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DEFOCUSING FOURTH-ORDER COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study the initial value problem for some defocusing coupled nonlinear fourth-order Schrödinger equations. We show global well-posedness and scattering in the energy space.

1. Introduction

This manuscript is concerned with the initial value problem for some defocusing fourth-order Schrödinger system with power-type nonlinearities

$$\begin{gathered} i\dot{u}_j + \Delta^2 u_j + \Big(\sum_{k=1}^m a_{jk}|u_k|^p\Big)|u_j|^{p-2}u_j = 0; \\ u_j(0,\cdot) = \psi_j, \end{gathered} \tag{1.1}$$

where $u_j : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ for $j \in [1, m]$ and $a_{jk} = a_{kj}$ are positive real numbers.

Fourth-order Schrödinger equations have been introduced by Karpman [9] and Karpman-Shagalov [10] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity.

The m-component classical coupled nonlinear Schrödinger system with power-type nonlinearity

$$i\dot{u}_j + \Delta u_j = \pm\sum_{k=1}^m a_{jk}|u_k|^p|u_j|^{p-2}u_j, \tag{1.2}$$

arises in many physical problems. This models physical systems in which the field has more than one component. For example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. Readers are referred to various works [8, 27] for the derivation and applications of this system. For mathematical point of view, well-posedness issues of $(CNLS)_p$ were investigated by many authors. Indeed, global existence of solutions and scattering hold [19, 3, 26, 24, 25].

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System (1.1) is a mixture of the two previous problems. A solution $\mathbf{u} := (u_1, \dots, u_m)$ to (1.1) formally satisfies respectively conservation of the mass and the energy

$$M(u_j) := \int_{\mathbb{R}^N} |u_j(t,x)|^2\,dx = M(\psi_j);$$

$$\begin{aligned} E(\mathbf{u}(t)) &:= \frac{1}{2}\sum_{j=1}^m \int_{\mathbb{R}^N} |\Delta u_j|^2\,dx + \frac{1}{2p}\sum_{j,k=1}^m a_{jk}\int_{\mathbb{R}^N} |u_j(t,x)|^p |u_k(t,x)|^p\,dx \\ &= E(\mathbf{u}(0)). \end{aligned}$$

To use the conservation laws, it is natural to study (1.1) in H^2, called energy space. Problem (1.1) is a natural extension of the classical one component fourth-order Schrödinger equation which was first studied in [5], where various properties of the equation in the subcritical regime were described. Related references [2] gave sharp dispersive estimates for the biharmonic Schrödinger operator which lead to the Strichartz estimates. The model case given by a pure power nonlinearity is of particular interest. Indeed, the question of well-posedness in the energy space H^2 was widely investigated. We denote for $p > 1$ the fourth-order Schrödinger problem

$$i\dot{u} + \Delta^2 u \pm u|u|^{p-1} = 0, \quad u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}. \tag{1.3}$$

This equation satisfies a scaling invariance. Indeed, if u is a solution to (1.3) with data u_0, then $u_\lambda := \lambda^{\frac{4}{p-1}} u(\lambda^4\,\cdot, \lambda\,\cdot)$ is a solution to (1.3) with data $\lambda^{\frac{4}{p-1}} u_0(\lambda\,\cdot)$. For $s_c := \frac{N}{2} - \frac{4}{p-1}$, the space $\dot{H}^{s_c}$ whose norm is invariant under the dilatation $u \mapsto u_\lambda$ is relevant in this theory. When $s_c = 2$ which is the energy critical case, the critical power is $p_c := \frac{N+4}{N-4}$, $N \geq 5$. Pausader [20] established global well-posedness in the defocusing subcritical case, namely $1 < p < p_c$. Moreover, he established global well-posedness and scattering for radial data in the defocusing critical case, namely $p = p_c$. The same result without radial condition was obtained by Miao, Xu and Zhao [17], for $N \geq 9$. See also [15], for similar results in the more general case $s_c \geq 1$. The focusing case was treated by the last authors in [16]. They obtained results similar to one proved by Kenig and Merle [12, 11] in the classical Schrödinger case. See [23] in the case of exponential nonlinearity.

In this note, which seems to be one of the first papers studying a system of nonlinear coupled fourth-order Schrödinger equations, we combine in some meaning the two problems (1.3) and $(CNLS)_p$. Thus, we have to overcome two difficulties. The first one is the presence of a bilaplacian in Schrödinger operator and the second one is the coupled nonlinearities. It is the purpose of this manuscript to obtaining global well-posedness in the energy space and scattering of (1.1) via Morawetz estimate.

The rest of the paper is organized as follows. The next section contains the main results and some technical tools needed in the sequel. The third and fourth sections are devoted to prove well-posedness of (1.1). In section five, scattering is established. In appendix, we give a proof of Morawetz estimate and a blow-up criterion.

We close this section with some notations. Define the product space

$$H := H^2(\mathbb{R}^N) \times \cdots \times H^2(\mathbb{R}^N) = [H^2(\mathbb{R}^N)]^m,$$

where $H^2(\mathbb{R}^N)$ is the usual Sobolev space endowed with the complete norm

$$\|u\|_{H^2(\mathbb{R}^N)} := \Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big)^{1/2}.$$

Let us denote the real number

$$p^* := \begin{cases} \frac{N}{N-4} & \text{if } N > 4; \\ \infty & \text{if } 1 \le N \le 4. \end{cases}$$

We mention that C will denote a constant which may vary from line to line and if A and B are non negative real numbers, $A \lesssim B$ means that $A \le CB$. For $1 \le r \le \infty$ and $(s,T) \in [1,\infty) \times (0,\infty)$, we denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\|\cdot\|_r := \|\cdot\|_{L^r}$, $\|\cdot\| := \|\cdot\|_2$ and

$$\|u\|_{L^s_T(L^r)} := \Big(\int_0^T \|u(t)\|_r^s\,dt\Big)^{1/s}, \quad \|u\|_{L^s(L^r)} := \Big(\int_0^{+\infty} \|u(t)\|_r^s\,dt\Big)^{1/s}.$$

For simplicity, we denote the usual Sobolev space $W^{s,p} := W^{s,p}(\mathbb{R}^N)$ and $H^s := W^{s,2}$. If X is an abstract space $C_T(X) := C([0,T],X)$ stands for the set of continuous functions valued in X and X_{rd} is the set of radial elements in X, moreover for an eventual solution to (1.1), we denote $T^* > 0$ its lifespan.

2. Background and main results

In what follows, we give the main results and collect some estimates needed in the sequel.

2.1. **Main results.**

First, local well-posedness of the fourth-order Schrödinger problem (1.1) is claimed.

Theorem 2.1. *Let $1 \le N \le 8$, $2 \le p \le p^*$ and $\Psi \in H$. Then, there exist $T^* > 0$ and a unique maximal solution to* (1.1), $\mathbf{u} \in C([0,T^*),H)$. *Moreover,*

1. $\mathbf{u} \in \big(L_{\rm loc}^{\frac{8p}{N(p-1)}}([0,T^*],W^{2,2p})\big)^{(m)}$;
2. $\mathbf{u}$ *satisfies conservation of the energy and the mass;*
3. $T^* = \infty$ *in the subcritical case* $(2 \le p < p^*)$.

Remark 2.2. The artificial condition $p \ge 2$, which is due to some technical difficulty, requires the restriction $N \le 8$.

Second, system (1.1) scatters in the energy space. Indeed, every global solution of (1.1) is asymptotic, as $t \to \pm\infty$, to a solution of the associated linear fourth-order Schrödinger system.

Theorem 2.3. *Let $4 < N < 8$ and $2 \le p < p^*$. Take $\mathbf{u} \in C(\mathbb{R},H)$ be a global solution to* (1.1). *Then*

$$\mathbf{u} \in \big(L^{\frac{8p}{N(p-1)}}(\mathbb{R},W^{2,2p})\big)^{(m)}$$

and there exists $(\psi_1^\pm,\dots,\psi_m^\pm) \in H$ such that

$$\lim_{t\to\pm\infty} \|\mathbf{u}(t) - (e^{it\Delta^2}\psi_1^\pm,\dots,e^{it\Delta^2}\psi_m^\pm)\|_{H^2} = 0.$$

Remark 2.4. When proving scattering, the intermediate result Proposition 4.2 gives a decay of global solutions to (1.1), in some Lebesgue norms.

Finally, in the critical case, global existence and scattering for small data hold in the energy space.

Theorem 2.5. *Let $4 < N \leq 8$ and $p = p^*$. There exists $\epsilon_0 > 0$ such that if $\Psi := (\psi_1, \dots, \psi_m) \in H$ satisfies $\xi(\Psi) := \sum_{j=1}^m \int_{\mathbb{R}^N} |\Delta\psi_j|^2 \, dx \leq \epsilon_0$, system (1.1) possesses a unique global solution $\mathbf{u} \in C(\mathbb{R}, H)$, which scatters.*

In the next subsection, we give some standard estimates needed in the paper.

2.2. **Tools.** We start with some properties of the free fourth-order Schrödinger kernel.

Proposition 2.6. *Denoting the free operator associated to the fourth-order fractional Schrödinger equation*

$$e^{it\Delta^2} u_0 := \mathcal{F}^{-1}(e^{it|y|^4}) * u_0,$$

yield

(1) $e^{it\Delta^2} u_0$ *is the solution to the linear problem associated to* (1.3);
(2) $e^{it\Delta^2} u_0 \mp i \int_0^t e^{i(t-s)\Delta^2} u|u|^{p-1} \, ds$ *is the solution to the problem* (1.3);
(3) $(e^{it\Delta^2})^* = e^{-it\Delta^2}$;
(4) $e^{it\Delta^2}$ *is an isometry of* L^2.

Now, we give the so-called Strichartz estimate [20].

Definition 2.7. A pair (q, r) of positive real numbers is said to be admissible if

$$2 \leq q, r \leq \infty, \quad (q, r, N) \neq (2, \infty, 4) \quad \text{and} \quad \frac{4}{q} = N\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

Proposition 2.8. *Let two admissible pairs (q, r), (a, b). There exists a positive real number $C := C_{q,a}$ such that for any $T > 0$,*

$$\|u\|_{L^q_T(W^{2,r})} \leq C\Big(\|u_0\|_{H^2} + \|i\dot{u} + \Delta^2 u\|_{L^{a'}_T(W^{2,b'})}\Big); \tag{2.1}$$

$$\|\Delta u\|_{L^q_T(L^r)} \leq C\Big(\|\Delta u_0\|_{L^2} + \|i\dot{u} + \Delta^2 u\|_{L^2_T(\dot{W}^{1,\frac{2N}{N+2}})}\Big). \tag{2.2}$$

The following Morawetz estimate which is essential in proving scattering, is proved in the appendix, in the spirit of [17, 18].

Proposition 2.9. *Let $5 \leq N \leq 8$, $2 \leq p \leq p^*$ and $\mathbf{u} \in C(I, H)$ be the solution to (1.1). Then,*

(1) *if $N > 5$,*
$$\sum_{j=1}^m \int_I \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_j(t,x)|^2 |u_j(t,y)|^2}{|x-y|^5} dx\, dy\, dt \lesssim_u 1; \tag{2.3}$$
(2) *if $N = 5$,*
$$\sum_{j=1}^m \int_I \int_{\mathbb{R}^5} |u_j(t,x)|^4 dx dt \lesssim_u 1. \tag{2.4}$$

Let us gather some useful Sobolev embeddings [1].

Proposition 2.10. *The continuous injections hold*

(1) $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *whenever $1 < p < q < \infty$, $s > 0$ and $\frac{1}{p} \leq \frac{1}{q} + \frac{s}{N}$;*

(2) $W^{s,p_1}(\mathbb{R}^N) \hookrightarrow W^{s-N(\frac{1}{p_1}-\frac{1}{p_2}),p_2}(\mathbb{R}^N)$ *if* $1 \le p_1 \le p_2 < \infty$.

Now, we give some fractional Gagliardo-Nirenberg inequality [7].

Lemma 2.11. *Let* $1 < p, p_1, p_2 < \infty$, $s, s_1 \in \mathbb{R}$ *and* $\mu \in [0,1]$. *Then, the fractional inequality*

$$\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|_{L^{p_0}}^{1-\mu} \|u\|_{\dot{H}^{s_1,p_1}}^{\mu},$$

holds whenever

$$\frac{N}{p} - s = (1-\mu)\frac{N}{p_0} + \mu(\frac{N}{p_1} - s_1) \quad \text{and} \quad s \le \mu s_1.$$

We close this subsection with some absorption result.

Lemma 2.12. *Let* $T > 0$ *and* $X \in C([0,T], \mathbb{R}_+)$ *such that*

$$X \le a + bX^\theta \quad \text{on} \quad [0,T],$$

where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})\frac{1}{(\theta b)^{\frac{1}{\theta}}}$ *and* $X(0) \le \frac{1}{(\theta b)^{\frac{1}{\theta-1}}}$. *Then*

$$X \le \frac{\theta}{\theta - 1} a \quad \text{on } [0,T].$$

Proof. The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{\frac{1}{1-\theta}}]$ and increasing on $[(b\theta)^{\frac{1}{1-\theta}}, \infty)$. The assumptions imply that $f((b\theta)^{\frac{1}{1-\theta}}) < 0$ and $f(\frac{\theta}{\theta-1}a) \le 0$. As $f(X(t)) \ge 0$, $f(0) > 0$ and $X(0) \le (b\theta)^{\frac{1}{1-\theta}}$, we conclude the proof by a continuity argument. □

3. Local well-posedness

This section is devoted to prove Theorem 2.1. The proof contains two steps. First we prove existence of a unique local solution to (1.1), second we establish global existence in the subcritical case.

3.1. **Local existence and uniqueness.**
We use a standard fixed point argument.

(1) *Subcritical case* $2 \le p < p^*$. For $T, R > 0$, we denote the space

$$E_{T,R} := \Big\{ \mathbf{u} \in (C_T(H^2) \cap L_T^{\frac{8p}{N(p-1)}}(W^{2,2p}))^{(m)} : \\ \|\mathbf{u}\|_{(L_T^\infty(L^2) \cap L_T^{\frac{8p}{N(p-1)}}(L^{2p}))^{(m)}} + \|\Delta \mathbf{u}\|_{(L_T^\infty(L^2) \cap L_T^{\frac{8p}{N(p-1)}}(L^{2p}))^{(m)}} \le R \Big\}$$

endowed with the distance

$$d(\mathbf{u}, \mathbf{v}) := \sum_{j=1}^m \Big(\|u_j - v_j\|_{L_T^\infty(L^2)} + \|u_j - v_j\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})} \Big).$$

Define the function

$$\phi(\mathbf{u})(t) := T(t)\Psi - i \sum_{k=1}^m a_{jk} \int_0^t T(t-s)\Big(|u_k|^p |u_1|^{p-2} u_1, \dots, |u_k|^p |u_m|^{p-2} u_m \Big)\, ds,$$

where $T(t)\Psi := (e^{it\Delta^2}\psi_1, \dots, e^{it\Delta^2}\psi_m)$. We prove the existence of some small $T, R > 0$ such that ϕ is a contraction of $E_{T,R}$.

• First step $3 < N \le 8$. Take $\mathbf{u},\mathbf{v} \in E_{T,R}$, applying the Strichartz estimate (2.1), we obtain

$$d(\phi(\mathbf{u}),\phi(\mathbf{v})) \lesssim \sum_{j,k=1}^{m} \| |u_k|^p |u_j|^{p-2}u_j - |v_k|^p|v_j|^{p-2}v_j\|_{L^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})}.$$

To derive the contraction, consider the function

$$f_{j,k} : \mathbb{C}^m \to \mathbb{C},\ (u_1,\dots,u_m) \mapsto |u_k|^p|u_j|^{p-2}u_j.$$

With the mean value Theorem,

$$\begin{aligned} &|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})| \\ &\lesssim \max\{|u_k|^{p-1}|u_j|^{p-1} + |u_k|^p|u_j|^{p-2}, |v_k|^p|v_j|^{p-2} + |v_k|^{p-1}|v_j|^{p-1}\}|\mathbf{u}-\mathbf{v}|. \end{aligned} \tag{3.1}$$

Using Hölder's inequality, Sobolev embedding and denoting the quantity

$$(\mathcal{I}) := \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})},$$

we compute via a symmetry argument

$$\begin{aligned} (\mathcal{I}) &\lesssim \big\|\big(|u_k|^{p-1}|u_j|^{p-1} + |u_k|^p|u_j|^{p-2}\big)|\mathbf{u}-\mathbf{v}|\big\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim T^{\frac{8p-2N(p-1)}{8p}} \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})} \big\||u_k|^{p-1}|u_j|^{p-1} + |u_k|^p|u_j|^{p-2}\big\|_{L_T^\infty(L^{\frac{p}{p-1}})} \\ &\lesssim T^{\frac{4p-N(p-1)}{4p}} \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})} \Big(\|u_k\|_{L_T^\infty(L^{2p})}^{p-1}\|u_j\|_{L_T^\infty(L^{2p})}^{p-1} \\ &\quad + \|u_k\|_{L_T^\infty(L^{2p})}^{p}\|u_j\|_{L_T^\infty(L^{2p})}^{p-2}\Big) \\ &\lesssim T^{\frac{4p-N(p-1)}{4p}} \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})} \Big(\|u_k\|_{L_T^\infty(H^2)}^{p-1}\|u_j\|_{L_T^\infty(H^2)}^{p-1} \\ &\quad + \|u_k\|_{L_T^\infty(H^2)}^{p}\|u_j\|_{L_T^\infty(H^2)}^{p-2}\Big). \end{aligned}$$

Thus

$$\begin{aligned} &\|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim T^{\frac{4p-N(p-1)}{4p}} \|\mathbf{u}\|_{L_T^\infty(H)}^{2(p-1)} \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})}. \end{aligned} \tag{3.2}$$

Then

$$d(\phi(\mathbf{u}),\phi(\mathbf{v})) \lesssim T^{\frac{4p-N(p-1)}{4p}} R^{2(p-1)} d(\mathbf{u},\mathbf{v}).$$

Moreover, taking $\mathbf{v} = 0$ in the previous inequality, yields

$$\|\phi(\mathbf{u})\|_{(L_T^\infty(L^2)\cap L_T^{\frac{8p}{N(p-1)}}(L^{2p}))^{(m)}} \lesssim \|\Psi\| + T^{\frac{4p-N(p-1)}{4p}} R^{2p-1}.$$

It remains to estimate the quantity

$$\begin{aligned} (A) &:= \|\Delta(f_{j,k}(\mathbf{u}))\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim \|D(f_{j,k})(\mathbf{u})\Delta\mathbf{u}\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} + \||\nabla\mathbf{u}|^2 D^2(f_{j,k})(\mathbf{u})\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim (\mathcal{I}_1) + (\mathcal{I}_2). \end{aligned}$$

Via Hölder inequality and Sobolev embedding, we obtain

$$(\mathcal{I}_1) \lesssim \|\Delta\mathbf{u}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})} \big\||u_k|^{p-1}|u_j|^{p-1} + |u_k|^p|u_j|^{p-2}\big\|_{L_T^{\frac{8p}{8p-2N(p-1)}}(L^{\frac{p}{p-1}})}$$

$$
\begin{aligned}
&\lesssim T^{\frac{4p-N(p-1)}{4p}}\|\Delta\mathbf{u}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})}\Big(\|u_k\|^{p-1}_{L_T^\infty(L^{2p})}\|u_j\|^{p-1}_{L_T^\infty(L^{2p})}\\
&\quad+\|u_k\|^{p}_{L_T^\infty(L^{2p})}\|u_j\|^{p-2}_{L_T^\infty(L^{2p})}\Big)\\
&\lesssim T^{\frac{4p-N(p-1)}{4p}}\|\Delta\mathbf{u}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})}\Big(\|u_k\|^{p-1}_{L_T^\infty(H^2)}\|u_j\|^{p-1}_{L_T^\infty(H^2)}\\
&\quad+\|u_k\|^{p}_{L_T^\infty(H^2)}\|u_j\|^{p-2}_{L_T^\infty(H^2)}\Big)\\
&\lesssim T^{\frac{4p-N(p-1)}{4p}}\|\Delta\mathbf{u}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})}\|\mathbf{u}\|^{2(p-1)}_{L_T^\infty(H)}.
\end{aligned}
$$

Using the interpolation inequality $\|\nabla\cdot\|_{2p}^2\lesssim\|\cdot\|_{2p}\|\Delta\cdot\|_{2p}$, we obtain

$$
\begin{aligned}
\||\nabla\mathbf{u}|^2(f_{j,k})_{ii}(\mathbf{u})\|_{L_x^{\frac{2p}{2p-1}}}&\lesssim\||\nabla\mathbf{u}|^2(|u_k|^{p-2}|u_j|^{p-1}+|u_k|^p|u_j|^{p-3})\|_{L_x^{\frac{2p}{2p-1}}}\\
&\lesssim\|\nabla\mathbf{u}\|^2_{L_x^{2p}}\Big(\|u_k\|^{p-2}_{L_x^{2p}}\|u_j\|^{p-1}_{L_x^{2p}}+\|u_k\|^{p}_{L_x^{2p}}\|u_j\|^{p-3}_{L_x^{2p}}\Big)\\
&\lesssim\|\Delta\mathbf{u}\|_{L_x^{2p}}\|\mathbf{u}\|_H^{2p-2}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(\mathcal{I}_2)&\lesssim\big\|\|\Delta\mathbf{u}\|_{L^{2p}}\|\mathbf{u}\|_H^{2p-2}\big\|_{L_T^{\frac{8p}{p(8-N)+N}}}\\
&\lesssim T^{\frac{4p-N(p-1)}{4p}}\|\Delta\mathbf{u}\|_{L_T^{\frac{8p}{N(p-1)}}(L^{2p})}\|\mathbf{u}\|^{2p-2}_{L_T^\infty(H)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\phi(\mathbf{u})\|_{(L_T^\infty(L^2)\cap L_T^{\frac{8p}{N(p-1)}}(L^{2p}))^{(m)}}+\|\Delta(\phi(\mathbf{u}))\|_{(L_T^{\frac{8p}{N(p-1)}}(L^{2p})\cap L_T^\infty(L^2))^{(m)}}\\
&\le C\Big(\|\Psi\|_H+T^{\frac{4p-N(p-1)}{4p}}R^{2p-1}\Big).
\end{aligned}
$$

Choosing $R>C\|\Psi\|_H$ and $T>0$ sufficiently small via the fact that $2\le p<p^*$, ϕ is a contraction of $E_{T,R}$.

• Second step $1\le N\le 3$. In this case, we use the Sobolev embedding $H^2\hookrightarrow L^\infty$. Applying Strichartz estimate to $\mathbf{u},\mathbf{v}\in E_{T,R}$ yields

$$
\begin{aligned}
d(\phi(\mathbf{u}),\phi(\mathbf{v}))&\lesssim\sum_{j,k=1}^m\||u_k|^p|u_j|^{p-2}u_j-|v_k|^p|v_j|^{p-2}v_j\|_{L_T^1(L^2)}\\
&\lesssim\sum_{j,k=1}^m\||u_k|^{p-1}|u_j|^{p-1}+|u_k|^p|u_j|^{p-2}\|_{L_T^\infty(L^\infty)}\|\mathbf{u}-\mathbf{v}\|_{L_T^1(L^2)}\qquad(3.3)\\
&\lesssim T\|\mathbf{u}\|^{2(p-1)}_{L_T^\infty(H^2)}\|\mathbf{u}-\mathbf{v}\|_{L_T^\infty(L^2)}\\
&\lesssim TR^{2(p-1)}d(\mathbf{u},\mathbf{v}).
\end{aligned}
$$

It remains to estimate

$$
\begin{aligned}
(B)&:=\|\Delta(f_{j,k}(\mathbf{u}))\|_{L_T^1(L^2)}\\
&\lesssim\|D(f_{j,k})(\mathbf{u})\Delta\mathbf{u}\|_{L_T^1(L^2)}+\||\nabla\mathbf{u}|^2D^2(f_{j,k})(\mathbf{u})\|_{L_T^1(L^2)}.
\end{aligned}
$$

Thanks Hölder and Sobolev inequalities, we obtain

$$
\|D(f_{j,k})(\mathbf{u})\Delta\mathbf{u}\|_{L_T^1(L^2)}\lesssim\|\Delta\mathbf{u}\|_{L_T^1(L^2)}\||u_k|^{p-1}|u_j|^{p-1}+|u_k|^p|u_j|^{p-2}\|_{L_T^\infty(L^\infty)}
$$

$$
\begin{aligned}
&\lesssim T\|\Delta \mathbf{u}\|_{L_T^\infty(L^2)}\|\mathbf{u}\|^{2(p-1)}_{L_T^\infty(H^2)}\\
&\lesssim TR^{2p-1}.
\end{aligned}
$$

Using the Sobolev injection $H^2 \hookrightarrow W^{1,4}$, we obtain

$$
\begin{aligned}
\||\nabla \mathbf{u}|^2(f_{j,k})_{ii}(\mathbf{u})\|_{L^1_T(L^2)} &\lesssim \||\nabla \mathbf{u}|^2(|u_k|^{p-2}|u_j|^{p-1}+|u_k|^p|u_j|^{p-3})\|_{L^1_T(L^2)}\\
&\lesssim T\|\nabla \mathbf{u}\|^2_{L_T^\infty(L^4)}\|\mathbf{u}\|^{2p-3}_{L_T^\infty(H^2)}\\
&\lesssim TR^{2p-1}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|\phi(\mathbf{u})\|_{(L_T^{\frac{8p}{N(p-1)}}(L^{2p})\cap L_T^\infty(L^2))^{(m)}} + \|\Delta(\phi(\mathbf{u}))\|_{(L_T^{\frac{8p}{N(p-1)}}(L^{2p})\cap L_T^\infty(L^2))^{(m)}}\\
&\le C\Big(\|\Psi\|_H + TR^{2p-1}\Big).
\end{aligned}
$$

Choosing $R > C\|\Psi\|_H$ and $T > 0$ sufficiently small, ϕ is a contraction of $E_{T,R}$.

Finally, thanks to a classical fixed point Theorem, We deduce the existence of a fixed point $\mathbf{u} \in B_T(R)$, which is a solution to (1.1). Moreover, uniqueness follows thanks to (3.2) and (3.3).

(2) *Critical case* $4 < N \le 8$ *and* $p = p^*$. The proof follows by arguing as in the subcritical case, where we take rather than $E_{T,R}$, the complete space

$$
F_{T,\rho} := \Big\{\mathbf{u} \in (L_T^{\frac{8p}{N(p-1)}}(W^{2,2p}))^{(m)} : \|\mathbf{u}\|_{(L_T^{\frac{8p}{N(p-1)}}(W^{2,2p}))^{(m)}} \le \rho\Big\}
$$

endowed with the distance

$$
d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|_{(L_T^{\frac{8p}{N(p-1)}}(L^{2p}))^{(m)}},
$$

via the fact that $\lim_{T\to 0}\|T(t)\Psi\|_{(L_T^{\frac{8p}{N(p-1)}}(L^{2p}))^{(m)}} = 0$ and the next result.

Lemma 3.1. *Let* $\Psi \in H$ *and suppose that* $\mathbf{u} \in (L_T^{\frac{8p}{N(p-1)}}(W^{2,2p}))^{(m)}$ *is a solution to* (1.1)*. Then, there exists* $0 < T' \le T$ *such that* $\mathbf{u} \in C_{T'}(H)$*.*

Proof. Using the previous computation via Duhamel formula (second point in Proposition 2.6), yields

$$
\|\mathbf{u}\|_{L_T^\infty(H)} \lesssim \|\Psi\|_H + \|\mathbf{u}\|^{2(p-1)}_{L_T^\infty(H)}\|\mathbf{u}\|_{(L_T^{\frac{8p}{N(p-1)}}(W^{2,2p}))^{(m)}}.
$$

The proof is complete thanks to Lemma 2.12. □

3.2. **Existence of global solutions.** We prove that the maximal solution of (1.1) is global in the subcritical case. The global existence is a consequence of energy conservation and previous calculations. Let $\mathbf{u} \in C([0,T^*),H)$ be the unique maximal solution of (1.1). We prove that $\mathbf{u}$ is global. By contradiction, suppose that $T^* < \infty$. For $0 < s < T^*$ consider the problem

$$
\begin{gathered}
i\dot{v}_j + \Delta^2 v_j = \Big(\sum_{k=1}^m |v_k|^p\Big)|v_j|^{p-2}v_j;\\
v_j(s,\cdot) = u_j(s,\cdot).
\end{gathered}
\tag{3.4}
$$

By the same arguments used in the local existence, we can find a real number $\tau > 0$ and a solution $\mathbf{v} = (v_1,\dots,v_m)$ to (3.4) on $C\big([s,s+\tau],H\big)$. Using the conservation of energy we see that τ does not depend on s. Thus, if we let s be close to T^* such that $T^* < s+\tau$, this fact contradicts the maximality of T^*.

4. Scattering

This section we establish Theorem 2.3 about the scattering of (1.1). For any time slab I, take the Strichartz space

$$S(I) := C(I, H^2) \cap L^{\frac{8p}{N(p-1)}}(I, W^{2,2p})$$

endowed the norm

$$\|u\|_{S(I)} := \|u\|_{L^\infty(I,H^2)} + \|u\|_{L^{\frac{8p}{N(p-1)}}(I,W^{2,2p})}.$$

The first intermediate result is as follows.

Lemma 4.1. *For any time slab I, we have*

$$\|\mathbf{u}(t) - e^{it\Delta^2}\Psi\|_{(S(I))^{(m)}} \lesssim \|\mathbf{u}\|^{\frac{2pN(p-1)-8p}{N(p-1)}}_{\left(L^\infty(I,L^{2p})\right)^{(m)}} \|\mathbf{u}\|^{\frac{8p-N(p-1)}{N(p-1)}}_{\left(L^{\frac{8p}{N(p-1)}}(I,W^{2,2p})\right)^{(m)}},$$

where $e^{it\Delta^2}(\Psi_1,\dots,\Psi_m) := (e^{it\Delta^2}\Psi_1,\dots,e^{it\Delta^2}\Psi_m)$.

Proof. Using Strichartz estimate, we have

$$\|\mathbf{u}(t) - e^{it\Delta^2}\Psi\|_{(S(I))^{(m)}} \lesssim \sum_{j,k=1}^{m} \|f_{j,k}(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,W^{2,\frac{2p}{2p-1}})}.$$

Thanks to Hölder inequality, we obtain

$$\|f_{j,k}(\mathbf{u})\|_{L_x^{\frac{2p}{2p-1}}} \lesssim \big\||u_k|^p|u_j|^{p-1}\big\|_{L_x^{\frac{2p}{2p-1}}} \lesssim \|u_k\|^p_{L_x^{2p}}\|u_j\|^{p-1}_{L_x^{2p}}.$$

Letting $\theta := \frac{8p-N(p-1)}{2N(p-1)}$, we obtain the inequality

$$\frac{1}{2} \le \theta \le p - \frac{1}{2}.$$

The left part of the inequality follows from $p \le p^*$. Denoting $X := p-1$, the right part of the claim is equivalent to

$$T(X) := NX^2 + (N-4)X - 4 \ge 0.$$

T has two roots $X_1 = -1 < 0 < X_2 = \frac{4}{N}$, since $2 \le p \le p^*$, it follows that $X = p-1 \ge X_2$. This proves of the inequality.

Now, using an interpolation argument, write

$$\begin{aligned}
&\|f_{j,k}(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})} \\
&\lesssim \Big\|\|u_k\|^p_{L^{2p}}\|u_j\|^{p-1}_{L^{2p}}\Big\|_{L^{\frac{8p}{p(8-N)+N}}(I)} \\
&\lesssim \|u_k\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_j\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\Big\|\|u_k\|^{\theta+\frac12}_{L^{2p}}\|u_j\|^{\theta-\frac12}_{L^{2p}}\Big\|_{L^{\frac{8p}{p(8-N)+N}}(I)} \\
&\lesssim \|u_k\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_j\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_k\|^{\theta+\frac12}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\|u_j\|^{\theta-\frac12}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}.
\end{aligned}$$

Then

$$\begin{aligned}
&\|f_{j,k}(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})} \\
&\lesssim \|u_k\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_j\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_k\|^{\theta+\frac12}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\|u_j\|^{\theta-\frac12}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})} \\
&\lesssim \|u_k\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_k\|^{\theta+\frac12}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\|u_j\|^{p-\frac12-\theta}_{L^\infty(I,L^{2p})}\|u_j\|^{\theta-\frac12}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})} \\
&\lesssim \|\mathbf{u}\|^{\frac{2pN(p-1)-8p}{N(p-1)}}_{(L^\infty(I,L^{2p}))^{(m)}}\|\mathbf{u}\|^{\frac{8p-N(p-1)}{N(p-1)}}_{(L^{\frac{8p}{N(p-1)}}(I,L^{2p}))^{(m)}}.
\end{aligned} \tag{4.1}$$

It remains to estimate the quantity $(\mathcal{I}) := \|\Delta(f_{j,k}(\mathbf{u}))\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})}$. We write

$$\begin{aligned}
(\mathcal{I}) &\lesssim \sum_{i=1}^m \|\Delta\mathbf{u}(f_{j,k})_i(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})} + \||\nabla\mathbf{u}|^2(f_{j,k})_{ii}(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})} \\
&\lesssim (\mathcal{I}_1)+(\mathcal{I}_2).
\end{aligned}$$

Using Hölder inequality, we obtain

$$\begin{aligned}
\|\Delta\mathbf{u}(f_{j,k})_i(\mathbf{u})\|_{L_x^{\frac{2p}{2p-1}}} &\lesssim \big\|\Delta\mathbf{u}\big(|u_k|^{p-1}|u_j|^{p-1}+|u_k|^p|u_j|^{p-2}\big)\big\|_{L_x^{\frac{2p}{2p-1}}} \\
&\lesssim \|\Delta\mathbf{u}\|_{(L_x^{2p})^m}\Big(\|u_k\|^{p-1}_{L_x^{2p}}\|u_j\|^{p-1}_{L_x^{2p}}+\|u_k\|^p_{L_x^{2p}}\|u_j\|^{p-2}_{L_x^{2p}}\Big).
\end{aligned}$$

Letting $\mu := \frac{4p-N(p-1)}{N(p-1)}$, we obtain the inequality

$$\mu \le \frac{p}{2} \le p-1.$$

Note that

$$\begin{aligned}
p \ge 2\mu &\Leftrightarrow p \ge \frac{8p-2N(p-1)}{N(p-1)} \\
&\Leftrightarrow Np(p-1) \ge 8p-2N(p-1) \\
&\Leftrightarrow T(X:=p-1) := NX^2+(3N-8)X-8 \ge 0.
\end{aligned}$$

T has two roots $p_1 < 0 < p_2$. Since $T(1) = 4(N-4) \ge 0$ and $X = p-1 \ge 1$. The inequality is proved.

Now, using Hölder inequality

$$\begin{aligned}
(\mathcal{I}_1) &\lesssim \Big\|\|\Delta\mathbf{u}\|_{(L^{2p})^m}\Big(\|u_k\|^{p-1}_{L^{2p}}\|u_j\|^{p-1}_{L^{2p}}+\|u_k\|^p_{L^{2p}}\|u_j\|^{p-2}_{L^{2p}}\Big)\Big\|_{L^{\frac{8p}{p(8-N)+N}}} \\
&\lesssim \|\Delta\mathbf{u}\|_{(L^{\frac{8p}{N(p-1)}}(I,L^{2p}))^{(m)}} \\
&\quad\times\Big(\|u_k\|^{p-1-\mu}_{L^\infty(I,L^{2p})}\|u_j\|^{p-1-\mu}_{L^\infty(I,L^{2p})}\big\|\|u_k\|^\mu_{L^{2p}}\|u_j\|^\mu_{L^{2p}}\big\|_{L^{\frac{8p}{8p-2N(p-1)}}} \\
&\quad+\|u_k\|^{p-2\mu}_{L^\infty(I,L^{2p})}\|u_j\|^{p-2}_{L^\infty(I,L^{2p})}\big\|\|u_k\|^{2\mu}_{L^{2p}}\big\|_{L^{\frac{8p}{8p-2N(p-1)}}}\Big) \\
&\lesssim \|\Delta\mathbf{u}\|_{(L^{\frac{8p}{N(p-1)}}(I,L^{2p}))^{(m)}} \\
&\quad\times\Big(\|u_k\|^{p-1-\mu}_{L^\infty(I,L^{2p})}\|u_j\|^{p-1-\mu}_{L^\infty(I,L^{2p})}\|u_k\|^\mu_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\|u_j\|^\mu_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}
\end{aligned}$$

$$+ \|u_k\|^{p-2\mu}_{L^\infty(I,L^{2p})}\|u_j\|^{p-2}_{L^\infty(I,L^{2p})}\|u_k\|^{2\mu}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\Big).$$

Then, $\mathcal{A} := \sum_{i,j,k=1}^m \|\Delta \mathbf{u}(f_{j,k})_i(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})}$ satisfies

$$\begin{aligned}\mathcal{A} &\lesssim \|\Delta\mathbf{u}\|_{\left(L^{\frac{8p}{N(p-1)}}(I,L^{2p})\right)^{(m)}}\Big(\sum_{k=1}^m \|u_k\|^{p-1-\mu}_{L^\infty(I,L^{2p})}\|u_k\|^{\mu}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\\ &\quad\times \sum_{j=1}^m \|u_j\|^{p-1-\mu}_{L^\infty(I,L^{2p})}\|u_j\|^{\mu}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\\ &\quad+ \sum_{k=1}^m \|u_k\|^{p-2\mu}_{L^\infty(I,L^{2p})}\|u_k\|^{2\mu}_{L^{\frac{8p}{N(p-1)}}(I,L^{2p})}\sum_{j=1}^m \|u_j\|^{p-2}_{L^\infty(I,L^{2p})}\Big).\end{aligned}$$

So,

$$\mathcal{A} \lesssim \|\mathbf{u}\|_{\left(L^{\frac{8p}{N(p-1)}}(I,W^{2,2p})\right)^{(m)}}\|\mathbf{u}\|^{\frac{2pN(p-1)-8p}{N(p-1)}}_{\left(L^\infty(I,L^{2p})\right)^{(m)}}\|\mathbf{u}\|^{\frac{8p-2N(p-1)}{N(p-1)}}_{\left(L^{\frac{8p}{N(p-1)}}(I,L^{2p})\right)^{(m)}}. \tag{4.2}$$

Similarly

$$\begin{aligned}\mathcal{B} &:= \sum_{j,k=1}^m \||\nabla\mathbf{u}|^2(f_{j,k})_{ii}(\mathbf{u})\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})}\\ &\lesssim \sum_{j,k=1}^m \Big\||\nabla\mathbf{u}|^2\Big(|u_k|^{p-2}|u_j|^{p-1}+|u_k|^p|u_j|^{p-3}\Big)\Big\|_{L^{\frac{8p}{p(8-N)+N}}(I,L^{\frac{2p}{2p-1}})}.\end{aligned}$$

Using the interpolation inequality $\|\nabla\cdot\|^2_{2p} \lesssim \|\cdot\|_{2p}\|\Delta\cdot\|_{2p}$, we obtain

$$\begin{aligned}&\||\nabla\mathbf{u}|^2(|u_k|^{p-2}|u_j|^{p-1}+|u_k|^p|u_j|^{p-3})\|_{L_x^{\frac{2p}{2p-1}}}\\ &\lesssim \|\nabla\mathbf{u}\|^2_{L_x^{2p}}\Big(\|u_k\|^{p-2}_{L_x^{2p}}\|u_j\|^{p-1}_{L_x^{2p}}+\|u_k\|^p_{L_x^{2p}}\|u_j\|^{p-3}_{L_x^{2p}}\Big)\\ &\lesssim \|\Delta\mathbf{u}\|_{L_x^{2p}}\|\mathbf{u}\|_{L_x^{2p}}\|\mathbf{u}\|^{2p-3}_{L_x^{2p}}\\ &\lesssim \|\Delta\mathbf{u}\|_{L_x^{2p}}\|\mathbf{u}\|^{2p-2}_{H}.\end{aligned}$$

Thus, arguing as previously, we obtain

$$\mathcal{B} \lesssim \|\mathbf{u}\|_{\left(L^{\frac{8p}{N(p-1)}}(I,W^{2,2p})\right)^{(m)}}\|\mathbf{u}\|^{\frac{2pN(p-1)-8p}{N(p-1)}}_{\left(L^\infty(I,L^{2p})\right)^{(m)}}\|\mathbf{u}\|^{\frac{8p-2N(p-1)}{N(p-1)}}_{\left(L^{\frac{8p}{N(p-1)}}(I,L^{2p})\right)^{(m)}}. \tag{4.3}$$

Finally, thanks to (4.1)–(4.3), it follows that

$$\|\mathbf{u}(t)-e^{it\Delta^2}\Psi\|_{(S(I))^{(m)}} \lesssim \|\mathbf{u}\|^{\frac{2pN(p-1)-8p}{N(p-1)}}_{\left(L^\infty(I,L^{2p})\right)^{(m)}}\|\mathbf{u}\|^{\frac{8p-N(p-1)}{N(p-1)}}_{\left(L^{\frac{8p}{N(p-1)}}(I,W^{2,2p})\right)^{(m)}}.$$

□

The next auxiliary result is about the decay of solutions.

Proposition 4.2. *For any $2 < r < \frac{2N}{N-4}$, we have*

$$\lim_{t\to\infty}\|\mathbf{u}(t)\|_{(L^r)^{(m)}} = 0.$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function and $\varphi_n := (\varphi_1^n, \dots, \varphi_m^n)$ be a sequence in H satisfying $\sup_n \|\varphi_n\|_H < \infty$ and

$$\varphi_n \rightharpoonup \varphi := (\varphi_1, \dots, \varphi_m) \in H.$$

Let $\mathbf{u}_n := (u_1^n, \dots, u_m^n)$ (respectively $\mathbf{u} := (u_1, \dots, u_m)$) be the solution in $C(\mathbb{R}, H)$ to (1.1) with initial data φ_n respectively φ). In what follows, we prove a claim.

Claim. For every $\epsilon > 0$, there exist $T_\epsilon > 0$ and $n_\epsilon \in \mathbb{N}$ such that

$$\|\chi(\mathbf{u}_n - \mathbf{u})\|_{(L^\infty_{T_\epsilon}(L^2))^{(m)}} < \epsilon \quad \text{for all } n > n_\epsilon. \tag{4.4}$$

Indeed, denoting the functions $\mathbf{v}_n := \chi \mathbf{u}_n$ and $\mathbf{v} = (v_1, \dots, v_m) := \chi \mathbf{u}$, we compute $v_j^n(0) = \chi \varphi_j^n$ and

$$\begin{aligned} i\dot{v}_j^n + \Delta^2 v_j^n &= \Delta^2 \chi u_j^n + 2\nabla\Delta\chi\nabla u_j^n + \Delta\chi\Delta u_j^n + 2\nabla\chi\nabla\Delta u_j^n \\ &\quad + 2\big(\nabla\Delta\chi\nabla u_j^n + \nabla\chi\nabla\Delta u_j^n + 2\sum_{i=1}^N \nabla\partial_i\chi\nabla\partial_i u_j^n\big) \\ &\quad + \chi\big(\sum_{k=1}^m |u_k^n|^p |u_j^n|^{p-2} u_j^n\big). \end{aligned}$$

Similarly, $v_j(0) = \chi\phi_j$ and

$$\begin{aligned} i\dot{v}_j + \Delta^2 v_j &= \Delta^2 \chi u_j + 2\nabla\Delta\chi\nabla u_j + \Delta\chi\Delta u_j + 2\nabla\chi\nabla\Delta u_j \\ &\quad + 2\big(\nabla\Delta\chi\nabla u_j + \nabla\chi\nabla\Delta u_j + 2\sum_{i=1}^N \nabla\partial_i\chi\nabla\partial_i u_j\big) \\ &\quad + \chi\big(\sum_{k=1}^m |u_k|^p |u_j|^{p-2} u_j\big). \end{aligned}$$

Denoting $\mathbf{w}_n = (w_1^n, \dots, w_m^n) := \mathbf{v}_n - \mathbf{v}$ and $\mathbf{z}_n = (z_1^n, \dots, z_m^n) := \mathbf{u}_n - \mathbf{u}$, we have

$$\begin{aligned} i\dot{w}_j^n + \Delta^2 w_j^n &= \Delta^2\chi z_j^n + 4\nabla\Delta\chi\nabla z_j^n + \Delta\chi\Delta z_j^n + 4\nabla\chi\nabla\Delta z_j^n \\ &\quad + 4\sum_{i=1}^N \nabla\partial_i\chi\nabla\partial_i z_j^n + \chi\big(\sum_{k=1}^m |u_k^n|^p|u_j^n|^{p-2}u_j^n - \sum_{k=1}^m |u_k|^p|u_j|^{p-2}u_j\big). \end{aligned}$$

Thanks to Strichartz estimate, we obtain

$$\begin{aligned} &\|\mathbf{w}_n\|_{\big(L^\infty_T(L^2)\cap L_T^{\frac{8p}{N(p-1)}}(L^{2p})\big)^{(m)}} \\ &\lesssim \|\chi(\varphi_n - \varphi)\|_{(L^2)^{(m)}} + \|\Delta^2\chi\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} + \|\nabla\Delta\chi\nabla\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} \\ &\quad + \|\nabla\chi\nabla\Delta\mathbf{z}_n\|_{(L^1(L^2))^{(m)}} + \|\nabla\partial_i\chi\nabla\partial_i\mathbf{z}_n\|_{(L^1(L^2))^{(m)}} \\ &\quad + \sum_{j,k=1}^m \big\|\chi\big(|u_k^n|^p|u_j^n|^{p-2}u_j^n - |u_k|^p|u_j|^{p-2}u_j\big)\big\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})}. \end{aligned}$$

Thanks to the Rellich Theorem, up to subsequence extraction, we have

$$\epsilon := \|\chi(\varphi_n - \varphi)\|_{L^2_x} \to 0 \quad \text{as } n \to \infty.$$

Moreover, by the conservation laws via properties of χ,

$$\begin{aligned} \mathcal{I}_1 &:= \|\Delta^2\chi\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} + \|\nabla\Delta\chi\nabla\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} + \|\nabla\chi\nabla\Delta\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} \\ &\quad + \|\nabla\partial_i\chi\nabla\partial_i\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} \end{aligned}$$

$$\lesssim \|\mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} + \|\nabla \mathbf{z}_n\|_{(L^1_T(L^2))^{(m)}} \lesssim CT,$$

where

$$C := \|\mathbf{u}\|_{(L^\infty(\mathbb{R},H^2))^{(m)}} + \|\mathbf{u}_n\|_{(L^\infty(\mathbb{R},H^2))^{(m)}}.$$

Arguing as previously, we have

$$\begin{aligned}
\mathcal{I}_2 &:= \|\chi(|u_k^n|^p|u_j^n|^{p-2}u_j^n - |u_k|^p|u_j|^{p-2}u_j)\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} \\
&\lesssim \|\chi(|u_k^n|^{p-1}|u_j^n|^{p-1} - |u_k|^p|u_j|^{p-2})|\mathbf{u}_n - \mathbf{u}|\|_{L_T^{\frac{8p}{p(8-N)+N}}(L^{\frac{2p}{2p-1}})} \\
&\lesssim \|\chi(\mathbf{u}_n - \mathbf{u})\|_{L_T^{\frac{8p}{p(8-N)+N}}((L^{2p})^{(m)})}\Big(\|u_k^n\|^{p-1}_{L_T^\infty(L^{2p})}\|u_j^n\|^{p-1}_{L_T^\infty(L^{2p})} \\
&\quad + \|u_k\|^p_{L_T^\infty(L^{2p})}\|u_j\|^{p-2}_{L_T^\infty(L^{2p})}\Big) \\
&\lesssim T^{\frac{8p-2N(p-1)}{8p}}\|\mathbf{w}_n\|_{L_T^{\frac{8p}{N(p-1)}}((L^{2p})^{(m)})}\Big(\|u_k^n\|^{2(p-1)}_{L_T^\infty(H^2)}\|u_j^n\|^{2(p-1)}_{L_T^\infty(H^2)} \\
&\quad + \|u_k\|^{2p}_{L_T^\infty(H^2)}\|u_j\|^{2(p-2)}_{L_T^\infty(H^2)}\Big) \\
&\lesssim T^{\frac{4p-N(p-1)}{4p}}\|\mathbf{w}_n\|_{L_T^{\frac{8p}{N(p-1)}}((L^{2p})^{(m)})}.
\end{aligned}$$

As a consequence

$$\begin{aligned}
\|\mathbf{w}_n\|_{\big(L_T^\infty(L^2)\cap L_T^{\frac{8p}{N(p-1)}}(L^{2p})\big)^{(m)}} &\lesssim \epsilon + CT + T^{\frac{4p-N(p-1)}{4p}}\|\mathbf{w}_n\|_{L^{\frac{8p}{N(p-1)}}((L^{2p})^{(m)})} \\
&\lesssim \frac{\epsilon + T}{1 - T^{\frac{4p-N(p-1)}{4p}}}.
\end{aligned}$$

The claim is proved.

By an interpolation argument it is sufficient to prove the decay for $r := 2 + \frac{4}{N}$. We recall the following Gagliardo-Nirenberg inequality

$$\|u_j(t)\|^{2+\frac{4}{N}}_{2+\frac{4}{N}} \le C\|u_j(t)\|^2_{H^2}\Big(\sup_x \|u_j(t)\|_{L^2(Q_1(x))}\Big)^{4/N}, \tag{4.5}$$

where $Q_a(x)$ denotes the square centered at x whose edge has length a. We proceed by contradiction. Assume that there exist a sequence (t_n) of positive real numbers and $\epsilon > 0$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\|u_j(t_n)\|_{L^{2+\frac{4}{N}}} > \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

By (4.5) and (4.6), there exist a sequence (x_n) in $\mathbb{R}^N$ and a positive real number denoted also by $\epsilon > 0$ such that

$$\|u_j(t_n)\|_{L^2(Q_1(x_n))} \ge \epsilon, \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

Let $\phi_j^n(x) := u_j(t_n, x + x_n)$. Using the conservation laws, we obtain

$$\sup_n \|\phi_j^n\|_{H^2} < \infty.$$

Then, up to a subsequence extraction, there exists $\phi_j \in H^2$ such that ϕ_j^n convergence weakly to ϕ_j in H^2. By Rellich Theorem, we have

$$\lim_{n\to\infty} \|\phi_j^n - \phi_j\|_{L^2(Q_1(0))} = 0.$$

Moreover, thanks to (4.7) we have, $\|\phi_j^n\|_{L^2(Q_1(0))} \geq \epsilon$. So, we obtain

$$\|\phi_j\|_{L^2(Q_1(0))} \geq \epsilon.$$

We denote by $\bar{u}_j \in C(\mathbb{R}, H^2)$ the solution of (1.1) with data ϕ_j and $u_j^n \in C(\mathbb{R}, H^2)$ the solution of (1.1) with data ϕ_j^n. Take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^N)$ which satisfies $0 \leq \chi \leq 1$, $\chi = 1$ on $Q_1(0)$ and $\operatorname{supp}(\chi) \subset Q_2(0)$. Using a continuity argument, there exists $T > 0$ such that

$$\inf_{t\in[0,T]} \|\chi \bar{u}_j(t)\|_{L^2(\mathbb{R}^N)} \geq \frac{\epsilon}{2}.$$

Now, taking account (4.4), there is a positive time denoted also T and $n_\epsilon \in \mathbb{N}$ such that

$$\|\chi(u_j^n - \bar{u}_j)\|_{L_T^\infty(L^2)} \leq \frac{\epsilon}{4} \quad \text{for all } n \geq n_\epsilon.$$

Hence, for all $t \in [0, T]$ and $n \geq n_\epsilon$,

$$\|\chi u_j^n(t)\|_{L^2} \geq \|\chi \bar{u}_j(t)\|_{L^2} - \|\chi(u_j^n - \bar{u}_j)(t)\|_{L^2} \geq \frac{\epsilon}{4}.$$

Using a uniqueness argument, it follows that $u_j^n(t,x) = u_j(t+t_n, x+x_n)$. Moreover, by the properties of χ and the last inequality, for all $t \in [0, T]$ and $n \geq n_\epsilon$,

$$\|u_j(t + t_n)\|_{L^2(Q_2(x_n))} \geq \frac{\epsilon}{4}.$$

This implies that

$$\|u_j(t)\|_{L^2(Q_2(x_n))} \geq \frac{\epsilon}{4}, \quad \text{for all } t \in [t_n, t_n + T] \text{ and all } n \geq n_\epsilon.$$

Moreover, as $\lim_{n\to\infty} t_n = \infty$, we can suppose that $t_{n+1} - t_n > T$ for $n \geq n_\epsilon$. Therefore, thanks to Morawetz estimates (2.3), we obtain for $N > 5$, the contradiction

$$\begin{aligned}
1 &\gtrsim \int_0^\infty \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_j(t,x)|^2 |u_j(t,y)|^2}{|x-y|^5} \, dx\, dy\, dt \\
&\gtrsim \sum_n \int_{t_n}^{t_n+T} \int_{Q_2(x_n)\times Q_2(x_n)} |u_j(t,x)|^2 |u_j(t,y)|^2 \, dx\, dy\, dt \\
&\gtrsim \sum_n T\big(\frac{\epsilon}{4}\big)^4 = \infty.
\end{aligned}$$

Using (2.4), for $N = 5$, we write

$$\begin{aligned}
1 &\gtrsim \int_0^\infty \|u_j(t)\|_{L^4(\mathbb{R}^5)}^4 dt \\
&\gtrsim \sum_n \int_{t_n}^{t_n+T} \|u_j(t)\|_{L^4(Q_2(x_n))}^4 dt \\
&\gtrsim \sum_n \int_{t_n}^{t_n+T} \|u_j(t)\|_{L^2(Q_2(x_n))}^4 dt \\
&\gtrsim \sum_n (\frac{\varepsilon}{4})^4 T = \infty.
\end{aligned}$$

This completes the proof of Proposition 4.2. □

Finally, we are ready to prove scattering. By the two previous lemmas, via the fact that $2 \le p < p^*$, we obtain

$$\|\mathbf{u}\|_{(S(t,\infty))^{(m)}} \lesssim \|\Psi\|_H + \epsilon(t)\|\mathbf{u}\|_{(S(t,\infty))^{(m)}}^{\frac{8p-N(p-1)}{N(p-1)}},$$

where $\epsilon(t) \to 0$, as $t \to \infty$. It follows from Lemma 2.12 that $\mathbf{u} \in (S(\mathbb{R}))^{(m)}$. Now, let $\mathbf{v}(t) = e^{-it\Delta^2}\mathbf{u}(t)$. Taking account of Duhamel formula,

$$\mathbf{v}(t) = \Psi + i\sum_{k=1}^{m} a_{jk}\int_0^t e^{-is\Delta^2}\big(|u_k|^p|u_1|^{p-2}u_1, \dots, |u_k|^p|u_m|^{p-2}u_m\big)\,ds.$$

Thanks to (4.1),(4.2) and (4.3),

$$f_{j,k}(\mathbf{u}) \in L^{\frac{8p}{p(8-N)}}(\mathbb{R}, W^{2,\frac{2p}{2p-1}}),$$

so, applying Strichartz estimate, for $0 < t < \tau$, we obtain

$$\|\mathbf{v}(t) - \mathbf{v}(\tau)\|_H \lesssim \sum_{j,k=1}^{m} \big\||u_k|^p|u_j|^{p-2}u_j\big\|_{L^{\frac{8p}{p(8-N)}}((t,\tau),W^{2,\frac{2p}{2p-1}})} \to 0$$

as $t, \tau \to \infty$. Taking $u^{\pm} := \lim_{t\to\pm\infty}\mathbf{v}(t)$, we obtain

$$\lim_{t\to\pm\infty}\|\mathbf{u}(t) - e^{it\Delta^2}u^{\pm}\|_{H^2} = 0.$$

Scattering is proved.

5. Global existence and scattering in the critical case

We establish the existence of a global solution to (1.1) in the critical case $p = p^*$ for small data when $4 < N \le 8$ as claimed in Theorem 2.5. Several norms have to be considered in the analysis of the critical case. Letting $I \subset \mathbb{R}$ a time slab, we define the norms

$$\|u\|_{M(I)} := \|\Delta u\|_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2N(N+4)}{N^2+16}})};$$
$$\|u\|_{W(I)} := \|\nabla u\|_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2N(N+4)}{N^2-2N+8}})};$$
$$\|u\|_{Z(I)} := \|u\|_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2(N+4)}{N-4}})};$$
$$\|u\|_{N(I)} := \|\nabla u\|_{L^2(I,L^{\frac{2N}{N+2}})}.$$

Let $M(\mathbb{R})$ be the completion of $C_c^\infty(\mathbb{R}^{N+1})$ with the norm $\|\cdot\|_{M(\mathbb{R})}$, and $M(I)$ be the set consisting of the restrictions to I of functions in $M(\mathbb{R})$. We adopt similar definitions for W and N. An important quantity closely related to the energy, is the functional ξ defined for $\mathbf{u} \in H$ by

$$\xi(\mathbf{u}) = \sum_{j=1}^{m}\int_{\mathbb{R}^N}|\Delta u_j|^2\,dx.$$

We give an auxiliary result.

Proposition 5.1. *Let $4 < N \le 8$ and $p = p^*$. There exists $\delta > 0$ such that for any initial data $\Psi \in H$ and any interval $I = [0,T]$, if*

$$\sum_{j=1}^{m}\|e^{it\Delta^2}\psi_j\|_{W(I)} < \delta,$$

then there exits a unique solution $\mathbf{u} \in C(I,H)$ *of* (1.1) *which satisfies* $\mathbf{u} \in \big(M(I) \cap L^{\frac{2(N+4)}{N}}(I\times\mathbb{R}^N)\big)^{(m)}$. *Moreover,*

$$\sum_{j=1}^m \|u_j\|_{W(I)} \le 2\delta;$$
$$\sum_{j=1}^m \|u_j\|_{M(I)} + \sum_{j=1}^m \|u_j\|_{L^\infty(I,H^2)} \le C(\|\Psi\|_{H^2} + \delta^{\frac{N+4}{N-4}}).$$

Furthermore, the solution depends continuously on the initial data in the sense that there exists δ_0 *depending on* δ, *such that for any* $\delta_1 \in (0,\delta_0)$, *if* $\sum_{j=1}^m \|\psi_j - \varphi_j\|_{H^2} \le \delta_1$ *and* $\mathbf{v}$ *is the local solution of* (1.1) *with initial data* $\varphi := (\varphi_1,\dots,\varphi_m)$, *then* $\mathbf{v}$ *is defined on* I *and for any admissible couple* (q,r),

$$\|\mathbf{u}-\mathbf{v}\|_{(L^q(I,L^r))^{(m)}} \le C\delta_1.$$

Proof. The proposition follows from a contraction mapping argument. For $\mathbf{u} \in (W(I))^{(m)}$, we let $\phi(\mathbf{u})$ given by

$$\phi(\mathbf{u})(t) := T(t)\Psi - i\sum_{k=1}^m a_{jk}\int_0^t T(t-s)\Big(|u_k|^{\frac{N}{N-4}}|u_1|^{\frac{8-N}{N-4}}u_1(s), \dots, |u_k|^{\frac{N}{N-4}}|u_m|^{\frac{8-N}{N-4}}u_m\Big)\,ds.$$

Define the set

$$X_{M,\delta} := \Big\{\mathbf{u} \in (M(I))^{(m)} : \sum_{j=1}^m \|u_j\|_{W(I)} \le 2\delta,\ \sum_{j=1}^m \|u_j\|_{L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})} \le 2M\Big\}$$

where $M := C\|\Psi\|_{(L^2)^{(m)}}$ and $\delta > 0$ is sufficiently small. Using Strichartz estimate, we obtain

$$\|\phi(\mathbf{u})-\phi(\mathbf{v})\|_{\big(L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})\big)^{(m)}} \lesssim \sum_{j,k=1}^m \big\|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\big\|_{L^{\frac{2(N+4)}{N+8}}(I,L^{\frac{2(N+4)}{N+8}})}.$$

Using Hölder inequality and denoting the quantity

$$(\mathcal{J}) := \big\|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\big\|_{L^{\frac{2(N+4)}{N+8}}(I,L^{\frac{2(N+4)}{N+8}})},$$

we obtain

$$\begin{aligned}(\mathcal{J}) &\lesssim \Big\|\Big(|u_k|^{\frac{4}{N-4}}|u_j|^{\frac{4}{N-4}} + |u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}\Big)|\mathbf{u}-\mathbf{v}|\Big\|_{L_T^{\frac{2(N+4)}{N+8}}(L^{\frac{2(N+4)}{N+8}})}\\
&\lesssim \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{2(N+4)}{N}}(L^{\frac{2(N+4)}{N}})}\Big(\|u_k\|^{\frac{4}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})}\|u_j\|^{\frac{4}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})}\\
&\quad + \|u_k\|^{\frac{N}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})}\|u_j\|^{\frac{8-N}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})}\Big).\end{aligned}$$

By Proposition 2.10, we have the Sobolev embedding

$$\|u\|_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2(N+4)}{N-4}})} \lesssim \|\nabla u\|_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2N(N+4)}{N^2-2N+8}})},$$

hence

$$(\mathcal{J}) \lesssim \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{2(N+4)}{N}}(L^{\frac{2(N+4)}{N}})}\Big(\|u_k\|^{\frac{4}{N-4}}_{W(I)}\|u_j\|^{\frac{4}{N-4}}_{W(I)} + \|u_k\|^{\frac{N}{N-4}}_{W(I)}\|u_j\|^{\frac{8-N}{N-4}}_{W(I)}\Big)$$

$$\lesssim \delta^{\frac{8}{N-4}} \|\mathbf{u}-\mathbf{v}\|_{L_T^{\frac{2(N+4)}{N}}(L^{\frac{2(N+4)}{N}})}.$$

Then

$$\|\phi(\mathbf{u})-\phi(\mathbf{v})\|_{\left(L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})\right)^{(m)}} \lesssim \delta^{\frac{8}{N-4}} \|\mathbf{u}-\mathbf{v}\|_{\left(L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})\right)^{(m)}}.$$

Moreover, taking in the previous inequality $\mathbf{v}=\mathbf{0}$, we obtain for small $\delta>0$,

$$\begin{aligned}
\|\phi(\mathbf{u})\|_{\left(L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})\right)^{(m)}} &\le C\|\Psi\|_{(L^2)^m} + 2\delta^{\frac{8}{N-4}} M \\
&\le (1+2\delta^{\frac{8}{N-4}})M \\
&\le 2M.
\end{aligned}$$

With a classical Picard argument, there exists $\mathbf{u}\in (L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}}))^m$ a solution to (1.1) satisfying

$$\|\mathbf{u}\|_{\left(L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})\right)^{(m)}} \le 2M.$$

Taking account of Strichartz estimate we obtain

$$\|\mathbf{u}\|_{(M(I))^{(m)}} \lesssim \|\Delta\Psi\|_{(L^2)^{(m)}} + \sum_{j,k=1}^{m} \|\nabla f_{j,k}(\mathbf{u})\|_{L_T^2(L^{\frac{2N}{N+2}})}.$$

Let $(\mathcal{J}_1) := \|\nabla f_{j,k}(\mathbf{u})\|_{L_T^2(L^{\frac{2N}{N+2}})}$. Using Hölder inequality and Sobolev embedding with, yields

$$\begin{aligned}
(\mathcal{J}_1) &\lesssim \Big\||\nabla\mathbf{u}|\Big(|u_k|^{\frac{4}{N-4}}|u_j|^{\frac{4}{N-4}} + |u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}\Big)\Big\|_{L_T^2(L^{\frac{2N}{N+2}})} \\
&\lesssim \|\nabla\mathbf{u}\|_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2N(N+4)}{N^2-2N+8}})} \Big(\|u_k\|^{\frac{4}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})} \|u_j\|^{\frac{4}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})} \\
&\quad + \|u_k\|^{\frac{N}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})} \|u_j\|^{\frac{8-N}{N-4}}_{L_T^{\frac{2(N+4)}{N-4}}(L^{\frac{2(N+4)}{N-4}})}\Big) \\
&\lesssim \|\mathbf{u}\|_{(W(I))^{(m)}} \Big(\|u_k\|^{\frac{4}{N-4}}_{W(I)}\|u_j\|^{\frac{4}{N-4}}_{W(I)} + \|u_k\|^{\frac{N}{N-4}}_{W(I)}\|u_j\|^{\frac{8-N}{N-4}}_{W(I)}\Big).
\end{aligned}$$

Then

$$\begin{aligned}
&\|\mathbf{u}\|_{(M(I))^{(m)}} \\
&\lesssim \|\Psi\|_H + \sum_{j,k=1}^{m} \|\mathbf{u}\|_{(W(I))^{(m)}} \Big(\|u_k\|^{\frac{4}{N-4}}_{W(I)}\|u_j\|^{\frac{4}{N-4}}_{W(I)} + \|u_k\|^{\frac{N}{N-4}}_{W(I)}\|u_j\|^{\frac{8-N}{N-4}}_{W(I)}\Big) \\
&\lesssim \|\Psi\|_H + \delta^{\frac{N+4}{N-4}}.
\end{aligned}$$

By Proposition 2.10, $M(I)\hookrightarrow W(I)$ and

$$\|\mathbf{u}\|_{(W(I))^{(m)}} \lesssim \|\mathbf{u}\|_{(M(I))^{(m)}}. \tag{5.1}$$

Thanks to Strichartz estimates, it follows that

$$\begin{aligned}
\|\mathbf{u}-e^{it\Delta^2}\Psi\|_{(W(I))^{(m)}} &\lesssim \Big\|\sum_{k=1}^{m}\int_0^t T(t-s)(f_{1,k}(\mathbf{u}),\dots,f_{m,k}(\mathbf{u}))\,ds\Big\|_{(W(I))^{(m)}} \\
&\lesssim \Big\|\sum_{k=1}^{m}\int_0^t T(t-s)(f_{1,k}(\mathbf{u}),\dots,f_{m,k}(\mathbf{u}))\,ds\Big\|_{(M(I))^{(m)}}
\end{aligned}$$

$$\lesssim \|\mathbf{u}\|^{\frac{N+4}{N-4}}_{(W(I))^{(m)}}.$$

So, by Lemma 2.12,

$$\|\mathbf{u}\|_{(W(I))^{(m)}} \le 2\delta.$$

Now, take an admissible couple (q,r) and denote $(\mathcal{J}_2) := \|\mathbf{u}-\mathbf{v}\|_{(L^q(I,L^r))^{(m)}} - \|\Psi-\varphi\|_{(L^2)^{(m)}}$. By Hölder inequality and Strichartz estimate, we have

$$\begin{aligned}
&(\mathcal{J}_2)\\
&\lesssim \sum_{j,k=1}^m \|f_{j,k}(\mathbf{u})-f_{j,k}(\mathbf{v})\|_{L^{\frac{2(N+4)}{N+8}}(I,L^{\frac{2(N+4)}{N+8}})}\\
&\lesssim \sum_{j,k=1}^m \Big\|\Big(|u_k|^{\frac{4}{N-4}}|u_j|^{\frac{4}{N-4}}+|u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}\Big)|\mathbf{u}-\mathbf{v}|\Big\|_{L^{\frac{2(N+4)}{N+8}}(I,L^{\frac{2(N+4)}{N+8}})}\\
&\lesssim \sum_{j,k=1}^m \|\mathbf{u}-\mathbf{v}\|_{L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})}\Big(\|u_k\|^{\frac{4}{N-4}}_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2(N+4)}{N-4}})}\|u_j\|^{\frac{4}{N-4}}_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2(N+4)}{N-4}})}\\
&\quad+\|u_k\|^{\frac{N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2(N+4)}{N-4}})}\|u_j\|^{\frac{8-N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}(I,L^{\frac{2(N+4)}{N-4}})}\Big)\\
&\lesssim \delta^{\frac{8}{N-4}}\|\mathbf{u}-\mathbf{v}\|_{\big(L^{\frac{2(N+4)}{N}}(I,L^{\frac{2(N+4)}{N}})\big)^{(m)}}.
\end{aligned}$$

The proof ends by taking δ small enough. □

We are ready to prove Theorem 2.5.

Proof of Theorem 2.5. Denote the homogeneous Sobolev space $\mathbf{H} = (\dot H^2)^{(m)}$. Using the previous proposition via (5.1), it suffices to prove that $\|\mathbf{u}\|_{\mathbf{H}}$ remains small on the whole interval of existence of $\mathbf{u}$, which is a consequence of the inequalities

$$\|\mathbf{u}\|^2_{\mathbf{H}} \le E(\mathbf{u}(t)) = E(\Psi) \le C\big(\xi(\Psi)+\xi(\Psi)^{\frac{N}{N-4}}\big).$$

Now, we prove scattering. Let $\mathbf{v}(t) = e^{-it\Delta^2}\mathbf{u}(t)$. Taking account of Duhamel formula

$$\mathbf{v}(t) := \Psi + i\sum_{k=1}^m\int_0^t e^{-is\Delta^2}\Big(|u_k|^{\frac{N}{N-4}}|u_1|^{\frac{8-N}{N-4}}u_1(s),\dots,|u_k|^{\frac{N}{N-4}}|u_m|^{\frac{8-N}{N-4}}u_m\Big)\,ds.$$

Therefore, for $0<t<\tau$,

$$\mathbf{v}(t)-\mathbf{v}(\tau) = i\sum_{k=1}^m\int_t^\tau e^{-is\Delta^2}\Big(|u_k|^{\frac{N}{N-4}}|u_1|^{\frac{8-N}{N-4}}u_1(s),\dots,|u_k|^{\frac{N}{N-4}}|u_m|^{\frac{8-N}{N-4}}u_m\Big)\,ds.$$

Applying Strichartz estimate and Sobolev embedding, we obtain

$$\begin{aligned}
\|\mathbf{v}(t)-\mathbf{v}(\tau)\|_H &\lesssim \sum_{j,k=1}^m \|(1+\nabla)(|u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}u_j)\|_{L^2((t,\tau),L^{\frac{2N}{N+2}})}\\
&\lesssim \|(1+|\nabla\mathbf{u}|)\Big(|u_k|^{\frac{4}{N-4}}|u_j|^{\frac{4}{N-4}}+|u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}\Big)\|_{L^2((t,\tau),L^{\frac{2N}{N+2}})}\\
&\lesssim \big(1+\|\nabla\mathbf{u}\|_{L^{\frac{2(N+4)}{N-4}}((t,\tau),L^{\frac{2N(N+4)}{N^2-2N+8}})}\big)\\
&\quad\times\Big(\|u_k\|^{\frac{4}{N-4}}_{L^{\frac{2(N+4)}{N-4}}((t,\tau),L^{\frac{2(N+4)}{N-4}})}\|u_j\|^{\frac{4}{N-4}}_{L^{\frac{2(N+4)}{N-4}}((t,\tau),L^{\frac{2(N+4)}{N-4}})}
\end{aligned}$$

$$
\begin{aligned}
&+ \|u_k\|^{\frac{N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}((t,\tau),L^{\frac{2(N+4)}{N-4}})} \|u_j\|^{\frac{8-N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}((t,\tau),L^{\frac{2(N+4)}{N-4}})}\Big) \\
&\lesssim (1 + \|\mathbf{u}\|_{(W(t,\tau))^{(m)}})\|\mathbf{u}\|^{\frac{8}{N-4}}_{(W(t,\tau))^{(m)}}.
\end{aligned}
$$

Using the proof of Proposition 5.1, we have $\mathbf{u} \in (W(\mathbb{R}))^m$. Hence,

$$\|\mathbf{v}(t) - \mathbf{v}(\tau)\|_H \overset{t,\tau\to\infty}{\to 0}.$$

Taking $u_\pm := \lim_{t\to\pm\infty} \mathbf{v}(t)$, we obtain

$$\lim_{t\to\pm\infty} \|\mathbf{u}(t) - e^{it\Delta^2} u_\pm\|_H = 0.$$

Scattering is proved. $\square$

6. Appendix

6.1. **Blow-up criterion.** We give a useful criterion for global existence in the critical case.

Proposition 6.1. *Let $4 < N \leq 8$, $p = p^*$ and $\mathbf{u} \in C([0,T), H)$ be a solution of (1.1) satisfying $\|\mathbf{u}\|_{(Z([0,T]))^{(m)}} < +\infty$. Then there exists*

$$K := K(\|\Psi\|_H, \|\mathbf{u}\|_{(Z([0,T]))^{(m)}})$$

such that

$$\|\mathbf{u}\|_{\big(L_T^{\frac{2(N+4)}{N}}(L^{\frac{2(N+4)}{N}})\big)^{(m)}} + \|\mathbf{u}\|_{\big(L_T^\infty(\mathbf{H})\big)^{(m)}} + \|\mathbf{u}\|_{(M([0,T]))^{(m)}} \leq K \tag{6.1}$$

and $\mathbf{u}$ can be extended to a solution $\tilde{\mathbf{u}} \in C([0,T'), H)$ of (1.1) for some $T' > T$.

Proof. Let $\eta > 0$ be a small real number and $M := \|\mathbf{u}\|_{(Z([0,T]))^{(m)}}$. The first step is to establish (6.1). In order to do so, we subdivide $[0,T]$ into n slabs I_j such that

$$n \sim (1 + \frac{M}{\eta})^{\frac{2(N+4)}{N-4}} \quad \text{and} \quad \|\mathbf{u}\|_{(Z(I_j))^{(m)}} \leq \eta.$$

Denote $(\mathcal{A}) := \|\mathbf{u}\|_{(M([t_j,t]))^{(m)}}$ and $I_j = [t_j, t_{j+1}]$. For $t \in I_j$, by Strichartz estimate and arguing as previously,

$$
\begin{aligned}
(\mathcal{A}) - \|\mathbf{u}(t_j)\|_{\mathbf{H}} &\lesssim \sum_{i,k=1}^m \|\nabla f_{i,k}(\mathbf{u})\|_{\big(L^2([t_j,t],L^{\frac{2N}{N+2}})\big)^{(m)}} \\
&\lesssim \sum_{j,k=1}^m \|\nabla \mathbf{u}\|_{L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2N(N+4)}{N^2-2N+8}})} \\
&\quad \times \Big(\|u_k\|^{\frac{4}{N-4}}_{L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2(N+4)}{N-4}})} \|u_j\|^{\frac{4}{N-4}}_{L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2(N+4)}{N-4}})} \\
&\quad + \|u_k\|^{\frac{N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2(N+4)}{N-4}})} \|u_j\|^{\frac{8-N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}[t_j,t],(L^{\frac{2(N+4)}{N-4}})} \Big) \\
&\lesssim \|\mathbf{u}\|_{(W([t_j,t]))^{(m)}} \|\mathbf{u}\|^{\frac{8}{N-4}}_{(Z([t_j,t]))^{(m)}} \\
&\lesssim \|\mathbf{u}\|_{(M([t_j,t]))^{(m)}} \|\mathbf{u}\|^{\frac{8}{N-4}}_{(Z([t_j,t]))^{(m)}} \lesssim \eta^{\frac{8}{N-4}} \|\mathbf{u}\|_{(M([t_j,t]))^{(m)}}.
\end{aligned}
$$

Take $(\mathcal{B}) := \|\mathbf{u}\|_{\big(L^{\frac{2(N+4)}{N}}([t_j,t],L^{\frac{2(N+4)}{N}})\big)^{(m)}}$. Applying Strichartz estimates, we obtain

$$
\begin{aligned}
&(\mathcal{B}) - C\|\mathbf{u}(t_j)\|_{(L^2)^{(m)}}\\
&\lesssim \sum_{j,k=1}^{m} \||u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}u_j\|_{L^{\frac{2(N+4)}{N+8}}([t_j,t],L^{\frac{2(N+4)}{N+8}})}\\
&\lesssim \sum_{j,k=1}^{m} \|u_k\|^{\frac{N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2(N+4)}{N-4}})} \|u_j\|^{\frac{8-N}{N-4}}_{L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2(N+4)}{N-4}})} \|u_j\|_{L^{\frac{2(N+4)}{N}}([t_j,t],L^{\frac{2(N+4)}{N}})}\\
&\lesssim \|\mathbf{u}\|^{\frac{8}{N-4}}_{\big(L^{\frac{2(N+4)}{N-4}}([t_j,t],L^{\frac{2(N+4)}{N-4}})\big)^{(m)}} \|\mathbf{u}\|_{\big(L^{\frac{2(N+4)}{N}}([t_j,t],L^{\frac{2(N+4)}{N}})\big)^{(m)}}\\
&\lesssim \eta^{\frac{8}{N-4}}\|\mathbf{u}\|_{\big(L^{\frac{2(N+4)}{N}}([t_j,t],L^{\frac{2(N+4)}{N}})\big)^{(m)}}.
\end{aligned}
$$

If η is sufficiently small, the conservation of the mass yields

$$
\|\mathbf{u}\|_{\big(L^{\frac{2(N+4)}{N}}([t_j,t],L^{\frac{2(N+4)}{N}})\big)^{(m)}} \le C\|\Psi\|_{(L^2)^{(m)}}
$$

and

$$
\|\mathbf{u}\|_{(M([t_j,t]))^{(m)}} \le C\|\mathbf{u}(t_j)\|_{\mathbf{H}}.
$$

Applying again Strichartz estimates, yields

$$
\|\mathbf{u}\|_{\big(L^\infty([t_j,t],\mathbf{H})\big)^{(m)}} \le C\|\mathbf{u}(t_j)\|_{\mathbf{H}}.
$$

In particular, $\|\mathbf{u}(t_{j+1})\|_{\mathbf{H}} \le C\|\mathbf{u}(t_j)\|_{\mathbf{H}}$. Finally,

$$
\|\mathbf{u}\|_{\big(L^\infty([t_j,t],\mathbf{H})\big)^{(m)}} + \|\mathbf{u}\|_{(M([t_j,t]))^{(m)}} \le 2C^n\|\Psi\|_{\mathbf{H}} < +\infty.
$$

The first step is done. Choose $t_0 \in I_n$, Duhamel's formula gives

$$
\begin{aligned}
\mathbf{u}(t) = e^{i(t-t_0)\Delta^2}\mathbf{u}(t_0) - i\sum_{k=1}^{m} a_{jk}\int_{t_0}^{t} e^{i(t-s)\Delta^2}\\
\times \Big(|u_k|^{\frac{N}{N-4}}|u_1|^{\frac{8-N}{N-4}}u_1, \dots, |u_k|^{\frac{N}{N-4}}|u_m|^{\frac{8-N}{N-4}}u_m\Big)\, ds.
\end{aligned}
$$

Thanks to Sobolev inequality and Strichartz estimate,

$$
\begin{aligned}
\|e^{i(t-t_0)\Delta^2}\mathbf{u}(t_0)\|_{(W([t_0,t]))^m} &\le \|\mathbf{u}\|_{(W([t_0,t]))^m} + C\sum_{j,k=1}^{m} \big\||u_k|^{\frac{N}{N-4}}|u_j|^{\frac{8-N}{N-4}}u_j\big\|_{N([t_0,t])}\\
&\le \|\mathbf{u}\|_{(W([t_0,t]))^m} + C\|\mathbf{u}\|^{\frac{N+4}{N-4}}_{(W([t_0,t]))^m}.
\end{aligned}
$$

The dominated convergence theorem ensures that $\|\mathbf{u}\|_{(W([t_0,T]))^m}$ can be made arbitrarily small as $t_0 \to T$, then

$$
\|e^{i(t-t_0)\Delta^2}\mathbf{u}(t_0)\|_{(W([t_0,T]))^m} \le \delta,
$$

where δ is as in Proposition 5.1. In particular, we can find $t_1 \in (0,T)$ and $T' > T$ such that

$$
\|e^{i(t-t_0)\Delta^2}\mathbf{u}(t_0)\|_{(W([t_1,T']))^m} \le \delta.
$$

Now, it follows from Proposition 5.1 that there exists $\mathbf{v} \in C([t_1,T'],H)$ such that $\mathbf{v}$ solves (1.1) with $p = \frac{N}{N-4}$ and $\mathbf{u}(t_1) = \mathbf{v}(t_1)$. By uniqueness, $\mathbf{u} = \mathbf{v}$ in $[t_1,T)$ and $\mathbf{u}$ can be extended in $[0,T']$. $\square$

6.2. **Morawetz estimate.** In what follows we give a classical proof, inspired by [4, 18], of Morawetz estimates. Let $\mathbf{u} := (u_1, \dots, u_m) \in H$ be a solution of

$$i\dot{u}_j + \Delta^2 u_j + \Big(\sum_{k=1}^m a_{jk}|u_k|^p\Big)|u_j|^{p-2}u_j = 0$$

in N_1-spatial dimensions and $\mathbf{v} := (v_1, \dots, v_m) \in H$ be a solution to

$$i\dot{v}_j + \Delta^2 v_j + \Big(\sum_{k=1}^m a_{jk}|v_k|^p\Big)|v_j|^{p-2}v_j = 0$$

in N_2-spatial dimensions. Define the tensor product $\mathbf{w} := (\mathbf{u} \otimes \mathbf{v})(t, z)$ for z in

$$\mathbb{R}^{N_1+N_2} := \{(x, y) \quad \text{s. t} \quad x \in \mathbb{R}^{N_1}, y \in \mathbb{R}^{N_2}\}$$

by the formula

$$(\mathbf{u} \otimes \mathbf{v})(t, z) = \mathbf{u}(t, x)\mathbf{v}(t, y).$$

Denote $F(\mathbf{u}) := \big(\sum_{k=1}^m a_{jk}|u_k|^p\big)|u_j|^{p-2}u_j$. A direct computation shows that $\mathbf{w} := (w_1, \dots, w_n) = \mathbf{u} \otimes \mathbf{v}$ solves the equation

$$i\dot{w}_j + \Delta^2 w_j + F(\mathbf{u}) \otimes v_j + F(\mathbf{v}) \otimes u_j := i\dot{w}_j + \Delta^2 w_j + h = 0 \tag{6.2}$$

where $\Delta^2 := \Delta_x^2 + \Delta_y^2$. Define the Morawetz action corresponding to $\mathbf{w}$ by

$$\begin{aligned} M_a^{\otimes_2} &:= 2\sum_{j=1}^m \int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \nabla a(z).\Im(\overline{u_j \otimes v_j(z)}\nabla(u_j \otimes v_j)(z))\,dz \\ &= 2\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \nabla a(z).\Im(\bar{\mathbf{w}}(z)\nabla(\mathbf{w})(z))\,dz, \end{aligned}$$

where $\nabla := (\nabla_x, \nabla_y)$. It follows from equation (6.2) that

$$\begin{aligned} &\Im(\dot{\bar{w}}_j\partial_i w_j) = \Re(-i\dot{\bar{w}}_j\partial_i w_j) \\ &= -\Re\big((\Delta^2\bar{w}_j + \sum_{k=1}^m a_{jk}|\bar{u}_k|^p|\bar{u}_j|^{p-2}\bar{u}_j\bar{v}_j + \sum_{k=1}^m a_{jk}|\bar{v}_k|^p|\bar{v}_j|^{p-2}\bar{v}_j\bar{u}_j)\partial_i w_j\big); \end{aligned}$$

$$\begin{aligned} &\Im(\bar{w}_j\partial_i\dot{w}_j) = \Re(-i\bar{w}_j\partial_i\dot{w}_j) \\ &= \Re\big(\partial_i(\Delta^2 w_j + \sum_{k=1}^m a_{jk}|u_k|^p|u_j|^{p-2}u_jv_j + \sum_{k=1}^m a_{jk}|v_k|^p|v_j|^{p-2}v_ju_j)\bar{w}_j\big). \end{aligned}$$

Moreover, denoting the quantity $\big\{h, w_j\big\}_p := \Re\big(h\nabla\bar{w}_j - w_j\nabla\bar{h}\big)$, we compute

$$\begin{aligned} \big\{h, w_j\big\}_p^i &= -\partial_i\Big(\sum_{k=1}^m a_{jk}|\bar{u}_k|^p|\bar{u}_j|^{p-2}\bar{u}_j\bar{v}_j + \sum_{k=1}^m a_{jk}|\bar{v}_k|^p|\bar{v}_j|^{p-2}\bar{v}_j\bar{u}_j\Big)w_j \\ &\quad + \Big(\sum_{k=1}^m a_{jk}|u_k|^p|u_j|^{p-2}u_jv_j + \sum_{k=1}^m a_{jk}|v_k|^p|v_j|^{p-2}v_ju_j\Big)\partial_i\bar{w}_j. \end{aligned}$$

It follows that

$$\begin{aligned} \dot{M}_a^{\otimes_2} &= 2\sum_{j=1}^m \int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \partial_i a\Re\big(\bar{w}_j\partial_i\Delta^2 w_j - \partial_i w_j\Delta^2\bar{w}_j\big)\,dz \\ &\quad - 2\sum_{j=1}^m \int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \partial_i a\big\{h, w_j\big\}_p^i\,dz \end{aligned}$$

$$
\begin{aligned}
&= -2\sum_{j=1}^{m}\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \big[\Delta a\Re(\bar{w}_j\Delta^2 w_j) + 2\Re(\partial_i a\partial_i \bar{w}_j\Delta^2 w_j)\big]\, dz \\
&\quad - \sum_{j=1}^{m} 2\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \partial_i a\big\{h, w_j\big\}_p^i\, dz \\
&:= \mathcal{I}_1 + \mathcal{I}_2 - 2\sum_{j=1}^{m}\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \partial_i a\big\{h, w_j\big\}_p^i\, dz.
\end{aligned}
$$

Similar computations as those in [18], give

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 = 2\sum_{j=1}^{m}\Re\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} &\Big\{2\big(\partial_{ik}^x\Delta_x a\partial_i\bar{u}_j\partial_k u_j|v_j|^2 + \partial_{ik}^y\Delta_y a\partial_i\bar{v}_j\partial_k v_j|u_j|^2\big) \\
&- \frac{1}{2}(\Delta_x^3 + \Delta_y^3)a|u_jv_j|^2 + \big(\Delta_x^2 a|\nabla u_j|^2|v_j|^2 + \Delta_y^2 a|\nabla v_j|^2|u_j|^2\big) \\
&- 4\big(\partial_{ik}^x a\partial_{i_1 i}\bar{u}_j\partial_{i_1 k}u_j|v_j|^2 + \partial_{ik}^y a\partial_{i_1 i}\bar{v}_j\partial_{i_1 k}v_j|u_j|^2\big)\Big\}\, dz.
\end{aligned}
$$

Now, we take $a(z) := a(x,y) = |x-y|$ where $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$. Then calculation done in [18], yield

$$
\dot{M}_a^{\otimes_2} \le 2\sum_{j=1}^{m}\Re\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \Big(-\frac{1}{2}(\Delta_x^3+\Delta_y^3)a|u_jv_j|^2 - 2\partial_i a\{h, w_j\}_p^i\Big)\, dz.
$$

Hence, we obtain

$$
\sum_{j=1}^{m}\int_0^T\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \Big((\Delta_x^3+\Delta_y^3)a|u_jv_j|^2 + 4\partial_i a\{h, w_j\}_p^i\Big)\, dz\, dt \le \sup_{[0,T]}|M_a^{\otimes_2}|.
$$

Then

$$
\begin{aligned}
\sum_{j=1}^{m}\int_0^T\int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} &\Big((\Delta_x^3+\Delta_y^3)a|u_jv_j|^2 + 4(1-\frac{1}{p})\Delta_x a\sum_{k=1}^{m} a_{jk}|u_k|^p|u_j|^p|v_j|^2 \\
&+ 4(1-\frac{1}{p})\Delta_y a\sum_{k=1}^{m} a_{jk}|v_k|^p|v_j|^p|u_j|^2\Big)\, dz\, dt \le \sup_{[0,T]}|M_a^{\otimes_2}|.
\end{aligned}
$$

Taking account of the equalities $\Delta_x a = \Delta_y a = (N-1)|x-y|^{-1}$ and

$$
\Delta_x^3 a = \Delta_y^3 a = \begin{cases} C\delta(x-y), & \text{if } N=5; \\ 3(N-1)(N-3)(N-5)|x-y|^{-5}, & \text{if } N>5, \end{cases}
$$

when $N=5$, choosing $u_j = v_j$, we obtain

$$
\sum_{j=1}^{m}\int_0^T\int_{\mathbb{R}^5} |u_j(t,x)|^4\, dx\, dt \lesssim \sup_{[0,T]}|M_a^{\otimes_2}|.
$$

If $N>5$, it follows that

$$
\sum_{j=1}^{m}\int_0^T\int_{\mathbb{R}^N\otimes\mathbb{R}^N} \frac{|u_j(t,x)|^2|u(y,t)|^2}{|x-y|^5}\, dx\, dy\, dt \lesssim \sup_{[0,T]}|M_a^{\otimes_2}|.
$$

This completes the proof.

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