

## NONLINEAR ELLIPTIC EQUATIONS AND SYSTEMS WITH LINEAR PART AT RESONANCE

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ABSTRACT. The famous result of Landesman and Lazer [10] dealt with resonance at a simple eigenvalue. Soon after publication of [10], Williams [14] gave an extension for repeated eigenvalues. The conditions in Williams [14] are rather restrictive, and no examples were ever given. We show that seemingly different classical result by Lazer and Leach [11], on forced harmonic oscillators at resonance, provides an example for this theorem. The article by Williams [14] also contained a shorter proof. We use a similar approach to study resonance for  $2 \times 2$  systems. We derive conditions for existence of solutions, which turned out to depend on the spectral properties of the linear part.

### 1. INTRODUCTION

The famous article by Landesman and Lazer [10] considered a semilinear Dirichlet problem at resonance (on a bounded domain  $D \subset \mathbb{R}^n$ )

$$\Delta u + \lambda_k u + g(u) = f(x), \quad x \in D, \quad u = 0 \quad \text{on } \partial D. \quad (1.1)$$

Assuming that  $g(u)$  has finite limits at  $\pm\infty$ , and  $\lambda_k$  is a simple eigenvalue of  $-\Delta$ , they gave a necessary and sufficient condition for the existence of solutions. This nonlinear version of the Fredholm alternative has generated an enormous body of research, and perhaps it can be seen as the beginning of the modern theory of nonlinear PDE's. Soon after publication of [10], a more elementary proof was given by Williams [14]. Both [10] and [14] were using Schauder's fixed point theorem. Williams [14] has observed that one can also handle the case of multiple eigenvalues under a straightforward extension of the Landesman and Lazer condition. No examples when this condition can be verified for repeated eigenvalues were ever given. Our first result is to observe that another famous result of Lazer and Leach [11], on forced harmonic oscillators at resonance, provides an example for this theorem, for the case of double eigenvalues of the periodic problem in one dimension. This appears to be the only known example in case  $\lambda_k$  is not simple. It relies on some special properties of the sine and cosine functions. Thus one has a uniform framework for these results, into which the existence result of de Figueiredo and Ni [2], its generalization by Iannacci et al [5] (and our two extensions) also fit in

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nically. We review these results, and connect them to the recent work of Korman and Li [9], and Korman [8].

We use a similar approach to give a rather complete discussion of  $2 \times 2$  elliptic systems at resonance, in case its linear part has constant coefficients. Requiring that finite limits at infinity exist, as in Landesman and Lazer, appears to be too restrictive for systems. Instead, we use more general inequality type conditions, which are rooted in Lazer and Leach [11]. We derive several sufficient conditions of this type for systems at resonance, which turned out to depend on the spectral properties of the linear part.

## 2. AN EXPOSITION AND EXTENSIONS OF KNOWN RESULTS

Given a bounded domain  $D \subset \mathbb{R}^n$ , with a smooth boundary, we denote by  $\lambda_k$  the eigenvalues of the Dirichlet problem

$$\Delta u + \lambda u = 0, \quad x \in D, \quad u = 0 \quad \text{on } \partial D, \quad (2.1)$$

and by  $\varphi_k(x)$  the corresponding eigenfunctions. For the *resonant* problem

$$\Delta u + \lambda_k u = f(x), \quad x \in D, \quad u = 0 \quad \text{on } \partial D, \quad (2.2)$$

with a given  $f(x) \in L^2(D)$ , the following well-known *Fredholm alternative* holds: the problem (2.2) has a solution if and only if

$$\int_D f(x) \varphi_k(x) dx = 0. \quad (2.3)$$

One could expect things to be considerably harder for the nonlinear problem

$$\Delta u + \lambda_k u + g(u) = f(x), \quad x \in D, \quad u = 0 \quad \text{on } \partial D. \quad (2.4)$$

However, in the classical papers of Lazer and Leach [11], and Landesman and Lazer [10] an interesting class of nonlinearities  $g(u)$  was identified, for which one still has an analog of the Fredholm alternative. Namely, one assumes that the finite limits  $g(-\infty)$  and  $g(\infty)$  exist, and

$$g(-\infty) < g(u) < g(\infty), \quad \text{for all } u \in R. \quad (2.5)$$

From (2.4),

$$\int_D g(u(x)) \varphi_k(x) dx = \int_D f(x) \varphi_k(x) dx, \quad (2.6)$$

which implies, in view of (2.5),

$$\begin{aligned} & g(-\infty) \int_{\varphi_k > 0} \varphi_k dx + g(\infty) \int_{\varphi_k < 0} \varphi_k dx \\ & < \int_D f(x) \varphi_k dx < g(\infty) \int_{\varphi_k > 0} \varphi_k dx + g(-\infty) \int_{\varphi_k < 0} \varphi_k dx. \end{aligned} \quad (2.7)$$

This is a necessary condition for solvability. It was proved by Landesman and Lazer [10] that this condition is also sufficient for solvability.

**Theorem 2.1** ([10]). *Assume that  $\lambda_k$  is a simple eigenvalue, while  $g(u) \in C(R)$  satisfies (2.5). Then for any  $f(x) \in L^2(D)$  satisfying (2.7), the problem (2.4) has a solution  $u(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$ .*

We shall prove the sufficiency part under a condition on  $g(u)$ , which is more general than (2.5). This condition had originated in Lazer and Leach [11], and it turned out to be appropriate when studying systems (see the next section). We assume that  $g(u) \in C(R)$  is bounded on  $R$ , and there exist numbers  $c, d, C$  and  $D$ , with  $c < d$  and  $C < D$ , such that

$$g(u) > D \quad \text{for } u > d, \quad (2.8)$$

$$g(u) < C \quad \text{for } u < c. \quad (2.9)$$

Define

$$L_2 = D \int_{\varphi_k > 0} \varphi_k \, dx + C \int_{\varphi_k < 0} \varphi_k \, dx, \quad L_1 = C \int_{\varphi_k > 0} \varphi_k \, dx + D \int_{\varphi_k < 0} \varphi_k \, dx.$$

Observe that  $L_2 > L_1$ , because  $D > C$ . We shall denote by  $\varphi_k^\perp$  the subspace of  $L^2(D)$ , consisting of functions satisfying (2.3).

**Theorem 2.2.** *Assume that  $\lambda_k$  is a simple eigenvalue, while  $g(u) \in C(R)$  is bounded on  $R$ , and satisfies (2.8) and (2.9). Then for any  $f(x) \in L^2(D)$  satisfying*

$$L_1 < \int_D f(x) \varphi_k \, dx < L_2, \quad (2.10)$$

problem (2.4) has a solution  $u(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$ .

*Proof.* Normalize  $\varphi_k(x)$ , so that  $\int_D \varphi_k^2(x) \, dx = 1$ . Denoting  $A_k = \int_D f(x) \varphi_k \, dx$ , we decompose  $f(x) = A_k \varphi_k(x) + e(x)$ , with  $e(x) \in \varphi_k^\perp$  (where  $\varphi_k^\perp$  is a subspace of  $L^2(D)$ ). Similarly, we decompose the solution  $u(x) = \xi_k \varphi_k(x) + U(x)$ , with  $U(x) \in \varphi_k^\perp$ . We rewrite (2.6), and then (2.4), as

$$\int_D g(\xi_k \varphi_k(x) + U(x)) \varphi_k(x) \, dx = A_k, \quad (2.11)$$

$$\begin{aligned} \Delta U + \lambda_k U &= -g(\xi_k \varphi_k + U) + \varphi_k \int_D g(\xi_k \varphi_k + U) \varphi_k \, dx + e, \quad x \in D \\ U &= 0 \quad \text{on } \partial D. \end{aligned} \quad (2.12)$$

Equations (2.11) and (2.12) constitute the classical Lyapunov-Schmidt reduction of the problem (2.4). To solve this system, we set up a map  $T : (\eta_k, V) \rightarrow (\xi_k, U)$ , taking the space  $R \times \varphi_k^\perp$  into itself, by solving the equation

$$\begin{aligned} \Delta U + \lambda_k U &= -g(\eta_k \varphi_k + V) + \varphi_k \int_D g(\eta_k \varphi_k + V) \varphi_k \, dx + e, \quad x \in D \\ U &= 0 \quad \text{on } \partial D, \end{aligned} \quad (2.13)$$

for  $U$ , and then setting

$$\xi_k = \eta_k + A_k - \int_D g(\eta_k \varphi_k(x) + U(x)) \varphi_k(x) \, dx. \quad (2.14)$$

The right hand side of (2.13) is orthogonal to  $\varphi_k$ , and so by the Fredholm alternative we can find infinitely many solutions  $U = U_0 + c\varphi_k$ . Then we can choose  $c$ , so that  $U \in \varphi_k^\perp$ .

Assume first that  $f(x) \in L^\infty(D)$ . By the elliptic theory, we can estimate  $\|U\|_{W^{2,p}(D)}$  by the  $L^p$  norm of the right hand side of (2.13) plus  $\|U\|_{L^p(D)}$ , for any  $p > 1$ . Since the homogeneous equation, corresponding to (2.13), has only the trivial solution in  $\varphi_k^\perp$ , the  $\|U\|_{L^p(D)}$  term can be dropped, giving us a uniform

estimate of  $\|U\|_{W^{2,p}(D)}$ . By the Sobolev embedding, for some constant  $c > 0$  (for  $p$  large enough)

$$\|U\|_{L^\infty(D)} \leq c \quad \text{uniformly in } (\eta_k, V) \in R \times \varphi_k^\perp. \quad (2.15)$$

This implies that if  $\eta_k$  is large and positive, the integral in (2.11) is greater than  $L_2$ . When  $\eta_k$  is large in absolute value and negative, the integral in (2.11) is smaller than  $L_1$ . By our condition (2.10), it follows that for  $\eta_k$  large and positive,  $\xi_k < \eta_k$ , while for  $\eta_k$  large in absolute value and negative,  $\xi_k > \eta_k$ . Hence, we can find a large  $N$ , so that if  $\eta_k \in (-N, N)$ , then  $\xi_k \in (-N, N)$ . The map  $T : (\eta_k, V) \rightarrow (\xi_k, U)$  is a continuous and compact map, taking a sufficiently large ball of  $R \times \varphi_k^\perp$  into itself. By Schauder's fixed point theorem (see e.g., Nirenberg [13]) it has a fixed point, which gives us a solution of the problem (2.4). (A fixed point of (2.14) is a solution of (2.11).)

In case  $f(x) \in L^2(D)$ , a little more care is needed to show that the integral in (2.14) is greater (smaller) than  $A_k$ , for  $\eta_k$  positive (negative) and large. Elliptic estimates give us

$$\|U\|_{L^2(D)} \leq c \quad \text{uniformly in } (\eta_k, V) \in R \times \varphi_k^\perp. \quad (2.16)$$

Set  $G = \sup_{x \in D, u \in R} |g(u)\varphi_k(x)|$ , and decompose

$$\int_D g(\eta_k \varphi_k(x) + U(x))\varphi_k(x) dx = \int_{\varphi_k > 0} g\varphi_k dx + \int_{\varphi_k < 0} g\varphi_k dx. \quad (2.17)$$

The first integral we decompose further, keeping the same integrand,

$$\int_{\varphi_k > 0} dx = \int_{0 < \varphi_k < \delta} dx + \int_{A_2} dx + \int_{A_3} dx \equiv I_1 + I_2 + I_3,$$

with  $A_2 = (\varphi_k > \delta) \cap (|U| > \frac{\eta_k \varphi_k}{2})$ , and  $A_3 = (\varphi_k > \delta) \cap (|U| < \frac{\eta_k \varphi_k}{2})$ . Given any  $\epsilon$ , we fix  $\delta$  so that the measure of the set  $\{0 < \varphi_k(x) < \delta\}$  is less than  $\epsilon$ . Then  $|I_1| < \epsilon G$ . In  $I_2$  we integrate over the set, where  $|U| > \frac{\eta_k \delta}{2}$ . Since  $U$  is bounded in  $L^2$  uniformly in  $\eta_k$ , the measure of this set will get smaller than  $\epsilon$  for  $\eta_k$  large, and then  $|I_2| < \epsilon G$ . In  $I_3$  we have  $g(u) > D$  for  $\eta_k$  large, and we integrate over the subset of the domain  $D_+ = \{x : \varphi_k(x) > 0\}$ , whose measure is smaller than that of  $D_+$  by no more than  $2\epsilon$ . Hence  $I_3 > D \int_{\varphi_k > 0} \varphi_k(x) dx - 2\epsilon G$ , and then

$$\int_{\varphi_k > 0} g\varphi_k dx > D \int_{\varphi_k > 0} \varphi_k(x) dx - 4\epsilon G.$$

Proceeding similarly with the second integral in (2.17), we conclude that

$$\int_D g(\eta_k \varphi_k(x) + U(x))\varphi_k(x) dx > L_2 - 8\epsilon G > A_k,$$

if  $\epsilon$  is small, which can be achieved for  $\eta_k > 0$  and large. Similarly, we show that this integral is smaller than  $A_k$  for  $\eta_k < 0$  and large.  $\square$

In case  $\lambda_k$  has a multidimensional eigenspace, a generalization of Landesman and Lazer [10] result follows by a similar argument, under a suitable condition. This was observed by S.A. Williams [14], back in 1970. Apparently no such examples in the PDE case for the Williams [14] condition were ever given. We remark that Williams [14] contained a proof of Landesman and Lazer [10] result, which is similar to the one above, see also a recent paper of Hastings and McLeod [4]. Our proof below is a little shorter than in [14].

**Theorem 2.3** ([14]). *Assume that  $g(u)$  satisfies (2.5),  $f(x) \in L^2(D)$ , while for any  $w(x)$  belonging to the eigenspace of  $\lambda_k$*

$$\int_D f(x)w(x) dx < g(\infty) \int_{w>0} w dx + g(-\infty) \int_{w<0} w dx. \quad (2.18)$$

*Then problem (2.4) has a solution. Condition (2.18) is also necessary for the existence of solutions.*

*Proof.* Let  $E \subset L^2(D)$  denote the eigenspace of  $\lambda_k$ , and let  $\varphi_1, \varphi_2, \dots, \varphi_m$  be its orthogonal basis, with  $\int_D \varphi_i^2(x) dx = 1$ ,  $1 \leq i \leq m$ . Denoting  $A_i = \int_D f(x)\varphi_i dx$ , we decompose  $f(x) = \sum_{i=1}^m A_i\varphi_i(x) + e(x)$ , with  $e(x) \in E^\perp$  (where  $E^\perp$  is a subspace of  $L^2(D)$ ). Similarly, we decompose the solution  $u(x) = \sum_{i=1}^m \xi_i\varphi_i(x) + U(x)$ , with  $U(x) \in E^\perp$ . We have

$$\int_D g\left(\sum_{i=1}^m \xi_i\varphi_i + U(x)\right)\varphi_i(x) dx = A_i, \quad i = 1, \dots, m, \quad (2.19)$$

$$\begin{aligned} \Delta U + \lambda_k U &= -g\left(\sum_{i=1}^m \xi_i\varphi_i + U\right) + \sum_{i=1}^m \varphi_i \int_D g\left(\sum_{i=1}^m \xi_i\varphi_i + U\right)\varphi_i dx + e \\ U &= 0 \quad \text{on } \partial D. \end{aligned} \quad (2.20)$$

Equations (2.19) and (2.20) constitute the classical Lyapunov-Schmidt reduction of problem (2.4). To solve this system, we set up a map  $T : (\eta_1, \dots, \eta_m, V) \rightarrow (\xi_1, \dots, \xi_m, U)$ , taking the space  $R^m \times E^\perp$  into itself, by solving the equation

$$\begin{aligned} \Delta U + \lambda_k U &= -g\left(\sum_{i=1}^m \eta_i\varphi_i + V\right) + \sum_{i=1}^m \varphi_i \int_D g\left(\sum_{i=1}^m \eta_i\varphi_i + V\right)\varphi_i dx + e \\ U &= 0 \quad \text{on } \partial D, \end{aligned} \quad (2.21)$$

for  $U$ , and then setting

$$\xi_i = \eta_i + A_i - \int_D g\left(\sum_{i=1}^m \eta_i\varphi_i(x) + U(x)\right)\varphi_i(x) dx, \quad i = 1, \dots, m. \quad (2.22)$$

The right-hand side of (2.21) is orthogonal to all  $\varphi_i$ , and so by the Fredholm alternative we can find infinitely many solutions  $U = U_0 + \sum_{i=1}^m c_i\varphi_i$ . Then we can choose  $c_i$ , so that  $U \in E^\perp$ .

As before, we get a bound on  $\|U\|_{L^2(D)}$ , uniformly in  $(\eta_1, \dots, \eta_m, V)$ . Denoting  $I_i = \int_D g\left(\sum_{i=1}^m \eta_i\varphi_i(x) + U(x)\right)\varphi_i(x) dx$ , we have

$$\sum_{i=1}^m \xi_i^2 = \sum_{i=1}^m \eta_i^2 + 2 \sum_{i=1}^m \eta_i(A_i - I_i) + \sum_{i=1}^m (A_i - I_i)^2. \quad (2.23)$$

Denoting  $w = \sum_{i=1}^m \frac{\eta_i}{\sqrt{\eta_1^2 + \dots + \eta_m^2}} \varphi_i$ , we have

$$\begin{aligned} \sum_{i=1}^m \eta_i(A_i - I_i) &= \sqrt{\eta_1^2 + \dots + \eta_m^2} \left( \int_D f w dx - \int_D g(\sqrt{\eta_1^2 + \dots + \eta_m^2} w + U) w dx \right) \\ &< -\epsilon \sqrt{\eta_1^2 + \dots + \eta_m^2}, \end{aligned}$$

for some  $\epsilon > 0$ , when  $\sqrt{\eta_1^2 + \cdots + \eta_m^2}$  is large, in view of condition (2.18). If we denote by  $h$  an upper bound on all of  $(A_i - I_i)^2$ , then from (2.23)

$$\sum_{i=1}^m \xi_i^2 < \sum_{i=1}^m \eta_i^2 - 2\epsilon\sqrt{\eta_1^2 + \cdots + \eta_m^2} + mh < \sum_{i=1}^m \eta_i^2,$$

for  $\sqrt{\eta_1^2 + \cdots + \eta_m^2}$  large. Then the map  $T$  is a compact and continuous map of a sufficiently large ball in  $R^m \times E^1$  into itself, and we have a solution by Schauder's fixed point theorem.  $\square$

Condition (2.18) implies the Landesman and Lazer [10] condition (2.7). Indeed, if  $\lambda_k$  is a simple eigenvalue, then  $w = b\varphi_k$ . If the constant  $b > 0$  ( $b < 0$ ), then (2.18) implies the right (left) inequality in (2.7).

In the ODE case a famous example of resonance with a two-dimensional eigenspace is the result of Lazer and Leach [11], which we describe next. We seek to find  $2\pi$  periodic solutions  $u = u(t)$  of

$$u'' + n^2u + g(u) = f(t), \quad (2.24)$$

with a given continuous  $2\pi$  periodic  $f(t) = f(t + 2\pi)$ , and  $n$  is a positive integer. The linear part of this problem is at resonance, because

$$u'' + n^2u = 0$$

has a two-dimensional  $2\pi$  periodic null space, spanned by  $\cos nt$  and  $\sin nt$ . The following result is included in Lazer and Leach [11].

**Theorem 2.4** ([11]). *Assume that  $g(u) \in C(R)$  satisfies (2.5). Define*

$$A = \int_0^{2\pi} f(t) \cos nt \, dt, \quad B = \int_0^{2\pi} f(t) \sin nt \, dt. \quad (2.25)$$

*Then a  $2\pi$  periodic solution of (2.24) exists if and only if*

$$\sqrt{A^2 + B^2} < 2(g(\infty) - g(-\infty)). \quad (2.26)$$

The following elementary lemma is easy to prove.

**Lemma 2.5.** *Consider a function  $\cos(nt - \delta)$ , with an integer  $n$  and any real  $\delta$ . Denote  $P = \{t \in (0, 2\pi) : \cos(nt - \delta) > 0\}$  and  $N = \{t \in (0, 2\pi) : \cos(nt - \delta) < 0\}$ . Then*

$$\int_P \cos(nt - \delta) \, dt = 2, \quad \int_N \cos(nt - \delta) \, dt = -2.$$

We show next that Theorem 2.4 of Lazer and Leach provides an example to Theorem 2.3 of Williams. Indeed, any eigenfunction corresponding to  $\lambda_n = n^2$  is of the form  $w = a \cos nt + b \sin nt$ ,  $a, b \in R$ , or  $w = \sqrt{a^2 + b^2} \cos(nt - \delta)$  for some  $\delta$ . The left hand side of (2.18) is  $aA + bB$ , while the integral on the right is equal to  $2\sqrt{a^2 + b^2}(g(\infty) - g(-\infty))$ , in view of Lemma 2.5. We then rewrite (2.18) in terms of a scalar product of two vectors

$$\left( \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right) \cdot (A, B) < 2(g(\infty) - g(-\infty)) \quad \text{for all } a \text{ and } b,$$

which is equivalent to the Lazer and Leach condition (2.26).

Another perturbation of a harmonic oscillator at resonance was considered by Lazer and Frederickson [3], and Lazer [12]:

$$u'' + g(u)u' + n^2u = f(t). \quad (2.27)$$

Here  $f(t) \in C(R)$  satisfies  $f(t + 2\pi) = f(t)$  for all  $t$ ,  $g(u) \in C(R)$ ,  $n \geq 1$  is an integer. Define  $G(u) = \int_0^u g(t) dt$ . We assume that the finite limits  $G(\infty)$  and  $G(-\infty)$  exist, and

$$G(-\infty) < G(u) < G(\infty) \quad \text{for all } u. \quad (2.28)$$

**Theorem 2.6.** *Assume that (2.28) holds, and let  $A$  and  $B$  be again defined by (2.25). Then the condition*

$$\sqrt{A^2 + B^2} < 2n(G(\infty) - G(-\infty)) \quad (2.29)$$

*is necessary and sufficient for the existence of  $2\pi$  periodic solution of (2.27).*

This result was proved in Lazer [12] for  $n = 1$ , and by Korman and Li [9], for  $n \geq 1$ . Observe that the condition for solvability now depends on  $n$ , unlike the Lazer and Leach condition (2.26). Also, Theorem 2.6 does not carry over to boundary value problems, see [9], unlike the result of Lazer and Leach.

Next, we discuss the result of de Figueiredo and Ni [2], involving resonance at the principal eigenvalue.

**Theorem 2.7** ([2]). *Consider the problem*

$$\Delta u + \lambda_1 u + g(u) = e(x), \quad x \in D, \quad u = 0 \quad \text{on } \partial D. \quad (2.30)$$

*Assume that  $e(x) \in L^\infty(D)$  satisfies  $\int_D e(x)\varphi_1(x) dx = 0$ , while the function  $g(u) \in C(R)$  is a bounded function, satisfying*

$$ug(u) > 0, \quad \text{for all } u \in R. \quad (2.31)$$

*Then the problem (2.30) has a solution  $u(x) \in W^{2,p}(D) \cap W_0^{1,p}(D)$ , for any  $p > 1$ .*

If, in addition to (2.31), we have

$$\liminf_{u \rightarrow \infty} g(u) > 0, \quad \text{and} \quad \limsup_{u \rightarrow -\infty} g(u) < 0, \quad (2.32)$$

then the previous Theorem 2.2 applies. The result of de Figueiredo and Ni [2] allows either one (or both) of these limits to be zero.

*Proof of Theorem 2.7.* Again we follow the proof of Theorem 2.2. As before, we set up the map  $T : (\eta_1, V) \rightarrow (\xi_1, U)$ , taking the space  $R \times \varphi_1^\perp$  into itself. We use (2.13), with  $k = 1$ , to compute  $U$ , while equation (2.14) takes the form

$$\xi_1 = \eta_1 - \int_D g(\eta_1\varphi_1(x) + U(x))\varphi_1(x) dx. \quad (2.33)$$

Since  $\|U\|_{L^\infty(D)}$  is bounded uniformly in  $(\eta_1, V)$ , we can find  $M > 0$ , so that for all  $x \in D$

$$\begin{aligned} \xi_1\varphi_1(x) + U(x) &> 0, \quad \text{for } \xi_1 > M \\ \xi_1\varphi_1(x) + U(x) &< 0, \quad \text{for } \xi_1 < -M. \end{aligned}$$

Hence,  $\xi_1 < \eta_1$  for  $\eta_1 > 0$  and large, while  $\xi_1 > \eta_1$  for  $\eta_1 < 0$  and  $|\eta_1|$  large. As before, the map  $T : (\eta_1, V) \rightarrow (\xi_1, U)$  is a continuous and compact map, taking a sufficiently large ball of  $R \times \varphi_1^\perp$  into itself, and the proof follows by Schauder's fixed point theorem.  $\square$

This result was generalized to unbounded  $g(u)$  by Iannacci, Nkashama and Ward [5]. We now extend Theorem 2.7 to the higher eigenvalues.

**Theorem 2.8.** *Consider the problem*

$$\Delta u + \lambda_k u + g(u) = e(x), \quad x \in D, \quad u = 0 \quad \text{on } \partial D. \quad (2.34)$$

Assume that  $\lambda_k$ ,  $k \geq 1$ , is a simple eigenvalue, and  $e(x) \in L^\infty(D)$  satisfies  $\int_D e(x)\varphi_k(x) dx = 0$ , while the function  $g(u) \in C(R)$  is a bounded function, satisfying (2.31) and (2.32). Then the problem (2.34) has a solution  $u(x) \in W^{2,p}(D) \cap W_0^{1,p}(D)$ , for any  $p > 1$ .

*Proof.* We follow the proof of Theorem 2.2, replacing problem (2.34) by its Lyapunov-Schmidt decomposition (2.11), (2.12), then setting up the map  $T$ , given by (2.13) and

$$\xi_k = \eta_k - \int_D g(\eta_k \varphi_k(x) + U(x)) \varphi_k(x) dx.$$

Since  $g(u)$  is bounded, the  $L^\infty$  estimate (2.15) holds. By our conditions on  $g(u)$ ,  $\xi_k < \eta_k$  ( $\xi_k > \eta_k$ ), provided that  $\eta_k > 0$  ( $\eta_k < 0$ ) and  $|\eta_k|$  is large. As in the proof of Theorem 2.2, we conclude that the map  $T$  has a fixed point.  $\square$

We now review another extension of the result of Iannacci et al [5] to the problem

$$\Delta u + \lambda_1 u + g(u) = \mu_1 \varphi_1 + e(x) \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial D, \quad (2.35)$$

with  $e(x) \in \varphi_1^\perp$ . Decompose the solution  $u(x) = \xi_1 \varphi_1 + U$ , with  $U \in \varphi_1^\perp$ . We wish to find a solution pair  $(u, \mu_1) = (u, \mu_1)(\xi_1)$ , i.e., the global solution curve. We proved the following result in Korman [8].

**Theorem 2.9.** *Assume that  $g(u) \in C^1(R)$  satisfies*

$$\begin{aligned} ug(u) &> 0 \quad \text{for all } u \in R, \\ g'(u) &\leq \gamma < \lambda_2 - \lambda_1 \quad \text{for all } u \in R. \end{aligned}$$

Then there is a continuous curve of solutions of (2.35):  $(u(\xi_1), \mu_1(\xi_1))$ ,  $u \in H^2(D) \cap H_0^1(D)$ , with  $-\infty < \xi_1 < \infty$ , and  $\int_D u(\xi_1)\varphi_1 dx = \xi_1$ . This curve exhausts the solution set of (2.35). The continuous function  $\mu_1(\xi_1)$  is positive for  $\xi_1 > 0$  and large, and  $\mu_1(\xi_1) < 0$  for  $\xi_1 < 0$  and  $|\xi_1|$  large. In particular,  $\mu_1(\xi_1^0) = 0$  at some  $\xi_1^0$ , i.e., we have a solution of

$$\Delta u + \lambda_1 u + g(u) = e(x) \quad \text{on } D, \quad u = 0 \quad \text{on } \partial D.$$

We see that the result of Iannacci et al [5] corresponds to just one point on this solution curve.

### 3. RESONANCE FOR SYSTEMS

We begin by considering linear systems of the type

$$\begin{aligned} \Delta u + au + bv &= f(x) \quad x \in D, \quad u = 0 \quad \text{on } \partial D \\ \Delta v + cu + dv &= g(x) \quad x \in D, \quad v = 0 \quad \text{on } \partial D, \end{aligned} \quad (3.1)$$

with given functions  $f(x)$  and  $g(x)$ , and constants  $a, b, c$  and  $d$ . As before, we denote by  $\lambda_k$  the eigenvalues of  $-\Delta$  with the Dirichlet boundary conditions (see (2.1)), and by  $\varphi_k(x)$  the corresponding eigenfunctions. We shall assume throughout this section that  $\int_D \varphi_k^2(x) dx = 1$ .

**Lemma 3.1.** *Assume that*

$$(a - \lambda_k)(d - \lambda_k) - bc \neq 0, \quad \text{for all } k \geq 1,$$

*i.e., all eigenvalues of  $-\Delta$  are not equal to the eigenvalues of the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .*

*Then for any pair  $(f, g) \in L^2(D) \times L^2(D)$  there exists a unique solution  $(u, v)$ , with  $u$  and  $v \in W^{2,2}(D) \cap W_0^{1,2}(D)$ . Moreover, for some  $c > 0$*

$$\|u\|_{W^{2,2}(D)} + \|v\|_{W^{2,2}(D)} \leq c (\|f\|_{L^2(D)} + \|g\|_{L^2(D)}).$$

*Proof.* Using Fourier series,  $f(x) = \sum_{k=1}^{\infty} f_k \varphi_k$ ,  $g(x) = \sum_{k=1}^{\infty} g_k \varphi_k$ ,  $u(x) = \sum_{k=1}^{\infty} u_k \varphi_k$  and  $v(x) = \sum_{k=1}^{\infty} v_k \varphi_k$ , we obtain the unique solution  $(u, v) \in L^2(D) \times L^2(D)$ . Using elliptic estimates for each equation separately, we obtain

$$\|u\|_{W^{2,2}(D)} + \|v\|_{W^{2,2}(D)} \leq c (\|u\|_{L^2(D)} + \|v\|_{L^2(D)} + \|f\|_{L^2(D)} + \|g\|_{L^2(D)}). \quad (3.2)$$

Since we have uniqueness of solution for (3.1), the extra terms  $\|u\|_{L^2(D)}$  and  $\|v\|_{L^2(D)}$  are removed in a standard way.  $\square$

The following two lemmas are proved similarly.

**Lemma 3.2.** *Consider the problem (here  $\mu$  is a constant)*

$$\begin{aligned} \Delta u + \lambda_k u &= f(x) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + \mu v &= g(x) & x \in D, & \quad v = 0 \quad \text{on } \partial D, \end{aligned} \quad (3.3)$$

*Assume that  $\mu \neq \lambda_n$ , for all  $n \geq 1$ ,  $f(x) \in \varphi_k^\perp$ ,  $g(x) \in L^2(D)$ . One can select a solution such that  $u(x) \in \varphi_k^\perp$ , and  $v(x) \in L^2(D)$ . Such a solution is unique, and the estimate (3.2) holds.*

**Lemma 3.3.** *Consider the problem*

$$\begin{aligned} \Delta u + \lambda_k u &= f(x) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + \lambda_m v &= g(x) & x \in D, & \quad v = 0 \quad \text{on } \partial D, \end{aligned} \quad (3.4)$$

*Assume that  $f(x) \in \varphi_k^\perp$ ,  $g(x) \in \varphi_m^\perp$ . One can select a solution such that  $u(x) \in \varphi_k^\perp$ , and  $v(x) \in \varphi_m^\perp$ . Such a solution is unique, and the estimate (3.2) holds.*

**Lemma 3.4.** *Consider the problem*

$$\begin{aligned} \Delta u + \lambda_k u + v &= f(x) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + \lambda_k v &= g(x) & x \in D, & \quad v = 0 \quad \text{on } \partial D, \end{aligned} \quad (3.5)$$

*a Assume that  $f(x) \in \varphi_k^\perp$ ,  $g(x) \in \varphi_k^\perp$ . One can select a solution such that  $u(x) \in \varphi_k^\perp$ , and  $v(x) \in \varphi_k^\perp$ . Such a solution is unique, and the estimate (3.2) holds.*

*Proof.* The second equation in (3.5) has infinitely many solutions of the form  $v = v_0 + c\varphi_k$ . We now select  $c$ , so that  $v \in \varphi_k^\perp$ . The first equation then takes the form

$$\Delta u + \lambda_k u = f - v_0 - c\varphi_k \in \varphi_k^\perp.$$

This equation has infinitely many solutions of the form  $u = u_0 + c_1\varphi_k$ . We select  $c_1$ , so that  $u \in \varphi_k^\perp$ .  $\square$

We now turn to nonlinear systems

$$\begin{aligned} \Delta u + au + bv + f(u, v) &= h(x) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + cu + dv + g(u, v) &= k(x) & x \in D, & \quad v = 0 \quad \text{on } \partial D, \end{aligned} \quad (3.6)$$

a with given  $h(x)$ ,  $k(x)$  and bounded  $f(u, v)$ ,  $g(u, v)$ . If there is no resonance, existence of solutions is easy.

**Theorem 3.5.** *Assume that  $(a - \lambda_n)(d - \lambda_n) - bc \neq 0$ , for all  $n \geq 1$ , and  $f(u, v)$ ,  $g(u, v)$  are bounded. Then for any pair  $(h, k) \in L^2(D) \times L^2(D)$  there exists a solution  $(u, v)$ , with  $u$  and  $v \in W^{2,2}(D) \cap W_0^{1,2}(D)$ .*

*Proof.* The map  $(w, z) \rightarrow (u, v)$ , obtained by solving

$$\begin{aligned} \Delta u + au + bv &= h(x) - f(w, z) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + cu + dv &= k(x) - g(w, z) & x \in D, & \quad v = 0 \quad \text{on } \partial D, \end{aligned}$$

in view of Lemma 3.1 (and Sobolev’s embedding), is a compact and continuous map of a sufficiently large ball around the origin in  $L^2(D) \times L^2(D)$  into itself, and Schauder’s fixed point theorem applies.  $\square$

Next, we discuss the case of resonance, when one of the eigenvalues of the coefficient matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $\lambda_k$ . We distinguish the following four possibilities: the second eigenvalue of  $A$  is not equal to  $\lambda_m$  for all  $m \geq 1$ , the second eigenvalue of  $A$  is equal to  $\lambda_m$  for some  $m \neq k$ , the second eigenvalue of  $A$  is equal to  $\lambda_k$ , and the matrix  $A$  is diagonalizable, and finally the second eigenvalue of  $A$  is equal to  $\lambda_k$ , and the matrix  $A$  is not diagonalizable. By a linear change of variables,  $(u, v) \rightarrow (u_1, v_1)$ ,  $\begin{bmatrix} u \\ v \end{bmatrix} = Q \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ , with a non-singular matrix  $Q$ , we transform the system (3.6) into

$$\Delta \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + Q^{-1}AQ \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = Q^{-1} \begin{bmatrix} h \\ k \end{bmatrix}$$

with the matrix  $Q^{-1}AQ$  being either diagonal, or the Jordan block  $\begin{bmatrix} \lambda_k & 1 \\ 0 & \lambda_k \end{bmatrix}$ . Let us assume that this change of variables has been performed, so that there are three canonical cases to consider.

We consider the system

$$\begin{aligned} \Delta u + \lambda_k u + f(u, v) &= h(x) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + \mu v + g(u, v) &= k(x) & x \in D, & \quad v = 0 \quad \text{on } \partial D. \end{aligned} \tag{3.7}$$

We assume that  $f(u, v)$ ,  $g(u, v) \in C(R \times R)$  are bounded on  $R \times R$ , and there exist numbers  $c, d, C$  and  $D$ , with  $c < d$  and  $C < D$ , such that

$$f(u, v) > D \quad \text{for } u > d, \quad \text{uniformly in } v \in R, \tag{3.8}$$

$$f(u, v) < C \quad \text{for } u < c, \quad \text{uniformly in } v \in R. \tag{3.9}$$

Define

$$L_2 = D \int_{\varphi_k > 0} \varphi_k \, dx + C \int_{\varphi_k < 0} \varphi_k \, dx, \quad L_1 = C \int_{\varphi_k > 0} \varphi_k \, dx + D \int_{\varphi_k < 0} \varphi_k \, dx.$$

Observe that  $L_2 > L_1$ , because  $D > C$ .

**Theorem 3.6.** *Assume that  $\lambda_k$  is a simple eigenvalue,  $\mu \neq \lambda_n$  for all  $n \geq 1$ , while  $f(u, v)$ ,  $g(u, v) \in C(R \times R)$  are bounded on  $R \times R$ , and satisfy (3.8) and (3.9). Assume that  $h(x)$ ,  $k(x) \in L^p(D)$ , for some  $p > n$ . Assume finally that*

$$L_1 < \int_D h(x)\varphi_k(x) \, dx < L_2. \tag{3.10}$$

Then the problem (3.7) has a solution  $(u, v)$ , with  $u, v \in W^{2,p}(D) \cap W_0^{1,p}(D)$ .

*Proof.* Denoting  $A_k = \int_D h(x)\varphi_k dx$ , we decompose  $h(x) = A_k\varphi_k(x) + e(x)$ , with  $e(x) \in \varphi_k^\perp$  (where  $\varphi_k^\perp$  is a subspace of  $L^2(D)$ ). Similarly, we decompose the solution  $u(x) = \xi_k\varphi_k(x) + U(x)$ , with  $U(x) \in \varphi_k^\perp$ . Multiply the first equation in (3.7) by  $\varphi_k$ , and integrate

$$\int_D f(\xi_k\varphi_k(x) + U(x), v)\varphi_k(x) dx = A_k. \tag{3.11}$$

Then the first equation in (3.7) becomes

$$\begin{aligned} \Delta U + \lambda_k U &= -f(\xi_k\varphi_k + U, v) + \varphi_k \int_D f(\xi_k\varphi_k + U, v)\varphi_k dx + e, \quad x \in D \\ U &= 0 \quad \text{on } \partial D. \end{aligned} \tag{3.12}$$

Equations (3.11) and (3.12) constitute the classical Lyapunov-Schmidt reduction of the first equation in (3.7). To solve (3.11), (3.12) and the second equation in (3.7), we set up a map  $T : (\alpha_k, W, Z) \rightarrow (\xi_k, U, V)$ , taking the space  $R \times \varphi_k^\perp \times L^2(D)$  into itself, by solving (separately) the linear equations

$$\begin{aligned} \Delta U + \lambda_k U &= -f(\alpha_k\varphi_k + W, Z) + \varphi_k \int_D f(\alpha_k\varphi_k + W, Z)\varphi_k dx + e, \\ \Delta V + \mu V &= -g(\alpha_k\varphi_k + W, Z) + k(x) \\ U = V &= 0 \quad \text{on } \partial D, \end{aligned} \tag{3.13}$$

and then setting

$$\xi_k = \alpha_k + A_k - \int_D f(\alpha_k\varphi_k + U, V)\varphi_k(x) dx. \tag{3.14}$$

By Lemma 3.2, the map  $T$  is well defined, and  $\|U\|_{W^{2,p}(D)}$  and  $\|V\|_{W^{2,p}(D)}$  are bounded, and then by the Sobolev embedding  $\|U\|_{C^1(D)}$  and  $\|V\|_{C^1(D)}$  are bounded uniformly in  $(\alpha_k, W, Z)$ . This implies that if  $\alpha_k$  is large and positive, the integral in (3.14) is greater than  $L_2 > A_k$ . When  $\alpha_k$  is large in absolute value and negative, the integral in (3.14) is smaller than  $L_1 < A_k$ . It follows that for  $\alpha_k$  large and positive,  $\xi_k < \alpha_k$ , while for  $\alpha_k$  large in absolute value and negative,  $\xi_k > \alpha_k$ . Hence, we can find a large  $N$ , so that if  $\alpha_k \in (-N, N)$ , then  $\xi_k \in (-N, N)$ . The map  $T$  is a continuous and compact map, taking a sufficiently large ball of  $R \times \varphi_k^\perp \times L^2(D)$  into itself. By Schauder's fixed point theorem it has a fixed point, which gives us a solution of (3.7).  $\square$

We consider next the system

$$\begin{aligned} \Delta u + \lambda_k u + f(u, v) &= h(x) \quad x \in D, \quad u = 0 \quad \text{on } \partial D \\ \Delta v + \lambda_m v + g(u, v) &= k(x) \quad x \in D, \quad v = 0 \quad \text{on } \partial D, \end{aligned} \tag{3.15}$$

a which includes, in particular, the case  $m = k$ . Assume that there exist numbers  $c_1, d_1, C_1$  and  $D_1$ , with  $c_1 < d_1$  and  $C_1 < D_1$ , such that

$$g(u, v) > D_1 \quad \text{for } v > d_1, \quad \text{uniformly in } u \in R, \tag{3.16}$$

$$g(u, v) < C_1 \quad \text{for } v < c_1, \quad \text{uniformly in } u \in R. \tag{3.17}$$

Define

$$M_2 = D_1 \int_{\varphi_m > 0} \varphi_m dx + C_1 \int_{\varphi_m < 0} \varphi_m dx,$$

$$M_1 = C_1 \int_{\varphi_m > 0} \varphi_m dx + D_1 \int_{\varphi_m < 0} \varphi_m dx.$$

Observe that  $M_2 > M_1$ , because  $D_1 > C_1$ .

**Theorem 3.7.** *Assume  $\lambda_k$  and  $\lambda_m$  are simple eigenvalues, while  $f(u, v)$ ,  $g(u, v)$  belong to  $C(R \times R)$  are bounded on  $R \times R$ , and satisfy (3.8), (3.9) and (3.16), (3.17). Assume that  $h(x)$ ,  $k(x) \in L^p(D)$ , for some  $p > n$ . Assume finally that*

$$L_1 < \int_D h(x)\varphi_k(x) dx < L_2, \quad \text{and} \quad M_1 < \int_D k(x)\varphi_m(x) dx < M_2. \quad (3.18)$$

Then problem (3.15) has a solution  $(u, v)$ , with  $u, v \in W^{2,p}(D) \cap W_0^{1,p}(D)$ .

*Proof.* Denoting  $A_k = \int_D h(x)\varphi_k dx$  and  $B_m = \int_D k(x)\varphi_m dx$ , we decompose  $h(x) = A_k\varphi_k(x) + e_1(x)$  and  $k(x) = B_m\varphi_m(x) + e_2(x)$ , with  $e_1(x) \in \varphi_k^\perp$  and  $e_2(x) \in \varphi_m^\perp$ . Similarly, we decompose the solution  $u(x) = \xi_k\varphi_k(x) + U(x)$  and  $v(x) = \eta_m\varphi_m(x) + V(x)$ , with  $U(x) \in \varphi_k^\perp$  and  $V(x) \in \varphi_m^\perp$ . As before, we write down the Lyapunov-Schmidt reduction of our problem (3.15)

$$\begin{aligned} \int_D f(\xi_k\varphi_k + U, \eta_m\varphi_m + V)\varphi_k(x) dx &= A_k \\ \int_D g(\xi_k\varphi_k + U, \eta_m\varphi_m + V)\varphi_m(x) dx &= B_m \\ \Delta U + \lambda_k U &= -f(\xi_k\varphi_k + U, \eta_m\varphi_m + V) \\ &\quad + \varphi_k \int_D f(\xi_k\varphi_k + U, \eta_m\varphi_m + V)\varphi_k dx + e_1, \\ \Delta V + \lambda_m V &= -g(\xi_k\varphi_k + U, \eta_m\varphi_m + V) \\ &\quad + \varphi_m \int_D g(\xi_k\varphi_k + U, \eta_m\varphi_m + V)\varphi_m dx + e_2, \\ U = V &= 0 \quad \text{on } \partial D. \end{aligned}$$

To solve this system, we set up a map  $T : (\alpha_k, W, \beta_m, Z) \rightarrow (\xi_k, U, \eta_m, V)$ , taking the space  $(R \times \varphi_k^\perp) \times (R \times \varphi_m^\perp)$  into itself, by solving (separately) the linear problems

$$\begin{aligned} \Delta U + \lambda_k U &= -f(\alpha_k\varphi_k + W, \beta_m\varphi_m + Z) \\ &\quad + \varphi_k \int_D f(\alpha_k\varphi_k + W, \beta_m\varphi_m + Z)\varphi_k dx + e_1, \\ \Delta V + \lambda_m V &= -g(\alpha_k\varphi_k + W, \beta_m\varphi_m + Z) \\ &\quad + \varphi_m \int_D g(\alpha_k\varphi_k + W, \beta_m\varphi_m + Z)\varphi_m dx + e_2, \\ U = V &= 0 \quad \text{on } \partial D, \end{aligned}$$

and then setting

$$\begin{aligned} \xi_k &= \alpha_k + A_k - \int_D f(\alpha_k\varphi_k + U, \beta_m\varphi_m + V)\varphi_k(x) dx \\ \eta_m &= \beta_m + B_m - \int_D g(\alpha_k\varphi_k + U, \beta_m\varphi_m + V)\varphi_m dx. \end{aligned} \quad (3.19)$$

By Lemma 3.3, the map  $T$  is well defined, and  $\|U\|_{C^1(D)}$  and  $\|V\|_{C^1(D)}$  are bounded uniformly in  $(\alpha_k, W, \beta_m, Z)$ . This implies that if  $\alpha_k$  is large and positive,  $\int_D f(\alpha_k\varphi_k + U, \beta_m\varphi_m + V)\varphi_k dx > L_2 > A_k$ . When  $\alpha_k$  is large in absolute value

and negative, this integral is smaller than  $L_1 < A_k$ . It follows that for  $\alpha_k$  large and positive,  $\xi_k < \alpha_k$ , while for  $\alpha_k$  large and negative,  $\xi_k > \alpha_k$ . Hence, we can find a large  $N$ , so that if  $\alpha_k \in (-N, N)$ , then  $\xi_k \in (-N, N)$ , and arguing similarly with the second line in (3.19), we see that if  $\beta_m \in (-N, N)$ , then  $\eta_m \in (-N, N)$  (possibly with a larger  $N$ ). The map  $T$  is a continuous and compact map, taking a sufficiently large ball of  $(R \times \varphi_k^\perp) \times (R \times \varphi_m^\perp)$  into itself. By Schauder's fixed point theorem it has a fixed point, which gives us a solution of (3.15).  $\square$

We now turn to the final case

$$\begin{aligned} \Delta u + \lambda_k u + v + f(u, v) &= h(x) & x \in D, & \quad u = 0 \quad \text{on } \partial D \\ \Delta v + \lambda_k v + g(u, v) &= k(x) & x \in D, & \quad v = 0 \quad \text{on } \partial D. \end{aligned} \tag{3.20}$$

Assume that there exist numbers  $c_2, d_2, C_2$  and  $D_2$ , with  $c_2 < d_2$  and  $C_2 < D_2$ , such that

$$g(u, v) > D_2 \quad \text{for } u > d_2, \quad \text{uniformly in } v \in R, \tag{3.21}$$

$$g(u, v) < C_2 \quad \text{for } u < c_2, \quad \text{uniformly in } v \in R. \tag{3.22}$$

Define

$$N_2 = D_2 \int_{\varphi_k > 0} \varphi_k \, dx + C_2 \int_{\varphi_k < 0} \varphi_k \, dx, \quad N_1 = C_2 \int_{\varphi_k > 0} \varphi_k \, dx + D_2 \int_{\varphi_k < 0} \varphi_k \, dx.$$

Observe that  $N_2 > N_1$ , because  $D_2 > C_2$ .

**Theorem 3.8.** *Assume that  $\lambda_k$  is a simple eigenvalue, while  $f(u, v), g(u, v) \in C(R \times R)$  are bounded on  $R \times R$ , and  $g$  satisfies (3.21), (3.22). Assume that  $h(x), k(x) \in L^p(D)$ , for some  $p > n$ . Assume finally that*

$$N_1 < \int_D k(x)\varphi_k \, dx < N_2. \tag{3.23}$$

Then problem (3.20) has a solution  $(u, v)$ , with  $u, v \in W^{2,p}(D) \cap W_0^{1,p}(D)$ .

*Proof.* Denoting  $A_k = \int_D h(x)\varphi_k \, dx$  and  $B_k = \int_D k(x)\varphi_k \, dx$ , we decompose  $h(x) = A_k\varphi_k(x) + e_1(x)$  and  $k(x) = B_k\varphi_k(x) + e_2(x)$ , with  $e_1, e_2 \in \varphi_k^\perp$ , and also decompose the solution  $u(x) = \xi_k\varphi_k(x) + U(x)$  and  $v(x) = \eta_k\varphi_k(x) + V(x)$ , with  $U, V \in \varphi_k^\perp$ . The Lyapunov-Schmidt reduction of our problem (3.20) is

$$\begin{aligned} \eta_k + \int_D f(\xi_k\varphi_k + U, \eta_k\varphi_k + V)\varphi_k(x) \, dx &= A_k \\ \int_D g(\xi_k\varphi_k + U, \eta_k\varphi_k + V)\varphi_k(x) \, dx &= B_k \\ \Delta U + \lambda_k U + V &= -f(\xi_k\varphi_k + U, \eta_k\varphi_k + V) \\ &\quad + \varphi_k \int_D f(\xi_k\varphi_k + U, \eta_k\varphi_k + V)\varphi_k \, dx + e_1, \\ \Delta V + \lambda_k V &= -g(\xi_k\varphi_k + U, \eta_k\varphi_k + V) \\ &\quad + \varphi_k \int_D g(\xi_k\varphi_k + U, \eta_k\varphi_k + V)\varphi_k \, dx + e_2, \\ U = V &= 0 \quad \text{on } \partial D. \end{aligned} \tag{3.24}$$

To solve this system, we set up a map  $T : (\alpha_k, W, \beta_k, Z) \rightarrow (\xi_k, U, \eta_k, V)$ , taking the space  $(R \times \varphi_k^\perp) \times (R \times \varphi_k^\perp)$  into itself, by solving the linear system

$$\begin{aligned} \Delta U + \lambda_k U + V &= -f(\alpha_k \varphi_k + W, \beta_k \varphi_k + Z) \\ &\quad + \varphi_k \int_D f(\alpha_k \varphi_k + W, \beta_k \varphi_k + Z) \varphi_k \, dx + e_1, \\ \Delta V + \lambda_k V &= -g(\alpha_k \varphi_k + W, \beta_k \varphi_k + Z) \\ &\quad + \varphi_k \int_D g(\alpha_k \varphi_k + W, \beta_k \varphi_k + Z) \varphi_k \, dx + e_2, \\ U = V &= 0 \quad \text{on } \partial D, \end{aligned}$$

and then setting

$$\begin{aligned} \xi_k &= \alpha_k + B_k - \int_D g(\alpha_k \varphi_k + U, \beta_k \varphi_k + V) \varphi_k(x) \, dx \\ \eta_k &= A_k - \int_D f(\alpha_k \varphi_k + U, \beta_k \varphi_k + V) \varphi_k \, dx. \end{aligned} \tag{3.25}$$

Fixed points of  $T$  give us solutions of (3.24), and hence of (3.20). By Lemma 3.4, the map  $T$  is well defined, and  $\|U\|_{C^1(D)}$  and  $\|V\|_{C^1(D)}$  are bounded uniformly in  $(\alpha_k, W, \beta_k, Z)$ . This implies that if  $\alpha_k$  is large and positive,  $\int_D g(\alpha_k \varphi_k + U, \beta_k \varphi_k + V) \varphi_k \, dx > N_2 > B_k$ . When  $\alpha_k$  is large in absolute value and negative, this integral is smaller than  $N_1 < B_k$ . It follows that for  $\alpha_k$  large and positive,  $\xi_k < \alpha_k$ , while for  $\alpha_k$  large and negative,  $\xi_k > \alpha_k$ . Hence, we can find a large  $N$ , so that if  $\alpha_k \in (-N, N)$ , then  $\xi_k \in (-N, N)$ . Since the right hand side of the second line in (3.25) is bounded, we see that if  $\beta_k \in (-N, N)$ , then  $\eta_k \in (-N, N)$  (possibly with a larger  $N$ ). The map  $T$  is a continuous and compact map, taking a sufficiently large ball of  $(R \times \varphi_k^\perp) \times (R \times \varphi_k^\perp)$  into itself. By Schauder's fixed point theorem it has a fixed point, which gives us a solution of (3.20).  $\square$

#### 4. APPENDIX: A DIRECT PROOF OF THE THEOREM OF LAZER AND LEACH

Many proofs of this classical result are available, including the one above, and a recent proof in the paper of Hastings and McLeod [4], which also has references to other proofs. In this appendix we present a proof which is consistent with our approach in the present paper.

*Proof of Theorem 2.4.* Let  $L_n^2 = \{u(t) \in L^2(R), u(t+2\pi) = u(t) \text{ for all } t : \int_0^{2\pi} u(t) \cos nt \, dt = \int_0^{2\pi} u(t) \sin nt \, dt = 0\}$ . As before, we decompose

$$\begin{aligned} f(t) &= \frac{A}{\pi} \cos nt + \frac{B}{\pi} \sin nt + e(t) \\ u(t) &= \xi \cos nt + \eta \sin nt + U(t), \end{aligned} \tag{4.1}$$

with  $e(t), U(t) \in L_n^2$ . Multiply (2.24) by  $\cos nt$  and  $\sin nt$  respectively, and integrate

$$\begin{aligned} \int_0^{2\pi} g(\xi \cos nt + \eta \sin nt + U(t)) \cos nt \, dt &= A \\ \int_0^{2\pi} g(\xi \cos nt + \eta \sin nt + U(t)) \sin nt \, dt &= B. \end{aligned} \tag{4.2}$$

Using these equations, and the ansatz (4.1) in (2.24)

$$\begin{aligned}
 U'' + n^2U &= -g(\xi \cos nt + \eta \sin nt + U(t)) \\
 &+ \frac{1}{\pi} \left( \int_0^{2\pi} g(\xi \cos nt + \eta \sin nt + U(t)) \cos nt \, dt \right) \cos nt \\
 &+ \frac{1}{\pi} \left( \int_0^{2\pi} g(\xi \cos nt + \eta \sin nt + U(t)) \sin nt \, dt \right) \sin nt + e(t).
 \end{aligned} \tag{4.3}$$

Equations (4.2) and (4.3) provide the Lyapunov-Schmidt reduction of (2.24).

To prove the necessity part, we multiply the first equation in (4.2) by  $\frac{A}{\sqrt{A^2+B^2}}$ , the second one by  $\frac{B}{\sqrt{A^2+B^2}}$ , and add, putting the result in the form

$$\int_0^{2\pi} g(\xi \cos nt + \eta \sin nt + U(t)) \cos(nt - \delta) \, dt = \sqrt{A^2 + B^2},$$

for some  $\delta$ . Using Lemma 2.5, the integral on the left is bounded from above by

$$g(\infty) \int_P \cos(nt - \delta) \, dt + g(-\infty) \int_N \cos(nt - \delta) \, dt = 2(g(\infty) - g(-\infty)).$$

Turning to the sufficiency part, we set up a map  $T : (a, b, V) \rightarrow (\xi, \eta, U)$ , taking  $R \times R \times L_n^2$  into itself, by solving

$$\begin{aligned}
 U'' + n^2U &= -g(a \cos nt + b \sin nt + V(t)) \\
 &+ \frac{1}{\pi} \left( \int_0^{2\pi} g(a \cos nt + b \sin nt + V(t)) \cos nt \, dt \right) \cos nt \\
 &+ \frac{1}{\pi} \left( \int_0^{2\pi} g(a \cos nt + b \sin nt + V(t)) \sin nt \, dt \right) \sin nt + e(t)
 \end{aligned} \tag{4.4}$$

for  $U$ , and then set

$$\begin{aligned}
 \xi &= a + A - \int_0^{2\pi} g(a \cos nt + b \sin nt + U(t)) \cos nt \, dt \\
 \eta &= b + B - \int_0^{2\pi} g(a \cos nt + b \sin nt + U(t)) \sin nt \, dt.
 \end{aligned} \tag{4.5}$$

The right hand side of (4.4) is in  $L_n^2$ , and hence (4.4) has infinitely many solutions of the form  $U = U_0 + c_1 \cos nt + c_2 \sin nt$ . We select the unique pair  $(c_1, c_2)$ , so that  $U \in L_n^2$ . By the elliptic theory, we then have (since  $g(u)$  is bounded)

$$\|U\|_{L^\infty} \leq c, \quad \text{with some } c > 0, \text{ uniformly in } (a, b, V). \tag{4.6}$$

We need to show that a sufficiently large ball in  $(a, b)$  plane is mapped into itself in  $(\xi, \eta)$  plane. We have

$$a \cos nt + b \sin nt = \sqrt{a^2 + b^2} \cos(nt - \delta_1), \quad \text{for some } \delta_1.$$

Then for  $a^2 + b^2$  large

$$\begin{aligned}
 &aA + bB - a \int_0^{2\pi} g(a \cos nt + b \sin nt + U(t)) \cos nt \, dt \\
 &- b \int_0^{2\pi} g(a \cos nt + b \sin nt + U(t)) \sin nt \, dt \\
 &\leq \sqrt{a^2 + b^2} \left[ \sqrt{A^2 + B^2} - \int_0^{2\pi} g\left(\sqrt{a^2 + b^2} \cos(nt - \delta_1) + U\right) \cos(nt - \delta_1) \, dt \right]
 \end{aligned}$$

$$< -\mu\sqrt{a^2 + b^2}, \quad \text{for some } \mu > 0,$$

because the integral in the brackets on a sufficiently large ball gets arbitrary close to  $2(g(\infty) - g(-\infty)) > \sqrt{A^2 + B^2}$ . Since  $g(u)$  is bounded, we can find  $h > 0$  so that

$$\begin{aligned} \left( A - \int_0^{2\pi} g(a \cos nt + b \cos nt + U(t)) \cos nt \, dt \right)^2 &< h, \\ \left( B - \int_0^{2\pi} g(a \cos nt + b \cos nt + U(t)) \sin nt \, dt \right)^2 &< h. \end{aligned}$$

Then from (4.5)

$$\xi^2 + \eta^2 < a^2 + b^2 - 2\mu\sqrt{a^2 + b^2} + 2h < a^2 + b^2,$$

for  $a^2 + b^2$  large. Then the map  $T$  is a compact and continuous map of a sufficiently large ball in  $R \times R \times L_n^2$  into itself, and we have a solution by Schauder's fixed point theorem.  $\square$

## 5. APPENDIX: PERTURBATIONS OF FORCED HARMONIC OSCILLATORS AT RESONANCE WITHOUT LAZER-LEACH CONDITION

We present next a result of de Figueiredo and Ni's [2] type for harmonic oscillators at resonance:

$$u'' + n^2u + g(u) = e(t). \quad (5.1)$$

**Theorem 5.1.** *Assume that  $g(u) \in C(R)$  is a bounded function, and*

$$ug(u) > 0 \quad \text{for all } u \in R, \quad (5.2)$$

$$\liminf_{u \rightarrow \infty} g(u) > 0, \quad \limsup_{u \rightarrow -\infty} g(u) < 0. \quad (5.3)$$

*Assume that  $e(t) \in C(R)$  is a  $2\pi$  periodic function, satisfying*

$$\int_0^{2\pi} e(t) \sin nt \, dt = \int_0^{2\pi} e(t) \cos nt \, dt = 0. \quad (5.4)$$

*Then problem (5.1) has a  $2\pi$  periodic solutions.*

*Proof.* We follow the proof of Theorem 2.4. As before, equations (4.2), with  $A = B = 0$ , and (4.3) provide the Lyapunov-Schmidt reduction of (5.1). To solve these equations, we again set up the map  $T : (a, b, V) \rightarrow (\xi, \eta, U)$ , taking  $R \times R \times L_n^2$  into itself, by solving (4.4) and then setting

$$\begin{aligned} \xi &= a - \int_0^{2\pi} g(a \cos nt + b \cos nt + U(t)) \cos nt \, dt \equiv a - I_1 \\ \eta &= b - \int_0^{2\pi} g(a \cos nt + b \cos nt + U(t)) \sin nt \, dt \equiv b - I_2. \end{aligned} \quad (5.5)$$

As before,

$$\|U\|_{L^\infty} \leq c, \quad \text{with some } c > 0, \text{ uniformly in } (a, b, V). \quad (5.6)$$

We have

$$\xi^2 + \eta^2 = a^2 + b^2 - 2(aI_1 + bI_2) + I_1^2 + I_2^2.$$

Using the condition (5.3) and the estimate (5.6), we can find a constant  $\mu > 0$  such that

$$aI_1 + bI_2 = \sqrt{a^2 + b^2} \int_0^{2\pi} g\left(\sqrt{a^2 + b^2} \cos(nt - \delta_1) + U\right) \cos(nt - \delta_1) dt \geq \mu \sqrt{a^2 + b^2},$$

for  $a^2 + b^2$  large. Denoting by  $h$  a bound on  $I_1$  and  $I_2$ , we have

$$\xi^2 + \eta^2 < a^2 + b^2 - 2\mu\sqrt{a^2 + b^2} + 2h < a^2 + b^2,$$

for  $a^2 + b^2$  large. Then the map  $T$  is a compact and continuous map of a sufficiently large ball in  $R \times R \times L_n^2$  into itself, and we have a solution by Schauder's fixed point theorem.  $\square$

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