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# EXISTENCE OF NON-OSCILLATORY SOLUTIONS TO FIRST-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This article presents sufficient conditions for the existence of non-oscillatory solutions to first-order differential equations having both delay and advance terms, known as mixed equations. Our main tool is the Banach contraction principle.

## 1. Introduction

In this article, we consider a first-order neutral differential equation

$$\begin{aligned}&\frac{d}{dt}[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)]\\&+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t+\sigma_2)=0,\end{aligned}\tag{1.1}$$

where $P_i \in C([t_0,\infty),\mathbb{R})$, $Q_i \in C([t_0,\infty),[0,\infty))$, $\tau_i > 0$ and $\sigma_i \geq 0$ for $i = 1, 2$. We give some new criteria for the existence of non-oscillatory solutions of (1.1).

Recently, the existence of non-oscillatory solutions of first-order neutral functional differential equations has been investigated by many authors. Yu and Wang [16] showed that the equation

$$\frac{d}{dt}\left[x(t)+px(t-c)\right]+Q(t)x(t-\sigma)=0,\quad t\geq t_0$$

has a non-oscillatory solution for $p \geq 0$. Later, in 1993, Chen et al [9] studied the same equation and they extended the results to the case $p \in \mathbb{R}\backslash\{-1\}$. Zhang et al [17] investigated the existence of non-oscillatory solutions of the first-order neutral delay differential equation with variable coefficients

$$\frac{d}{dt}[x(t)+P(t)x(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,\quad t\geq t_0\,.$$

They obtained sufficient conditions for the existence of non-oscillatory solutions depending on the four different ranges of $P(t)$. In [10], existence of non-oscillatory solutions of first-order neutral differential equations

$$\frac{d}{dt}[x(t)-a(t)x(t-\tau)]=p(t)f(x(t-\sigma))$$

was studied.

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On the other hand, there has been research activities about the oscillatory behavior of first and higher order neutral differential equations with advanced terms. For instance, in [1] and [5], n-th order neutral differential equations with advanced term of the form

$$[x(t)+ax(t-\tau)+bx(t+\tau)]^{(n)}+\delta\left(q(t)x(t-g)+p(t)x(t+h)\right)=0$$

and

$$[x(t)+\lambda ax(t-\tau)+\mu bx(t+\tau)]^{(n)}+\delta\Big(\int_c^d q(t,\xi)x(t-\xi)d\xi+\int_c^d p(t,\xi)x(t+\xi)d\xi\Big)=0,$$

were studied, respectively.

This article was motivated by the above studies. To the best of our knowledge, this current paper is the only paper regarding to the existence of non-oscillatory solutions of neutral differential equation with advanced term. Some other papers for the existence of non-oscillatory solutions of first, second and higher order neutral functional differential and difference equations; see [13, 18, 6, 7, 8, 15] and the references contained therein. We refer the reader to the books [14, 12, 4, 11, 2, 3] on the subject of neutral differential equations.

Let $m=\max\{\tau_1,\sigma_1\}$. By a solution of (1.1) we mean a function $x\in C([t_1-m,\infty),\mathbb{R})$, for some $t_1\geq t_0$, such that $x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)$ is continuously differentiable on $[t_1,\infty)$ and (1.1) is satisfied for $t\geq t_1$.

As it is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called non-oscillatory.

The following theorem will be used to prove the theorems.

**Theorem 1.1** (Banach's Contraction Mapping Principle). *A contraction mapping on a complete metric space has exactly one fixed point.*

## 2. Main Results

To show that an operator $S$ satisfies the conditions for the contraction mapping principle, we consider different cases for the ranges of the coefficients $P_1(t)$ and $P_2(t)$.

**Theorem 2.1.** *Assume that* $0\leq P_1(t)\leq p_1<1$, $0\leq P_2(t)\leq p_2<1-p_1$ *and*

$$\int_{t_0}^{\infty}Q_1(s)ds<\infty,\quad \int_{t_0}^{\infty}Q_2(s)ds<\infty, \tag{2.1}$$

*then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* Because of (2.1), we can choose a $t_1>t_0$,

$$t_1\geq t_0+\max\{\tau_1,\sigma_1\} \tag{2.2}$$

sufficiently large such that

$$\int_t^{\infty}Q_1(s)ds\leq\frac{M_2-\alpha}{M_2},\quad t\geq t_1, \tag{2.3}$$

$$\int_t^{\infty}Q_2(s)ds\leq\frac{\alpha-(p_1+p_2)M_2-M_1}{M_2},\quad t\geq t_1, \tag{2.4}$$

where $M_1$ and $M_2$ are positive constants such that

$$(p_1+p_2)M_2+M_1<M_2\quad\text{and}\quad\alpha\in\big((p_1+p_2)M_2+M_1,M_2\big).$$

Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_1 \le x(t) \le M_2,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define an operator $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ + \int_t^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]ds, & t \ge t_1, \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Obviously, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.3) and (2.4), respectively, it follows that

$$(Sx)(t) \le \alpha + \int_t^\infty Q_1(s)x(s-\sigma_1)ds \le \alpha + M_2 \int_t^\infty Q_1(s)ds \le M_2$$

and

$$\begin{aligned}(Sx)(t) &\ge \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) - \int_t^\infty Q_2(s)x(s+\sigma_2)ds \\ &\ge \alpha - p_1M_2 - p_2M_2 - M_2\int_t^\infty Q_2(s)ds \ge M_1.\end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$,

$$\begin{aligned}&|(Sx_1)(t) - (Sx_2)(t)| \\ &\le P_1(t)|x_1(t-\tau_1) - x_2(t-\tau_1)| + P_2(t)|x_1(t+\tau_2) - x_2(t+\tau_2)| \\ &\quad + \int_t^\infty \left(Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)| + Q_2(s)|x_1(s+\sigma_2) - x_2(s+\sigma_2)|\right) ds\end{aligned}$$

or

$$\begin{aligned}&|(Sx_1)(t) - (Sx_2)(t)| \\ &\le \|x_1 - x_2\|\Big(p_1 + p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\, ds\Big) \\ &\le \Big(p_1 + p_2 + \frac{M_2-\alpha}{M_2} + \frac{\alpha - (p_1+p_2)M_2 - M_1}{M_2}\Big)\|x_1 - x_2\| \\ &= \lambda_1\|x_1 - x_2\|,\end{aligned}$$

where $\lambda_1 = (1 - \frac{M_1}{M_2})$. This implies that

$$\|Sx_1 - Sx_2\| \le \lambda_1\|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_1 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.2.** *Assume that* $0 \le P_1(t) \le p_1 < 1$, $p_1 - 1 < p_2 \le P_2(t) \le 0$ *and* (2.1) *hold, then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* Because of (2.1), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.2) such that

$$\int_t^\infty Q_1(s)ds \le \frac{(1+p_2)N_2 - \alpha}{N_2}, \quad t \ge t_1, \tag{2.5}$$

$$\int_t^\infty Q_2(s)ds \le \frac{\alpha - p_1N_2 - N_1}{N_2}, \quad t \ge t_1, \tag{2.6}$$

where $N_1$ and $N_2$ are positive constants such that

$$N_1 + p_1N_2 < (1+p_2)N_2 \quad \text{and} \quad \alpha \in (N_1 + p_1N_2, (1+p_2)N_2).$$

Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_1 \le x(t) \le N_2,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define an operator $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ + \int_t^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\,ds, & t \ge t_1, \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Obviously, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.5) and (2.6), respectively, it follows that

$$(Sx)(t) \le \alpha - p_2N_2 + N_2\int_t^\infty Q_1(s)ds \le N_2,$$
$$(Sx)(t) \ge \alpha - p_1N_2 - N_2\int_t^\infty Q_2(s)ds \ge N_1.$$

This proves that $S\Omega \subset \Omega$. To apply the contraction mapping principle, it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$,

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \|x_1 - x_2\|\Big(p_1 - p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\,ds\Big) \\ &\le \lambda_2\|x_1 - x_2\|, \end{aligned}$$

where $\lambda_2 = (1 - \frac{N_1}{N_2})$. This implies

$$\|Sx_1 - Sx_2\| \le \lambda_2\|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_2 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.3.** *Assume that* $1 < p_1 \le P_1(t) \le p_{1_0} < \infty$, $0 \le P_2(t) \le p_2 < p_1 - 1$ *and* (2.1) *hold, then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* In view of (2.1), we can choose a $t_1 > t_0$,

$$t_1 + \tau_1 \ge t_0 + \sigma_1, \tag{2.7}$$

sufficiently large such that

$$\int_t^\infty Q_1(s)ds \le \frac{p_1M_4 - \alpha}{M_4}, \quad t \ge t_1, \tag{2.8}$$

$$\int_t^\infty Q_2(s)ds \le \frac{\alpha - p_{1_0}M_3 - (1+p_2)M_4}{M_4}, \quad t \ge t_1, \tag{2.9}$$

where $M_3$ and $M_4$ are positive constants such that

$$p_{1_0}M_3 + (1+p_2)M_4 < p_1M_4 \quad \text{and} \quad \alpha \in \big(p_{1_0}M_3 + (1+p_2)M_4, p_1M_4\big).$$

Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_3 \le x(t) \le M_4,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define a mapping $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \frac{1}{P_1(t+\tau_1)}\{\alpha - x(t+\tau_1) - P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ +\int_{t+\tau_1}^{\infty}[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\,ds\}, & t \ge t_1, \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Clearly, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.8) and (2.9), respectively, it follows that

$$(Sx)(t) \le \frac{1}{P_1(t+\tau_1)}\Big(\alpha + M_4\int_t^{\infty} Q_1(s)ds\Big) \le \frac{1}{p_1}\Big(\alpha + M_4\int_t^{\infty} Q_1(s)ds\Big) \le M_4$$

and

$$\begin{aligned} (Sx)(t) &\ge \frac{1}{P_1(t+\tau_1)}\Big(\alpha - (1+p_2)M_4 - M_4\int_t^{\infty} Q_2(s)ds\Big) \\ &\ge \frac{1}{p_{1_0}}\Big(\alpha - (1+p_2)M_4 - M_4\int_t^{\infty} Q_2(s)ds\Big) \ge M_3. \end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$,

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \frac{1}{p_1}\|x_1 - x_2\|\Big(1 + p_2 + \int_t^{\infty}(Q_1(s) + Q_2(s))\,ds\Big) \\ &\le \lambda_3\|x_1 - x_2\|, \end{aligned}$$

where $\lambda_3 = (1 - \frac{p_{1_0}M_3}{p_1M_4})$. This implies

$$\|Sx_1 - Sx_2\| \le \lambda_3\|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_3 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.4.** *Assume that* $1 < p_1 \le P_1(t) \le p_{1_0} < \infty$, $1 - p_1 < p_2 \le P_2(t) \le 0$ *and* (2.1) *hold, then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* In view of (2.1), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.7) such that

$$\int_t^{\infty} Q_1(s)ds \le \frac{(p_1+p_2)N_4 - \alpha}{N_4}, \quad t \ge t_1, \tag{2.10}$$

$$\int_t^{\infty} Q_2(s)ds \le \frac{\alpha - p_{1_0}N_3 - N_4}{N_4}, \quad t \ge t_1, \tag{2.11}$$

where $N_3$ and $N_4$ are positive constants such that

$$p_{1_0}N_3 + N_4 < (p_1+p_2)N_4 \quad \text{and} \quad \alpha \in \big(p_{1_0}N_3 + N_4, (p_1+p_2)N_4\big).$$

Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_3 \le x(t) \le N_4,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define a mapping $S:\Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \frac{1}{P_1(t+\tau_1)}\{\alpha - x(t+\tau_1) - P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ +\int_{t+\tau_1}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\,ds\}, & t \ge t_1, \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Clearly, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.10) and (2.11), respectively, it follows that

$$\begin{aligned} (Sx)(t) &\le \frac{1}{P_1(t+\tau_1)}\Big(\alpha - p_2N_4 + N_4\int_t^\infty Q_1(s)ds\Big) \\ &\le \frac{1}{p_1}\Big(\alpha - p_2N_4 + N_4\int_t^\infty Q_1(s)ds\Big) \le N_4 \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\ge \frac{1}{P_1(t+\tau_1)}\Big(\alpha - N_4 - N_4\int_t^\infty Q_2(s)ds\Big) \\ &\ge \frac{1}{p_{1_0}}\Big(\alpha - N_4 - N_4\int_t^\infty Q_2(s)ds\Big) \ge N_3. \end{aligned}$$

This proves that $S\Omega \subset \Omega$. To apply the contraction mapping principle it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$,

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \frac{1}{p_1}\|x_1 - x_2\|\Big(1 - p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\,ds\Big) \\ &\le \lambda_4\|x_1 - x_2\|, \end{aligned}$$

where $\lambda_4 = (1 - \frac{p_{1_0}N_3}{p_1N_4})$. This implies

$$\|Sx_1 - Sx_2\| \le \lambda_4\|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_4 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.5.** *Assume that* $-1 < p_1 \le P_1(t) \le 0$, $0 \le P_2(t) \le p_2 < 1 + p_1$ *and* (2.1) *hold, then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* Because of (2.1), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.2) such that

$$\int_t^\infty Q_1(s)ds \le \frac{(1+p_1)M_6 - \alpha}{M_6}, \quad t \ge t_1, \tag{2.12}$$

and

$$\int_t^\infty Q_2(s)ds \le \frac{\alpha - p_2M_6 - M_5}{M_6}, \quad t \ge t_1\,, \tag{2.13}$$

where $M_5$ and $M_6$ are positive constants such that

$$M_5 + p_2M_6 < (1+p_1)M_6 \quad \text{and} \quad \alpha \in (M_5 + p_2M_6, (1+p_1)M_6)\,.$$

Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_5 \le x(t) \le M_6,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define an operator $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ + \int_t^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\,ds, & t \ge t_1, \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Obviously, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.12) and (2.13), respectively, it follows that

$$(Sx)(t) \le \alpha - p_1 M_6 + M_6 \int_t^\infty Q_1(s)ds \le M_6,$$
$$(Sx)(t) \ge \alpha - p_2 M_6 - M_6 \int_t^\infty Q_2(s)ds \ge M_5.$$

This proves that $S\Omega \subset \Omega$. To apply the contraction mapping principle it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$, $t \ge t_1$,

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \|x_1 - x_2\|\Big( - p_1 + p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\,ds\Big) \\ &\le \lambda_5 \|x_1 - x_2\|, \end{aligned}$$

where $\lambda_5 = (1 - \frac{M_5}{M_6})$. This implies

$$\|Sx_1 - Sx_2\| \le \lambda_5 \|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_5 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.6.** *Assume that* $-1 < p_1 \le P_1(t) \le 0$, $-1 - p_1 < p_2 \le P_2(t) \le 0$ *and* (2.1) *hold, then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* Because of (2.1), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.2) such that

$$\int_t^\infty Q_1(s)ds \le \frac{(1 + p_1 + p_2)N_6 - \alpha}{N_6}, \quad t \ge t_1, \tag{2.14}$$

and

$$\int_t^\infty Q_2(s)ds \le \frac{\alpha - N_5}{N_6}, \quad t \ge t_1, \tag{2.15}$$

where $N_5$ and $N_6$ are positive constants such that

$$N_5 < (1 + p_1 + p_2)N_6 \quad \text{and} \quad \alpha \in (N_5, (1 + p_1 + p_2)N_6).$$

Let $\Lambda$ be the set of continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_5 \le x(t) \le N_6,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define an operator $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ + \int_t^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\, ds, & t \geq t_1, \\ (Sx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

Obviously, $Sx$ is continuous. For $t \geq t_1$ and $x \in \Omega$, from (2.14) and (2.15), respectively, it follows that

$$(Sx)(t) \leq \alpha - p_1 N_6 - p_2 N_6 + N_6 \int_t^\infty Q_1(s)ds \leq N_6,$$
$$(Sx)(t) \geq \alpha - N_6 \int_t^\infty Q_2(s)ds \geq N_5.$$

This proves that $S\Omega \subset \Omega$. To apply the contraction mapping principle it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$,

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\leq \|x_1 - x_2\| \Big( -p_1 - p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\, ds \Big) \\ &\leq \lambda_6 \|x_1 - x_2\|, \end{aligned}$$

where $\lambda_6 = (1 - \frac{N_5}{N_6})$. This implies

$$\|Sx_1 - Sx_2\| \leq \lambda_6 \|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_6 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.7.** *Assume that $-\infty < p_{1_0} \leq P_1(t) \leq p_1 < -1$, $0 \leq P_2(t) \leq p_2 < -p_1 - 1$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.*

*Proof.* In view of (2.1), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.7) such that

$$\int_t^\infty Q_1(s)ds \leq \frac{p_{1_0} M_7 + \alpha}{M_8}, \quad t \geq t_1, \tag{2.16}$$

and

$$\int_t^\infty Q_2(s)ds \leq \frac{(-p_1 - 1 - p_2)M_8 - \alpha}{M_8}, \quad t \geq t_1, \tag{2.17}$$

where $M_7$ and $M_8$ are positive constants such that

$$-p_{1_0} M_7 < (-p_1 - 1 - p_2)M_8 \quad \text{and} \quad \alpha \in (-p_{1_0} M_7, (-p_1 - 1 - p_2)M_8).$$

Let $\Lambda$ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_7 \leq x(t) \leq M_8,\ t \geq t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define a mapping $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \frac{-1}{P_1(t+\tau_1)} \{\alpha + x(t+\tau_1) + P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ - \int_{t+\tau_1}^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\, ds\}, & t \geq t_1 \\ (Sx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

Clearly, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.17) and (2.16), respectively, it follows that

$$(Sx)(t) \le \frac{-1}{p_1}\left(\alpha + M_8 + p_2 M_8 + M_8 \int_t^\infty Q_2(s)ds\right) \le M_8$$

and

$$(Sx)(t) \ge \frac{-1}{p_{1_0}}\left(\alpha - M_8 \int_t^\infty Q_1(s)ds\right) \ge M_7.$$

This implies that $S\Omega \subset \Omega$. To apply the contraction mapping principle it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$,

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \frac{-1}{p_1}\|x_1 - x_2\|\Big(1 + p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\,ds\Big) \\ &\le \lambda_7 \|x_1 - x_2\|, \end{aligned}$$

where $\lambda_7 = (1 - \frac{p_{1_0} M_7}{p_1 M_8})$. This implies

$$\|Sx_1 - Sx_2\| \le \lambda_7 \|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_7 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. □

**Theorem 2.8.** *Assume that* $-\infty < p_{1_0} \le P_1(t) \le p_1 < -1$, $p_1 + 1 < p_2 \le P_2(t) \le 0$ *and* (2.1) *hold, then* (1.1) *has a bounded non-oscillatory solution.*

*Proof.* In view of (2.1), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.7) such that

$$\int_t^\infty Q_1(s)ds \le \frac{p_{1_0} N_7 + p_2 N_8 + \alpha}{N_8}, \quad t \ge t_1, \tag{2.18}$$

and

$$\int_t^\infty Q_2(s)ds \le \frac{(-p_1 - 1)N_8 - \alpha}{N_8}, \quad t \ge t_1, \tag{2.19}$$

where $N_7$ and $N_8$ are positive constants such that

$$-p_{1_0} N_7 - p_2 N_8 < (-p_1 - 1)N_8 \quad \text{and} \quad \alpha \in (-p_{1_0} N_7 - p_2 N_8, (-p_1 - 1)N_8).$$

Let $\Lambda$ be the set of continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_7 \le x(t) \le N_8,\ t \ge t_0\}.$$

It is clear that $\Omega$ is a bounded, closed and convex subset of $\Lambda$. Define a mapping $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \frac{-1}{P_1(t+\tau_1)}\{\alpha + x(t+\tau_1) + P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ -\int_{t+\tau_1}^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\,ds\}, & t \ge t_1, \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Clearly, $Sx$ is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.19) and (2.18), respectively, it follows that

$$(Sx)(t) \le \frac{-1}{p_1}\Big(\alpha + N_8 + N_8 \int_t^\infty Q_2(s)ds\Big) \le N_8$$

and

$$(Sx)(t) \ge \frac{-1}{p_{1_0}}\Big(\alpha + p_2 N_8 - N_8 \int_t^\infty Q_1(s)ds\Big) \ge N_7.$$

These prove that $S\Omega \subset \Omega$. To apply the contraction mapping principle it remains to show that $S$ is a contraction mapping on $\Omega$. Thus, if $x_1, x_2 \in \Omega$, $t \geq t_1$,

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\leq \frac{-1}{p_1}\|x_1 - x_2\|\Big(1 - p_2 + \int_t^\infty (Q_1(s) + Q_2(s))\, ds\Big)\\ &\leq \lambda_8 \|x_1 - x_2\|,\end{aligned}$$

where $\lambda_8 = (1 - \frac{p_{1_0} N_7}{p_1 N_8})$. This implies

$$\|Sx_1 - Sx_2\| \leq \lambda_8 \|x_1 - x_2\|,$$

where the supremum norm is used. Since $\lambda_8 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof. $\square$

**Example 2.9.** Consider the equation

$$\begin{aligned}&\Big[x(t) - \frac{1}{2}x(t - 2\pi) + \big[\frac{1}{2} - \exp(-\frac{t}{2})\big]x(t + 5\pi)\Big]'\\ &+ \frac{1}{2}\exp(-\frac{t}{2})x(t - 4\pi) - \exp(-\frac{t}{2})x(t + \frac{5\pi}{2}) = 0, \quad t > -2\ln(1/2)\end{aligned} \tag{2.20}$$

and note that

$$P_1(t) = -\frac{1}{2}, \quad P_2(t) = \frac{1}{2} - \exp(-\frac{t}{2}), \quad Q_1(t) = \frac{1}{2}\exp(-\frac{t}{2}), \quad Q_2(t) = \exp(-\frac{t}{2}).$$

A straightforward verification yields that the conditions of Theorem 2.5 are valid. We note that $x(t) = 2 + \sin t$ is a non-oscillatory solution of (2.20).

**Example 2.10.** Consider the equation

$$\begin{aligned}&\Big[x(t) - \frac{1}{\exp(1)}\big[\frac{3}{4} - \exp(-t)\big]x(t - 1) - \exp(1/4)\big[\frac{1}{4} + \exp(-t)\big]x(t + \frac{1}{4})\Big]'\\ &+ \exp(-t - 1)x(t - 1) - \exp(-t + \frac{1}{4})x(t + \frac{1}{4}) = 0, \quad t \geq \frac{3}{2}\end{aligned} \tag{2.21}$$

and note that

$$\begin{gathered}P_1(t) = -\frac{1}{\exp(1)}\big[\frac{3}{4} - \exp(-t)\big], \quad P_2(t) = -\exp(\frac{1}{4})\big[\frac{1}{4} + \exp(-t)\big],\\ Q_1(t) = \exp(-t - 1), \quad Q_2(t) = \exp(-t + \frac{1}{4}).\end{gathered}$$

It is easy to verify that the conditions of Theorem 2.6 are valid. We note that $x(t) = 1 + \exp(-t)$ is a non-oscillatory solution of (2.21).

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