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GROUND STATES OF LINEARLY COUPLED SCHRÖDINGER SYSTEMS

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Abstract. This article concerns the standing waves of a linearly coupled Schrödinger system which arises from nonlinear optics and condensed matter physics. The coefficients of the system are spatially dependent and have a mixed behavior: they are periodic in some directions and tend to positive constants in other directions. Under suitable assumptions, we prove that the system has a positive ground state. In addition, when the L^∞-norm of the coupling coefficient tends to zero, the asymptotic behavior of the ground states is also obtained.

1. Introduction and statement of main results

Nonlinear Schrödinger systems of the form

$$
\begin{gathered}
-i\frac{\partial}{\partial t}\Psi_1 = \Delta\Psi_1 - V_1(x)\Psi_1 + \mu_1|\Psi_1|^2\Psi_1 + \beta|\Psi_2|^2\Psi_1 + \gamma\Psi_2 \\
-i\frac{\partial}{\partial t}\Psi_2 = \Delta\Psi_2 - V_2(x)\Psi_2 + \mu_2|\Psi_2|^2\Psi_2 + \beta|\Psi_1|^2\Psi_2 + \gamma\Psi_1 \\
x \in \mathbb{R}^N,\ t > 0, \\
\Psi_j = \Psi_j(x,t) \in \mathbb{C}, \quad t > 0,\ j = 1,2
\end{gathered}
\tag{1.1}
$$

model several interesting phenomena in physics. Physically, Ψ_j are two components of a quantum system, μ_j and β are the intraspecies and interspecies scattering lengths, γ is the Rabi frequency related to the external electric field. The sign of the scattering length β determines whether the interaction is repulsive or attractive. We refer to [1, 16, 17, 18, 29] and references therein for more information on the physical background of (1.1).

To study standing waves of the system (1.1), we set $\Psi_j(x,t) = e^{i\lambda t}u_j(x)$ for $j = 1,2$. Then (1.1) is reduced to the following elliptic system

$$
\begin{gathered}
-\Delta u_1 + (V_1(x) + \lambda)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 + \gamma u_2 \quad \text{in } \mathbb{R}^N, \\
-\Delta u_2 + (V_2(x) + \lambda)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 + \gamma u_1 \quad \text{in } \mathbb{R}^N, \\
u_j(x) \to 0 \quad \text{as } |x| \to \infty,\ j = 1,2.
\end{gathered}
\tag{1.2}
$$

In the presence of only nonlinearly coupling terms (i.e., $\gamma = 0$), (1.2) has been studied extensively in recent years for the existence, multiplicity and asymptotic

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behavior of nontrivial solutions. We make no attempt here to give a complete survey of all related results and only refer the reader to [4, 7, 8, 21, 23, 24, 27, 28, 30] and references therein. However, In the presence of only linearly coupling terms (i.e., $\beta = 0$), (1.2) has not been much studied and we are only aware of a few papers in this direction ([2, 3, 5, 11, 12, 20]).

On the other hand, the single elliptic equation

$$-\Delta u + V(x)u = \mu(x)|u|^{p-2}u, \tag{1.3}$$

has been deeply studied in the literature. Among other solutions, ground states are physically and mathematically of particular interest. We refer the reader to [6, 9, 10, 13, 14, 15, 19, 22, 25, 26, 31, 32] for related results. Here we only mention that (1.3) has a positive ground state if $0 < V(x) \le \lim_{|x|\to\infty} V(x) < \infty$, $0 < \lim_{|x|\to\infty} \mu(x) \le \mu(x)$ ([14, 19, 22, 31]) or if V, μ are positive and periodic in each variable ([13, 19]). The potentials considered in [10] have a mixed behavior. More precisely, the potentials are periodic in some directions and tend to positive constants in other directions.

Partially motivated by [3, 10], we deal with in this paper ground states of linearly coupled Schrödinger equations in which all of the physical parameters are spatially dependent, i.e., we will consider the system

$$\begin{gathered}
-\Delta u_1 + V_1(x)u_1 = \mu_1(x)|u_1|^{p-2}u_1 + \gamma(x)u_2 \quad \text{in } \mathbb{R}^N, \\
-\Delta u_2 + V_2(x)u_2 = \mu_2(x)|u_2|^{p-2}u_2 + \gamma(x)u_1 \quad \text{in } \mathbb{R}^N, \\
u_j \in H^1(\mathbb{R}^N), \quad j = 1, 2,
\end{gathered} \tag{1.4}$$

where $N \ge 2$, $2 < p < 2^*$, $2^* = 2N/(N-2)$ for $N > 2$ and $2^* = +\infty$ for $N = 2$, and the coefficients V_j, μ_j, γ are continuous functions on $\mathbb{R}^N$.

For system (1.4), a nontrivial solution is a solution (u_1, u_2) with $(u_1, u_2) \ne (0, 0)$. A nontrivial solution of (1.4) will be called a ground state if it has the least energy among all nontrivial solutions. A positive ground state means a ground state with each component being positive. We remark that each component of a nontrivial solution of (1.4) must be nonzero. Write $N = k + l$ with $1 \le k \le N - 1$ and $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^N$. We study (1.4) under the following assumptions.

(H1) The five functions V_j, μ_j, γ are τ_i-periodic in x_i, $\tau_i > 0$, $i = 1, \dots, k$.

(H2) $0 < V_j(x) \le V_{j\infty} := \lim_{|x''|\to\infty} V_j(x) < \infty$ for all $x \in \mathbb{R}^N$, $j = 1, 2$;
$0 < \mu_{j\infty} := \lim_{|x''|\to\infty} \mu_j(x) \le \mu_j(x)$ for all $x \in \mathbb{R}^N$, $j = 1, 2$;
$0 < \gamma_\infty := \lim_{|x''|\to\infty} \gamma(x) \le \gamma(x)$ for all $x \in \mathbb{R}^N$.

(H3) $\left|\gamma(V_1V_2)^{-1/2}\right|_\infty < 1$, where $|\cdot|_\infty$ is the usual norm in $L^\infty(\mathbb{R}^N)$.

The first main result in our paper is as follows and is for the existence of ground states of (1.4).

Theorem 1.1. *If* (H1)–(H3) *hold, then* (1.4) *has a positive ground state.*

We assume (H3) to guarantee that the Nehari manifold is bounded away from zero, see Lemma 2.1 for details.

The second aim of our paper is to describe the asymptotic behavior of ground states when L^∞-norm of the linearly coupling coefficient of (1.4) tends to zero. For this purpose, we replace (H1), (H2) and (H3) with

(H1') The functions V_j, μ_j, γ_n are τ_i-periodic in x_i, $\tau_i > 0$, $i = 1, \dots, k$.

(H2') $0 < V_j(x) \le V_{j\infty} := \lim_{|x''|\to\infty} V_j(x) < \infty$ for all $x \in \mathbb{R}^N$, $j = 1, 2$;
$0 < \mu_{j\infty} := \lim_{|x''|\to\infty} \mu_j(x) \le \mu_j(x)$ for all $x \in \mathbb{R}^N$, $j = 1, 2$;
$0 < \gamma_{n\infty} := \lim_{|x''|\to\infty} \gamma_n(x) \le \gamma_n(x)$ for all $x \in \mathbb{R}^N$, $n = 1, 2, \dots$.

(H3') $|\gamma_n|_\infty \to 0$ as $n \to \infty$.

Under the assumptions (H1'), (H2') and (H3'), we see from Theorem 1.1 that, for n sufficiently large, (1.4) with $\gamma = \gamma_n$ has a positive ground state (u_{n1}, u_{n2}). Next we show that one component of (u_{n1}, u_{n2}) converges to zero in $H^1(\mathbb{R}^N)$.

Theorem 1.2. *Assume* (H1')–(H3') *are satisfied. Let* (u_{n1}, u_{n2}) *be the positive ground state of* (1.4) *with* $\gamma = \gamma_n$, *then we have either* $u_{n1} \to 0$ *in* $H^1(\mathbb{R}^N)$ *or* $u_{n2} \to 0$ *in* $H^1(\mathbb{R}^N)$.

This article is organized as follows. In Section 2, we prove some preliminary results including basic properties of the Nehari manifold. Section 3 is devoted to the existence of ground states for (1.4), while Section 4 concerns the asymptotic behavior of ground states when L^∞-norm of the coupling coefficient tends to zero.

2. Preliminaries

By (H1) and (H2),

$$\|u\|_j = \Big(\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V_j u^2\big)\Big)^{1/2}, \quad j = 1, 2$$

are equivalent norms in $H^1(\mathbb{R}^N)$. Set $\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and, for $\vec{u} = (u_1, u_2) \in \mathcal{H}$, denote

$$\|\vec{u}\| = \Big(\|u_1\|_1^2 + \|u_2\|_2^2\Big)^{1/2}.$$

Then $\|\cdot\|$ is equivalent to the standard norm in $\mathcal{H}$. Throughout this paper, the notation $\|\cdot\|$ will always refer to this norm.

It is well known that solutions of (1.4) correspond to critical points of the energy functional $I : \mathcal{H} \to \mathbb{R}$ defined by

$$I(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p) - \int_{\mathbb{R}^N} \gamma u_1 u_2.$$

Denote by $\mathcal{N}$ the so-called Nehari manifold associated with I, namely,

$$\mathcal{N} = \big\{\vec{u} \in \mathcal{H} \setminus \{(0,0)\} : J(\vec{u}) := \langle I'(\vec{u}), \vec{u}\rangle = 0\big\}.$$

Some useful properties of the Nehari manifold are given next.

Lemma 2.1. *There exists* $\delta > 0$ *such that, for* $\vec{u} \in \mathcal{N}$, $\|\vec{u}\| \ge \delta$.

Proof. For $\vec{u} \in \mathcal{N}$, we use the Hölder inequality and Sobolev inequality to deduce

$$\begin{aligned}\Big(1 - \big|\gamma(V_1V_2)^{-1/2}\big|_\infty\Big)\|\vec{u}\|^2 &\le \|\vec{u}\|^2 - 2\int_{\mathbb{R}^N} \gamma u_1 u_2 \\ &= \int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p) \le C\|\vec{u}\|^p,\end{aligned}$$

which implies the desired result. $\square$

Note that for $\vec{u} \in \mathcal{N}$,

$$I(\vec{u}) = (\frac{1}{2} - \frac{1}{p})\Big(\|\vec{u}\|^2 - 2\int_{\mathbb{R}^N} \gamma u_1 u_2\Big) \geq (\frac{1}{2} - \frac{1}{p})\Big(1 - \big|\gamma (V_1 V_2)^{-1/2}\big|_\infty\Big)\|\vec{u}\|^2.$$

Therefore, as a consequence of Lemma 2.1, we have

Lemma 2.2. $c = \inf_{\vec{u}\in\mathcal{N}} I(\vec{u}) > 0$.

Lemma 2.3. *If c is achieved, then* (1.4) *has a positive ground state.*

Proof. We first claim that any minimizer of c is a ground state of (1.4). Indeed, if $\vec{u} = (u_1, u_2) \in \mathcal{N}$ is a minimizer of c, then there exists $\lambda \in \mathbb{R}$ such that

$$I'(\vec{u}) = \lambda J'(\vec{u}).$$

From $\langle I'(\vec{u}), \vec{u}\rangle = 0$ and

$$\begin{aligned}
\langle J'(\vec{u}), \vec{u}\rangle &= 2\Big(\|\vec{u}\|^2 - 2\int_{\mathbb{R}^N} \gamma u_1 u_2\Big) - p\int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p) \\
&= -(p-2)(\|\vec{u}\|^2 - 2\int_{\mathbb{R}^N} \gamma u_1 u_2) \\
&\leq -(p-2)\Big(1 - \big|\gamma(V_1V_2)^{-1/2}\big|_\infty\Big)\|\vec{u}\|^2 \\
&\leq -(p-2)\Big(1 - \big|\gamma(V_1V_2)^{-1/2}\big|_\infty\Big)\delta^2 < 0
\end{aligned}$$

it follows that $\lambda = 0$. Then $I'(\vec{u}) = 0$ and so $\vec{u}$ is a ground state of (1.4).

Next we prove that (1.4) has a positive ground state. Assume $\vec{u} = (u_1, u_2) \in \mathcal{N}$ is a minimizer of c and let $t > 0$ be such that $(t|u_1|, t|u_2|) \in \mathcal{N}$. Then

$$t^{p-2} = \frac{\|\vec{u}\|^2 - 2\int_{\mathbb{R}^N} \gamma |u_1||u_2|}{\int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p)} \leq \frac{\|\vec{u}\|^2 - 2\int_{\mathbb{R}^N} \gamma u_1 u_2}{\int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p)} = 1,$$

which implies

$$\begin{aligned}
c \leq I(t|u_1|, t|u_2|) &= (\frac{1}{2} - \frac{1}{p})t^p \int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p) \\
&\leq (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p) \\
&= I(\vec{u}) = c.
\end{aligned}$$

This means $t = 1$ and $(|u_1|, |u_2|)$ is also a minimizer of c. From the above claim, $(|u_1|, |u_2|)$ is a ground state of (1.4). Note that none of $|u_1|$ and $|u_2|$ can be identically zero. Using the strong maximum principle, we have $|u_1| > 0$ and $|u_2| > 0$. Therefore $(|u_1|, |u_2|)$ is a positive ground state of (1.4). The proof is complete. □

3. Proof of Theorem 1.1

In this section we prove the existence of ground states of (1.4). From Lemma 2.3, it suffices to prove that c is achieved. For this purpose, we need to compare the value of c with

$$c_\infty = \inf_{\vec{u}\in\mathcal{N}_\infty} I_\infty(\vec{u}),$$

where the functional $I_\infty : \mathcal{H} \to \mathbb{R}$ is defined by

$$I_\infty(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u_1|^2 + V_{1\infty}u_1^2 + |\nabla u_2|^2 + V_{2\infty}u_2^2\big)$$

$$-\frac{1}{p}\int_{\mathbb{R}^N}(\mu_{1\infty}|u_1|^p+\mu_{2\infty}|u_2|^p)-\int_{\mathbb{R}^N}\gamma_\infty u_1u_2,$$

and

$$\mathcal{N}_\infty=\{\vec{u}\in\mathcal{H}\setminus\{(0,0)\}:\langle I'_\infty(\vec{u}),\vec{u}\rangle=0\}$$

is the Nehari manifold associated with I_∞.

Lemma 3.1. *Assume that* (H1)–(H3) *are satisfied and suppose in addition that at least one of the five functions V_j,μ_j,γ is not a constant, then $c<c_\infty$.*

Proof. By [3, Lemma 3.2], c_∞ is achieved at some $\vec{u}_\infty=(u_{1\infty},u_{2\infty})$ with each component being positive. Let $t>0$ be such that $t\vec{u}_\infty\in\mathcal{N}$. Then

$$\begin{aligned}
c_\infty&=I_\infty(\vec{u}_\infty)\ge I_\infty(t\vec{u}_\infty)\\
&=I(t\vec{u}_\infty)+\frac{t^2}{2}\int_{\mathbb{R}^N}\big[(V_{1\infty}-V_1)\,u_{1\infty}^2+(V_{2\infty}-V_2)\,u_{2\infty}^2\big]\\
&\quad+\frac{t^p}{p}\int_{\mathbb{R}^N}[(\mu_1-\mu_{1\infty})\,|u_{1\infty}|^p+(\mu_2-\mu_{2\infty})\,|u_{2\infty}|^p]\\
&\quad+t^2\int_{\mathbb{R}^N}(\gamma-\gamma_\infty)\,u_{1\infty}u_{2\infty}\\
&>I(t\vec{u}_\infty)\ge c.
\end{aligned}$$

The proof is complete. □

Now we are in a position to prove the first main result.

Proof of Theorem 1.1. By Lemma 2.3, it suffices to prove that the infimum

$$c=\inf_{\vec{u}\in\mathcal{N}}I(\vec{u})$$

is achieved. If the five functions V_j,μ_j,γ are all constants, then the result has been proved in [3, Lemma 3.2]. In what follows we always assume that at least one of the five functions V_j,μ_j,γ is not a constant. By Ekeland's variational principle, there exists $\{\vec{u}_m\}\subset\mathcal{N}$ with $\vec{u}_m=(u_{m1},u_{m2})$ such that

$$I(\vec{u}_m)\to c\quad\text{and}\quad(I|_{\mathcal{N}})'(\vec{u}_m)\to0\quad\text{as } m\to\infty.$$

Then there exists a sequence $\{\lambda_m\}$ of real numbers such that

$$I'(\vec{u}_m)-\lambda_mJ'(\vec{u}_m)\to0\quad\text{as } m\to\infty.$$

From (H3) and

$$I(\vec{u}_m)=\big(\frac12-\frac1p\big)\Big(\|\vec{u}_m\|^2-2\int_{\mathbb{R}^N}\gamma u_{m1}u_{m2}\Big)=c+o(1),$$

we see that $\{\vec{u}_m\}$ is bounded in $\mathcal{H}$. Then

$$\begin{aligned}
o(1)&=\langle I'(\vec{u}_m)-\lambda_mJ'(\vec{u}_m),\vec{u}_m\rangle\\
&=\lambda_m(p-2)\Big(\|\vec{u}_m\|^2-2\int_{\mathbb{R}^N}\gamma u_{m1}u_{m2}\Big)\\
&=\lambda_m(2pc+o(1)),
\end{aligned}$$

which implies that $\lambda_m=o(1)$ and so $I'(\vec{u}_m)\to0$ as $m\to\infty$.

Since $\{\vec{u}_m\}\subset\mathcal{H}$ is bounded, up to a subsequence, it can be assumed that

$$(u_{m1},u_{m2})\rightharpoonup(u_1,u_2)\quad\text{in }\mathcal{H},$$

$$
\begin{gathered}
(u_{m1}, u_{m2}) \to (u_1, u_2) \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N) \times L^p_{\rm loc}(\mathbb{R}^N), \\
(u_{m1}, u_{m2}) \to (u_1, u_2) \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered}
$$

Then $\vec{u} = (u_1, u_2)$ is a critical point of I. We have the following two cases.

Case 1. $\vec{u} \neq (0,0)$. In this case, $\vec{u} \in \mathcal{N}$ and

$$
\begin{aligned}
c \le I(\vec{u}) &= \big(\frac{1}{2} - \frac{1}{p}\big) \int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p) \\
&\le \lim_{m\to\infty} \big(\frac{1}{2} - \frac{1}{p}\big) \int_{\mathbb{R}^N} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \\
&= \lim_{m\to\infty} I(\vec{u}_m) = c.
\end{aligned}
$$

Then c is achieved by $\vec{u}$.

Case 2. $\vec{u} = (0,0)$. In this case, we decompose $\mathbb{R}^N$ into N-dimensional intervals $\{Q_j\}_{j\in\mathbb{N}\cup\{0\}}$ with each of them having sides of size $(\tau_1, \dots, \tau_k, 1, \dots, 1)$ and chosen in such a way that 0 is the center of Q_0. For each m, we set

$$
d_m = \sup_{j\in\mathbb{N}\cup\{0\}} \Big[\int_{Q_j} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \Big]^{1/p}.
$$

Then there exists $\eta > 0$ such that

$$
d_m \ge \eta > 0 \tag{3.1}
$$

for all m. Indeed, using the Sobolev embeddings $H^1(Q_j) \hookrightarrow L^p(Q_j)$ and the boundedness of $\{\vec{u}_m\} \subset \mathcal{H}$ leads to

$$
\begin{aligned}
\frac{2p}{p-2}(c + o(1)) &= \int_{\mathbb{R}^N} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \\
&= \sum_{j=0}^{\infty} \int_{Q_j} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \\
&\le d_m^{p-2} \sum_{j=0}^{\infty} \Big[\int_{Q_j} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \Big]^{2/p} \\
&\le C d_m^{p-2} \|\vec{u}_m\|^2 \\
&\le C d_m^{p-2},
\end{aligned}
$$

which implies (3.1).

From

$$
\sum_{j=0}^{\infty} \int_{Q_j} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \le C,
$$

we see that d_m is achieved. Let $y_m \in \mathbb{R}^N$ be the center of the interval Q_m^* satisfying

$$
d_m = \Big[\int_{Q_m^*} (\mu_1|u_{m1}|^p + \mu_2|u_{m2}|^p) \Big]^{1/p}.
$$

It follows from (3.1) and $(u_{m1}, u_{m2}) \rightharpoonup (0,0)$ in $\mathcal{H}$ that $\{y_m\}$ is unbounded. Without loss of generality, we assume $|y_m| \to \infty$ as $m \to \infty$. Set $\tilde{u}_{mj} = u_{mj}(\cdot + y_m)$ for $j = 1, 2$ and assume that

$$
\begin{gathered}
(\tilde{u}_{m1}, \tilde{u}_{m2}) \rightharpoonup (\tilde{u}_1, \tilde{u}_2) \quad \text{in } \mathcal{H}, \\
(\tilde{u}_{m1}, \tilde{u}_{m2}) \to (\tilde{u}_1, \tilde{u}_2) \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N) \times L^p_{\rm loc}(\mathbb{R}^N),
\end{gathered}
$$

$$(\tilde{u}_{m1}, \tilde{u}_{m2}) \to (\tilde{u}_1, \tilde{u}_2) \quad \text{a.e. in } \mathbb{R}^N.$$

Then $(\tilde{u}_1, \tilde{u}_2) \neq (0,0)$ as showed by

$$\begin{aligned}
0 < \eta &\leq \varliminf_{m\to\infty} \Big[\int_{Q_m^*} (\mu_1 |u_{m1}|^p + \mu_2 |u_{m2}|^p) \Big]^{1/p} \\
&\leq C \varliminf_{m\to\infty} \Big[\int_{Q_m^*} (|u_{m1}|^p + |u_{m2}|^p) \Big]^{1/p} \\
&= C \lim_{m\to\infty} \Big[\int_{Q_0} (|\tilde{u}_{m1}|^p + |\tilde{u}_{m2}|^p) \Big]^{1/p} \\
&= C \Big[\int_{Q_0} (|\tilde{u}_1|^p + |\tilde{u}_2|^p) \Big]^{1/p}.
\end{aligned}$$

Next we claim that $\{|y''_m|\}$ is bounded. Suppose by contradiction that $|y''_m| \to \infty$ as $m \to \infty$. Then, using (H2), we deduce from $I'(u_{m1}, u_{m2}) \to 0$ that

$$I'_\infty(\tilde{u}_1, \tilde{u}_2) = 0.$$

Therefore, since $(\tilde{u}_1, \tilde{u}_2) \neq (0,0)$, $(\tilde{u}_1, \tilde{u}_2) \in \mathcal{N}_\infty$ and we have

$$\begin{aligned}
c_\infty \leq I_\infty(\tilde{u}_1, \tilde{u}_2) &= \big(\frac{1}{2} - \frac{1}{p}\big) \int_{\mathbb{R}^N} (\mu_{1\infty} |\tilde{u}_1|^p + \mu_{2\infty} |\tilde{u}_2|^p) \\
&\leq \varliminf_{m\to\infty} \big(\frac{1}{2} - \frac{1}{p}\big) \int_{\mathbb{R}^N} (\mu_{1\infty} |\tilde{u}_{m1}|^p + \mu_{2\infty} |\tilde{u}_{m2}|^p) \\
&\leq \lim_{m\to\infty} I(\vec{u}_m) = c,
\end{aligned}$$

a contradiction to Lemma 3.1.

Decompose $\mathbb{R}^N$ into bigger N-dimensional intervals $\{\hat{Q}_j\}_{j\in\mathbb{N}\cup\{0\}}$ and assume that $Q_m^* \subset \hat{Q}_m^*$ with $(y'_m, 0)$ being the center of $\hat{Q}_m^*$. Setting $\hat{u}_{mj} = u_{mj}(\cdot + (y'_m, 0))$ for $j = 1, 2$, we see from the assumption (H1) that

$$I(\hat{u}_{m1}, \hat{u}_{m2}) \to c \quad \text{and} \quad I'(\hat{u}_{m1}, \hat{u}_{m2}) \to 0 \quad \text{as } m \to \infty.$$

Since $\{(\hat{u}_{m1}, \hat{u}_{m2})\}$ is bounded in $\mathcal{H}$, we may assume that

$$\begin{gathered}
(\hat{u}_{m1}, \hat{u}_{m2}) \rightharpoonup (\hat{u}_1, \hat{u}_2) \quad \text{in } \mathcal{H}, \\
(\hat{u}_{m1}, \hat{u}_{m2}) \to (\hat{u}_1, \hat{u}_2) \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N) \times L^p_{\rm loc}(\mathbb{R}^N), \\
(\hat{u}_{m1}, \hat{u}_{m2}) \to (\hat{u}_1, \hat{u}_2) \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered}$$

Then $(\hat{u}_1, \hat{u}_2)$ is a critical point of I. Furthermore,

$$\begin{aligned}
0 < \eta &\leq \varliminf_{m\to\infty} \Big[\int_{Q_m^*} (\mu_1 |u_{m1}|^p + \mu_2 |u_{m2}|^p) \Big]^{1/p} \\
&\leq C \varliminf_{m\to\infty} \Big[\int_{\hat{Q}_m^*} (|u_{m1}|^p + |u_{m2}|^p) \Big]^{1/p} \\
&= C \lim_{m\to\infty} \Big[\int_{\hat{Q}_0} (|\hat{u}_{m1}|^p + |\hat{u}_{m2}|^p) \Big]^{1/p} \\
&= C \Big[\int_{\hat{Q}_0} (|\hat{u}_1|^p + |\hat{u}_2|^p) \Big]^{1/p},
\end{aligned}$$

which implies $(\hat{u}_1, \hat{u}_2) \neq (0,0)$. Using the same arguments as in Case 1, we see that c is achieved by $(\hat{u}_1, \hat{u}_2)$.

In both cases we reach the conclusion. The proof is complete. □

4. Proof of Theorem 1.2

In this section, we describe the asymptotic behavior of the ground states of (1.4) when the L^∞-norm of the coupling coefficient tends to zero. To underline the dependence on the coupling coefficient, for (1.4) with $\gamma = \gamma_n$, the corresponding functional and the Nehari manifold will be denoted by I_{γ_n} and $\mathcal{N}_{\gamma_n}$ respectively. Then we see from Theorem 1.1 that, for n sufficiently large, the infimum

$$c_{\gamma_n} = \inf_{\vec{u}\in\mathcal{N}_{\gamma_n}} I_{\gamma_n}(\vec{u})$$

is achieved at (u_{n1}, u_{n2}). We also use the functional $I_0 : \mathcal{H} \to \mathbb{R}$ defined by

$$I_0(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} (\mu_1|u_1|^p + \mu_2|u_2|^p)$$

and its corresponding Nehari manifold

$$\mathcal{N}_0 = \{\vec{u} \in \mathcal{H} \setminus \{(0,0)\} : \langle I_0'(\vec{u}), \vec{u}\rangle = 0\}\,.$$

Define $c_0 = \inf_{\vec{u}\in\mathcal{N}_0} I_0(\vec{u})$, then we have

$$c_0 = \min_{j\in\{1,2\}} \Phi_j(w_j),$$

where

$$\Phi_j(u) = \frac{1}{2}\|u\|_j^2 - \frac{1}{p}\int_{\mathbb{R}^N} \mu_j|u|^p$$

and w_j is the positive ground state of the single elliptic equation

$$-\Delta u + V_j u = \mu_j|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Lemma 4.1. *$c_{\gamma_n} \to c_0$ as $n \to \infty$.*

Proof. Since $(w_1, 0)$ and $(0, w_2)$ are contained in $\mathcal{N}_{\gamma_n}$, we have

$$c_{\gamma_n} \le \min\{I_{\gamma_n}(w_1, 0), I_{\gamma_n}(0, w_2)\} = \min_{j\in\{1,2\}} \Phi_j(w_j) = c_0. \tag{4.1}$$

Then it is easy to see that $\{(u_{n1}, u_{n2})\}$ is bounded in $\mathcal{H}$.

We define a sequence $\{t_n\}$ of positive numbers by

$$t_n^{p-2} = \frac{\|(u_{n1}, u_{n2})\|^2}{\int_{\mathbb{R}^N} (\mu_1|u_{n1}|^p + \mu_2|u_{n2}|^p)}.$$

Then $(t_n u_{n1}, t_n u_{n2}) \in \mathcal{N}_0$. We claim that $t_n = 1 + o(1)$ as $n \to \infty$. Indeed, since (u_{n1}, u_{n2}) is a positive ground state of (1.4) with $\gamma = \gamma_n$, we have

$$\begin{aligned}
&\Big(1 - \big|\gamma_n(V_1V_2)^{-1/2}\big|_\infty\Big)\|(u_{n1}, u_{n2})\|^2 \\
&\le \|(u_{n1}, u_{n2})\|^2 - 2\int_{\mathbb{R}^N} \gamma_n u_{n1}u_{n2} \\
&= \int_{\mathbb{R}^N} (\mu_1|u_{n1}|^p + \mu_2|u_{n2}|^p) \le C\|(u_{n1}, u_{n2})\|^p.
\end{aligned}$$

Then, since $|\gamma_n|_\infty \to 0$, we see that $\|(u_{n1}, u_{n2})\|^2$ and $\int_{\mathbb{R}^N} (\mu_1|u_{n1}|^p + \mu_2|u_{n2}|^p)$ have a positive lower bound. From this, it can be seen that $t_n = 1 + o(1)$ as $n \to \infty$. Therefore,

$$c_0 \le I_0(t_n u_{n1}, t_n u_{n2}) = \big(\frac{1}{2} - \frac{1}{p}\big)t_n^p \int_{\mathbb{R}^N} (\mu_1|u_{n1}|^p + \mu_2|u_{n2}|^p)$$

$$
\begin{aligned}
&= \big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^N}(\mu_1|u_{n1}|^p+\mu_2|u_{n2}|^p)+o(1)\\
&= c_{\gamma_n}+o(1),
\end{aligned}
$$

which combined with (4.1) completes the proof. □

Proof of Theorem 1.2. We use an argument of contradiction and, up to a subsequence, suppose that

$$
\|u_{nj}\|_j \ge \alpha > 0 \tag{4.2}
$$

for $j=1,2$. Define two sequences $\{s_n\}$ and $\{t_n\}$ of positive numbers by

$$
s_n^{p-2}=\frac{\|u_{n1}\|_1^2}{\int_{\mathbb{R}^N}\mu_1|u_{n1}|^p},\quad t_n^{p-2}=\frac{\|u_{n2}\|_2^2}{\int_{\mathbb{R}^N}\mu_2|u_{n2}|^p}.
$$

Since (u_{n1},u_{n2}) is a positive ground state of (1.4) with $\gamma=\gamma_n$ and $|\gamma_n|_\infty\to 0$, we have

$$
\begin{aligned}
\|u_{n1}\|_1^2 &= \int_{\mathbb{R}^N}\mu_1|u_{n1}|^p+\int_{\mathbb{R}^N}\gamma_n u_{n1}u_{n2}=\int_{\mathbb{R}^N}\mu_1|u_{n1}|^p+o(1),\\
\|u_{n2}\|_2^2 &= \int_{\mathbb{R}^N}\mu_2|u_{n2}|^p+\int_{\mathbb{R}^N}\gamma_n u_{n1}u_{n2}=\int_{\mathbb{R}^N}\mu_2|u_{n2}|^p+o(1),
\end{aligned}
$$

which combined with (4.2) implies that $s_n=1+o(1)$ and $t_n=1+o(1)$ as $n\to\infty$. Then we have

$$
\begin{aligned}
2c_0 &\le \Phi_1(s_nu_{n1})+\Phi_2(t_nu_{n2})\\
&= \big(\frac{1}{2}-\frac{1}{p}\big)s_n^p\int_{\mathbb{R}^N}\mu_1|u_{n1}|^p+\big(\frac{1}{2}-\frac{1}{p}\big)t_n^p\int_{\mathbb{R}^N}\mu_2|u_{n2}|^p\\
&= \big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^N}(\mu_1|u_{n1}|^p+\mu_2|u_{n2}|^p)+o(1)\\
&= c_{\gamma_n}+o(1),
\end{aligned}
$$

which contradicts the result in Lemma 4.1. This contradiction implies that either $u_{n1}\to 0$ in $H^1(\mathbb{R}^N)$ or $u_{n2}\to 0$ in $H^1(\mathbb{R}^N)$. □

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