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SOME PROPERTIES OF MEROMORPHIC SOLUTIONS FOR q-DIFFERENCE EQUATIONS

HONG YAN XU, SAN YANG LIU, XIU MIN ZHENG

ABSTRACT. The main purpose of this article is to investigate some properties on the meromorphic solutions of some types of q-difference equations, which can be seen the q-difference analogues of Painevé equations. We obtain estimates of the exponent of convergence of poles of $\Delta_q f(z) := f(qz) - f(z)$, which extends some earlier results by Chen et al.

1. Introduction and statement of main results

Throughout this paper, the term "meromorphic" will mean meromorphic in the complex plane $\mathbb{C}$. Also, we shall assume that readers are familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, etc. (see Hayman [13], Yang [25] and Yi and Yang [26]). We use $\sigma(f), \lambda(f)$ and $\lambda(1/f)$ to denote the order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$ respectively, and we also use $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{1}{t} dt.$$

Throughout this article, where the set F of logarithmic density will be not necessarily the same at each occurrence.

A century ago, Painlevé and his colleagues [21] considered the class

$$w''(z) = F(z; w; w'),$$

where F is rational in w and w' and (locally) analytic in z. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions. These equations are now known as the Painlevé equations. The first two of these equations are P_I and P_{II}:

$$w'' = 6w^2 + z, \quad w'' = 2w^2 + zw + \alpha,$$

where α is a constant. However, after that, essentially nothing happened until about 1980, and just after that differential Painlevé equations became an important research subject.

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In the 1990s, the discrete Painlevé equations have become important research problems (see [6, 8]). For example, the following equations

$$y_{n+1}+y_{n-1}=\frac{an+b}{y_n}+c, \quad y_{n+1}+y_{n-1}=\frac{an+b}{y_n}+\frac{c}{y_n^2},$$

are some known as the special discretization of discrete P_I, and the equation

$$y_{n+1}+y_{n-1}=\frac{(an+b)y_n+c}{1-y_n^2},$$

is known as the special discretization of the discrete P_{II}, where a,b,c are constants, $n\in N$.

Recently, a number of papers (see [5, 14, 17]) focused on complex difference equations and difference analogues of Nevanlinn's theory. Around 2006s, Halburd and Korhonen [10, 11, 12] used Nevanlinna value distribution theory to single out the difference Painlevé I and II equations from the following form

$$w(z+1)+w(z-1)=R(z,w), \tag{1.1}$$

where $R(z,w)$ is rational in w and meromorphic in z. They obtained that if (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed by a linear change in w to some difference equations, which include difference Painlevé I equations

$$w(z+1)+w(z-1)=\frac{az+b}{w(z)}+c, \tag{1.2}$$

$$w(z+1)+w(z-1)=\frac{(az+b)}{w(z)}+\frac{c}{w(z)^2}, \tag{1.3}$$

and difference Painlevé II equation

$$w(z+1)+w(z-1)=\frac{(az+b)w(z)+c}{1-w(z)^2}. \tag{1.4}$$

Chen et al [3, 4, 22] studied some properties of finite order transcendental meromorphic solutions of (1.2)–(1.4), and obtained a lot of interesting results. In 2007, Barnett, Halburd, Korhonen and Morgan [1] firstly established an analogue of the Logarithmic Derivative Lemma on q-difference operators. Closely related to difference expressions are q-difference expressions, where the usual shift $f(z+c)$ of a meromorphic function will be replaced by the q-difference $f(qz)$, $q\in\mathbb{C}\backslash\{0,1\}$. By this way, there were lots of results about difference operators, difference equations, q-difference operators, q-difference equations, and so on (see [7, 9, 18, 19, 20, 24, 27, 28, 29]).

In 2015, Qi and Yang [23] investigated the equations

$$f(qz)+f(\frac{z}{q})=\frac{az+b}{f(z)}+c, \tag{1.5}$$

$$f(qz)+f(\frac{z}{q})=\frac{(az+b)f(z)+c}{1-f(z)^2}, \tag{1.6}$$

which can be seen q-difference analogues of (1.2) and (1.4), and obtained some theorems as follows.

Theorem 1.1 ([23, Theorem 1.1]). *Let $f(z)$ be a transcendental meromorphic solution with zero order of* (1.5)*, and a,b,c be three constants such that a,b cannot vanish simultaneously. Then*

(i) *$f(z)$ has infinitely many poles.*
(ii) *If $a \neq 0$ and any $d \in \mathbb{C}$, then $f(z) - d$ has infinitely many zeros.*
(iii) *If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.*

Theorem 1.2 ([23, Theorem 1.3])**.** *Let a, b, c be constants with $ac \neq 0$, and let $f(z)$ be a transcendental meromorphic solution with zero order of equation* (1.6)*. Then $f(z)$ has infinitely many poles and $f(z) - d$ has infinitely many zeros, where $d \in \mathbb{C}$.*

In this article, we further investigated some properties of transcendental meromorphic solutions of the equations (1.5), (1.6) and

$$f(qz) + f(\frac{z}{q}) = \frac{az + b}{f(z)} + \frac{c}{f(z)^2}, \tag{1.7}$$

and obtained the following theorems, which extends the previous results given by Qi and Yang [23].

Theorem 1.3. *Let a, b, c be constants with $|a| + |b| \neq 0$. Suppose that $f(z)$ is a zero order transcendental meromorphic solution of* (1.5)*. Then*

(i) *if $a \neq 0$, $p(z)$ is a polynomial of degree $k \geq 0$ and $|q| \neq 1$, then $f(z) - p(z)$ has infinitely many zeros and $\lambda(f - p) = \sigma(f)$;*
if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 2z^2 - cz - b = 0\}$;
(ii) $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta_q f}) = \sigma(\Delta_q f) = \sigma(f)$.

Theorem 1.4. *Let a, b, c be constants with $|a| + |b| + |c| \neq 0$. Suppose that $f(z)$ is a zero order transcendental meromorphic solution of* (1.6)*. Then*

(i) *if $a \neq 0$, $p(z)$ is a polynomial of degree $k \geq 0$ and $|q| \neq 1$, then $f(z) - p(z)$ has infinitely many zeros and $\lambda(f - p) = \sigma(f)$; if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 2z^3 + (b - 2)z + c = 0\}$; if $c \neq 0$, then $\lambda(f) = \sigma(f)$;*
(ii) $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta_q f}) = \sigma(\Delta_q f) = \sigma(f)$.

Theorem 1.5. *Let a, b, c be constants with $|a| + |b| + |c| \neq 0$. Suppose that $f(z)$ is a zero order transcendental meromorphic solution of* (1.7)*. Then*

(i) *if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 2z^3 - bz - c = 0\}$;*
(ii) $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta_q f}) = \sigma(\Delta_q f) = \sigma(f)$.

2. Some Lemmas

The following result can be called an analogue of q-difference Clunie lemma, recently proved by Barnett et al. [1, Theorem 2.1]. Here a q-difference polynomial of f for $q \in \mathbb{C} \backslash \{0, 1\}$ is a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz), \dots, f(q^n z)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r, f))$ on a set of logarithmic density 1.

Lemma 2.1 ([16, Theorem 2.5])**.** *Let f be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z, f) P_q(z, f) = Q_q(z, f),$$

where $U_q(z,f), P_q(z,f)$ and $Q_q(z,f)$ are q-difference polynomials such that the total degree $\deg U_q(z,f) = n$ in $f(z)$ and its q-shifts, whereas $\deg Q_q(z,f) \le n$. Moreover, we assume that $U_q(z,f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then

$$m(r, P_q(z,f)) = o(T(r,f)),$$

on a set of logarithmic density 1.

Lemma 2.2 ([1, Theorem 2.5])**.** *Let f be a nonconstant zero-order meromorphic solution of $P_q(z,f) = 0$, where $P_q(z,f)$ is a q-difference polynomial in $f(z)$. If $P_q(z,a) \not\equiv 0$ for slowly moving target $a(z)$, then*

$$m(r, \frac{1}{f-a}) = o(T(r,f)),$$

on a set of logarithmic density 1.

Lemma 2.3 ([27, Theorem 1.1 and 1.3])**.** *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1+o(1))T(r, f(z)), \quad N(r, f(qz)) = (1+o(1))N(r, f(z)),$$

on a set of lower logarithmic density 1.

Lemma 2.4 (Valiron-Mohon'ko [15])**.** *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies that

$$T(r, R(z, f(z))) = dT(r,f) + O(\Psi(r)),$$

where $d = \max\{m,n\}$ and $\Psi(r) = \max_{i,j}\{T(r,a_i), T(r,b_j)\}$.

3. Proof of Theorem 1.3

Suppose that $f(z)$ is a zero order transcendental meromorphic solution of (1.5).

(i) $a \neq 0$. Let $p(z)$ is a polynomial of degree k and $p(z) = a_k z^k + \dots$. Let $g(z) = f(z) - p(z)$. Substituting $f(z) = g(z) + p(z)$ into equation (1.5), we have

$$g(qz) + p(qz) + g(\frac{z}{q}) + p(\frac{z}{q}) = \frac{az+b}{g(z)+p(z)} + c.$$

It follows that

$$\begin{aligned} P_q(z,g) :=& [g(qz) + p(qz) + g(\frac{z}{q}) + p(\frac{z}{q})][g(z)+p(z)] \\ &- (az+b) - c[g(z)+p(z)] = 0. \end{aligned} \tag{3.1}$$

Then, we have

$$P_q(z,0) = [p(qz) + p(\frac{z}{q})]p(z) - (az+b) - cp(z). \tag{3.2}$$

If $p(z) \equiv 0$, then $P_q(z,0) = -(az+b) \not\equiv 0$.

If $k = 0$ and $a_0 \equiv \alpha \in \mathbb{C} \setminus \{0\}$, then $P_q(z,0) = 2\alpha^2 - (az+b) - c\alpha \not\equiv 0$.

If $k \geq 1$ and $a_k \neq 0$ is a constant. Then, we have from (3.2) that

$$P_q(z,0) = [p(qz) + p(\frac{z}{q})]p(z) - (az+b) - cp(z) = (q^k + \frac{1}{q^k})a_k^2 z^{2k} + \dots. \tag{3.3}$$

Since $|q| \neq 1$, we have $q^k + \frac{1}{q^k} \neq 0$, then $P_q(z,0) \not\equiv 0$. Thus, we have by Lemma 2.2 that

$$m(r, \frac{1}{g}) = S(r,g).$$

Then, we obtain

$$N\Big(r, \frac{1}{f(z)-p(z)}\Big) = N\big(r, \frac{1}{g(z)}\big) = T(r,g) + S(r,g) = T(r,f) + S(r,f). \tag{3.4}$$

Hence, it follows that $\lambda(f-p) = \sigma(f)$.

If $a = 0$ and $p(z) = \beta \notin E$, then we have

$$P_q(z,0) = 2\beta^2 - c\beta - b \not\equiv 0.$$

Set $g(z) = f(z) - \beta$, by using the same argument as above, we can obtain $\lambda(f-\beta) = \sigma(f)$. Therefore, we can obtain that the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 2z^2 - cz - b = 0\}$.

(ii) From (1.5), we have

$$f(z)[f(qz) + f(\frac{z}{q})] = az + b + cf(z). \tag{3.5}$$

It follows from Lemma 2.1 that

$$m\big(r, f(qz) + f(\frac{z}{q})\big) = S(r,f). \tag{3.6}$$

By applying Lemma 2.4 for (1.5), we have

$$T\big(r, f(qz) + f(\frac{z}{q})\big) = T(r,f) + S(r,f). \tag{3.7}$$

And by Lemma 2.3 we obtain

$$N\big(r, f(qz) + f(\frac{z}{q})\big) \leq N(r, f(qz)) + N\big(r, f(\frac{z}{q})\big) = 2(1+o(1))N(r,f) \tag{3.8}$$

on a set of lower logarithmic density 1. Thus, combining (3.6) and (3.7), we have

$$T(r,f) \leq 2(1+o(1))N(r,f) + S(r,f).$$

Hence, we have

$$\sigma(f(z)) \leq \lambda\big(\frac{1}{f(z)}\big). \tag{3.9}$$

Next, we prove that $\lambda(\frac{1}{\Delta_q f(z)}) \geq \lambda(\frac{1}{f(z)})$. Set $z = qw$, then we can rewrite (1.5) as the form

$$f(q^2 w) + f(w) = \frac{aqw + b}{f(qw)} + c. \tag{3.10}$$

Then it follows that

$$f(qw)[f(q^2 w) + f(w)] = aqw + b + cf(qw). \tag{3.11}$$

Since $\Delta_q f(w) = f(qw) - f(w)$, we have $f(qw) = \Delta_q f(w) + f(w)$ and $f(q^2 w) = \Delta_q f(qw) + \Delta_q f(w) + f(w)$. Substituting them into (3.11), we obtain

$$[\Delta_q f(w) + f(w)][\Delta_q f(qw) + \Delta_q f(w) + 2f(w)] = (aqw+b) + c[\Delta_q f(w) + f(w)],$$

i.e.,

$$\begin{aligned} -2f(w)^2 =& [\Delta_q f(qw) + 3\Delta_q f(w) - c]f(w) - (aqw + b) \\ &+ [\Delta_q f(qw) + \Delta_q f(w) - c]\Delta_q f(w). \end{aligned} \tag{3.12}$$

Since $f(z)$ is a zero order transcendental meromorphic function and $z = qw$, by Lemma 2.3, we obtain that $f(w)$ is of zero order. Thus, by Lemma 2.3 again, we have that $\Delta_q f(w), \Delta_q f(qw)$ are of zero order. Set $\Delta_q^2 f(w) := \Delta_q(\Delta_q f(w))$, so we have $\Delta_q f(qw) = \Delta_q^2 f(w) + \Delta_q f(w)$. Since $\Delta_q f(w)$ is of zero order, and by Lemma 2.3 we have

$$N(r, \Delta_q^2 f(w)) \le 2N(r, \Delta_q f(w)) + S(r, f). \tag{3.13}$$

It follows that

$$N(r, \Delta_q f(qw)) \le 3N(r, \Delta_q f(w)) + S(r, f). \tag{3.14}$$

Thus, from (19) and (3.14) we have

$$\begin{aligned} 2N(r, f(w)) =& N\Big(r, [\Delta_q f(qw) + 3\Delta_q f(w) - c]f(w) - (aqw + b) \\ &+ [\Delta_q f(qw) + \Delta_q f(w) - c]\Delta_q f(w)\Big) \\ \le& N(r, f(w)) + 9N(r, \Delta_q f(w)) + O(\log r) + S(r, f). \end{aligned}$$

That is,

$$N(r, f(w)) \le 9N(r, \Delta_q f(w)) + S(r, f). \tag{3.15}$$

Then, it follows that

$$\lambda\Big(\frac{1}{\Delta_q f(w)}\Big) \ge \lambda\Big(\frac{1}{f(w)}\Big). \tag{3.16}$$

Since $f(w)$ is of zero order, by Lemma 2.3 and $z = qw$ we have

$$\begin{gathered} \lambda\Big(\frac{1}{\Delta_q f(w)}\Big) = \lambda\Big(\frac{1}{\Delta_q f(\frac{z}{q})}\Big) = \lambda\Big(\frac{1}{\Delta_q f(z)}\Big), \\ \lambda\Big(\frac{1}{f(w)}\Big) = \lambda\Big(\frac{1}{f(\frac{z}{q})}\Big) = \lambda\Big(\frac{1}{f(z)}\Big). \end{gathered}$$

Hence,

$$\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \ge \lambda\Big(\frac{1}{f(z)}\Big). \tag{3.17}$$

From this inequality and (3.9) we have

$$\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \ge \lambda\Big(\frac{1}{f(z)}\Big) \ge \sigma(f(z)). \tag{3.18}$$

So, we have $T(r, \Delta_q f(z)) \le 2T(r, f(z)) + S(r, f)$ by Lemma 2.3; that is, $\sigma(f(z)) \ge \sigma(\Delta_q f(z))$. Thus, combining this and (3.18), we have

$$\lambda\Big(\frac{1}{f(z)}\Big) = \lambda\Big(\frac{1}{\Delta_q f(z)}\Big) = \sigma(\Delta_q f(z)) = \sigma(f(z)).$$

The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.4

Suppose that $f(z)$ is a zero order transcendental meromorphic solution of (1.6). We will consider the following two cases.

(i) $a \neq 0$. If $p(z)$ is a polynomial of degree k and $p(z) = a_k z^k + \dots$. Let $g_1(z) = f(z) - p(z)$. Substituting $f(z) = g_1(z) + p(z)$ into equation (1.6), we have

$$g_1(qz) + p(qz) + g_1(\frac{z}{q}) + p(\frac{z}{q}) = \frac{(az+b)[g_1(z)+p(z)]+c}{1-[g_1(z)+p(z)]^2}.$$

It follows that

$$\begin{aligned} P_q(z,g_1) :=&[g_1(qz) + p(qz) + g_1(\frac{z}{q}) + p(\frac{z}{q})][g_1(z)+p(z)]^2 \\ &+ (az+b)[g_1(z)+p(z)] + c \\ &- [g_1(qz) + p(qz) + g_1(\frac{z}{q}) + p(\frac{z}{q})] = 0. \end{aligned} \tag{4.1}$$

From this equality, we have

$$P_q(z,0) = [p(qz) + p(\frac{z}{q})]p(z)^2 + (az+b)p(z) + c - p(qz) - p(\frac{z}{q}). \tag{4.2}$$

If $k = 0$ and $a_0 \equiv \alpha \in \mathbb{C} \setminus \{0\}$, then $P_q(z,0) = 2\alpha^3 + \alpha(az+b) + c - 2\alpha \not\equiv 0$.

If $k \geq 1$ and $a_k \neq 0$ is a constant. Then, we have from (4.1) that

$$\begin{aligned} P_q(z,0) &= [p(qz) + p(\frac{z}{q})]p(z)^2 + (az+b)p(z) + c - p(z) - p(\frac{z}{q}) \\ &= \big(q^k + \frac{1}{q^k}\big)a_k^3 z^{3k} + \dots. \end{aligned}$$

Since $|q| \neq 1$, we have $q^k + \frac{1}{q^k} \neq 0$, then $P_q(z,0) \not\equiv 0$. Thus, we have by Lemma 2.2 that

$$m(r, \frac{1}{g_1}) = S(r, g_1).$$

Then, we obtain

$$N\Big(r, \frac{1}{f(z)-p(z)}\Big) = N\Big(r, \frac{1}{g_1(z)}\Big) = T(r,g_1) + S(r,g_1) = T(r,f) + S(r,f).$$

It follows hat $\lambda(f-p) = \sigma(f)$.

If $a = 0$ and $p(z) = \beta \notin E$, then we have

$$P_q(z,0) = 2\beta^3 + (b-2)\beta + c \not\equiv 0.$$

Set $g_1(z) = f(z) - \beta$, by using the same argument as above, we can obtain $\lambda(f-\beta) = \sigma(f)$. Therefore, we can obtain that the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 2z^3 + (b-2)z + c = 0\}$.

If $c \neq 0$, then we have from (1.6) that

$$P_q(z,f) := f(z)^2[f(qz) + f(\frac{z}{q})] + (az+b)f(z) + c - f(qz) - f(\frac{z}{q}).$$

Hence, we obtain

$$P_q(z,0) \equiv c \not\equiv 0.$$

Using a similar method as above, we obtain $\lambda(f) = \sigma(f)$.

(ii) From (1.6), we have

$$f(z)^2[f(qz) + f(\frac{z}{q})] = f(qz) + f(\frac{z}{q}) - (az+b)f(z) - c. \tag{4.3}$$

It follows from (3.5) that

$$m\Big(r, f(qz)+f(\frac{z}{q})\Big)=S(r,f). \tag{4.4}$$

By applying Lemma 2.4 for (1.6), we have

$$T\Big(r, f(qz)+f(\frac{z}{q})\Big)=2T(r,f)+S(r,f). \tag{4.5}$$

By Lemma 2.3 we obtain

$$N\Big(r, f(qz)+f(\frac{z}{q})\Big)\leq N(r,f(qz))+N\Big(r,f(\frac{z}{q})\Big)=2(1+o(1))N(r,f) \tag{4.6}$$

on a set of lower logarithmic density 1. Thus, combining (31) and (32), we have

$$T(r,f)\leq 2(1+o(1))N(r,f)+S(r,f).$$

Hence,

$$\sigma(f(z))\leq \lambda\Big(\frac{1}{f(z)}\Big). \tag{4.7}$$

Next, we prove that $\lambda\Big(\frac{1}{\Delta_q f(z)}\Big)\geq\lambda\Big(\frac{1}{f(z)}\Big)$. Set $z=qw$, then we can rewrite (1.6) in the form

$$f(q^2w)+f(w)=\frac{(aqw+b)f(qw)+c}{1-f(qw)^2}. \tag{4.8}$$

Then it follows that

$$f(qw)^2[f(q^2w)+f(w)]=f(q^2w)+f(w)-(aqw+b)f(qw)-c. \tag{4.9}$$

Since $\Delta_q f(w)=f(qw)-f(w)$, we have $f(qw)=\Delta_q f(w)+f(w)$ and $f(q^2w)=\Delta_q f(qw)+\Delta_q f(w)+f(w)$. Substituting these two equalities in (4.9), we obtain

$$\begin{aligned}&[\Delta_q f(w)+f(w)]^2[\Delta_q f(qw)+\Delta_q f(w)+2f(w)]\\&=\Delta_q f(qw)+\Delta_q f(w)+2f(w)-(aqw+b)[\Delta_q f(w)+f(w)]-c,\end{aligned}$$

Thus, we obtain

$$-2f(w)^3=A(w)f(w)+B(w)\Delta_q f(w)+C(w), \tag{4.10}$$

where

$$\begin{gathered}A(w)=[\Delta_q f(qw)+5\Delta_q f(w)]f(w)+4(\Delta_q f(w))^2\\+2\Delta_q f(w)\Delta_q f(qw)+aqw+b-2,\\B(w)=\Delta_q f(qw)\Delta_q f(w)+(\Delta_q f(w))^2+(aqw+b)-1,\\C(w)=c-\Delta_q f(qw).\end{gathered}$$

Thus, by Lemma 2.3 and from (3.14) we have

$$\begin{aligned}3N(r,f(w))=&N(r,A(w)f(w)+B(w)\Delta_q f(w)+C(w))\\\leq&2N(r,f(w))+19N(r,\Delta_q f(w))+O(\log r)+S(r,f).\end{aligned}$$

That is,

$$N(r,f(w))\leq 19N(r,\Delta_q f(w))+S(r,f). \tag{4.11}$$

Then, it follows from (37) that

$$\lambda\Big(\frac{1}{\Delta_q f(w)}\Big)\geq\lambda\Big(\frac{1}{f(w)}\Big). \tag{4.12}$$

Since $f(w)$ is of zero order, by Lemma 2.3 and $z = qw$ we have

$$\lambda\Big(\frac{1}{\Delta_q f(w)}\Big) = \lambda\Big(\frac{1}{\Delta_q f(\frac{z}{q})}\Big) = \lambda\Big(\frac{1}{\Delta_q f(z)}\Big),$$
$$\lambda\Big(\frac{1}{f(w)}\Big) = \lambda\Big(\frac{1}{f(\frac{z}{q})}\Big) = \lambda\Big(\frac{1}{f(z)}\Big).$$

Hence, we obtain

$$\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \geq \lambda\Big(\frac{1}{f(z)}\Big). \tag{4.13}$$

Thus, from this inequality and (4.7) we have

$$\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \geq \lambda\Big(\frac{1}{f(z)}\Big) \geq \sigma(f(z)). \tag{4.14}$$

Then we have $T(r, \Delta_q f(z)) \leq 2T(r, f(z)) + S(r, f)$ by Lemma 2.3; that is, $\sigma(f(z)) \geq \sigma(\Delta_q f(z))$. Thus, combining this and (4.14), we have

$$\lambda\Big(\frac{1}{f(z)}\Big) = \lambda\Big(\frac{1}{\Delta_q f(z)}\Big) = \sigma(\Delta_q f(z)) = \sigma(f(z)).$$

The proof of Theorem 1.4 is complete.

5. Proof of Theorem 1.5

For the convenience of the reader, we use the notation form the proof of Theorem 1.3(i). Suppose that $f(z)$ is a zero order transcendental meromorphic solution of (1.7). We will consider the following two cases.

(i) Let $a = 0$ and $p(z) = \beta \notin E$. Using the same methods as in the proof of Theorem 1.3(i), we have

$$P_q(z, 0) = 2\beta^3 - b\beta - c \not\equiv 0.$$

Thus, $\lambda(f - \beta) = \sigma(f)$. Hence, the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 2z^3 - bz - c = 0\}$.

(ii) From (1.7), we have

$$f(z)^2[f(qz) + f(\frac{z}{q})] = (az + b)f(z) + c. \tag{5.1}$$

It follows from Lemma 2.1 that

$$m\Big(r, f(qz) + f(\frac{z}{q})\Big) = S(r, f). \tag{5.2}$$

By Lemma 2.3 we obtain

$$N\Big(r, f(qz) + f(\frac{z}{q})\Big) \leq N(r, f(qz)) + N\Big(r, f(\frac{z}{q})\Big) = 2(1 + o(1))N(r, f) \tag{5.3}$$

on a set of lower logarithmic density 1.

If $c \neq 0$, by applying Lemma 2.4 for (1.7), we have

$$T\Big(r, f(qz) + f(\frac{z}{q})\Big) = 2T(r, f) + S(r, f). \tag{5.4}$$

Thus, it follows from (5.2)–(5.4) that

$$T(r, f) \leq (1 + o(1))N(r, f) + S(r, f). \tag{5.5}$$

If $c = 0$, by applying Lemma 2.4 for (1.7) and since $|a| + |b| + |c| = |a| + |b| \neq 0$, we have

$$T\Big(r, f(qz) + f(\frac{z}{q})\Big) = T(r, f) + S(r, f). \tag{5.6}$$

Thus, it follows from (5.2), (5.3) and (5.6) that

$$T(r, f) \leq 2(1 + o(1))N(r, f) + S(r, f). \tag{5.7}$$

From (5.5) and (5.7) we have

$$\sigma(f(z)) \leq \lambda\Big(\frac{1}{f(z)}\Big). \tag{5.8}$$

Next, we prove that $\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \geq \lambda\Big(\frac{1}{f(z)}\Big)$. Set $z = qw$, by using the same argument as in Theorem 1.4(ii), we have

$$-2f(w)^3 = A(w)f(w) + B(w)\Delta_q f(w) + C(w), \tag{5.9}$$

where

$$\begin{gathered}
A(w) = [\Delta_q f(qw) + 5\Delta_q f(w)]f(w) + 4(\Delta_q f(w))^2 + 2\Delta_q f(w)\Delta_q f(qw) - aqw - b, \\
B(w) = [\Delta_q f(qw) + \Delta_q f(w)]\Delta_q f(w) - aqw - b, \\
C(w) = -c.
\end{gathered}$$

Thus, by Lemma 2.3 and from (3.14) we have

$$\begin{aligned}
3N(r, f(w)) &= N(r, A(w)f(w) + B(w)\Delta_q f(w) + C(w)) \\
&\leq 2N(r, f(w)) + 15N(r, \Delta_q f(w)) + O(\log r) + S(r, f).
\end{aligned}$$

That is,

$$N(r, f(w)) \leq 15N(r, \Delta_q f(w)) + S(r, f). \tag{5.10}$$

Then, it follows that

$$\lambda\Big(\frac{1}{\Delta_q f(w)}\Big) \geq \lambda\Big(\frac{1}{f(w)}\Big). \tag{5.11}$$

Since $f(w)$ is of zero order, by Lemma 2.3 and $z = qw$ we have

$$\begin{gathered}
\lambda\Big(\frac{1}{\Delta_q f(w)}\Big) = \lambda\Big(\frac{1}{\Delta_q f(\frac{z}{q})}\Big) = \lambda\Big(\frac{1}{\Delta_q f(z)}\Big), \\
\lambda\Big(\frac{1}{f(w)}\Big) = \lambda\Big(\frac{1}{f(\frac{z}{q})}\Big) = \lambda\Big(\frac{1}{f(z)}\Big).
\end{gathered}$$

Hence, we obtain

$$\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \geq \lambda\Big(\frac{1}{f(z)}\Big). \tag{5.12}$$

Thus, by (5.8) we have

$$\lambda\Big(\frac{1}{\Delta_q f(z)}\Big) \geq \lambda\Big(\frac{1}{f(z)}\Big) \geq \sigma(f(z)). \tag{5.13}$$

So, we have $T(r, \Delta_q f(z)) \leq 2T(r, f(z)) + S(r, f)$ by Lemma 2.3; that is, $\sigma(f(z)) \geq \sigma(\Delta_q f(z))$. Thus, combining this and (5.13), we have

$$\lambda\Big(\frac{1}{f(z)}\Big) = \lambda\Big(\frac{1}{\Delta_q f(z)}\Big) = \sigma(\Delta_q f(z)) = \sigma(f(z)).$$

The proof of Theorem 1.5 is complete.

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Hong Yan Xu (corresponding author)
School of mathematics and statistics, Xidian University, Xi'an, Shaanxi 710126, China.
Department of Informatics and Engineering, Jingdezhen Ceramic Institute (Xiang Hu Xiao Qu), Jingdezhen, Jiangxi 333403, China
E-mail address: xhyhhh@126.com

San Yang Liu
School of mathematics and statistics, Xidian University, Xi'an, Shaanxi 710126, China
E-mail address: liusanyang@126.com

Xiu Min Zheng
Department of Mathematics, Jiangxi Normal University, Nanchan, Jiangxi 330022, China
E-mail address: zhengxiumin2008@sina.com