}$$

On the other hand we have

$$I(1) = (\frac{1}{q} - \frac{1}{p})C_N,$$

where C_N is the volume of the unit ball in $\mathbb{R}^N$. We need to show that for small values of $s \neq 0$, we have

$$(\frac{1}{q} - \frac{1}{p})\frac{\big(\int |1+\lambda_2 sw|^q\,dx\big)^{\frac{p}{p-q}}}{\big(\int |1+sw|^p\,dx\big)^{\frac{q}{p-q}}} < (\frac{1}{q} - \frac{1}{p})C_N.$$

We can rewrite the latter inequality as follows

$$\Big(\int |1+\lambda_2 sw|^q\,dx\Big)^p < C_N^{p-q}\Big(\int |1+sw|^p\,dx\Big)^q.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(s) := \Big(\int |1+\lambda_2 sw|^q\,dx\Big)^p - C_N^{p-q}\Big(\int |1+sw|^p\,dx\Big)^q.$$

Note that $f(0) = 0$. It also follows from $\int_{B_1} w\,dx = 0$ that $f'(0) = 0$. An easy computation shows that

$$f''(0) = pq(q-1)\lambda_2^2\, C_N^{(p-1)}\int |w|^2\,dx - C_N^{p-q} qp(p-1) C_N^{q-1}\int |w|^2\,dx,$$

from which we have that $f^{''}(0) < 0$ if and only if $\lambda_2 < p-1$. This indeed shows that $f(s) < f(0)$ for s sufficiently close to zero from which the desired result follows. □

4. Elliptic systems

In this section we are interested in obtaining positive solutions of the gradient system (1.2). We assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a sufficiently smooth function. We also assume that the function $f_u := \frac{\partial}{\partial u} f$ satisfies the following properties:

- (A1) For each $r \in [0,1]$ the function $(u,v) \to f(r,u,v)$ is convex.
- (A2) For each $r \in (0,1]$ and $u, v \ge 0$ one has $\partial_r f_u, \partial_r f_v, f_u, f_v, f_{uu}, f_{vv}, f_{uv}$ are nonnegative.
- (A3) There exists $p_1, p_2 > 2$ and positive functions $a_1, a_2 \in L^1(0,1)$ such that

$$0 \le f(r,u,v) \le \big(a_1(r)|u|^{p_1} + a_2(r)|v|^{p_2}\big).$$

(A4) There exists $\mu > 2$ such that
$$\mu f(r,u,v) \geq f_u(r,u,v)u + f_v(r,u,v)v,$$
for all $(r,u,v) \in [0,1] \times \mathbb{R}^2$.

Now consider the Banach space $V = \big(H^1_{\rm rad}(B_1) \times H^1_{\rm rad}(B_1)\big) \cap \big(L^{p_1}_{a_1}(B_1) \times L^{p_2}_{a_2}(B_1)\big)$, where
$$L^{p_i}_{a_i}(B_1) := \big\{u : \int_{B_1} a_i(|x|)|u|^{p_i}\,dx < \infty\big\}, \quad i = 1,2,$$
and V is equipped with the norm
$$\|(u,v)\| := \|u\|_{H^1} + \|v\|_{H^1} + \Big(\int_{B_1} a_1(|x|)|u|^{p_1}\Big)^{1/p_1} + \Big(\int_{B_1} a_2(|x|)|v|^{p_2}\Big)^{1/p_2}.$$
For $(u,v) \in V$ define the linear symmetric operator $B : \mathrm{Dom}(B) \subset V \to V^*$ by $B(u,v) := (-\Delta u + u, -\Delta v + v)$ where
$$\mathrm{Dom}(B) = \{(u,v) \in V; \frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} = 0, \quad \text{and} \quad B(u,v) \in V^*\}.$$
Note that B is a positive operator as
$$\langle B(u,v),(u,v)\rangle_{V\times V^*} = \int_{B_1} |\nabla u|^2\,dx + \int_{B_1} |u|^2\,dx + \int_{B_1} |\nabla v|^2\,dx + \int_{B_1} |v|^2\,dx.$$
Note that one can rewrite the system (1.2) as $B(u,v) = DF(u,v)$, where the convex function $F : V \to \mathbb{R}$ is defined by
$$F(u,v) = \int_{B_1} f\big(|x|, u(x), v(x)\big)\,dx$$
As in the previous case we define
$$G(u,v) = \begin{cases} F^*\big(B(u,v)\big) & (u,v) \in K \times K \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.1}$$
where $F^* : V^* \to (-\infty,+\infty]$ is the Fenchel dual of F. We have the following result.

Theorem 4.1. *Assume that conditions* (A1)–(A4) *hold. Then the functional* $J : V \to (-\infty,+\infty]$ *defined by*
$$J(u,v) = G(u,v) - F(u,v),$$
has a nontrivial critical point (u_0,v_0) *which is indeed a solution for the system* (1.2).

Proof. The proof is very similar to the previous case when we dealt with an equation. Here we just sketch the proof. Let $W_i = L^{p_i}_{a_i}(B_1)$ for $i = 1,2$. It follows from A_3 that the functional $F : W_1 \times W_2 \to \mathbb{R}$ defined by
$$F(u,v) = \int_{B_1} f\big(|x|, u(x), v(x)\big)\,dx,$$
is C^1. The pairing between W_i and W_i^* is nothing but $\langle u, u^*\rangle_{W_i\times W_i} = \int_{B_1} u(x)u^*(x)\,dx$ for $u \in W_i$ and $u^* \in W_i^*$. It also follows from A_3 that for all $(u^*,v^*) \in W_1^* \times W_2^*$,
$$\begin{aligned} &F^*(u^*,v^*) \\ &= \sup_{u,v}\{\langle u,u^*\rangle + \langle v,v^*\rangle - F(u,v)\} \end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{u,v}\Big\{\langle u,u^*\rangle+\langle v,v^*\rangle-\int a_1(|x|)|u(x)|^{p_1}\,dx-\int a_2(|x|)|v(x)|^{p_2}\,dx\Big\}\\
&\geq C\int a_1(|x|)^{1-p_1'}|u^*(x)|^{p_1'}\,dx+C\int a_2(|x|)^{1-p_2'}|v^*(x)|^{p_2'}\,dx,
\end{aligned}
$$

where $C>0$ is a constant and $1/p_i+1/p_i'=1$. One can now easily deduce from the same argument as in the Lemma 3.5 that the functional J has a nontrivial critical point (u_0,v_0). We claim that the linear system

$$
\begin{gathered}
-\Delta u+u=f_u(|x|,u_0,v_0),\quad x\in B_1\\
-\Delta v+v=f_v(|x|,u_0,v_0),\quad x\in B_1\\
\frac{\partial u}{\partial\nu}=\frac{\partial u}{\partial\nu}=0,\quad x\in\partial B_1,
\end{gathered}
\tag{4.2}
$$

has a solution $(u,v)\in K\times K$. Since the linear symmetric operator $B:V\to V^*$ is non-negative, assuming the claim is true, it then follows from Theorem 1.2 that (u_0,v_0) is indeed a solution of the system (1.2). We shall now prove the claim. First note that $f_u,f_v\geq 0$ by assumption on f. We can then apply standard methods to obtain nonnegative smooth radial solutions of (4.2). We now show the solutions are increasing. To do this, one first writes the system (4.2) in radial coordinates and then taking a derivative in $r=|x|$ gives

$$
\begin{gathered}
-\Delta u_r+\Big(\frac{N-1}{r^2}+1\Big)u_r=\partial_r f_u+f_{uu}(u_0)_r+f_{uv}(v_0)_r,\quad 0<r<1,\\
-\Delta v_r+\Big(\frac{N-1}{r^2}+1\Big)v_r=\partial_r f_v+f_{vu}(u_0)_r+f_{vv}(v_0)_r,\quad 0<r<1,\\
u_r(1)=v_r(1)=0,
\end{gathered}
\tag{4.3}
$$

where $u_r(r)=u'(r)$. Note that since $u_0,v_0\in K$ and after noting the assumptions on f one sees the right hand sides of (4.3) is nonnegative. One can then argue as in the proof of Lemma 3.6 to see that $u_r,v_r\geq 0$ in $(0,1)$. From this we can conclude that $(u,v)\in K\times K$. □

Acknowledgments. C. C. and A.M. acknowledge the support from the National Sciences and Engineering Research Council of Canada.

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