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FIRST-ORDER SELFADJOINT SINGULAR DIFFERENTIAL OPERATORS IN A HILBERT SPACE OF VECTOR FUNCTIONS

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Abstract. In this article, we give a representation of all selfadjoint extensions of the minimal operator generated by first-order linear symmetric multipoint singular differential expression, with operator coefficient in the direct sum of Hilbert spaces of vector-functions defined at the semi-infinite intervals. To this end we use the Calkin-Gorbachuk method. Finally, the geometry of spectrum set of such extensions is researched.

1. Introduction

In the first years of the previous century, von Neumann [11] and Stone [10] investigated the theory of selfadjoint extensions of linear densely defined closed symmetric operators in a Hilbert spaces. Applications to scalar linear even order symmetric differential operators and description of all selfadjoint extensions in terms of boundary conditions due to Glazman in his seminal work [5] and in the book of Naimark [8]. In this sense the famous Glazman-Krein-Naimark (or Everitt-Krein-Glazman-Naimark) Theorem in the mathematical literature it is to be noted. In the mathematical literature there is another method co-called Calkin-Gorbachuk method. (see [6, 9]).

Our motivation for this article originates from the interesting researches of Everitt, Markus, Zettl, Sun, O'Regan, Agarwal [2, 3, 4, 12] in scalar cases. Throughout this paper we consider Zettl and Suns's view about these topics [12]. A selfadjoint ordinary differential operator in Hilbert space is generated by two things:

1. a symmetric (formally selfadjoint) differential expression;
2. a boundary condition which determined selfadjoint differential operators;

And also for a given selfadjoint differential operator, a basic question is: What is its spectrum?

In this work in Section 3 the representation of all selfadjoint extensions of a multipoint symmetric quasi-differential operator, generated by first-order symmetric differential-operator expression (for the definition see [4]) in the direct sum of Hilbert spaces of vector-functions defined at the semi-infinite intervals in terms of

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boundary conditions are described. In sec. 4 the structure of spectrum of these selfadjoint extensions is investigated.

2. Statement of the problem

In the direct sum $\mathcal{H} = L^2(H,(-\infty,a_1))\oplus L^2(H,(a_2,\infty))$, H is a separable Hilbert space, and $a_1, a_2 \in \mathbb{R}$ will be considered for the multipoint differential-operator expression in the form

$$l(u) = (l_1(u_1), l_2(u_2)),$$
$$l_k(u_k) = i\rho_k u_k' + \frac{1}{2} i\rho_k' u_k + A_k u_k, \quad k = 1, 2,$$

where

(1) $\rho_1 : (-\infty, a_1) \to (0,\infty)$, $\rho_2 : (a_2,\infty) \to (0,\infty)$;
(2) $\rho_1 \in AC_{loc}(-\infty, a_1)$ and $\rho_2 \in AC_{loc}(a_2,\infty)$;
(3) $\int_{-\infty}^{a_1} \frac{ds}{\rho_1(s)} = \infty$, $\int_{a_2}^{\infty} \frac{ds}{\rho_2(s)} = \infty$;
(4) $A_k^* = A_k : D(A_k) \subset H \to H$, $k = 1, 2$.

The minimal operators L_0^1 and L_0^2 corresponding to differential-operator expressions l_1 and l_2 in $L^2(H,(-\infty,a_1))$ and $L^2(H,(a_2,\infty))$, respectively, can be defined by a standard processes, see[7]. The operators $L_1 = (L_0^1)^*$ and $L_2 = (L_0^2)^*$ are maximal operators corresponding to l_1 and l_2 in $L^2(H,(-\infty,a_1))$ and $L^2(H,(a_2,\infty))$, respectively. In this case the operators

$$L_0 = L_0^1 \oplus L_0^2 \quad \text{and} \quad L = L^1 \oplus L^2$$

will be indicating the minimal and maximal operators corresponding to differential expression on $\mathcal{H}$, respectively.

It is clear that

$$D(L^1) = \{u_1 \in L^2(H,(-\infty,a_1)) : l_1(u_1) \in L^2(H,(-\infty,a_1)\},$$
$$D(L_0^1) = \{u_1 \in D(L^1) : (\sqrt{\rho_1}u_1)(a_1) = 0\}$$

and

$$D(L^2) = \{u_2 \in L^2(H,(a_2,\infty)) : l_2(u_2) \in L^2(H,(a_2,\infty)\},$$
$$D(L_0^2) = \{u_2 \in D(L^2) : (\sqrt{\rho_2}u_2)(a_2) = 0\}.$$

3. Description of Selfadjoint Extensions

In this section using the Calkin-Gorbachuk method will be investigated the general representation of selfadjoint extensions of minimal operator L_0. Firstly we prove the following result.

Lemma 3.1. *The deficiency indices of the operators L_0^1 and L_0^2 are of the form*

$$(m(L_0^1), n(L_0^1)) = (0, \dim H), \quad (m(L_0^2), n(L_0^2)) = (\dim H, 0).$$

Proof. Now for simplicity we assume that $A_1 = A_2 = 0$. It is clear that the general solutions of differential equations

$$i\rho_1(t)u_{1\pm}'(t) + \frac{1}{2} i\rho_1'(t)u_{1\pm}(t) \pm iu_{1\pm}(t) = 0, \quad t < a_1,$$
$$i\rho_2(t)u_{2\pm}'(t) + \frac{1}{2} i\rho_2'(t)u_{2\pm}(t) \pm iu_{2\pm}(t) = 0, \quad t > a_2$$

in $L^2(H,(-\infty,a_1))$ and $L^2(H,(a_2,+\infty))$ are in the form

$$u_{1\pm}(t)=\exp\Big(\pm\int_t^{c_1}\frac{2\pm\rho_1'(s)}{2\rho_1(s)}ds\Big)f_1,\quad f_1\in H,\ t<a_1,\ c_1<a_1$$

and

$$u_{2\pm}(t)=\exp\Big(\mp\int_{c_2}^{t}\frac{2\pm\rho_2'(s)}{2\rho_2(s)}ds\Big)f_2,\quad f_2\in H,\ t>a_2,\ c_2>a_2$$

respectively. From these representations we have

$$\begin{aligned}
&\|u_{1+}\|^2_{L^2(H,(-\infty,a_1))}\\
&=\int_{-\infty}^{a_1}\|u_{1+}(t)\|_H^2dt\\
&=\int_{-\infty}^{a_1}\exp\Big(\int_t^{c_1}\frac{2+\rho_1'(s)}{\rho_1(s)}ds\Big)dt\|f_1\|_H^2\\
&=\int_{-\infty}^{a_1}\frac{\rho_1(c_1)}{\rho_1(t)}\exp\Big(\int_t^{c_1}\frac{2}{\rho_1(s)}ds\Big)dt\|f_1\|_H^2\\
&=\frac{\rho_1(c_1)}{2}\int_{-\infty}^{a_1}\exp\Big(\int_t^{c_1}\frac{2}{\rho_1(s)}ds\Big)d\Big(-\int_t^{c_1}\frac{2}{\rho_1(s)}ds\Big)\|f_1\|_H^2\\
&=-\frac{\rho_1(c_1)}{2}\Big[\exp\Big(\int_{a_1}^{c_1}\frac{2}{\rho_1(s)}ds\Big)-\exp\Big(\int_{-\infty}^{c_1}\frac{2}{\rho_1(s)}ds\Big)\Big]\|f_1\|_H^2=\infty.
\end{aligned}$$

Consequently,

$$\dim\ker(L_0^1+iE)=0$$

On the other hand it is clear that

$$\begin{aligned}
&\|u_{1-}\|^2_{L^2(H,(-\infty,a_1))}\\
&=\int_{-\infty}^{a_1}\|u_{1-}(t)\|_H^2dt\\
&=\int_{-\infty}^{a_1}\exp\Big(-\int_t^{c_1}\frac{2+\rho_1'(s)}{\rho_1(s)}ds\Big)dt\|f_1\|_H^2\\
&=\int_{-\infty}^{a_1}\frac{\rho_1(c_1)}{\rho_1(t)}\exp\Big(-\int_t^{c_1}\frac{2}{\rho_1(s)}ds\Big)dt\|f_1\|_H^2\\
&=\frac{\rho_1(c_1)}{2}\int_{-\infty}^{a_1}\exp\Big(-\int_t^{c_1}\frac{2}{\rho_1(s)}ds\Big)d\Big(-\int_t^{c_1}\frac{2}{\rho_1(s)}ds\Big)\|f_1\|_H^2\\
&=\frac{\rho_1(c_1)}{2}\Big[\exp\Big(-\int_{a_1}^{c_1}\frac{2}{\rho_1(s)}ds\Big)-\exp\Big(-\int_{-\infty}^{c_1}\frac{2}{\rho_1(s)}ds\Big)\Big]\|f_1\|_H^2\\
&=\frac{\rho_1(c_1)}{2}\exp\Big(-\int_{a_1}^{c_1}\frac{2}{\rho_1(s)}ds\Big)\|f_1\|_H^2<\infty.
\end{aligned}$$

Therefore,

$$u_{1-}(t)=\exp\Big(\int_{a_1}^{t}\frac{2-\rho_1'(s)}{2\rho_1(s)}ds\Big)f_1\in L^2(H,(-\infty,a_1)).$$

Hence

$$\dim\ker(L_0^1-iE)=\dim H$$

In a similar way it can be shown that

$$m(L_0^2)=\dim\ker(L_0^2+iE)=\dim H\quad\text{and}\quad n(L_0^2)=\dim\ker(L_0^2-iE)=0$$

This completes the proof. □

Consequently, the minimal operator L_0 has selfadjoint extensions; see [6]. To describe these extensions we need to obtain the space of boundary values.

Definition 3.2 ([6])**.** Let H be any Hilbert space and $S : D(S) \subset H \to H$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$ having equal finite or infinite deficiency indices. A triplet $(\mathcal{B}, \gamma_1, \gamma_2)$, where $\mathcal{B}$ is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathcal{B}$, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$(S^*f, g)_H - (f, S^*g)_H = (\gamma_1(f), \gamma_2(g))_{\mathcal{B}} - (\gamma_2(f), \gamma_1(g))_{\mathcal{B}}$$

while for any $F_1, F_2 \in \mathcal{B}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

Lemma 3.3. *Let*

$$\begin{gathered}
\gamma_1 : D(L) \to H, \quad \gamma_1(u) = \frac{1}{i\sqrt{2}}\big((\sqrt{\rho_1}u_1)(a_1) + (\sqrt{\rho_2}u_2)(a_2)\big), \\
\gamma_2 : D(L) \to H, \quad \gamma_2(u) = \frac{1}{\sqrt{2}}\left((\sqrt{\rho_1}u_1)(a_1) - (\sqrt{\rho_2}u_2)(a_2)\right),
\end{gathered}$$

where $u = (u_1, u_2) \in D(L)$. Then the triplet (H, γ_1, γ_2) is a space of boundary values of the minimal operator L_0 in $\mathcal{H}$.

Proof. For any $u = (u_1, u_2)$, $v = (v_1, v_2) \in D(L)$

$$\begin{aligned}
&(Lu, v)_{\mathcal{H}} - (u, Lv)_{\mathcal{H}} \\
&= (L_1u_1, v_1)_{L^2(H,(-\infty,a_1))} + (L_2u_2, v_2)_{L^2(H,(a_2,\infty))} - (u_1, L_1v_1)_{L^2(H,(-\infty,a_1))} \\
&\quad - (u_2, L_2v_2)_{L^2(H,(a_2,\infty))} \\
&= \big[(i\rho_1u_1' + \frac{i}{2}\rho_1'u_1 + A_1u_1, v_1)_{L^2(H,(-\infty,a_1))} \\
&\quad - (u_1, i\rho_1v_1' + \frac{i}{2}\rho_1'v_1 + A_1v_1)_{L^2(H,(-\infty,a_1))}\big] \\
&\quad + \big[(i\rho_2u_2' + \frac{i}{2}\rho_2'u_2 + A_2u_2, v_2)_{L^2(H,(a_2,\infty))} \\
&\quad - (u_2, i\rho_2v_2' + \frac{i}{2}\rho_2'v_2 + A_2v_2)_{L^2(H,(a_2,\infty))}\big] \\
&= i\big[(\rho_1u_1', v_1)_{L^2(H,(-\infty,a_1))} + (u_1, \rho_1v_1')_{L^2(H,(-\infty,a_1))}\big] \\
&\quad + \frac{i}{2}\big[(\rho_1'u_1, v_1)_{L^2(H,(-\infty,a_1))} + (u_1, \rho_1'v_1)_{L^2(H,(-\infty,a_1))}\big] \\
&\quad + i\big[(\rho_2u_2', v_2)_{L^2(H,(a_2,\infty))} + (u_2, \rho_2v_2')_{L^2(H,(a_2,\infty))}\big] \\
&\quad + \frac{i}{2}\big[(\rho_2'u_2, v_2)_{L^2(H,(a_2,\infty))} + (u_2, \rho_2'v_2)_{L^2(H,(a_2,\infty))}\big] \\
&= i\big[(\rho_1u_1', v_1)_{L^2(H,(-\infty,a_1))} + (u_1, \rho_1v_1')_{L^2(H,(-\infty,a_1))}\big] + i(\rho_1'u_1, v_1)_{L^2(H,(-\infty,a_1))} \\
&\quad + i\big[(\rho_2u_2', v_2)_{L^2(H,(a_2,\infty))} + (u_2, \rho_1v_2')_{L^2(H,(a_2,\infty))}\big] + i(\rho_2'u_2, v_2)L_{(H,(a_2,\infty))} \\
&= i\big[(\rho_1u_1' + \rho_1u_1, v_1)_{L^2(H,(-\infty,a_1))} + (\rho_1u_1, v_1')_{L^2(H,(-\infty,a_1))}\big] \\
&\quad + i\big[(\rho_2u_2' + \rho_2u_2, v_2)_{L^2(H,(a_2,\infty))} + (\rho_2u_2, v_2')_{L^2(H,(a_2,\infty))}\big] \\
&= i[((\rho_1u_1), v_1)']_{L^2(H,(-\infty,a_1))} + i\,[((\rho_2u_2), v_2)']_{L^2(H,(a_2,\infty))}
\end{aligned}$$

$$= i\big[((\sqrt{\rho_2}u_2)(a_2), (\sqrt{\rho_2}v_2)(a_2))_H - ((\sqrt{\rho_1}u_1)(a_1), (\sqrt{\rho_1}v_1)(a_1))_H \big]$$
$$= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H.$$

Now for any element $f_1, f_2 \in H$ let us find the function $u = (u_1, u_2) \in D(L)$ such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}}((\sqrt{\rho_1}u_1)(a_1) + (\sqrt{\rho_2}u_2)(a_2)) = f_1,$$
$$\gamma_2(u) = \frac{1}{\sqrt{2}}((\sqrt{\rho_1}u_1)(a_1) - (\sqrt{\rho_2}u_2)(a_2)) = f_2$$

From here the following two expressions are obtained

$$(\sqrt{\rho_1}u_1)(a_1) = (if_1 + f_2)/\sqrt{2}, \quad (\sqrt{\rho_2}u_2)(a_2) = (if_1 - f_2)/\sqrt{2}.$$

If we choose the functions $u_1(\cdot)$, $u_2(\cdot)$ in the following forms

$$u_1(t) = \frac{1}{\sqrt{\rho_1(t)}} e^{t-a_1}(if_1 + f_2)/\sqrt{2}, \quad t < a_1,$$
$$u_2(t) = \frac{1}{\sqrt{\rho_2(t)}} e^{a_2-t}(if_1 - f_2)/\sqrt{2}, \quad t > a_2,$$

then it is clear that $u = (\sqrt{\rho_1}u_1, \sqrt{\rho_2}u_2) \in D(L)$ and $\gamma_1(u) = f_1$, $\gamma_2(u) = f_2$. □

Theorem 3.4. *If $\widetilde{L}$ is a selfadjoint extension of the minimal operator L_0 in $\mathcal{H}$, then it is generated by the differential-operator expression $l(\cdot)$ and the following boundary condition*

$$(\sqrt{\rho_2}u_2)(a_2) = W(\sqrt{\rho_1}u_1)(a_1),$$

where $W : H \to H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_W$ and vice versa.

4. Spectrum of the Selfadjoint Extensions

In this section the structure of the spectrum of the selfadjoint extensions L_W of the minimal operator L_0 in $\mathcal{H}$ will be investigated. First let us prove the following results.

Theorem 4.1. *The point spectrum of the selfadjoint extension L_W is empty, i.e. $\sigma_p(L_W) = \emptyset$.*

Proof. Consider the eigenvalue problem

$$l(u) = \lambda u, \quad u = (u_1, u_2) \in \mathcal{H}, \quad \lambda \in \mathbb{R}$$

with the boundary condition

$$(\sqrt{\rho_2}u_2)(a_2) = W(\sqrt{\rho_1}u_1)(a_1).$$

From here the following expressions are obtained

$$i\rho_1(t)u_1'(t) + \frac{1}{2}i\rho_1'(t)u_1(t) + A_1u_1(t) = \lambda u_1(t), \quad t < a_1,$$
$$i\rho_2(t)u_2'(t) + \frac{1}{2}i\rho_2'(t)u_2(t) + A_2u_2(t) = \lambda u_2(t), \quad t > a_2,$$
$$(\sqrt{\rho_2}u_2)(a_2) = W(\sqrt{\rho_1}u_1)(a_1).$$

The general solutions of these equations are in the form

$$u_1(t;\lambda)=\sqrt{\frac{\rho_1(c)}{\rho_1(t)}}\exp\Big(-i(A_1-\lambda)\int_t^c\frac{ds}{\rho_1(s)}\Big)f_\lambda^1,\quad f_\lambda^1\in H,\quad t<a_1,\ c<a_1,$$
$$u_2(t;\lambda)=\sqrt{\frac{\rho_2(c)}{\rho_2(t)}}\exp\Big(i(A_2-\lambda)\int_c^t\frac{ds}{\rho_2(s)}\Big)f_\lambda^2,\quad f_\lambda^2\in H,\quad t>a_2,\ c>a_2,$$
$$(\sqrt{\rho_2}u_2)(a_2)=W(\sqrt{\rho_1}u_1)(a_1).$$

It is clear that for the $f_\lambda^1\neq 0$ and $f_\lambda^2\neq 0$ the solutions are $u_1(\cdot;\lambda)\notin L^2(H,(-\infty,a_1))$ and $u_2(\cdot;\lambda)\notin L^2(H,(a_2,\infty))$. Therefore for every unitary operator W we have $\sigma_p(L_W)=\emptyset$. □

Since the residual spectrum for any selfadjoint operator in any Hilbert space is empty, then we have to investigate the continuous spectrum of selfadjoint extensions L_W of the minimal operator L_0 is investigated. On the other hand from the general theory of linear selfadjoint operators in Hilbert spaces for the resolvent set $\rho(L_W)$ of any selfadjoint extension L_W is true

$$\rho(L_W)\supset\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda\neq 0\}.$$

For the continuous spectrum of selfadjoint extensions we have the following statement.

Theorem 4.2. *The continuous spectrum of any selfadjoint extension L_W in of the form*

$$\sigma_c(L_W)=\mathbb{R}.$$

Proof. For $\lambda\in\mathbb{C}$, $\lambda_i=\operatorname{Im}\lambda>0$ and $f=(f_1,f_2)\in\mathcal{H}$ the norm of function $R_\lambda(L_W)f(t)$ in $\mathcal{H}$ we have

$$\begin{aligned}
&\|R_\lambda(L_W))f(t)\|_{\mathcal{H}}^2\\
&=\|\frac{1}{\rho_1(t)}\exp\Big(i(\lambda-A_1)\int_t^{a_1}\frac{ds}{\rho_1(s)}\Big)f_\lambda^1\\
&\quad+\frac{i}{\sqrt{\rho_1(t)}}\int_t^{a_1}\exp\Big(i(A_1-\lambda)\int_s^t\frac{d\tau}{\rho_1(\tau)}\Big)\frac{f_1(s)}{\sqrt{\rho_1(s)}}ds\|_{L^2(H,(-\infty,a_1))}^2\\
&\quad+\|\frac{i}{\sqrt{\rho_2(t)}}\int_t^{\infty}\exp\Big(i(A_2-\lambda)\int_s^t\frac{d\tau}{\rho_2(\tau)}\Big)\frac{f_2(s)}{\sqrt{\rho_2(s)}}ds\|_{L^2(H,(a_2,\infty))}^2\\
&\geq\|\frac{i}{\sqrt{\rho_2(t)}}\int_t^{\infty}\exp\Big(i(A_2-\lambda)\int_s^t\frac{d\tau}{\rho_2(\tau)}\Big)\frac{f_2(s)}{\sqrt{\rho_2(s)}}ds\|_{L^2(H,(a_2,\infty))}^2.
\end{aligned}$$

The vector functions $f^*(t;\lambda)$ in the form

$$f^*(t;\lambda)=\Big(0,\frac{1}{\sqrt{\rho_2(t)}}\exp\Big(-i(\overline{\lambda}-A_2)\int_{a_2}^t\frac{ds}{\rho_2(s)}\Big)f\Big),$$

with $\lambda\in\mathbb{C}$, $\lambda_i=\operatorname{Im}\lambda>0$, $f\in H$ belong to $\mathcal{H}$. Indeed,

$$\begin{aligned}
\|f^*(t;\lambda)\|_{\mathcal{H}}^2&=\int_{a_2}^{\infty}\frac{1}{\rho_2(t)}\|\exp\Big(-i(\overline{\lambda}-A_2)\int_{a_2}^t\frac{ds}{\rho_2(s)}\Big)f\|_H^2dt\\
&=\int_{a_2}^{\infty}\frac{1}{\rho_2(t)}\exp\Big(-2\lambda_i\int_{a_2}^t\frac{ds}{\rho_2(s)}\Big)dt\|f\|_H^2
\end{aligned}$$

$$= \frac{1}{2\lambda_i}\|f\|_H^2 < \infty.$$

For such functions $f^*(\cdot;\lambda)$ we have

$$\begin{aligned}
&\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathcal{H}}^2 \\
&\geq \Big\|\frac{i}{\sqrt{\rho_2(t)}}\int_t^\infty \frac{1}{\rho_2(s)}\exp\Big(i(A_2-\lambda)\int_s^t \frac{d\tau}{\rho_2(\tau)} \\
&\quad - i(\overline{\lambda}-A_2)\int_{a_2}^s \frac{d\tau}{\rho_2(\tau)}\Big) f\,ds\Big\|_{L^2(H,(a_2,\infty))}^2 \\
&= \Big\|\frac{1}{\sqrt{\rho_2(t)}}\exp\Big(-i\lambda\int_{a_2}^t \frac{d\tau}{\rho_2(\tau)} + iA_2\int_{a_2}^t \frac{d\tau}{\rho_2(\tau)}\Big) \\
&\quad \times \int_t^\infty \frac{1}{\rho_2(s)}\exp\Big(-2\lambda_i\int_{a_2}^s \frac{d\tau}{\rho_2(\tau)}\Big) f\,ds\Big\|_{L^2(H,(a_2,\infty))}^2 \\
&= \Big\|\frac{1}{\sqrt{\rho_2(t)}}\exp\Big(\lambda_i\int_{a_2}^t \frac{d\tau}{\rho_2(\tau)}\Big)\int_t^\infty \frac{1}{\rho_2(s)}\exp\Big(-2\lambda_i\int_{a_2}^s \frac{d\tau}{\rho_2(\tau)}\Big)ds\Big\|_{L^2(H,(a_2,\infty))}^2 \\
&\quad \times \|f\|_H^2 \\
&= \Big\|\frac{1}{2\lambda_i}\exp\Big(-\lambda_i\int_{a_2}^t \frac{d\tau}{\rho_2(\tau)}\Big)\Big\|_{L^2(H,(a_2,\infty))}^2 \|f\|_H^2 \\
&= \frac{1}{4\lambda_i^2}\int_{a_2}^\infty \frac{1}{\rho_2(t)}\exp\Big(-2\lambda_i\int_{a_2}^t \frac{d\tau}{\rho_2(\tau)}\Big)dt\,\|f\|_H^2 \\
&= \frac{1}{8\lambda_i^3}\|f\|_H^2.
\end{aligned}$$

From this we have

$$\|R_\lambda(L_W)f^*(\cdot;\lambda)\|_{\mathcal{H}} \geq \frac{\|f\|_H^2}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} = \frac{1}{2\lambda_i}\|f^*(t;\lambda)\|_{\mathcal{H}}.$$

Then for $\lambda_i = \operatorname{Im}\lambda > 0$ and $f \neq 0$ the following inequality is valid

$$\frac{\|R_\lambda(L_W)f^*(\cdot,\lambda)\|_{\mathcal{H}}}{\|f^*(\lambda;t)\|_{\mathcal{H}}} \geq \frac{1}{2\lambda_i}.$$

On the other hand it is clear that

$$\|R_\lambda(L_W)\| \geq \frac{\|R_\lambda(L_W)f^*(\cdot;\lambda)\|_{\mathcal{H}}}{\|f^*(\cdot;\lambda)\|_{\mathcal{H}}}, \quad f \neq 0.$$

Consequently, for $\lambda \in \mathbb{C}$ and $\lambda_i = \operatorname{Im}\lambda > 0$ we have

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i}.$$

□

Remark 4.3. In the special case $\rho_k = 1$, $k = 1, 2$, similar results have been obtained in [1].

As an example all selfadjoint extensions L_φ of the minimal operator L_0, generated by the multipoint differential expression

$$l(u) = (l_1(u_1), l_2(u_2))$$

$$
\begin{aligned}
= \Big(& itu_1'(t,x) + \frac{1}{2} i u_1(t,x) - \frac{\partial^2 u_1}{\partial x^2}(t,x), i\sqrt{t}u_2'(t,x) \\
& + \frac{1}{4\sqrt{t}} i u_2(t,x) - \frac{\partial^2 u_2}{\partial x^2}(t,x)\Big),
\end{aligned}
$$

with boundary conditions

$$
\begin{gathered}
u_1(t,0) = u_1(t,1), \quad u_1'(t,0) = u_1'(t,1), \quad t < -1, \\
u_2(t,0) = u_2(t,1), \quad u_2'(t,0) = u_2'(t,1), \quad t > 1
\end{gathered}
$$

in the direct sum $L^2((-\infty,-1)\times(0,1)) \oplus L^2((1,\infty)\times(0,1))$ in terms of boundary conditions are described the boundary condition

$$
(t^{1/4}u_2(t))(1,x) = e^{i\varphi}(\sqrt{t}u_1(t))(-1,x), \quad \varphi \in [0,2\pi),\ x \in (0,1).
$$

Moreover, the spectrum of such extension is

$$
\sigma(L_\varphi) = \sigma_c(L_\varphi) = \mathbb{R}.
$$

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