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EXISTENCE OF SOLUTIONS FOR SINGULAR (p,q)-KIRCHHOFF TYPE SYSTEMS WITH MULTIPLE PARAMETERS

SAYYED HASHEM RASOULI

Abstract. This article concerns the existence of positive solutions for singular (p,q)-Kirchhoff type systems with multiple parameters. Our approach is based on the method of sub- and super-solutions.

1. Introduction

In this article, we are interested in the existence of positive solutions for the singular (p,q)-Kirchhoff type system

$$
\begin{gathered}
-M_1\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = a(x)\Big[\alpha_1\Big(f(v)-\frac{1}{u^\eta}\Big)+\beta_1\Big(h(u)-\frac{1}{u^\eta}\Big)\Big], \quad x\in\Omega,\\
-M_2\Big(\int_\Omega |\nabla v|^q dx\Big)\Delta_q v = b(x)\Big[\alpha_2\Big(g(u)-\frac{1}{v^\theta}\Big)+\beta_2\Big(k(v)-\frac{1}{v^\theta}\Big)\Big], \quad x\in\Omega,\\
u=v=0, \quad x\in\partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $M_i:\mathbb{R}^+\to\mathbb{R}^+$, $i=1,2$ are two continuous and increasing functions such that $M_i(t)\geq m_i>0$ for all $t\in\mathbb{R}^+$, $\Delta_r z = div(|\nabla z|^{r-2}\nabla z)$, for $r>1$ denotes the r-Laplacian operator, $\alpha_1,\alpha_2,\beta_1,\beta_2$ are positive parameters, Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 1$ with sufficiently smooth boundary and $\eta,\theta\in(0,1)$. Here $a(x),b(x)\in C(\overline{\Omega})$ are weight functions such that $a(x)\geq a_0>0$, $b(x)\geq b_0>0$ for all $x\in\overline{\Omega}$, $f,g,h,k\in C([0,\infty)$ are nondecreasing functions and $f(0),g(0),h(0),k(0)>0$.

Problem (1.1) is called nonlocal because of the term $-M(\int_\Omega |\nabla u|^r dx)$ which implies that the first two equations in (1.1) are no longer pointwise equalities. This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Also, such a problem has physical motivation. Moreover, system (1.1) is related to the stationary version of the Kirchhoff equation

$$
\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.2}
$$

presented by Kirchhoff [13]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in (1.2) have the following meanings: L is the length

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of the string, h is the area of cross section, E is the Youngs modulus of the material, ρ is the mass density, and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert's equation. Later on, Woinowsky-Krieger (Nash-Modeer) incorporated this correction in the classical Euler-Bernoulli equation for the beam (plate) with hinged ends. See, for example, [5, 6] and the references therein.

Nonlocal problems also appear in other fields: for example, biological systems where u and v describe a process which depends on the average of itself (for instance, population density). See [3, 4, 11, 19, 20] and the references therein. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [1, 9, 17, 12, 22, 21, 8], in which the authors have used different methods to prove the existence of solutions.

Let $F(s,t) = (f(t) - \frac{1}{s^\eta}) + (h(s) - \frac{1}{s^\eta})$, and $G(s,t) = (g(s) - \frac{1}{t^\eta}) + (k(t) - \frac{1}{t^\eta})$. Then $\lim_{(s,t)\to(0,0)} F(s,t) = -\infty = \lim_{(s,t)\to(0,0)} G(s,t)$, and hence we refer to (1.1) as an infinite semipositone problem. See [2], where the authors studied the corresponding non-singular finite system when $M_1(t) = M_2(t) \equiv 1$, and $a(x) = b(x) \equiv 1$. It is well documented that the study of positive solutions to such semipositone problems is mathematically very challenging [7], [18]. In this paper, we study the even more challenging semipositone system with $\lim_{(s,t)\to(0,0)} F(s,t) = -\infty = \lim_{(s,t)\to(0,0)} G(s,t)$. We do not need the boundedness of the Kirchhoff functions M_1, M_2, as in [10]. Using the sub and supersolutions techniques, we prove the existence of positive solutions to the system (1.1). To our best knowledge, this is an interesting and new research topic for singular (p,q)-Kirchhoff type systems. One can refer to [14, 15, 16] for some recent existence results of infinite semipositone systems.

To precisely state our existence result we consider the eigenvalue problem

$$\begin{gathered}
-\Delta_r \phi = \lambda\, |\phi|^{r-2}\phi, \quad x \in \Omega, \\
\phi = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{1.3}$$

Let $\phi_{1,r}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (1.3) such that $\phi_{1,r}(x) > 0$ in Ω, and $\|\phi_{1,r}\|_\infty = 1$ for $r = p,q$. Let $m, \sigma, \delta > 0$ be such that

$$\sigma \le \phi_{1,r}^{\frac{r}{r-1+s}} \le 1, \quad x \in \Omega - \overline{\Omega_\delta}, \tag{1.4}$$

$$|\nabla\phi_{1,r}|^r \ge m, \quad x \in \overline{\Omega_\delta}, \tag{1.5}$$

for $r = p,q$, and $s = \eta, \theta$, where $\overline{\Omega_\delta} := \{x \in \Omega | d(x,\partial\Omega) \le \delta\}$. (This is possible since $|\nabla\phi_{1,r}|^r \ne 0$ on $\partial\Omega$ while $\phi_{1,r} = 0$ on $\partial\Omega$ for $r = p,q$). We will also consider the unique solution $\zeta_r \in W_0^{1,r}(\Omega)$ of the boundary-value problem

$$\begin{gathered}
-\Delta_r \zeta_r = 1, \quad x \in \Omega, \\
\zeta_r = 0, \quad x \in \partial\Omega.
\end{gathered}$$

It is known that $\zeta_r > 0$ in Ω and $\frac{\partial\zeta_r}{\partial n} < 0$ on $\partial\Omega$.

2. Existence of solutions

In this section, we shall establish our existence result via the method of sub-super-solution [2]. For the system

$$
\begin{gathered}
-M_1\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = h_1(x,u,v), \quad x\in\Omega,\\
-M_2\Big(\int_\Omega |\nabla v|^q dx\Big)\Delta_q v = h_2(x,u,v), \quad x\in\Omega,\\
u=v=0, \quad x\in\partial\Omega,
\end{gathered}
$$

a pair of functions $(\psi_1,\psi_2)\in W^{1,p}\cap C(\overline{\Omega})\times W^{1,q}\cap C(\overline{\Omega})$ and $(z_1,z_2)\in W^{1,p}\cap C(\overline{\Omega})\times W^{1,q}\cap C(\overline{\Omega})$ are called a subsolution and supersolution if they satisfy $(\psi_1,\psi_2)=(0,0)=(z_1,z_2)$ on $\partial\Omega$,

$$
\begin{gathered}
M_1\Big(\int_\Omega |\nabla \psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx \le \int_\Omega h_1(x,\psi_1,\psi_2)w\,dx,\\
M_2\Big(\int_\Omega |\nabla \psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx \le \int_\Omega h_2(x,\psi_1,\psi_2)w\,dx
\end{gathered}
$$

and

$$
\begin{gathered}
M_1\Big(\int_\Omega |\nabla z_1|^p dx\Big)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx \ge \int_\Omega h_1(x,z_1,z_2)w\,dx,\\
M_2\Big(\int_\Omega |\nabla z_2|^q dx\Big)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx \ge \int_\Omega h_2(x,z_1,z_2)w\,dx,
\end{gathered}
$$

for all $w\in W=\{w\in C_0^\infty(\Omega)|w\ge 0, x\in\Omega\}$. Then the following result holds.

Lemma 2.1 ([8])**.** *Suppose there exist sub- and super-solutions (ψ_1,ψ_2) and (z_1,z_2) respectively of (1.1) such that $(\psi_1,\psi_2)\le(z_1,z_2)$. Then (1.1) has a solution (u,v) such that $(u,v)\in[(\psi_1,\psi_2),(z_1,z_2)]$.*

We use the following hypotheses:

(H1) $f,g,h,k\in C([0,\infty)$ are nondecreasing functions such that $f(0)>0$, $g(0)>0$, $h(0)>0$, $k(0)>0$,

$$
\begin{gathered}
\lim_{s\to+\infty} f(s)=\lim_{s\to+\infty} h(s)=\lim_{s\to+\infty} g(s)=\lim_{s\to+\infty} k(s)=+\infty,\\
\lim_{s\to+\infty}\frac{h(s)}{s^{p-1}}=\lim_{s\to+\infty}\frac{k(s)}{s^{q-1}}=0.
\end{gathered}
$$

(H2) for all $A>0$,

$$
\lim_{s\to\infty}\frac{f(Ag(s)^{\frac{1}{q-1}})}{s^{p-1+\eta}}=0.
$$

Our main result read as follows.

Theorem 2.2. *Let (H1)–(H2) hold. Then (1.1) has a large positive solution (u,v) provided $\alpha_1+\beta_1$ and $\alpha_2+\beta_2$ are large.*

Proof. Let $\gamma_0=\min\{f(0),h(0)\}>0$ and

$$
\begin{aligned}
\gamma_1=\min\Big\{&f\Big((\alpha_2+\beta_2)^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}(\frac{q-1+\theta}{q})\sigma\Big),\\
&h\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)\Big\}.
\end{aligned}
$$

For fixed $r_1 \in (\frac{1}{p-1+\eta}, \frac{1}{p-1})$ and $r_2 \in (\frac{1}{q-1+\theta}, \frac{1}{q-1})$, we shall verify that with

$$\psi_1 = \frac{(\alpha_1+\beta_1)^{r_1}(p-1+\eta)}{p}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1+\eta}},$$
$$\psi_2 = \frac{(\alpha_2+\beta_2)^{r_2}(q-1+\theta)}{q}(\frac{b_0}{m_2})^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q}{q-1+\theta}},$$

and $(\psi_1,\psi_2)$is a sub-solution of (1.1). Let $w \in W$. Then a calculation shows that

$$\nabla\psi_1 = (\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\phi_{1,p}^{\frac{1-\eta}{p-1+\eta}}\nabla\phi_{1,p},$$

and we have

$$\begin{aligned}
&M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx\\
&= \frac{a_0(\alpha_1+\beta_1)^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\\
&\quad\times\int_\Omega \phi_{1,p}^{1-\frac{\eta p}{p-1+\eta}}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla w dx\\
&= \frac{a_0(\alpha_1+\beta_1)^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\\
&\quad\int_\Omega |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\big\{\nabla(\phi_{1,p}^{1-\frac{\eta p}{p-1+\eta}}w) - w\nabla(\phi_{1,p}^{1-\frac{\eta p}{p-1+\eta}})\big\}dx\\
&= \frac{a_0(\alpha_1+\beta_1)^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\\
&\quad\times\Big\{\int_\Omega \big[\lambda_{1,p}\phi_{1,p}^{p-\frac{\eta p}{p-1+\eta}} - |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla(\phi_{1,p}^{1-\frac{\eta p}{p-1+\eta}})\big]w dx\Big\}\\
&= \frac{a_0(\alpha_1+\beta_1)^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\\
&\quad\times\Big\{\int_\Omega \big[\lambda_{1,p}\phi_{1,p}^{p-\frac{\eta p}{p-1+\eta}} - |\nabla\phi_{1,p}|^p(1-\frac{\eta p}{p-1+\eta})\phi_{1,p}^{-\frac{\eta p}{p-1+\eta}}\big]w dx\Big\}\\
&\le a_0(\alpha_1+\beta_1)^{(p-1)r_1}\Big\{\int_\Omega \big[\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}} - \frac{(1-\eta)(p-1)}{p-1+\eta}\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\eta p}{p-1+\eta}}}\big]w dx\Big\}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx\\
&\le b_0(\alpha_2+\beta_2)^{(q-1)r_2}\Big\{\int_\Omega \big[\lambda_{1,q}\phi_{1,q}^{\frac{q(q-1)}{q-1+\theta}} - \frac{(1-\theta)(q-1)}{q-1+\theta}\frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\theta q}{q-1+\theta}}}\big]w dx\Big\}.
\end{aligned}$$

Thus (ψ_1,ψ_2) is a sub-solution if

$$\begin{aligned}
&a_0(\alpha_1+\beta_1)^{r_1}\big\{\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}} - \frac{(1-\eta)(p-1)}{p-1+\eta}\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\eta p}{p-1+\theta}}}\big\}\\
&\le a(x)\Big[\alpha_1\Big(f(\psi_2)-\frac{1}{\psi_1^\eta}\Big)+\beta_1\Big(h(\psi_1)-\frac{1}{\psi_1^\eta}\Big)\Big].
\end{aligned}$$

and

$$
\begin{aligned}
& b_0(\alpha_2+\beta_2)^{r_2}\big\{\lambda_{1,q}\phi_{1,q}^{\frac{q(q-1)}{q-1+\theta}} - \frac{(1-\theta)(q-1)}{q-1+\theta}\frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\theta q}{q-1+\theta}}}\big\} \\
& \le b(x)\Big[\alpha_2\Big(g(\psi_1)-\frac{1}{\psi_2^\theta}\Big)+\beta_2\Big(k(\psi_2)-\frac{1}{\psi_2^\theta}\Big)\Big].
\end{aligned}
$$

First we consider the case when $x\in\overline{\Omega_\delta}$. Since $1-(p-1)r_1-r_1\eta<0$, for $\alpha_1+\beta_1\gg 1$, we have

$$
\begin{aligned}
& -(\alpha_1+\beta_1)^{(p-1)r_1}\frac{(1-\eta)(p-1)}{p-1+\eta}\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{p\eta}{p-1+\eta}}} \\
& \le(\alpha_1+\beta_1)\Big[-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\Big)^\eta}\Big].
\end{aligned}
$$

Also in $\bar\Omega_\delta$ (in fact in Ω), since $(p-1)r_1<1$, if $\alpha_1+\beta_1\gg 1$ and $\alpha_2+\beta_2\gg 1$,

$$
\begin{aligned}
(\alpha_1+\beta_1)^{(p-1)r_1}\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}} &\le(\alpha_1+\beta_1)\gamma_0 \\
&=\alpha_1\gamma_0+\beta_1)\gamma_0 \\
&\le\alpha_1 f\Big((\alpha_2+\beta_2)^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}(\frac{q-1+\theta}{q})\phi_{1,q}^{\frac{q}{q-1+\theta}}\Big) \\
&\quad+\beta_1 h\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\Big).
\end{aligned}
$$

It follows that in $\overline{\Omega_\delta}$ for $\alpha_1+\beta_1\gg 1$ and $\alpha_2+\beta_2\gg 1$, we have

$$
\begin{aligned}
& a_0(\alpha_1+\beta_1)^{r_1}\Big[\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}}-\frac{(1-\eta)(p-1)}{p-1+\eta}\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\eta p}{p-1+\eta}}}\Big] \\
& = a_0\Big[(\alpha_1+\beta_1)^{r_1}\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}}-(\alpha_1+\beta_1)^{r_1}\frac{(1-\eta)(p-1)}{p-1+\eta}\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\eta p}{p-1+\eta}}}\Big] \\
& \le a(x)\Big[\alpha_1 f\Big((\alpha_2+\beta_2)^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}(\frac{q-1+\theta}{q})\phi_{1,q}^{\frac{q}{q-1+\theta}}\Big)+\beta_1 h\Big((\alpha_1 \\
&\quad+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\Big) \\
&\quad-\frac{(\alpha_1+\beta_1)}{\big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\big)^\eta}\Big] \\
& = a(x)\Big[\alpha_1\Big(f(\psi_2)-\frac{1}{\psi_1^\eta}\Big)+\beta_1\Big(h(\psi_1)-\frac{1}{\psi_1^\eta}\Big)\Big].
\end{aligned}
$$

On the other hand, on $\Omega-\overline{\Omega_\delta}$, we have $\sigma\le\phi_{1,r}^{\frac{r}{r-1+s}}\le 1$, for $r=p,q$ and $s=\eta,\theta$. Also, since $(p-1)r_1<1$, for $\alpha_1+\beta_1\gg 1$ and $\alpha_2+\beta_2\gg 1$,

$$
\begin{aligned}
& a_0(\alpha_1+\beta_1)^{r_1}\Big[\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}}-\frac{(1-\eta)(p-1)}{p-1+\eta}\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\eta p}{p-1+\eta}}}\Big] \\
& \le a(x)(\alpha_1+\beta_1)^{r_1}\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\eta}}
\end{aligned}
$$

$$
\begin{aligned}
&\le a(x)(\alpha_1+\beta_1)\Big(\gamma_1-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)^{\eta}}\Big)\\
&= a(x)\Big[\alpha_1\Big(\gamma_1-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)^{\eta}}\Big)\\
&\quad+\beta_1\Big(\gamma_1-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)^{\eta}}\Big)\Big]\\
&\le a(x)\Big\{\alpha_1\Big[f\Big((\alpha_2+\beta_2)^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}(\frac{q-1+\theta}{q})\sigma\Big)\\
&\quad-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)^{\eta}}\Big]\\
&\quad+\beta_1\Big[h\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)\\
&\quad-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\sigma\Big)^{\eta}}\Big]\Big\}\\
&\le a(x)\Big\{\alpha_1\Big[f\Big((\alpha_2+\beta_2)^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}(\frac{q-1+\theta}{q})\phi_{1,q}^{\frac{q}{q-1+\theta}}\Big)\\
&\quad-\frac{1}{\big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\big)^{\eta}}\Big]\\
&\quad+\beta_1\Big[h\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\Big)\\
&\quad-\frac{1}{\Big((\alpha_1+\beta_1)^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}(\frac{p-1+\eta}{p})\phi_{1,p}^{\frac{p}{p-1+\eta}}\Big)^{\eta}}\Big]\Big\}\\
&= a(x)\Big[\alpha_1\Big(f(\psi_2)-\frac{1}{\psi_1^{\eta}}\Big)+\beta_1\Big(h(\psi_1)-\frac{1}{\psi_1^{\eta}}\Big)\Big].
\end{aligned}
$$

Hence, if $\alpha_1+\beta_1\gg 1$ and $\alpha_2+\beta_2\gg 1$, we see that

$$
\begin{aligned}
&M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx\\
&\le \int_\Omega a(x)\Big[\alpha_1\Big(f(\psi_2)-\frac{1}{\psi_1^{\eta}}\Big)+\beta_1\Big(h(\psi_1)-\frac{1}{\psi_1^{\eta}}\Big)\Big]w\,dx.
\end{aligned}
$$

Similarly, for $\alpha_1+\beta_1\gg 1$ and $\alpha_2+\beta_2\gg 1$, we get

$$
\begin{aligned}
&M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx\\
&\le \int_\Omega b(x)\Big[\alpha_2\Big(g(\psi_1)-\frac{1}{\psi_2^{\theta}}\Big)+\beta_2\Big(k(\psi_2)-\frac{1}{\psi_2^{\theta}}\Big)\Big]w\,dx.
\end{aligned}
$$

This means that, (ψ_1,ψ_2) is a positive subsolution of (1.1).

Now, we construct a supersolution $(z_1,z_2)\ge(\psi_1,\psi_2)$. By (H1) and (H2) we can choose $C\gg 1$ so that

$$
\frac{m_1}{\|a\|_\infty}\ge\frac{\alpha_1 f\Big([\frac{\|b\|_\infty(\alpha_2+\beta_2)}{m_2}]^{\frac{1}{q-1}}[g(C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}}\|\zeta_q\|_\infty\Big)+\beta_1 h\big(C\|\zeta_p\|_\infty\big)}{C^{p-1}}.
$$

Let

$$(z_1, z_2) = \Big(C\zeta_p \, , \, \Big[\frac{\|b\|_\infty (\alpha_2 + \beta_2)}{m_2} \Big]^{\frac{1}{q-1}} [g(C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}} \zeta_q \Big).$$

We shall show that (z_1, z_2) is a supersolution of (1.1). Then

$$\begin{aligned}
& M_1 \Big(\int_\Omega |\nabla z_1|^p dx \Big) \int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w \, dx \\
&= C^{p-1} M_1 \Big(\int_\Omega |\nabla z_1|^p dx \Big) \int_\Omega |\nabla \zeta_p|^{p-2} \nabla \zeta_p \cdot \nabla w \, dx \\
&= C^{p-1} M_1 \Big(\int_\Omega |\nabla z_1|^p dx \Big) \int_\Omega w \, dx \\
&\geq m_1 C^{p-1} \int_\Omega w \, dx \\
&\geq \|a\|_\infty \int_\Omega \Big[\alpha_1 f \Big(\Big[\frac{\|b\|_\infty (\alpha_2 + \beta_2)}{m_2} \Big]^{\frac{1}{q-1}} [g(C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}} \|\zeta_q\|_\infty \Big) \\
&\quad + \beta_1 h\big(C\|\zeta_p\|_\infty \big) \Big] w dx \\
&\geq \int_\Omega a(x) \Big[\alpha_1 \Big(f(z_2) - \frac{1}{z_1^\eta} \Big) + \beta_1 \Big(h(z_1) - \frac{1}{z_1^\eta} \Big) \Big] w \, dx.
\end{aligned}$$

Again by (H2) for C large enough we have

$$g(C\|\zeta(x)\|_\infty) \geq k \Big(\Big[\frac{\|b\|_\infty (\alpha_2 + \beta_2)}{m_2} \Big]^{\frac{1}{q-1}} [g(C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}} \|\zeta_q\|_\infty \Big).$$

Hence

$$\begin{aligned}
& M_2 \Big(\int_\Omega |\nabla z_2|^q dx \Big) \int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w \, dx \\
&= \frac{\|b\|_\infty (\alpha_2 + \beta_2)}{m_2} M_2 \Big(\int_\Omega |\nabla z_2|^q dx \Big) \int_\Omega g(C\|\zeta_p\|_\infty) w \, dx \\
&\geq \int_\Omega b(x) \Big\{ \alpha_2 g(C\|\zeta_p\|_\infty) + \beta_2 g(C\|\zeta_p\|_\infty) \Big\} w \, dx \\
&\geq \int_\Omega b(x) \Big\{ \alpha_2 g(C\|\zeta_p\|_\infty) \\
&\quad + \beta_2 k \Big(\Big[\frac{\|b\|_\infty (\alpha_2 + \beta_2)}{m_2} \Big]^{\frac{1}{q-1}} [g(C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}} \|\zeta_q\|_\infty \Big\} w \, dx \\
&\geq \int_\Omega b(x) \Big(\alpha_2 g(z_1) + \beta_2 k(z_2) \Big) w \, dx \\
&\geq \int_\Omega b(x) \Big[\alpha_2 \Big(g(z_1) - \frac{1}{z_2^\theta} \Big) + \beta_2 \Big(k(z_2) - \frac{1}{z_2^\theta} \Big) \Big] w \, dx.
\end{aligned}$$

i.e., (z_1, z_2) is a supersolution of (1.1). Furthermore, C can be chosen large enough so that $(z_1, z_2) \geq (\psi_1, \psi_2)$. Thus, there exists a positive solution (u, v) of (1.1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof. $\square$

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Sayyed Hashem Rasouli

Department of Mathematics, Faculty of Basic Sciences, Babol University of Technology, Babol, Iran

E-mail address: s.h.rasouli@nit.ac.ir