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SECOND-ORDER BOUNDARY ESTIMATE FOR THE SOLUTION TO INFINITY LAPLACE EQUATIONS

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Abstract. In this article, we establish a second-order estimate for the solutions to the infinity Laplace equation

$$-\Delta_\infty u = b(x)g(u), \quad u > 0, \quad x \in \Omega, \ u|_{\partial\Omega} = 0,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $g \in C^1((0,\infty),(0,\infty))$, g is decreasing on $(0,\infty)$ with $\lim_{s\to 0^+} g(s) = \infty$ and g is normalized regularly varying at zero with index $-\gamma$ ($\gamma > 1$), $b \in C(\bar\Omega)$ is positive in Ω, may be vanishing on the boundary. Our analysis is based on Karamata regular variation theory.

1. Introduction and statement of main results

The operator Δ_∞ is the so-called ∞-Laplacian

$$\Delta_\infty u := \langle D^2uDu, Du\rangle = \sum_{i,j=1}^N D_iuD_{ij}uD_ju.$$

The infinity Laplacian equation $\Delta_\infty u = 0$ is the properly interpreted Euler-Lagrange equation associated with minimizing the functional $(u, X) \mapsto \|\nabla u\|_{L^\infty(X)}$ for $X \subset \mathbb{R}^N$. It was introduced and first studied by Aronsson [3] in 1967. Notice that the infinity Laplacian is a quasilinear and highly degenerate elliptic operator, and this degeneracy accounts for the non-existence, in general, of smooth solutions to Dirichlet problems. Several approaches were developed to overcome this problem, including the notion of viscosity solutions (see [13]) and the method of comparison with cones, developed by Crandall, Evans and Gariepy [14]. It was only in 1993 that Jensen [20] showed a continuous function u is a viscosity solution to $\Delta_\infty u = 0$ if and only if it is a so-called absolutely minimizing Lipschitz extension. Jensen also proved uniqueness in this setting. Peres, Schramm, Sheffield and Wilson [31] introduced a new perspective by applying game theory to these problems. Using the game random-tug-of-war, they proved the most general existence and uniqueness results to date for solving equations involving the operator Δ_∞. Recently, the infinity Laplacian equation has been discussed extensively by many authors in previous literature, see [4, 7] and the references therein.

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The main concern of the present paper is the second-order estimate for the solution near the boundary to the singular boundary-value problem

$$-\Delta_\infty u = b(x)g(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where where the operator Δ_∞ is the ∞-Laplacian, a highly degenerate elliptic operator given by

$$\Delta_\infty u := \langle D^2 u Du, Du\rangle = \sum_{i,j=1}^{N} D_i u D_{ij} u D_j u,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, the functions b g satisfy

- (H1) $b \in C(\bar{\Omega})$ and is positive in Ω,
- (H2) there exist $k \in \Lambda$ and $B_0 \in \mathbb{R}$ such that
$$b(x) = k^4(d(x))(1 + B_0 d(x) + o(d(x))) \quad \text{near } \partial\Omega,$$
where $d(x) = \operatorname{dist}(x, \partial\Omega)$, Λ denotes the set of all positive non-decreasing functions in $C^1(0, \delta_0)$ which satisfy
$$\lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) := C_k \in (0,1], \quad K(t) = \int_0^t k(s)ds;$$
- (H3) $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$ and g is decreasing on $(0,\infty)$;
- (H4) there exist $\gamma > 1$ and a function $f \in C^1(0, a_1] \cap C[0, a_1]$ for $a_1 > 0$ small enough such that
$$\frac{-g'(s)s}{g(s)} := \gamma + f(s) \quad \text{with} \quad \lim_{s\to 0^+} f(s) = 0, \ s \in (0, a_1],$$
i.e.,
$$g(s) = c_0 s^{-\gamma} \exp\Big(\int_s^{a_1} \frac{f(\nu)}{\nu} d\nu\Big), \quad s \in (0, a_1], \ c_0 > 0;$$
- (H5) there exists $\eta \geq 0$ such that
$$\lim_{s\to 0^+} \frac{f'(s)s}{f(s)} = \eta.$$

Lu and Wang [22, 23] first investigated the inhomogeneous Dirichlet problem

$$\Delta_\infty u = f(x,u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = m, \tag{1.2}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and $m \in C(\partial\Omega)$. When the right hand side $f(x,u)$ is independent of u, they show that the Dirichlet problem (1.2) admits a unique solution $u \in C(\bar{\Omega})$, in the viscosity sense. Bhattacharya and Mohammed [6] is the first paper that addresses problem (1.2) in which the inhomogeneous term f depends on both the variables x and u. The paper considers the existence or nonexistence of solutions to problem (1.2) for the f with the sign and the monotonicity restrictions. Later, [7] removes the sign and the monotonicity restrictions, and presents fairly general sufficient conditions on f to ensure the existence of viscosity solutions to problem (1.2). In particular, [6] discusses the bounds and boundary behavior of solutions to problem (1.1) when b is a positive constant in Ω and $f(u) = u^{-\gamma}, \gamma > 0$. The author [26] further investigate the boundary asymptotic behavior of solutions to problem (1.1) for a wide range of functions $b(x)$ and $f(u)$.

Boundary asymptotic behavior of solutions to singular elliptic boundary value problem has been studied extensively in the context of the classical Laplace operator, i.e.

$$-\Delta u = b(x)g(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.3}$$

It is well known that problem (1.3) has been discussed and extended by many authors in many contexts, for instance, the existence, uniqueness, regularity and boundary behavior of solutions, see, [1, 27] and the references therein.

For $b \equiv 1$ in Ω and g satisfying (H3), Crandall, Rabinowitz and Tartar [15], Fulks and Maybee [16] derived that problem (1.1) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, in [15], the following result was established: If $\phi_1 \in C[0, \delta_0] \cap C^2(0, \delta_0]$ is the local solution to problem

$$-\phi_1''(t) = g(\phi_1(t)), \quad \phi_1(t) > 0, \quad 0 < t < \delta_0, \quad \phi_1(0) = 0, \tag{1.4}$$

then there exist positive constants c_1 and c_2 such that

$$c_1\phi_1(d(x)) \le u(x) \le c_2\phi_1(d(x)) \quad \text{near } \partial\Omega.$$

In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, u has the property

$$c_1(d(x))^{2/(1+\gamma)} \le u(x) \le c_2(d(x))^{2/(1+\gamma)} \quad \text{near } \partial\Omega. \tag{1.5}$$

Later, for $b \equiv 1$ on Ω, $g(u) = u^{-\gamma}$ with $\gamma > 0$, Berhanu, Cuccu and Porru [5] obtained the following results on a sufficiently small neighborhood of $\partial\Omega$;

(i) for $\gamma = 1$,

$$u(x) = \phi_1(d(x))\left(1 + A(x)(-\ln(d(x)))^{-\beta}\right) \quad \text{near } \partial\Omega,$$

where ϕ_1 is the solution of problem (1.3) with $\gamma = 1$, $\phi_1(t) \approx t\sqrt{-2\ln t}$ near $t = 0$, $\beta \in (0, 1/2)$ and A is bounded;

(ii) for $\gamma \in (1, 3)$,

$$u(x) = \Big(\frac{(1+\gamma)^2}{2(\gamma-1)}\Big)^{1/(1+\gamma)}(d(x))^{2/(1+\gamma)}\left(1 + A(x)(d(x))^{2(\gamma-1)/(1+\gamma)}\right) \quad \text{near } \partial\Omega;$$

(iii) for $\gamma = 3$,

$$u(x) = \sqrt{2d(x)}\big(1 - A(x)d(x)\ln(d(x))\big) \quad \text{near } \partial\Omega.$$

For $\gamma > 3$, McKenna and Reichel [25] proved that

$$\Big|\frac{u(x)}{(d(x))^{2/(1+\gamma)}} - \Big(\frac{(1+\gamma)^2}{2(\gamma-1)}\Big)^{1/(1+\gamma)}\Big| < c_4(d(x))^{(\gamma+3)/(1+\gamma)} \quad \text{near } \partial\Omega.$$

On the other hand, Cîrstea and Rădulescu [9, 10, 11] introduced a new unified approach via the Karamata regular variation theory, to study the boundary behavior and uniqueness of solutions for elliptic problems. Later, using this approach, Zhang [35] and the author [27] continued to prove the second-order asymptotic behavior of solutions to problem (1.3). However, the investigation of the second order expansion of viscosity solutions to problem (1.1) is just getting started.

With motivation from the above works, in this article we want to consider the two-term asymptotic expansion of the viscosity solution u of problem (1.1) near $\partial\Omega$ for suitable conditions on $b(x)$ and $f(u)$.

Let $\beta > 0$, we define

$$\Lambda_{1,\beta} = \big\{k \in \Lambda, \lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{d}{dt}\big(\frac{K(t)}{k(t)}\big) - C_k\Big) = D_{1k} \in \mathbb{R}\big\};$$

$$\Lambda_2 = \big\{k \in \Lambda,\ \lim_{t\to 0^+} t^{-1}\Big(\frac{d}{dt}\big(\frac{K(t)}{k(t)}\big) - C_k\Big) = D_{2k} \in \mathbb{R}\big\}.$$

The key to our estimates in this paper is the solution of the problem

$$\int_0^{\phi(t)} \frac{ds}{\big(g(s)\big)^{1/3}} = t,\ t > 0. \tag{1.6}$$

Our main results are summarized as follows.

Theorem 1.1. *Let* (H1)–(H5) *be satisfied. Suppose that* $k \in \Lambda_{1,\beta}$, $\eta > 0$ *in* (H5) *and* $C_k(\gamma+3) > 4$, *then for the viscosity solution* u *of problem* (1.1) *and all* x *in a neighborhood of* $\partial\Omega$, *it holds that*

$$u(x) = \xi_0\phi(K^{4/3}(d(x)))\left(1 + A_0(-\ln(d(x)))^{-\beta} + o((-\ln(d(x)))^{-\beta})\right), \tag{1.7}$$

where ϕ *is uniquely determined by* (1.6) *and*

$$\xi_0 = \Big((\frac{3}{4})^3\frac{\gamma+3}{C_k(\gamma+3)-4}\Big)^{1/(3+\gamma)}, \quad A_0 = -(\frac{3}{4})^2\frac{D_{1k}}{C_k(\gamma+3)-4}. \tag{1.8}$$

Theorem 1.2. *Let* (H1)–(H5) *be satisfied. Suppose that* $\eta = 0$ *in* (H5) *and* $C_k(\gamma+3) > 4$.

(i) *If* $k \in \Lambda_{1,\beta}$ *and*
(H6) *there exist* $\sigma \in \mathbb{R}$ *such that*

$$\lim_{s\to 0^+}(-\ln s)^{\beta} f(s) = \sigma,$$

where β *is the parameter used in the definition of* $\Lambda_{1,\beta}$.
then for the viscosity solution u *of problem* (1.1) *and all* x *in a neighborhood of* $\partial\Omega$, *it holds that*

$$u(x) = \xi_0\phi(K^{4/3}(d(x)))\left(1 + A_1(-\ln(d(x)))^{-\beta} + o((-\ln(d(x)))^{-\beta})\right), \tag{1.9}$$

where ϕ *is uniquely determined by* (1.6), ξ_0 *is in* (1.8) *and*

$$A_1 = -\frac{(\frac{4}{3})^3 D_{1k} - A_2}{C_k(\gamma+3)-4} \quad \text{with} \quad A_2 = -A_3\sigma\Big((\frac{4}{3})^4(\gamma+1)^{-2} + \xi_0^{-(\gamma+3)}\ln\xi_0\Big),$$
$$A_3 = 4^{-\beta}(C_k(\gamma+3))^{\beta}.$$

(ii) *Suppose that* $k \in \Lambda_2$, *then (i) still holds, where*

$$A_1 = (\frac{3}{4})^3\frac{A_2}{C_k(\gamma+3)-4}.$$

Remark 1.3 ((Existence and uniqueness [6, Cor. 6.3])**.** Let $g : (0,\infty) \to (0,\infty)$ be non-increasing and $b \in C(\Omega)$ be a positive function such that $\sup_{x\in\Omega} b(x) < \infty$. The singular boundary value problem (1.1) admits a unique solution.

The outline of this paper is as follows. In section 2 we give some preparation. The proofs of Theorems 1.1 and 1.2 will be given in section 3.

2. Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in the theory of stochastic process (see [32, **?**] and the references therein.). The theory of regular variation has been applied in Tauberian theorems, Abelian theorems, analytic theorems, and analytic number theorems etc.. The regular variation theory enables us to obtain significant information about the qualitative behavior of large solutions in a general framework. In this section, we give a brief account of the definition and properties of regularly varying functions involved in our paper (see [32, 33]).

Definition 2.1. A positive measurable function g defined on $(0, a)$, for some $a > 0$, is called *regularly varying at zero* with index ρ, written as $g \in RVZ_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t\to 0^+} \frac{g(\xi t)}{g(t)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, g is called *slowly varying at zero*.

From the above definition we easily deduce that if L is slowly varying at zero, then $t^\rho L(t) \in RVZ_\rho$. Some basic examples of slowly varying functions at zero are

(i) every measurable function on $(0, a)$ which has a positive limit at zero;
(ii) $(-\ln t)^p$ and $\big(\ln(-\ln t)\big)^p$, $p \in \mathbb{R}$;
(iii) $e^{(-\ln t)^p}$, $0 < p < 1$.

Definition 2.2. A positive measurable function f defined on $[a, \infty)$, for some $a > 0$, is called *regularly varying at infinity* with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.2}$$

In particular, when $\rho = 0$, f is called *slowly varying at infinity*.

Proposition 2.3 (Uniform convergence theorem)**.** *If $g \in RVZ_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2 < a$.*

Proposition 2.4 (Representation theorem)**.** *A function L is slowly varying at zero if and only if it can be written in the form*

$$L(t) = y(t) \exp\Big(\int_t^{a_1} \frac{f(\nu)}{\nu} d\nu\Big), \quad t \in (0, a_1), \tag{2.3}$$

for some $a_1 \in (0, a)$, where the functions f and y are measurable and for $t \to 0^+$, $f(t) \to 0$ and $y(t) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(t) = c_0 \exp\Big(\int_t^{a_1} \frac{f(\nu)}{\nu} d\nu\Big), \quad t \in (0, a_1), \tag{2.4}$$

is *normalized* slowly varying at zero and

$$g(t) = c_0 t^\rho \hat{L}(t), \quad t \in (0, a_1), \tag{2.5}$$

is *normalized* regularly varying at zero with index ρ (and written $g \in NRVZ_\rho$).

A function $g \in RVZ_\rho$ belongs to $NRVZ_\rho$ if and only if

$$g \in C^1(0, a_1) \quad \text{for some } a_1 > 0 \text{ and } \lim_{t\to 0^+} \frac{tg'(t)}{g(t)} = \rho. \tag{2.6}$$

Proposition 2.5. *If the functions L, L_1 are slowly varying at zero, then*

(i) L^ρ *(for every $\rho \in \mathbb{R}$), $c_1 L + c_2 L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(t) \to 0$ as $t \to 0^+$) are also slowly varying at zero.*

(ii) *For every $\rho > 0$ and $t \to 0^+$,*

$$t^\rho L(t) \to 0, \quad t^{-\rho} L(t) \to \infty.$$

(iii) *For $\rho \in \mathbb{R}$ and $t \to 0^+$, $\ln(L(t))/\ln t \to 0$ and $\ln(t^\rho L(t))/\ln t \to \rho$.*

Proposition 2.6. *If $g_1 \in RVZ_{\rho_1}$, $g_2 \in RVZ_{\rho_2}$ with $\lim_{t\to 0^+} g_2(t) = 0$, then $g_1 \circ g_2 \in RVZ_{\rho_1 \rho_2}$.*

Proposition 2.7 (Asymptotic behavior)**.** *If a function L is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,*

(i) $\int_0^t s^\rho L(s) ds \cong (\rho + 1)^{-1} t^{1+\rho} \ L(t)$, *for $\rho > -1$;*

(ii) $\int_t^a s^\rho L(s) ds \cong (-\rho - 1)^{-1} t^{1+\rho} \ L(t)$, *for $\rho < -1$.*

Next, we recall the precise definition of viscosity solutions for problem (1.1).

Definition 2.8. A function $u \in C(\Omega)$ is a viscosity subsolution of the PDE $\Delta_\infty u = -b(x)g(u)$ in Ω if for every $\varphi \in C^2(\Omega)$, with the property that $u - \varphi$ has a local maximum at some $x_0 \in \Omega$, then

$$\Delta_\infty \varphi(x_0) \geq -b(x_0) g(u(x_0)).$$

Definition 2.9. A function $u \in C(\Omega)$ is a viscosity supsolution of the PDE $\Delta_\infty u = -b(x)g(u)$ in Ω if for every $\varphi \in C^2(\Omega)$, with the property that $u - \varphi$ has a local minimum at some $x_0 \in \Omega$, then

$$\Delta_\infty \varphi(x_0) \leq -b(x_0) g(u(x_0)).$$

Definition 2.10. A function $u \in C(\Omega)$ is a viscosity solution of the PDE $\Delta_\infty u = -b(x)g(u)$ in Ω if it is both a subsolution and a supersolution.

Remark 2.11. It is easy to prove that if $u \in C^2(\Omega)$ is a classical subsolution (supersolution) of the PDE $\Delta_\infty u = -b(x)g(u)$, then u is a viscosity subsolution (supersolution) of the PDE $\Delta_\infty u = -b(x)g(u)$.

Our results in this section are summarized as follows.

Lemma 2.12. *Let $k \in \Lambda$. Then*

(i) $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$, $\lim_{t\to 0^+} \frac{tk(t)}{K(t)} = C_k^{-1}$, *i.e., $K \in NRVZ_{C_k^{-1}}$;*

(ii) $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \frac{1-C_k}{C_k}$, *i.e., $k \in NRVZ_{(1-C_k)/C_k}$, $\lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k$;*

(iii) *when $k \in \Lambda_{1,\beta}$, $\lim_{t\to 0^+} (-\ln t)^\beta \big(\frac{K(t)k'(t)}{k^2(t)} - (1 - C_k)\big) = -D_{1k}$;*

(iv) *when $k \in \Lambda_2$, $\lim_{t\to 0^+} t^{-1} \big(\frac{K(t)k'(t)}{k^2(t)} - (1 - C_k)\big) = -D_{2k}$.*

The proof of the above lemma is similar to the proof of [35, Lemma 2.1], so we omit it.

Lemma 2.13. *If g satisfies* (H3)-(H5)*, then:*

(i) $\int_0^a \frac{ds}{(g(s))^{1/3}} < \infty$, *for some $a > 0$;*

(ii)

$$\lim_{t\to 0^+}\Big(\big(g(t)\big)^{1/3}\Big)'\int_0^t \frac{ds}{\big(g(s)\big)^{1/3}} = -\frac{\gamma}{\gamma+3}\,,\quad \lim_{t\to 0^+}\frac{\big(g(t)\big)^{1/3}\int_0^t \frac{ds}{(g(s))^{1/3}}}{t} = \frac{3}{\gamma+3}.$$

Proof. (i) Assumption (H4) implies that $g\in NRVZ_{-\gamma}$ with $\gamma>1$, so $g(s)=c_0s^{-\gamma}\hat{L}(s),\ s\in(0,a_1)$, where $\hat{L}$ is normalized slowly varying at zero and $c_0>0$. (i) is obvious due to Propositions 2.7(i) and 2.5(ii).

(ii) Also we have

$$\frac{\big(g(t)\big)^{1/3}}{t}\int_0^t \frac{ds}{\big(g(s)\big)^{1/3}} \sim \frac{3}{\gamma+3}\frac{t^{-\frac{\gamma}{3}}}{(L(t))^{1/3}}\frac{t^{\frac{\gamma+3}{3}}\,(L(t))^{1/3}}{t} = \frac{3}{\gamma+3},$$
$$\Big(\big(g(t)\big)^{1/3}\Big)'\int_0^t \frac{ds}{\big(g(s)\big)^{1/3}} \sim \frac{1}{3}\frac{tg'(t)}{g(t)}\frac{3}{\gamma+3} = -\frac{\gamma}{\gamma+3}.$$

□

Lemma 2.14. *Let g satisfy* (H3)–(H5). *If $\eta=0$ in (H5) and (H6) holds. Then*

(i) $\lim_{t\to0^+}(-\ln t)^{\beta}\Big(\frac{tg'(t)}{g(t)}+\gamma\Big)=\sigma_1$, *where*

$$\sigma_1=\begin{cases}0, & \text{if } \eta>0,\\ -\sigma, & \text{if } \eta=0;\end{cases}$$

(ii)

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{\int_0^t\frac{ds}{(g(s))^{1/3}}}{\frac{t}{(g(t))^{1/3}}}-\frac{3}{\gamma+1}\Big)=\sigma_2;$$

where

$$\sigma_2=\begin{cases}0, & \text{if } \eta>0,\\ -\frac{3\sigma}{(\gamma+3)^2}, & \text{if } \eta=0;\end{cases}$$

(iii)

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\big((g(t))^{1/3}\big)'\int_0^t\frac{ds}{\big(g(s)\big)^{1/3}}+\frac{\gamma}{\gamma+3}\Big)=\sigma_3;$$

where

$$\sigma_3=\begin{cases}0, & \text{if } \eta>0,\\ -\frac{\sigma}{(\gamma+3)^2}, & \text{if } \eta=0;\end{cases}$$

(iv)

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{g(\xi_0t)}{\xi_0g(t)}-\xi_0^{-(\gamma+1)}\Big)=\sigma_4.$$

where

$$\sigma_4=\begin{cases}0, & \text{if } \eta>0,\\ -\sigma\xi_0^{-(\gamma+1)}\ln\xi_0, & \text{if } \eta=0.\end{cases}$$

Proof. When $f \in NRVZ_\eta$ with $\eta > 0$, by Proposition 2.5 (ii), it follows that $\lim_{t\to 0^+}(-\ln t)^\beta f(t) = 0$, and when $\eta = 0$, by (H6), $\lim_{t\to 0^+}(-\ln t)^\beta f(t) = \sigma$.

(i) By $\frac{tg'(t)}{g(t)} + \gamma = -f(t)$, we see that (i) holds.

(ii) By (H4) and a simple calculation, we obtain

$$s\Big(\frac{1}{\big(g(s)\big)^{1/3}}\Big)' = \frac{\gamma}{3\big(g(s)\big)^{1/3}} + \frac{f(s)}{3\big(g(s)\big)^{1/3}}, \quad s \in (0, a_1]. \tag{2.7}$$

Since $g \in NRVZ_{-\gamma}$ with $\gamma > 1$, by Proposition 2.5 (ii), we have $\lim_{t\to 0^+}\frac{t}{\big(g(t)\big)^{1/3}} = 0$. Integrating (2.7) from 0 to t, by parts, we obtain

$$\frac{t}{\big(g(t)\big)^{1/3}} = (\frac{\gamma}{3}+1)\int_0^t \frac{ds}{\big(g(s)\big)^{1/3}} + \frac{1}{3}\int_0^t \frac{f(s)}{\big(g(s)\big)^{1/3}}ds, \quad t \in (0, a_1],$$

i.e.,

$$\frac{\int_0^t \frac{ds}{(g(s))^{1/3}}}{\frac{t}{(g(t))^{1/3}}} - \frac{3}{\gamma+3} = -\frac{f(t)}{\gamma+3}\frac{\int_0^t \frac{f(s)}{(g(s))^{1/3}}ds}{t\frac{f(t)}{(g(t))^{1/3}}}, \quad t \in (0, a_1].$$

Since $g \in NRVZ_{-\gamma}$, $f \in NRVZ_\eta$, we obtain by Proposition 2.7 that

$$\lim_{t\to 0^+}\frac{\int_0^t \frac{f(s)}{(g(s))^{1/3}}ds}{t\frac{f(t)}{(g(t))^{1/3}}} = \frac{1}{\frac{\gamma}{3}+\eta+1}.$$

Thus,

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^\beta\Big(\frac{\int_0^t \frac{ds}{(g(s))^{1/3}}}{\frac{t}{(g(t))^{1/3}}} - \frac{3}{\gamma+3}\Big)\\
&= -\frac{1}{\gamma+3}\lim_{t\to 0^+}(-\ln t)^\beta f(t)\lim_{t\to 0^+}\frac{\int_0^t \frac{f(s)}{(g(s))^{1/3}}ds}{t\frac{f(t)}{(g(t))^{1/3}}} = \sigma_2.
\end{aligned}$$

(iii) By a simple calculation, we have

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^\beta\Big(\Big(\big(g(t)\big)^{1/3}\Big)'\int_0^t \frac{ds}{\big(g(s)\big)^{1/3}} + \frac{\gamma}{\gamma+3}\Big)\\
&= \lim_{t\to 0^+}(-\ln t)^\beta\Big(\frac{1}{3}\frac{tg'(t)}{g(t)}\frac{\int_0^t \frac{ds}{(g(s))^{1/3}}}{\frac{t}{(g(t))^{1/3}}} + \frac{\gamma}{\gamma+3}\Big)\\
&= \lim_{t\to 0^+}(-\ln t)^\beta\Big(\frac{1}{3}\Big(\frac{tg'(t)}{g(t)}+\gamma\Big)\Big(\frac{\int_0^t \frac{ds}{(g(s))^{1/3}}}{\frac{t}{(g(t))^{1/3}}} - \frac{3}{\gamma+3}\Big)\\
&\quad + \frac{1}{\gamma+3}\Big(\frac{tg'(t)}{g(t)}+\gamma\Big) - \frac{\gamma}{3}\Big(\frac{\int_0^t \frac{ds}{(g(s))^{1/3}}}{\frac{t}{(g(t))^{1/3}}} - \frac{3}{\gamma+3}\Big)\Big).
\end{aligned}$$

Hence, by (i)-(ii), we obtain

$$\lim_{t\to 0^+}(-\ln t)^\beta\Big(\Big(\big(g(t)\big)^{1/3}\Big)'\int_0^t \frac{ds}{\big(g(s)\big)^{1/3}}+\frac{\gamma}{\gamma+3}\Big)=\sigma_3.$$

(iv) When $\xi_0=1$, the result is obvious. Now suppose that $\xi_0\neq 1$. By (H4), we obtain

$$\frac{g(\xi_0 t)}{\xi_0 g(t)}-\xi_0^{-(\gamma+1)}=\xi_0^{-(\gamma+1)}\Big(\exp\Big(\int_{\xi_0 t}^t \frac{f(\nu)}{\nu}d\nu\Big)-1\Big).$$

Note that

$$\lim_{t\to 0^+}\frac{f(ts)}{s}=0 \quad\text{and}\quad \lim_{t\to 0^+}\frac{f(ts)}{f(t)s}=s^{\eta-1}$$

uniformly with respect to $s\in[1,\xi_0]$ or $s\in[\xi_0,1]$. So,

$$\lim_{t\to 0^+}\int_{\xi_0 t}^t \frac{f(\nu)}{\nu}d\nu=\lim_{t\to 0^+}\int_{\xi_0}^1\frac{f(ts)}{s}ds=0$$

and

$$\lim_{t\to 0^+}\int_{\xi_0}^1\frac{f(ts)}{f(t)s}ds=\int_{\xi_0}^1 s^{\eta-1}ds=\chi,$$

where

$$\chi=\begin{cases}-\ln\xi_0, & \text{if } \eta=0;\\ \frac{1}{\eta}(1-\xi_0^\eta), & \text{if } \eta>0.\end{cases}$$

Since $e^r-1\sim r$ as $r\to 0$, it follows that

$$\frac{g(\xi_0 t)}{\xi_0 g(t)}-\xi_0^{-(\gamma+1)}\sim\xi_0^{-(\gamma+1)}\int_{\xi_0 t}^t\frac{f(\nu)}{\nu}d\nu \quad\text{as } t\to 0.$$

Hence,

$$\begin{aligned}&\lim_{t\to 0^+}(-\ln t)^\beta\Big(\frac{g(\xi_0 t)}{\xi_0 g(t)}-\xi_0^{-(\gamma+1)}\Big)\\ &=\xi_0^{-(\gamma+1)}\lim_{t\to 0^+}(-\ln t)^\beta f(t)\lim_{t\to 0^+}\int_{\xi_0}^1\frac{f(ts)}{f(t)s}ds=\sigma_4.\end{aligned}$$

□

Lemma 2.15. *Let g satisfy* (H3)-(H4) *and ϕ be the solution to the problem*

$$\int_0^{\phi(t)}\frac{ds}{(g(s))^{1/3}}=t,\quad \forall t>0.$$

Then

(i) $\phi'(t)=\big(g(\phi(t))\big)^{1/3}$, $\phi(t)>0$, $t>0$, $\phi(0)=0$ *and*
$$\phi''(t)=\frac{1}{3}\big(g(\phi(t))\big)^{-\frac{1}{3}}g'(\phi(t)),\quad t>0;$$
(ii) $\phi\in NRVZ_{\frac{3}{3+\gamma}}$;
(iii) $\phi'\in NRVZ_{-\frac{\gamma}{3+\gamma}}$;
(iv) $\lim_{t\to 0^+}\frac{\ln(\phi(t))}{\ln t}=\frac{3}{3+\gamma}$ *and* $\lim_{t\to 0^+}\frac{\ln(g(\phi(t)))}{-\ln t}=\frac{3\gamma}{3+\gamma}$;
(v) $\lim_{t\to 0^+}\frac{\ln t}{\ln(\phi(K^{4/3}(t)))}=\frac{C_k(\gamma+3)}{4}$, *if $k\in\Lambda$;*
(vi) $\lim_{t\to 0^+}(-\ln t)^\beta\frac{t}{\phi(K^{4/3}(t))}=0$, *if $k\in\Lambda$ and $C_k(\gamma+3)>4$.*

Proof. By the definition of ϕ and a direct calculation, we can prove (i).

(ii) Let $u=\phi(t)$, by Lemma 2.13, we have

$$\lim_{t\to 0^+}\frac{t\phi''(t)}{\phi'(t)}=\frac{1}{3}\lim_{t\to 0^+}\frac{tg'(\phi(t))}{\big(g(\phi(t))\big)^{\frac{2}{3}}}=\lim_{u\to 0^+}\Big(\big(g(u)\big)^{1/3}\Big)'\int_0^u\frac{ds}{\big(g(s)\big)^{1/3}}=-\frac{\gamma}{\gamma+3},$$

and

$$\lim_{t\to 0^+}\frac{t\phi'(t)}{\phi(t)}=\lim_{t\to 0^+}\frac{t\big(g(\phi(t))\big)^{1/3}}{\phi(t)}=\lim_{u\to 0^+}\frac{\big(g(u)\big)^{1/3}}{u}\int_0^u\frac{ds}{\big(g(s)\big)^{1/3}}=\frac{3}{\gamma+3},$$

i.e., $\phi'=g\circ\phi\in NRVZ_{-\frac{\gamma}{\gamma+1}}$ and $\phi\in NRVZ_{\frac{1}{\gamma+1}}$ and (iii) follows.

(v) Since $K\in NRVZ_{C_k^{-1}}$ and $\phi\in NRVZ_{3/(\gamma+3)}$, we see by Proposition 2.5 (iii) that (v) holds.

(vi) By (iv) and Proposition 2.6, $\phi\circ K^{4/3}\in NRVZ_{4/(C_k(\gamma+3))}$ and $\frac{t}{\phi(K^{4/3}(t))}\in NRVZ_{\frac{C_k(\gamma+3)-4}{C_k(\gamma+3)}}$. Since $C_k(\gamma+3)>4$, (vi) follows by Proposition 2.5 (ii). □

Lemma 2.16. *Suppose that* (H1)–(H5) *are satisfied, and* $C_k(\gamma+3)>4$*. If* $k\in\Lambda_{1,\beta}$*,* $\eta>0$ *in (H5) and* ϕ *is the solution of the problem*

$$\int_0^{\phi(t)}\frac{ds}{(g(s))^{1/3}}=t,\quad \forall t>0,$$

then

(i)
$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{K^{4/3}(t)\phi''(K^{4/3}(t))}{\phi'(K^{4/3}(t))}+\frac{\gamma}{\gamma+3}\Big)=0;$$

(ii)
$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{g(\xi_0\phi(K^{4/3}(t)))}{\xi_0 g(\phi(K^{4/3}(t)))}-\xi_0^{-(\gamma+1)}\Big)=0.$$

Proof. (i) By the definition of ϕ, Lemma 2.14 (iii) and Lemma 2.15 (iv), we arrive at

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{K^{4/3}(t)\phi''(K^{4/3}(t))}{\phi'(K^{4/3}(t))}+\frac{\gamma}{\gamma+3}\Big)\\
&=\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\Big(\big(g(\phi(K^{4/3}(t)))\big)^{1/3}\Big)'\int_0^{\phi(K^{4/3}(t))}\frac{ds}{g(s)}+\frac{\gamma}{\gamma+3}\Big)\\
&=\lim_{t\to 0^+}(-\ln\phi(K^{4/3}(t)))^{\beta}\Big(\big(g^{1/3}(\phi(K^{4/3}(t)))\big)'\int_0^{\phi(K^{4/3}(t))}\frac{ds}{\big(g(s)\big)^{1/3}}+\frac{\gamma}{\gamma+3}\Big)\\
&\quad\times\lim_{t\to 0^+}\Big(\frac{\ln t}{\ln\phi(K^{4/3}(t))}\Big)^{\beta}=0.
\end{aligned}$$

(ii) By Lemma 2.14 (iv) and Lemma 2.15 (iv), we infer that

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{g(\xi_0\phi(K^{4/3}(t)))}{\xi_0 g(\phi(K^{4/3}(t)))}-\xi_0^{-(\gamma+1)}\Big)\\
&=\lim_{t\to 0^+}(-\ln(\phi(K^{4/3}(t))))^{\beta}\Big(\frac{g(\xi_0\phi(K^{4/3}(t)))}{\xi_0 g(\phi(K^{4/3}(t)))}-\xi_0^{-(\gamma+1)}\Big)
\end{aligned}$$

$$\times \lim_{t\to 0^+}\Big(\frac{\ln t}{\ln \phi(K^{4/3}(t))}\Big)^{\beta} = 0.$$

□

Lemma 2.17. *Suppose that* (H1)–(H5) *are satisfied, and* $C_k(\gamma+3) > 4$*. If* $\eta = 0$ *in (H5), (H6) holds and* ϕ *is the solution to the problem*

$$\int_0^{\phi(t)} \frac{ds}{(g(s))^{1/3}} = t, \quad \forall t > 0,$$

then

(i)

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{K^{4/3}(t)\phi''(K^{4/3}(t))}{\phi'(K^{4/3}(t))} + \frac{\gamma}{\gamma+3}\Big) = -\frac{A_3\sigma}{(\gamma+3)^2};$$

(ii)

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{g(\xi_0\phi(K^{4/3}(t)))}{\xi_0 g(\phi(K^{4/3}(t)))} - \xi_0^{-(\gamma+1)}\Big) = -A_3\sigma\xi_0^{-(\gamma+1)}\ln\xi_0,$$

where $A_3 = 4^{-\beta}(C_k(3+\gamma))^{\beta}$*.*

Proof. (i) By the definition of ϕ, Lemma 2.14 (iii) and Lemma 2.15 (iv), we find that

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{K^{4/3}(t)\phi''(K^{4/3}(t))}{\phi'(K^{4/3}(t))} + \frac{\gamma}{\gamma+3}\Big)\\
&= \lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\Big(\big(g(\phi(K^{4/3}(t)))\big)^{1/3}\Big)'\int_0^{\phi(K^{4/3}(t))}\frac{ds}{\big(g(s)\big)^{1/3}} + \frac{\gamma}{\gamma+3}\Big)\\
&= \lim_{t\to 0^+}(-\ln\phi(K^{4/3}(t)))^{\beta}\Big(\Big(\big(g(\phi(K^{4/3}(t)))\big)^{1/3}\Big)'\int_0^{\phi(K^{4/3}(t))}\frac{ds}{\big(g(s)\big)^{1/3}} + \frac{\gamma}{\gamma+3}\Big)\\
&\quad\times \lim_{t\to 0^+}\Big(\frac{\ln t}{\ln\phi(K^{4/3}(t))}\Big)^{\beta}\\
&= -\frac{A_3\sigma}{(\gamma+3)^2}.
\end{aligned}$$

(ii) By Lemma 2.14 (iv) and Lemma 2.15 (iv), we obtain that

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{g(\xi_0\phi(K^{4/3}(t)))}{\xi_0 g(\phi(K^{4/3}(t)))} - \xi_0^{-(\gamma+1)}\Big)\\
&= \lim_{t\to 0^+}(-\ln\phi(K^{4/3}(t)))^{\beta}\Big(\frac{g(\xi_0\phi(K^{4/3}(t)))}{\xi_0 g(\phi(K^{4/3}(t)))} - \xi_0^{-(\gamma+1)}\Big)\lim_{t\to 0^+}\Big(\frac{\ln t}{\ln\phi(K^{4/3}(t))}\Big)^{\beta}\\
&= -A_3\sigma\xi_0^{-(\gamma+1)}\ln\xi_0.
\end{aligned}$$

□

3. Proofs of main results

In this section, we prove Theorems 1.1 and 1.2. First we need the following result.

Lemma 3.1 (The comparison principle [6, Lemma 4.3])**.** *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,t)$ is non-decreasing in t. Assume further that f has one sign (either positive or negative) in $\Omega \times \mathbb{R}$. If $u, v \in C(\bar{\Omega})$ are such that*

$$\Delta_\infty u \geq f(x,u), \quad \Delta_\infty v \leq f(x,v), \quad u \leq v \text{ on } \partial\Omega,$$

then $u \leq v$ in Ω.

3.1. **Proof of Theorem 1.1.** Fix $\varepsilon > 0$. For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since Ω is C^2-smooth, choose $\delta_1 \in (0, \delta_0)$ such that $d \in C^2(\Omega_{\delta_1})$ and

$$|\nabla d(x)| = 1, \quad \Delta d(x) = -(N-1)H(\bar{x}) + o(1), \quad \forall x \in \Omega_{\delta_1}. \tag{3.1}$$

where, for $x \in \Omega_{\delta_1}$, $\bar{x}$ denotes the unique point of the boundary such that $d(x) = |x - \bar{x}|$ and $H(\bar{x})$ denotes the mean curvature of the boundary at that point.

If h is a C^2-function on $(0, \delta_1)$, a simple computation shows that

$$\Delta_\infty h(d(x)) = (h'(d(x)))^2 h''(d(x)).$$

Let

$$w_\pm = \xi_0 \phi(K^{4/3}(d(x))) \left(1 + (A_0 \pm \varepsilon)(-\ln(d(x)))^{-\beta}\right), \quad x \in \Omega_{\delta_1}.$$

By the Lagrange mean value theorem, we obtain that there exist $\lambda_\pm \in (0,1)$ and

$$\Phi_\pm(d(x)) = \xi_0 \phi(K^{4/3}(d(x))) \left(1 + \lambda_\pm (A_0 \pm \varepsilon)(-\ln(d(x)))^{-\beta}\right)$$

such that for $x \in \Omega_{\delta_1}$,

$$\begin{aligned} &g(w_\pm(x)) \\ &= g(\xi_0 \phi(K^{4/3}(d(x)))) + \xi_0 (A_0 \pm \varepsilon) \phi(K^{4/3}(d(x))) g'(\Phi_\pm(d(x)))(-\ln(d(x)))^{-\beta}. \end{aligned}$$

Since $g \in NRVZ_{-\gamma}$, by Proposition 2.3, we obtain

$$\lim_{d(x)\to 0} \frac{g(\xi_0 \phi(K^{4/3}(d(x))))}{g(\Phi_\pm(d(x)))} = \lim_{d(x)\to 0} \frac{g'(\xi_0 \phi(K^{4/3}(d(x))))}{g'(\Phi_\pm(d(x)))} = 1.$$

Define $r = d(x)$ and

$$\begin{aligned} I_1(r) = (-\ln r)^\beta \Big(&(\frac{4}{3})^4 \frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))} + (\frac{4}{3})^3 \frac{K(r)k'(r)}{k^2(r)} \\ &+ \frac{g(\xi_0 \phi(K^{4/3}(r)))}{\xi_0^3 g(\phi(K^{4/3}(r)))} + \frac{4}{9}(\frac{4}{3})^2 \Big), \\ I_{2\pm}(r) = 3(A_0 \pm \varepsilon) \Big(&(\frac{4}{3})^4 \frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))} + (\frac{4}{3})^3 \frac{K(r)k'(r)}{k^2(r)} \\ &+ \frac{1}{3}\xi_0^{-2} \frac{g'(\Phi_\pm(r))}{g'(\xi_0 \phi(K^{4/3}(r)))} \frac{\phi(K^{4/3}(r)) g'(\xi_0 \phi(K^{4/3}(r)))}{\big(\phi'(K^{4/3}(r))\big)^3} + \frac{4}{9}(\frac{4}{3})^2 \Big); \end{aligned}$$

$$
\begin{aligned}
I_{3\pm}(r) = {}& (\frac{4}{3})^2\beta(A_0\pm\varepsilon)^2(-\ln r)^{-\beta}\left((A_0\pm\varepsilon)(-\ln r)^{-\beta}+3\right)\\
&\times\Big((\frac{4}{3})^2\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))}+\frac{4}{3}\frac{K(r)k'(r)}{k^2(r)}+\frac{4}{9}\Big)\\
&+2(\frac{4}{3})^3\frac{K(r)}{rk(r)}r^2(-\ln r)^{-1}\left(1+(A_0\pm\varepsilon)(-\ln r)^{-\beta}\right);\\
I_{4\pm}(r) = {}& (\frac{4}{3})^2(A_0\pm\varepsilon)\beta\left(1+(A_0\pm\varepsilon)(-\ln r)^{-\beta}\right)^2\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\frac{K(r)}{rk(r)}\\
&\times\Big((A_0\pm\varepsilon)\frac{K(r)}{rk(r)}+\frac{2}{3}(-\ln r)^{-1}\Big(4\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))}\\
&+1+\frac{16K(r)k'(r)}{3(k(r))^2}\Big)\Big)\\
&+\xi_0^{-2}(A_0\pm\varepsilon)(B_0\pm\varepsilon)r\frac{g'(\Phi_\pm(r))}{g'(\xi_0\phi(K^2(r)))}\frac{\phi(K^2(r))g'(\xi_0\phi(K^2(r)))}{(\phi'(K^2(r)))^3};\\
I_{5\pm}(r) = {}& (A_0\pm\varepsilon)^2\beta^2(-\ln r)^{-\beta-2}\left(1+(A_0\pm\varepsilon)(-\ln r)^{-\beta}\right)\\
&\times\Big(\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\Big)^2\Big((\frac{K(r)}{k(r)})^2\Big((\frac{4}{3})^3\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))}\\
&+\frac{4}{9}+\frac{4K(r)k'(r)}{3(k(r))^2}\Big)-\frac{8}{3}(\frac{K(r)}{rk(r)})^3+\frac{8}{3}(\beta+1)(\frac{K(r)}{rk(r)})^3(-\ln r)^{-1}\Big)\\
&+\xi_0^{-3}r\frac{g(\xi_0\phi(K^{4/3}(r)))}{g(\phi(K^{4/3}(r)))};\\
I_{6\pm}(r) = {}& (A_0\pm\varepsilon)^3\beta^3(-\ln r)^{-2\beta-3}\Big(\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\Big)^2(\frac{K(r)}{rk(r)})^3\\
&\times\Big(\frac{8}{3}+\left((\beta+1)(-\ln r)^{-1}-1\right)\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\frac{K(r)}{rk(r)}\Big).
\end{aligned}
$$

By (2.1), (2.6), Lemmas 2.12, 2.15 and 2.16, combining with the choices of ξ_0, A_0 in Theorem 1.1, we obtain the following lemma.

Lemma 3.2. *Suppose that* (H1)–(H5) *are satisfied, and* $C_k(\gamma+3)>4$. *If* $k\in\Lambda_{1,\beta}$ *and* $\eta>0$ *in* (H5), *then*

- (i) $\lim_{r\to0}I_1(r)=-\frac{4}{3}D_{1k}$;
- (ii) $\lim_{r\to0}I_{2\pm}(r)=(\frac{4}{3})^3(A_0\pm\varepsilon)(4-C_k(\gamma+3))$;
- (iii) $\lim_{d(x)\to0}I_{3\pm}(r)=\lim_{d(x)\to0}I_{4\pm}(r)=\lim_{d(x)\to0}I_{5\pm}(r)=\lim_{d(x)\to0}I_{6\pm}(r)=0$;
- (iv) $\lim_{d(x)\to0}\left(I_1(r)+I_{2\pm}(r)+I_{3\pm}(r)+I_{4\pm}(r)+I_{5\pm}(r)+I_{6\pm}(r)\right)=\pm(\frac{4}{3})^3\varepsilon(4-C_k(\gamma+3))$.

Proof of Theorem 1.1. Let $v\in C(\bar\Omega)$ be the unique solution of the problem

$$-\Delta_\infty v=1,\quad v>0,\quad x\in\Omega,\quad v|_{\partial\Omega}=0. \tag{3.2}$$

By [6, Theorem 7.7], we see that

$$c_1d(x)\le v(x)\le c_2d(x),\quad \forall x\in\Omega\quad\text{near }\partial\Omega. \tag{3.3}$$

where c_1, c_2 are positive constants.

By (H1), (H2), Lemma 2.12 and $K \in C[0,\delta_0)$ with $K(0)=0$, we see that there exist $\delta_{1\varepsilon}, \delta_{2\varepsilon} \in \big(0, \min\{1,\delta_1\}\big)$ (which is corresponding to ε) sufficiently small such that

(i) $0 \le K^{4/3}(r) \le \delta_{1\varepsilon}$, $r \in (0, \delta_{2\varepsilon})$;
(ii) $k^4(d(x))(1+(B_0-\varepsilon)d(x)) \le b(x) \le k^4(d(x))(1+(B_0+\varepsilon)d(x))$, $x \in \Omega_{\delta_{1\varepsilon}}$;
(iii) $I_1(r)+I_{2+}(r)+I_{3+}(r)+I_{4+}(r)+I_{5+}(r)+I_{6+}(r) \le 0$, for all $(x,r) \in \Omega_{\delta_{1\varepsilon}} \times (0,\delta_{2\varepsilon})$;
(iv) $I_1(r)+I_{2-}(r)+I_{3-}(r)+I_{4-}(r)+I_{5-}(r)+I_{6-}(r) \ge 0$ for all $(x,r) \in \Omega_{\delta_{1\varepsilon}} \times (0,\delta_{2\varepsilon})$.

Now we define

$$\bar{u}_\varepsilon = \xi_0 \phi(K^{4/3}(d(x)))\left(1+(A_0+\varepsilon)(-\ln(d(x)))^{-\beta}\right), \quad x \in \Omega_{\delta_{1\varepsilon}}.$$

Then for $x \in \Omega_{\delta_{1\varepsilon}}$,

$$\begin{aligned} &g(\bar{u}_\varepsilon(x)) \\ &= g(\xi_0\phi(K^{4/3}(d(x)))) + \xi_0(A_0+\varepsilon)\phi(K^{4/3}(d(x)))g'(\Phi_+(d(x)))(-\ln(d(x)))^{-\beta}, \end{aligned}$$

where $\lambda_+ \in (0,1)$ and

$$\Phi_+(d(x)) = \xi_0\phi(K^{4/3}(d(x)))\left(1+\lambda_+(A_0+\varepsilon)(-\ln(d(x)))^{-\beta}\right), \quad x \in \Omega_{\delta_{1\varepsilon}}.$$

By Lemma 3.2 and a direct calculation ($h=\phi(\xi_0 K^{4/3}(t))$), we see that for $x \in \Omega_{\delta_{1\varepsilon}}$,

$$\begin{aligned} &\Delta_\infty \bar{u}_\varepsilon(x) + k^4(d(x))(1+(B_0+\varepsilon)d(x))g(\bar{u}_\varepsilon(x)) \\ &= \xi_0^3\big(\phi'(K^{4/3}(d(x)))\big)^3 k^4(d(x))(-\ln(d(x)))^{-\beta}\big(I_1(r)+I_{2+}(r)+I_{3+}(r) \\ &\quad + I_{4+}(r)+I_{5+}(r)+I_{6+}(r)\big) \le 0, \end{aligned}$$

where $r=d(x)$, i.e., $\bar{u}_\varepsilon$ is a classical supersolution of (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Hence, $\bar{u}_\varepsilon$ is a viscosity supersolution of (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we show that

$$\underline{u}_\varepsilon = \xi_0\phi(K^{4/3}(d(x)))\left(1+(A_0-\varepsilon)(-\ln(d(x)))^{-\beta}\right), \quad x \in \Omega_{\delta_{1\varepsilon}},$$

is a classical subsolution of (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Hence, $\underline{u}_\varepsilon$ is a viscosity subsolution of (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

Let $u \in C(\Omega)$ be the unique solution to problem (1.1). We assert that there exists M large enough such that

$$u(x) \le Mv(x) + \bar{u}_\varepsilon(x), \quad \underline{u}_\varepsilon(x) \le u(x) + Mv(x), \quad x \in \Omega_{\delta_{1\varepsilon}}, \tag{3.4}$$

where v is the solution of problem (3.2).

In fact, we can choose M large enough such that

$$u(x) \le \bar{u}_\varepsilon(x) + Mv(x) \quad \text{and} \quad \underline{u}_\varepsilon(x) \le u(x) + Mv(x)$$

on $\{x \in \Omega : d(x) = \delta_{1\varepsilon}\}$. By (H3) we see that $\bar{u}_\varepsilon(x)+Mv(x)$ and $u(x)+Mv(x)$ are also supersolutions of equation (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Since $u = \bar{u}_\varepsilon + Mv = u + Mv = \underline{u}_\varepsilon = 0$ on $\partial\Omega$, (3.4) follows by (H3) and Lemma 3.1. Hence, for $x \in \Omega_{\delta_{1\varepsilon}}$,

$$\begin{aligned} &A_0 - \varepsilon - \frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^{4/3}(d(x)))} \le (-\ln(d(x)))^\beta\Big(\frac{u(x)}{\xi_0\phi(K^{4/3}(d(x)))} - 1\Big), \\ &(-\ln(d(x)))^\beta\Big(\frac{u(x)}{\xi_0\phi(K^{4/3}(d(x)))} - 1\Big) \le A_0 + \varepsilon + \frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^{4/3}(d(x)))}. \end{aligned}$$

Consequently, by (3.3) and Lemma 2.15 (v),

$$A_0-\varepsilon\le\liminf_{d(x)\to0}(-\ln(d(x)))^{\beta}\Big(\frac{u(x)}{\xi_0\phi(K^{4/3}(d(x)))}-1\Big),$$
$$\limsup_{d(x)\to0}(-\ln(d(x)))^{\beta}\Big(\frac{u(x)}{\xi_0\phi(K^{4/3}(d(x)))}-1\Big)\le A_0+\varepsilon.$$

Thus, letting $\varepsilon\to0$, we obtain (1.7). □

Proof of Theorem 1.2. As before, fix $\varepsilon>0$. For any $\delta>0$, we define $\Omega_\delta=\{x\in\Omega:0<d(x)<\delta\}$. Since Ω is C^2-smooth, choose $\delta_1\in(0,\delta_0)$ such that $d\in C^2(\Omega_{\delta_1})$ and (3.1) holds. Let

$$w_\pm=\xi_0\phi(K^{4/3}(d(x)))\left(1+(A_1\pm\varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x\in\Omega_{\delta_1}.$$

By the Lagrange mean value theorem, we obtain that there exist $\lambda_\pm\in(0,1)$ and

$$\Phi_\pm(d(x))=\xi_0\phi(K^{4/3}(d(x)))\left(1+\lambda_\pm(A_1\pm\varepsilon)(-\ln(d(x)))^{-\beta}\right)$$

such that for $x\in\Omega_{\delta_1}$,

$$\begin{aligned}&g(w_\pm(x))\\&=g(\xi_0\phi(K^{4/3}(d(x))))+\xi_0(A_1\pm\varepsilon)\phi(K^{4/3}(d(x)))g'(\Phi_\pm(d(x)))(-\ln(d(x)))^{-\beta}.\end{aligned}$$

Since $g\in NRVZ_{-\gamma}$, by Proposition 2.3 we obtain

$$\lim_{d(x)\to0}\frac{g(\xi_0\phi(K^{4/3}(d(x))))}{g(\Phi_\pm(d(x)))}=\lim_{d(x)\to0}\frac{g'(\xi_0\phi(K^{4/3}(d(x))))}{g'(\Phi_\pm(d(x)))}=1.$$

Define $r=d(x)$ and

$$\begin{aligned}I_1(r)&=(-\ln r)^{\beta}\Big((\frac43)^4\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))}+(\frac43)^3\frac{K(r)k'(r)}{k^2(r)}\\&\quad+\frac{g(\xi_0\phi(K^{4/3}(r)))}{\xi_0^3g(\phi(K^{4/3}(r)))}+\frac49(\frac43)^2\Big);\\
I_{2\pm}(r)&=3(A_0\pm\varepsilon)\Big((\frac43)^4\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))}+(\frac43)^3\frac{K(r)k'(r)}{k^2(r)}\\&\quad+\frac13\xi_0^{-2}\frac{g'(\Phi_\pm(r))}{g'(\xi_0\phi(K^{4/3}(r)))}\frac{\phi(K^{4/3}(r))g'(\xi_0\phi(K^{4/3}(r)))}{\big(\phi'(K^{4/3}(r))\big)^3}+\frac49(\frac43)^2\Big);\\
I_{3\pm}(r)&=(\frac43)^2\beta(A_0\pm\varepsilon)^2(-\ln r)^{-\beta}\left((A_0\pm\varepsilon)(-\ln r)^{-\beta}+3\right)\\&\quad\times\Big((\frac43)^2\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))}+\frac43\frac{K(r)k'(r)}{k^2(r)}+\frac49\Big)\\&\quad+2(\frac43)^3\frac{K(r)}{rk(r)}r^2(-\ln r)^{-1}\big(1+(A_0\pm\varepsilon)(-\ln r)^{-\beta}\big);\end{aligned}$$

$$
\begin{aligned}
I_{4\pm}(r) =& (\frac{4}{3})^2(A_0 \pm \varepsilon)\beta \big(1+(A_0 \pm \varepsilon)(-\ln r)^{-\beta}\big)^2 \frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\frac{K(r)}{rk(r)} \\
&\times \Big((A_0 \pm \varepsilon)\frac{K(r)}{rk(r)} + \frac{2}{3}(-\ln r)^{-1}\Big(4\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))} \\
&+ 1 + \frac{16K(r)k'(r)}{3(k(r))^2}\Big)\Big) \\
&+ \xi_0^{-2}(A_0 \pm \varepsilon)(B_0 \pm \varepsilon) r \frac{g'(\Phi_{\pm}(r))}{g'(\xi_0\phi(K^2(r)))}\frac{\phi(K^2(r))g'(\xi_0\phi(K^2(r)))}{(\phi'(K^2(r)))^3}. \\
I_{5\pm}(r) =& (A_0 \pm \varepsilon)^2\beta^2(-\ln r)^{-\beta-2}\big(1+(A_0 \pm \varepsilon)(-\ln r)^{-\beta}\big) \\
&\times \Big(\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\Big)^2\Big(\Big(\frac{K(r)}{k(r)}\Big)^2\Big((\frac{4}{3})^3\frac{K^{4/3}(r)\phi''(K^{4/3}(r))}{\phi'(K^{4/3}(r))} + \frac{4}{9} \\
&+ \frac{4K(r)k'(r)}{3(k(r))^2}\Big) - \frac{8}{3}\big(\frac{K(r)}{rk(r)}\big)^3 + \frac{8}{3}(\beta+1)\big(\frac{K(r)}{rk(r)}\big)^3(-\ln r)^{-1}\Big) \\
&+ \xi_0^{-3} r \frac{g(\xi_0\phi(K^{4/3}(r)))}{g(\phi(K^{4/3}(r)))}; \\
I_{6\pm}(r) =& (A_0 \pm \varepsilon)^3\beta^3(-\ln r)^{-2\beta-3}\Big(\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\Big)^2\Big(\frac{K(r)}{rk(r)}\Big)^3 \\
&\times \Big(\frac{8}{3} + \big((\beta+1)(-\ln r)^{-1} - 1\big)\frac{\phi(K^{4/3}(r))}{K^{4/3}(r)\phi'(K^{4/3}(r))}\frac{K(r)}{rk(r)}\Big).
\end{aligned}
$$

By (2.1), (2.6), Lemmas 2.12, 2.15 and 2.17, combining with the choices of ξ_0, A_1, A_2, A_3 in Theorem 1.2, we obtain the following lemma.

Lemma 3.3. *Suppose that* (A1)–(A5) *are satisfied, and* $C_k(\gamma+3) > 4$. *If* $\eta = 0$ *in* (H5)*, and (H6) holds. Then*

(i) $\lim_{r\to 0} I_1(r) = -(\frac{4}{3})^3 D_{1k} + A_2$*, if* $k \in \Lambda_{1,\beta}$*;*
(ii) $\lim_{r\to 0} I_1(r) = A_2$*, if* $k \in \Lambda_2$*;*
(iii) $\lim_{r\to 0} I_{2\pm}(r) = (\frac{4}{3})^3(A_1 \pm \varepsilon)(4 - C_k(\gamma+3))$*;*
(iv) $\lim_{d(x)\to 0} I_{3\pm}(r) = \lim_{d(x)\to 0} I_{4\pm}(r) = \lim_{d(x)\to 0} I_{5\pm}(r) = \lim_{d(x)\to 0} I_{6\pm}(r) = 0$*;*
(v) $\lim_{d(x)\to 0}(I_1(r) + I_{2\pm}(r) + I_{3\pm}(r) + I_{4\pm}(r) + I_{5\pm}(r) + I_{6\pm}(r)) = \pm(\frac{4}{3})^3\varepsilon(4 - C_k(\gamma+3))$.

Proof of Theorem 1.2. As in the proof of Theorem 1.1, suppose that

$$\bar{u}_\varepsilon = \xi_0\phi(K^{4/3}(d(x)))\left(1+(A_1+\varepsilon)(-\ln(d(x)))^{-\beta}\right), \quad x \in \Omega_{\delta_{1\varepsilon}}.$$

Then, by Lemma 3.3 and a direct calculation, for $x \in \Omega_{\delta_{1\varepsilon}}$, we have

$$
\begin{aligned}
&\Delta\bar{u}_\varepsilon(x) + k^4(d(x))\big(1+(B_0+\varepsilon)d(x)\big)g(\bar{u}_\varepsilon(x)) \\
&= \xi_0^3\big(\phi'(K^{4/3}(d(x)))\big)^3 k^4(d(x))(-\ln(d(x)))^{-\beta}\big(I_1(r) + I_{2+}(r) + I_{3+}(r) \\
&\quad + I_{4+}(r) + I_{5+}(r) + I_{6+}(r)\big) \le 0,
\end{aligned}
$$

where $r = d(x)$, i.e., $\bar{u}_\varepsilon$ is a classical supersolution of equation (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Hence, $\bar{u}_\varepsilon$ is a viscosity supersolution of equation (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we show that

$$\underline{u}_\varepsilon = \xi_0\phi(K^{4/3}(d(x)))\left(1+(A_1-\varepsilon)(-\ln(d(x)))^{-\beta}\right), \quad x \in \Omega_{\delta_{1\varepsilon}},$$

is a classical subsolution of (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Hence, $\underline{u}_\varepsilon$ is a viscosity subsolution of (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

As in the proof of Theorem 1.1, for $x \in \Omega_{\delta_{1\varepsilon}}$, we obtain

$$A_1 - \varepsilon - \frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^2(d(x)))} \le (-\ln(d(x)))^\beta\Big(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\Big),$$
$$(-\ln(d(x)))^\beta\Big(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\Big) \le A_1 + \varepsilon + \frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^2(d(x)))}.$$

Consequently, by (3.3) and Lemma 2.15 (v),

$$A_1 - \varepsilon \le \liminf_{d(x)\to 0}(-\ln(d(x)))^\beta\Big(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\Big),$$
$$\limsup_{d(x)\to 0}(-\ln(d(x)))^\beta\Big(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\Big) \le A_1 + \varepsilon.$$

Thus letting $\varepsilon \to 0$, we obtain (1.9). The proof is complete. □

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