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GLOBAL CARLEMAN ESTIMATE FOR THE PLATE EQUATION AND APPLICATIONS TO INVERSE PROBLEMS

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Abstract. In this article, we establish a Carleman estimate for the plate equation in order to solve an inverse problem retrieving the zeroth-order term for a plate equation from boundary measurements. We prove the local stability result for this inverse problem. Our proof relies on Carleman estimate.

1. Introduction

In this article, we discuss the local Lipschitz stability in determining a coefficient of the zeroth-order term for a plate equation from boundary measurements. Physically speaking, we are required to determine a coefficient $p(x)$ from measurements of boundary displacement. The stability result for the inverse problem is based on a global Carleman estimate for a plate equation.

Inverse Problem: Determine $p(x)$ for $x \in I$ such that

$$\begin{gathered} u_{tt} + u_{xxxx} + pu = 0 \quad \text{in } Q, \\ u(0,t) = 0 = u(1,t) \quad \text{in } (0,T), \\ u_x(0,t) = 0 = u_x(1,t) \quad \text{in } (0,T), \\ u(x,0) = a(x) \quad \text{in } I, \\ u_t(x,0) = b(x) \quad \text{in } I, \end{gathered} \tag{1.1}$$

from the observed data $u|_{\{1\}\times(0,T)}$, where $I = (0,1)$, $T > 0$ and $Q = I \times (0,T)$.

The inverse problem for the wave equation has drawn the attention of many authors. In [22] it is discussed the global Lipschitz stability in determining a coefficient of the zeroth-order term for a wave equation from data of the solution in a sub-domain over a time interval. In [2], uniqueness and Lipschitz stability are obtained for the inverse problem of retrieving a stationary potential for the wave equation with Dirichlet data and discontinuous principal coefficient from a single time dependent Neumann boundary measurement. Doubonva citeD1 solved an inverse problem of retrieving a stationary potential for the wave equation with Dirichlet data from a single time-dependent Neumann boundary measurement on a suitable part of the boundary. The uniqueness and the stability are also proved for this problem when a Neumann measurement is only located on a part of the boundary satisfying a rotated exit condition. Bellassoued [4] studied the global

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logarithmic stability in determination of a coefficient of the zeroth-order term in a wave equation from data of the solution in a subboundary over a time interval. The similar inverse problems for parabolic equations can be found in the survey paper by Yamamoto [35] and the references therein. The inverse problem for dispersive equations is also interesting and was studied in [3, 5, 29] for Schrödinger equations. Baudouin [1] studied the same problem for the Korteweg-de Vries equation. To our best knowledge, very little work is concerned with the inverse problem for plate equations, our work is motivated by [22]. The key ingredient we follow here to determine the coefficient relies on a Carleman estimate for the plate operator.

In this article we use the following assumptions:

(H1) $x_0 < 0$, $\beta \in (0, 1)$, $T > \sup_{x\in I} |x - x_0|$.

We define the functions

$$\psi(x,t) = (x - x_0)^2 - \beta t^2, \quad \varphi(x,t) = e^{\mu\psi(x,t)},$$
$$l(x,t) = \lambda\varphi(x,t), \quad \theta = e^{l(x,t)}.$$

Let

$$L^\infty_M(I) = \{p \in L^\infty(I) | \|p\|_{L^\infty(I)} \le M\}.$$

Let P be the operator $Py := y_{tt} + y_{xxxx} + py$ with domain

$$\begin{aligned}\mathcal{U} := \Big\{ & y \in C^1(-T,T;H^2(I)) \cap C^2(-T,T;L^2(I)) \cap C(-T,T;H^4(I)) : \\ & y(t,0) = y(t,1) = y_x(t,0) = y_x(t,1) = 0,\ t \in (-T,T), \\ & y(\pm T,x) = y_t(\pm T,x) = 0,\ x \in I,\ Py \in L^2(-T,T;L^2(I)) \Big\},\end{aligned}$$

where $p \in L^\infty(I)$. Throughout this paper, C stands for a generic positive constant whose value can change from line to line.

(H2) $a \in H^2_0(I)$ satisfies $|a(x)| \ge r_0 > 0$ for all $x \in I$, $b \in L^2(I)$.

The main result in this paper is the following local stability for the inverse problem.

Theorem 1.1. *Let assumptions* (H1), (H2) *be satisfied. Then there exists a constant* $C^\star = C^\star(T, x_0, a, b, M) > 0$ *such that for all* $p, q \in L^\infty_M(I)$, *we have*

$$\|p - q\|_{L^2(I)} \le C^\star \int_0^T [(u(p) - u(q))^2_{xxt} + (u(p) - u(q))^2_{xxxt}](1,t)dt \tag{1.2}$$

where $u(p), u(q)$ *are the solutions of* (1.1) *depending on* p, q.

Remark 1.2. Stability estimates play a special role in the theory of inverse problems of mathematical physics that are ill posed in the classical sense. They determine the choice of regularization parameters and the rate at which solutions of regularized problems converge to an exact solution.

Remark 1.3. From Theorem 1.1, we can obtain the mapping $p \to u(p)|_{\{1\}\times(0,T)}$ is one to one, and the mapping $u(p)|_{\{1\}\times(0,T)} \to p$ is continuous.

To obtain the stability of (1.2), the following Carleman estimate is essential.

Theorem 1.4. *There exist three constants $\mu_0 > 1$, $\lambda_0 > 0$ and $C_1 > 0$ such that for $\mu = \mu_0$ and for every $\lambda \geq \lambda_0$ and $y \in \mathcal{U}$, we have*

$$\begin{aligned}
&\int_{-T}^{T}\int_I (\lambda\mu^2\varphi\theta^2 y_{xxx}^2 + \lambda^3\mu^4\varphi^3\theta^2 y_{xx}^2 + \lambda^5\mu^6\varphi^5\theta^2 y_x^2 + \lambda^7\mu^8\varphi^7\theta^2 y^2 \\
&+ \lambda^3\mu^4\varphi^3\theta^2 y_t^2 + \lambda\mu^2\varphi\theta^2 y_{xt}^2)\,dx\,dt \qquad (1.3)\\
&\leq C_1[\int_{-T}^{T}\int_I \theta^2|Py|^2\,dx\,dt + \int_0^T (\lambda^3\mu^3\varphi^3\theta^2 y_{xx}^2 + \lambda\mu\varphi\theta^2 y_{xxx}^2)(1,t)dt].
\end{aligned}$$

Carleman estimate is an L^2-weighted estimate with large parameter for a solution to a PDE and it is one of the major tools used in the study of unique continuation, observability, and controllability problems for various kinds of PDEs. Its history may date back to Carleman [6] for a two-dimensional elliptic equation. Then, many authors have considered this estimate such as Egorov [9], Hörmander [19, 20], Isakov [24, 25, 26], Tataru [31], Taylor [32] and Trèves [33]. There have already been rich amounts of work for the Carleman estimates of second order parabolic equations, see [11, 23, 35]. Global Carleman estimate for fourth order parabolic equation we established in[12, 13, 14, 15, 18, 36]. A similar estimate for the hyperbolic equation can be found in [10, 21]. In [17, 30] there is a Carleman estimate for the KdV equation without the interior observation. Then a global Carleman estimate for the KdV equation was established in [7]. In [16] there are global Carleman estimates for forward stochastic fourth order parabolic equation and backward stochastic fourth order parabolic equation.

The remainder of this article is organized as follows. In Section 2, we establish the Carleman estimate (1.3), Section 3 we prove Theorem 1.1.

2. Proof of Theorem 1.4

As in [28], it is sufficient to prove (1.3) for $\widetilde{P}y = y_{tt} + y_{xxxx}$ with $y \in \mathcal{U}$. In fact, assume that we have proved (1.3) for $\widetilde{P}y$, we have

$$\int_Q \theta^2|\widetilde{P}y|^2\,dx\,dt \leq \int_Q \theta^2|Py|^2\,dx\,dt + \|p\|_{L^\infty(I)}^2 \int_Q \theta^2 y^2\,dx\,dt.$$

By choosing $\lambda_0 = \lambda_0(\mu, T) > 0$ large, when $\lambda > \lambda_0$, it is possible to absorb $\|p\|_{L^\infty(I)}^2 \int_Q \theta^2 y^2\,dx\,dt$ with the left-hand side of (1.3), concluding that (1.3) also holds for Py.

Set $u = \theta y, \widetilde{P}y = f$. Direct computations show that

$$\theta(y_{tt} + y_{xxxx}) = u_{tt} + A_0 u_t + A_1 u + A_2 u_x + A_3 u_{xx} + A_4 u_{xxx} + u_{xxxx}$$

where

$$\begin{gathered}
A_0 = -2l_t, \quad A_1 = l_t^2 - l_{tt} + l_x^4 + 4l_x l_{xxx} - l_{xxxx} - 6l_x^2 l_{xx} + 3l_{xx}^2, \\
A_2 = 12 l_x l_{xx} - 4l_x^3 - 4l_{xxx}, \quad A_3 = 6l_x^2 - 6l_{xx}, \quad A_4 = -4l_x.
\end{gathered}$$

Set

$$\begin{gathered}
I_1 = u_{xxxx} + B_1 u + B_3 u_{xx} + u_{tt}, \\
I_2 = B_2 u_x + B_4 u_{xxx} + au + B_0 u_t, \\
Ru = \theta f - I_1 - I_2 = S_0 u + S_1 u_x + S_2 u_{xx},
\end{gathered}$$

where

$$
\begin{gathered}
a=-12l_x^2l_{xx},\quad B_0=-2l_t,\quad B_1=l_x^4,\\
B_2=-4l_x^3,\quad B_3=6l_x^2,\quad B_4=-4l_x,\\
S_0=l_t^2-l_{tt}+4l_xl_{xxx}-l_{xxxx}+6l_x^2l_{xx}+3l_{xx}^2,\\
S_1=12l_xl_{xx}-4l_{xxx},\quad S_2=-6l_{xx}.
\end{gathered}
$$

Step 1. We shall prove the equality

$$
\begin{aligned}
I_1\cdot I_2=&\,u^2\{\cdots\}+u_x^2\{\cdots\}+u_{xx}^2\{\cdots\}+u_{xxx}^2\{\cdots\}+u_t^2\{\cdots\}\\
&+u_{xt}^2\{\cdots\}+u_{xt}u_{xxx}\{\cdots\}+u_tu_x\{\cdots\}+u_tu_{xxx}\{\cdots\}\\
&+\{\cdots\}_x+\{\cdots\}_{xx}+\{\cdots\}_{xxx}+\{\cdots\}_{xxxx}+\{\cdots\}_t+\{\cdots\}_{tt},
\end{aligned}\tag{2.1}
$$

where

$$
\begin{aligned}
\{\cdots\}_x=\{&\frac{3}{2}B_{2xx}u_x^2-\frac{3}{2}B_2u_{xx}^2+\frac{1}{2}B_4u_{xxx}^2+4a_xu_x^2-2a_{xxx}u^2\\
&+B_0u_tu_{xxx}+\frac{1}{2}B_1B_2u^2+\frac{3}{2}(B_1B_4)_{xx}u^2-\frac{3}{2}B_1B_4u_x^2+\frac{1}{2}B_2B_3u_x^2\\
&+\frac{1}{2}B_3B_4u_{xx}^2-(aB_3)_xu^2+B_0B_3u_tu_x-\frac{1}{2}B_2u_t^2-\frac{3}{2}B_{4xx}u_t^2+\frac{3}{2}B_4u_{xt}^2\}_x,
\end{aligned}
$$

$$
\{\cdots\}_{xx}=\{-\frac{3}{2}B_{2x}u_x^2+3a_{xx}u^2-2au_x^2-\frac{3}{2}(B_1B_4)_xu^2+\frac{1}{2}aB_3u^2+\frac{3}{2}B_{4x}u_t^2\}_{xx},
$$

$$
\{\cdots\}_{xxx}=\{\frac{1}{2}B_2u_x^2-2a_xu^2+\frac{1}{2}B_1B_4u^2-\frac{1}{2}B_4u_t^2\}_{xxx},
$$

$$
\{\cdots\}_{xxxx}=\{\frac{1}{2}au^2\}_{xxxx},
$$

$$
\{\cdots\}_t=\{\frac{1}{2}B_0B_1u^2-\frac{1}{2}B_0B_3u_x^2+B_2u_tu_x+\frac{1}{2}B_0u_t^2-a_tu^2+B_4u_tu_{xxx}\}_t,
$$

$$
\{\cdots\}_{tt}=\{\frac{1}{2}au^2\}_{tt},
$$

$$
\begin{aligned}
u^2\{\cdots\}=u^2\{&\frac{1}{2}a_{xxxx}-\frac{1}{2}(B_1B_2)_x-\frac{1}{2}(B_1B_4)_{xxx}+aB_1-\frac{1}{2}(B_0B_1)_t\\
&+\frac{1}{2}(aB_3)_{xx}+\frac{1}{2}a_{tt}\},
\end{aligned}
$$

$$
u_x^2\{\cdots\}=u_x^2\{-\frac{1}{2}B_{2xxx}-2a_{xx}+\frac{3}{2}(B_1B_4)_x-\frac{1}{2}(B_2B_3)_x-aB_3+\frac{1}{2}(B_0B_3)_t\},
$$

$$
u_{xx}^2\{\cdots\}=u_{xx}^2\{\frac{3}{2}B_{2x}+a-\frac{1}{2}(B_3B_4)_x\},
$$

$$
u_{xxx}^2\{\cdots\}=u_{xxx}^2\{-\frac{1}{2}B_{4x}\},
$$

$$
u_t^2\{\cdots\}=u_t^2\{\frac{1}{2}B_{2x}-\frac{1}{2}B_{0t}-a+\frac{1}{2}B_{4xxx}\},
$$

$$
u_{xt}^2\{\cdots\}=u_{xt}^2\{-\frac{3}{2}B_{4x}\},
$$

$$
u_{xt}u_{xxx}\{\cdots\}=u_{xt}u_{xxx}\{-B_0\},
$$

$$
u_tu_x\{\cdots\}=u_tu_x\{-(B_0B_3)_x-B_{2t}\},
$$

$$
u_tu_{xxx}\{\cdots\}=u_tu_{xxx}\{-B_{4t}-B_{0x}\}.
$$

Indeed, (2.1) can be obtained from the following equations

$$
\begin{aligned}
u_{xxxx}B_2u_x &= \frac{1}{2}(B_2u_x^2)_{xxx} - \frac{3}{2}(B_{2x}u_x^2)_{xx} + \frac{3}{2}(B_{2xx}u_x^2 - B_2u_{xx}^2)_x \\
&\quad + \frac{3}{2}B_{2x}u_{xx}^2 - \frac{1}{2}B_{2xxx}u_x^2, \\
u_{xxxx}B_4u_{xxx} &= \frac{1}{2}[(B_4u_{xxx}^2)_x - B_{4x}u_{xxx}^2],
\end{aligned}
$$

$$
\begin{aligned}
u_{xxxx}au &= \frac{1}{2}(au^2)_{xxxx} - 2(a_xu^2)_{xxx} + (3a_{xx}u^2 - 2au_x^2)_{xx} \\
&\quad + (4a_xu_x^2 - 2a_{xxx}u^2)_x + au_{xx}^2 - 2a_{xx}u_x^2 + \frac{1}{2}a_{xxxx}u^2, \\
u_{xxxx}bu_{xx} &= \frac{1}{2}(bu_{xx}^2)_{xx} - (b_xu_{xx}^2)_x - bu_{xxx}^2 + \frac{1}{2}b_{xx}u_{xx}^2, \\
u_{xxxx}B_0u_t &= (B_0u_tu_{xxx})_x - B_{0x}u_tu_{xxx} - B_0u_{xt}u_{xxx}, \\
B_1uB_2u_x &= \frac{1}{2}[(B_1B_2u^2)_x - (B_1B_2)_xu^2], \\
B_1uB_4u_{xxx} &= \frac{1}{2}(B_1B_4u^2)_{xxx} - \frac{3}{2}[(B_1B_4)_xu^2]_{xx} \\
&\quad + \frac{3}{2}[(B_1B_4)_{xx}u^2 - B_1B_4u_x^2]_x + \frac{3}{2}(B_1B_4)_xu_x^2 - \frac{1}{2}(B_1B_4)_{xxx}u^2, \\
B_1ubu_{xx} &= \frac{1}{2}(bB_1u^2)_{xx} - [(bB_1)_xu^2]_x - bB_1u_x^2 + \frac{1}{2}(bB_1)_{xx}u^2, \\
B_1uB_0u_t &= \frac{1}{2}[(B_0B_1u^2)_t - (B_0B_1)_tu^2], \\
B_3u_{xx}B_2u_x &= \frac{1}{2}[(B_2B_3u_x^2)_x - (B_2B_3)_xu_x^2], \\
B_3u_{xx}B_4u_{xxx} &= \frac{1}{2}[(B_3B_4u_{xx}^2)_x - (B_3B_4)_xu_{xx}^2], \\
B_3u_{xx}au &= \frac{1}{2}(aB_3u^2)_{xx} - [(aB_3)_xu^2]_x - aB_3u_x^2 + \frac{1}{2}(aB_3)_{xx}u^2, \\
B_3u_{xx}B_0u_t &= -(B_0B_3)_xu_tu_x + \frac{1}{2}(B_0B_3)_tu_x^2 - \frac{1}{2}(B_0B_3u_x^2)_t + (B_0B_3u_xu_t)_x, \\
u_{tt}B_2u_x &= (B_2u_tu_x)_t - B_{2t}u_tu_x - \frac{1}{2}[(B_2u_t^2)_x - B_{2x}u_t^2], \\
u_{tt}B_0u_t &= \frac{1}{2}[(B_0u_t^2)_t - B_{0t}u_t^2], \\
u_{tt}au &= \frac{1}{2}(au^2)_{tt} - (a_tu^2)_t - au_t^2 + \frac{1}{2}a_{tt}u^2, \\
u_{tt}B_4u_{xxx} &= (B_4u_tu_{xxx})_t - B_{4t}u_tu_{xxx} - \frac{1}{2}(B_4u_t^2)_{xxx} + \frac{3}{2}(B_{4x}u_t^2)_{xx} \\
&\quad - \frac{3}{2}[B_{4xx}u_t^2 - B_4u_{xt}^2]_x - \frac{3}{2}B_{4x}u_{xt}^2 + \frac{1}{2}B_{4xxx}u_t^2, \\
u_{tt}bu_{xx} &= (bu_tu_{xx})_t + b_{xt}u_tu_x - \frac{1}{2}b_{tt}u_x^2 + \frac{1}{2}(b_tu_x^2)_t \\
&\quad - (b_tu_tu_x)_x - \frac{1}{2}(bu_t^2)_{xx} + (b_xu_t^2)_x + bu_{xt}^2 - \frac{1}{2}b_{xx}u_t^2.
\end{aligned}
$$

Step 2. We shall prove the estimate

$$
\begin{aligned}
&\int_{-T}^{T}\int_I(\lambda\mu^2\varphi\theta^2y_{xxx}^2+\lambda^3\mu^4\varphi^3\theta^2y_{xx}^2+\lambda^5\mu^6\varphi^5\theta^2y_x^2+\lambda^7\mu^8\varphi^7\theta^2y^2\\
&+\lambda^3\mu^4\varphi^3\theta^2y_t^2+\lambda\mu^2\varphi\theta^2y_{xt}^2)\,dx\,dt\\
&\le C_1\Big[\int_{-T}^{T}\int_I\theta^2|Py|^2\,dx\,dt+\int_{-T}^{T}(\lambda^3\mu^3\varphi^3\theta^2y_{xx}^2+\lambda\mu\varphi\theta^2y_{xxx}^2)(1,t)dt\Big].
\end{aligned}
$$

Indeed,

$$
\begin{gathered}
u_{xxx}^2\{\cdots\}=u_{xxx}^2\{2l_{xx}\},\quad u_{xx}^2\{\cdots\}=u_{xx}^2\{6l_x^2l_{xx}\},\\
u_x^2\{\cdots\}=u_x^2\{102l_x^4l_{xx}+r_1\},\quad u^2\{\cdots\}=u^2\{2l_x^6l_{xx}+r_2\},\\
u_t^2\{\cdots\}=u_t^2\{6l_x^2l_{xx}+l_{tt}-2l_{xxxx}\},\quad u_{xt}^2\{\cdots\}=u_{xt}^2\{6l_{xx}\},\\
u_{xt}u_{xxx}\{\cdots\}=u_{xt}u_{xxx}\{2l_t\},\quad u_tu_x\{\cdots\}=u_tu_x\{24l_{xt}l_x^2+24l_tl_xl_{xx}\},\\
u_tu_{xxx}\{\cdots\}=u_tu_{xxx}\{6l_{xt}\},
\end{gathered}
$$

where

$$
\begin{gathered}
r_1=60l_{xx}^3+180l_xl_{xx}l_{xxx}+30l_x^2l_{xxxx}-6l_{tt}l_x^2-12l_tl_xl_{xt},\\
r_2=-6(l_x^2l_{xx})_{xxxx}+2(l_x^5)_{xxx}+(l_tl_x^4)_t-36(l_x^4l_{xx})_{xx}-6(l_x^2l_{xx})_{tt}.
\end{gathered}
$$

Now, we estimate the term $\int_{-T}^{T}\int_I(\{\cdots\}_x+\{\cdots\}_{xx}+\{\cdots\}_{xxx}+\{\cdots\}_{xxxx}+\{\cdots\}_t+\{\cdots\}_{tt})\,dx\,dt$ in (2.1). Indeed, noting that

$$
\begin{gathered}
y(0,t)=y(1,t)=y_x(0,t)=y_x(1,t)=0\quad\forall t\in(-T,T),\\
y(x,\pm T)=y_t(x,\pm T)=0\quad\forall x\in I,
\end{gathered}
$$

this implies

$$
\begin{gathered}
u(0,t)=u(1,t)=u_x(0,t)=u_x(1,t)=0\quad\forall t\in(-T,T)\\
u(x,\pm T)=u_t(x,\pm T)=0\quad\forall x\in I.
\end{gathered}
$$

Thus

$$
\begin{gathered}
\int_{-T}^{T}\int_I(\{\cdots\}_t+\{\cdots\}_{tt})\,dx\,dt=0,\\
\int_{-T}^{T}\int_I(\{\cdots\}_x+\{\cdots\}_{xx}+\{\cdots\}_{xxx}+\{\cdots\}_{xxxx})\,dx\,dt\\
=\int_{-T}^{T}\{u_{xx}^2(-10l_x^3)+u_{xxx}^2(-2l_x)\}(\cdot,t)|_0^1dt=:V(1)-V(0).
\end{gathered}
\tag{2.2}
$$

If we choose $\lambda \geq \lambda_0 = \lambda_0(\mu, T)$ with λ_0 large enough such that $\lambda\mu^{-1}\varphi \geq 1$, then it holds

$$\begin{gathered}
|r_1| \leq C(\lambda^3\mu^6\varphi^3 + \lambda^3\mu^4\varphi^3 + \lambda^3\mu^5\varphi^3) \leq C\lambda^5\varphi^5(\mu^2 + \mu^4 + \mu^3) \leq C\lambda^5\mu^5\varphi^5, \\
|r_2| \leq C(\lambda^3\mu^6\varphi^3 + \lambda^3\mu^8\varphi^3 + \lambda^5\mu^8\varphi^5) \leq C\lambda^7\varphi^7(\mu^2 + \mu^4 + \mu^6) \leq C\lambda^7\mu^7\varphi^7, \\
\begin{aligned}
|S_0|^2 &\leq C(\lambda^2\mu^2\varphi^2 + \lambda^2\mu^4\varphi^2 + \lambda^4\mu^8\varphi^4 + \lambda^6\mu^8\varphi^6 + \lambda^4\mu^4\varphi^4 + \lambda^2\mu^8\varphi^2) \\
&\leq C\lambda^7\varphi^7(\mu^{-3} + \mu^{-1} + \mu^5 + \mu^7 + \mu + \mu^3) \\
&\leq C\lambda^7\mu^7\varphi^7,
\end{aligned} \\
|S_1|^2 \leq C(\lambda^4\mu^6\varphi^4 + \lambda^2\mu^6\varphi^2) \leq C\lambda^5\mu^5\varphi^5, \\
|S_2|^2 \leq C\lambda^2\mu^4\varphi^2 \leq C\lambda^3\mu^3\varphi^3
\end{gathered} \tag{2.3}$$

and

$$\begin{gathered}
u_{xt}^2\{\cdots\} + u_{xxx}^2\{\cdots\} + u_{xt}u_{xxx}\{\cdots\} \geq C(\lambda\mu^2\varphi u_{xxx}^2 + \lambda\mu^2\varphi u_{xt}^2), \\
u_x^2\{\cdots\} + u_t^2\{\cdots\} + u_tu_x\{\cdots\} \geq C(\lambda^5\mu^6\varphi^5u_x^2 + \lambda^3\mu^4\varphi^3u_t^2), \\
u_{xxx}^2\{\cdots\} + u_t^2\{\cdots\} + u_tu_{xxx}\{\cdots\} \geq C(\lambda\mu^2\varphi u_{xxx}^2 + \lambda^3\mu^4\varphi^3u_t^2).
\end{gathered} \tag{2.4}$$

Since $I_1 + I_2 = \theta f - S_0u - S_1u_x - S_2u_{xx}$, it is clear that

$$\begin{aligned}
2\int_{-T}^T\int_I I_1I_2 \; dx\,dt &\leq \|I_1 + I_2\|_{L^2(-T,T;L^2(I))}^2 \\
&= \|\theta f + S_0u + S_1u_x + S_2u_{xx}\|_{L^2(Q)}^2 \\
&\leq C\int_{-T}^T\int_I(\theta^2f^2 + S_0^2u^2 + S_1^2u_x^2 + S_2^2u_{xx}^2)\,dx\,dt.
\end{aligned} \tag{2.5}$$

From (2.2)-(2.5), we can obtain

$$\begin{aligned}
&\int_{-T}^T\int_I[\lambda\mu^2\varphi u_{xxx}^2 + \lambda^3\mu^4\varphi^3u_{xx}^2 + \lambda^5\mu^6\varphi^5u_x^2 + \lambda^7\mu^8\varphi^7u^2 + \lambda^3\mu^4\varphi^3u_t^2 \\
&+ \lambda\mu^2\varphi u_{xt}^2]\,dx\,dt + V(1) - V(0) \\
&\leq C\int_{-T}^T\int_I(\theta^2f^2 + \lambda^7\mu^7\varphi^7u^2 + \lambda^5\mu^5\varphi^5u_x^2 + \lambda^3\mu^3\varphi^3u_{xx}^2)\,dx\,dt.
\end{aligned}$$

Noting that $-V(0) \geq 0$ and that

$$|V(1)| \leq C\int_0^T(\lambda^3\mu^3\varphi^3u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)(1,t)dt,$$

and choosing λ_0 large enough, when $\lambda \geq \lambda_0$, we conclude that

$$\begin{aligned}
&\int_{-T}^T\int_I(\lambda\mu^2\varphi u_{xxx}^2 + \lambda^3\mu^4\varphi^3u_{xx}^2 + \lambda^5\mu^6\varphi^5u_x^2 + \lambda^7\mu^8\varphi^7u^2 \\
&+ \lambda^3\mu^4\varphi^3u_t^2 + \lambda\mu^2\varphi u_{xt}^2)\,dx\,dt \\
&\leq C[\int_{-T}^T\int_I\theta^2f^2\,dx\,dt + \int_0^T(\lambda^3\mu^3\varphi^3u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)(1,t)dt].
\end{aligned}$$

Returning u to θy, we can obtain (1.3).

3. Proof of Theorem 1.1

In this section, we use the Bukhgeim-Klibanov method to study the inverse problem. We are now in a position to prove the stability result Theorem 1.1, the proof follows the ideas used in [22, 34]. The idea is to reduce the nonlinear inverse problem to some perturbed inverse problem which will be solved with the help of a global Carleman estimate. Firstly, we consider the system

$$\begin{gathered} y_{tt} + y_{xxxx} + py = G \quad \text{in } Q, \\ y(0,t) = 0 = y(1,t) \quad \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) \quad \text{in } (0,T), \\ y(x,0) = 0 \quad \text{in } I, \\ y_t(x,0) = 0 \quad \text{in } I\,. \end{gathered} \tag{3.1}$$

Proposition 3.1. *Let assumptions* (H1), (H2) *be satisfied. Assume that there exists a function $g_0 \in L^2(0,T)$ such that*

$$\begin{gathered} G \in H^1(0,T;L^2(I)), \quad \|G_t\|_{L^2(Q)} \le M, \\ g_0(t)|G(x,0)| \ge |G_t(x,t)|, \quad (x,t) \in Q. \end{gathered} \tag{3.2}$$

Then there exists a constant $C_1 = C_1(M,T,r_0)$ such that

$$\|G\|_{H^1(0,T;L^2(I))} \le C_1 \int_0^T (y_{xxt}^2 + y_{xxxt}^2)(1,t)dt. \tag{3.3}$$

Proof. Setting $y_1 = y_t$, we have

$$\begin{gathered} y_{1tt} + y_{1xxxx} + py_1 = G_t \quad \text{in } Q, \\ y_1(0,t) = 0 = y_1(1,t) \quad \text{in } (0,T), \\ y_{1x}(0,t) = 0 = y_{1x}(1,t) \quad \text{in } (0,T), \\ y_1(x,0) = 0 \quad \text{in } I, \\ y_{1t}(x,0) = G(x,0) \quad \text{in } I\,. \end{gathered}$$

Since $y(x,0) = y_t(x,0) = 0$, from [27] we have

$$y \in C([0,T], H^4(I)) \cap C^1(0,T;H^2(I)) \cap C^2(0,T;L^2(I))$$

and there exists a constant $C = C(T,M) > 0$ such that

$$\|y\|_{H^2(I\times(0,T))} \le C\|G\|_{H^1(0,T;L^2(I))}.$$

We extend the function y from $I \times (0,T)$ by the formula $y(x,t) = y(x,-t), (x,t) \in I \times (-T,0)$ and denote the extension by the same symbol y. We know $y \in C([-T,T], H^4(I)) \cap C^1([-T,T]; H^2(I)) \cap C^2([-T,T]; L^2(I))$ and there exists a constant $C = C(T,M) > 0$ such that

$$\|y\|_{H^2(I\times(-T,T))} \le C\|G\|_{H^1(0,T;L^2(I))}.$$

We extend the function G_t on $I \times (-T,T)$ as the even function in t and denote the extension by the same symbol G_t. Then $G_t \in L^2(-T,T;L^2(I))$.

By assumption (H1), there exists a $\beta \in (0,1)$ such that $\beta > \sup_{x\in I} |x - x_0|/T^2$. Therefore, by definition of ψ, φ, we have

$$\varphi(x,0) \ge 1, \quad \varphi(x,-T) = \varphi(x,T) < 1, \quad x \in [0,1].$$

Therefore for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$, such that

$$\varphi(x,t) \ge 1-\varepsilon, \quad (x,t) \in [0,1]\times[-\delta,\delta],$$
$$\varphi(x,t) \le 1-2\varepsilon, \quad (x,t) \in [0,1]\times([-T,-T+2\delta]\cup[T-2\delta,T]).$$

To apply Theorem 1.4, we introduce a cut-off function χ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty(\mathbb{R})$, and

$$\chi(t) = \begin{cases} 0, & [-T,-T+\delta]\cup[T-\delta,T] \\ 1, & [-T+2\delta, T-2\delta]. \end{cases}$$

We set

$$u = e^{\lambda\varphi}\chi y_t, w = \chi y_t, D(y) = \int_0^T (y_{xxt}^2 + y_{xxxt}^2)(1,t)dt.$$

Direct computations show that

$$\begin{aligned} Pu &= u(\lambda\varphi_{tt} - \lambda^2\varphi_t^2 - \lambda^4\varphi_x^4 + 6\lambda^3\varphi_x^2\varphi_{xx} - 3\lambda^2\varphi_{xx}^2 + \lambda\varphi_{xxxx} \\ &\quad - 4\lambda^2\varphi_x\varphi_{xxx}) + 2\lambda\varphi_t u_t + u_x(4\lambda^3\varphi_x^3 - 12\lambda^2\varphi_x\varphi_{xx} + 4\lambda\varphi_{xxx}) \\ &\quad + u_{xx}(-6\lambda^2\varphi_x^2 + 6\lambda\varphi_{xx}) + u_{xxx}\cdot 4\lambda\varphi_x + e^{\lambda\varphi}\chi_{tt}y_t + 2e^{\lambda\varphi}\chi_t y_{tt} + e^{\lambda\varphi}\chi G_t. \end{aligned}$$

Multiplying Pu by u_t and integrating it over $I\times(-T,0)$, it follows that

$$\begin{aligned} \int_{-T}^0\int_I Pu\cdot u_t\,dx\,dt &= \frac{1}{2}\int_I u_t(x,0)^2dx + \frac{1}{2}\int_I u_{xx}^2(0,x)dx \\ &\ge \frac{1}{2}\int_I u_t(x,0)^2dx = \frac{1}{2}\int_I G(x,0)^2e^{2\lambda\varphi(x,0)}dx. \end{aligned} \tag{3.4}$$

On the other hand, by the Cauchy inequality, we have

$$\begin{aligned} &\int_{-T}^0\int_I Pu\cdot u_t\,dx\,dt \\ &\le C[\int_{-T}^0\int_I (G_t^2\chi^2\theta^2 + \theta^2\chi_{tt}^2y_t^2 + \theta^2\chi_t^2y_{tt}^2)\,dx\,dt \\ &\quad + \int_{-T}^0\int_I (\lambda^7\mu^8\varphi^7u^2 + \lambda^5\mu^6\varphi^5u_x^2 + \lambda^3\mu^4\varphi^3u_{xx}^2 + \lambda\mu^2\varphi u_{xxx}^2 \\ &\quad + \lambda^3\mu^4\varphi^3u_t^2 + \lambda\mu^2\varphi u_{xt}^2)\,dx\,dt] =: J_1 + J_2. \end{aligned}$$

It is easy to see that

$$\begin{aligned} J_1 &\le C\Big[\int_{-T}^T\int_I G_t^2\chi^2\theta^2\,dx\,dt + \Big(\int_{-T+\delta}^{-T+2\delta} + \int_{T-2\delta}^{T-\delta}\Big)(y_t^2 + y_{tt}^2)\theta^2\,dx\,dt\Big] \\ &\le C\Big[\int_{-T}^T\int_I G_t^2\chi^2\theta^2\,dx\,dt + e^{2\lambda(1-2\varepsilon)}\|y\|_{H^2(I\times(-T,T))}^2\Big] \\ &\le C\Big[\int_{-T}^T\int_I G_t^2\chi^2\theta^2\,dx\,dt + e^{2\lambda(1-2\varepsilon)}\|G\|_{H^1(0,T;L^2(I))}^2\Big]. \end{aligned}$$

Noting $Pw = \chi G_t + 2y_{tt}\chi_t + \chi_{tt}y_t$, from Theorem 1.4 it follows that

$$\begin{aligned} &\int_{-T}^T\int_I (\lambda\mu^2\varphi\theta^2w_{xxx}^2 + \lambda^3\mu^4\varphi^3\theta^2w_{xx}^2 + \lambda^5\mu^6\varphi^5\theta^2w_x^2 + \lambda^7\mu^8\varphi^7\theta^2w^2 \\ &+ \lambda^3\mu^4\varphi^3\theta^2w_t^2 + \lambda\mu^2\varphi\theta^2w_{xt}^2)\,dx\,dt \end{aligned}$$

$$\begin{aligned}
&\le C_1\Big[\int_{-T}^{T}\int_I \theta^2|Pw|^2\,dx\,dt+\int_0^T(\lambda^3\mu^3\varphi^3\theta^2w_{xx}^2+\lambda\mu\varphi\theta^2w_{xxx}^2)(1,t)dt\Big]\\
&\le C[\int_{-T}^{T}\int_I\theta^2|\chi G_t+2y_{tt}\chi_t+\chi_{tt}y_t|^2\,dx\,dt\\
&\quad+\int_0^T(\lambda^3\mu^3\varphi^3\theta^2y_{xxt}^2+\lambda\mu\varphi\theta^2y_{xxxt}^2)(1,t)dt]\\
&\le C\Big[\int_{-T}^{T}\int_I G_t^2\theta^2\,dx\,dt+e^{2\lambda(1-2\varepsilon)}\|G\|^2_{H^1(0,T;L^2(I))}+D(y)\Big];
\end{aligned}$$

thus

$$\begin{aligned}
J_2&\le\int_{-T}^{0}\int_I(\lambda^7\mu^8\varphi^7u^2+\lambda^5\mu^6\varphi^5u_x^2+\lambda^3\mu^4\varphi^3u_{xx}^2+\lambda\mu^2\varphi u_{xxx}^2\\
&\quad+\lambda^3\mu^4\varphi^3u_t^2+\lambda\mu^2\varphi u_{xt}^2)\,dx\,dt\\
&\le\int_{-T}^{T}\int_I(\lambda^7\mu^8\varphi^7u^2+\lambda^5\mu^6\varphi^5u_x^2+\lambda^3\mu^4\varphi^3u_{xx}^2+\lambda\mu^2\varphi u_{xxx}^2\\
&\quad+\lambda^3\mu^4\varphi^3u_t^2+\lambda\mu^2\varphi u_{xt}^2)\,dx\,dt\\
&\le C\int_{-T}^{T}\int_I(\lambda\mu^2\varphi\theta^2w_{xxx}^2+\lambda^3\mu^4\varphi^3\theta^2w_{xx}^2+\lambda^5\mu^6\varphi^5\theta^2w_x^2+\lambda^7\mu^8\varphi^7\theta^2w^2\\
&\quad+\lambda^3\mu^4\varphi^3\theta^2w_t^2+\lambda\mu^2\varphi\theta^2w_{xt}^2)\,dx\,dt\\
&\le C\Big[\int_{-T}^{T}\int_I G_t^2\theta^2\,dx\,dt+e^{2\lambda(1-2\varepsilon)}\|G\|^2_{H^1(0,T;L^2(I))}+D(y)\Big].
\end{aligned}$$

From the above estimates, we know that

$$\begin{aligned}
&\int_{-T}^{0}\int_I Pu\cdot u_t\,dx\,dt\\
&\le C\Big[\int_{-T}^{T}\int_I G_t^2\theta^2\,dx\,dt+e^{2\lambda(1-2\varepsilon)}\|G\|^2_{H^1(0,T;L^2(I))}+D(y)\Big].
\end{aligned}\tag{3.5}$$

Consequently, (3.4) and (3.5) imply

$$\begin{aligned}
&\frac{1}{2}\int_I G(x,0)^2e^{2\lambda\varphi(x,0)}dx\\
&\le C\Big[\int_{-T}^{T}\int_I G_t^2\theta^2\,dx\,dt+e^{2\lambda(1-2\varepsilon)}\|G\|^2_{H^1(0,T;L^2(I))}+D(y)\Big].
\end{aligned}\tag{3.6}$$

Next we estimate the first term of the right-hand side of (3.6).

$$\begin{aligned}
&\int_{-T}^{T}\int_I G_t^2\theta^2\,dx\,dt\\
&=\int_I\Big(\int_{-T}^{T}|G_t(x,t)|^2e^{2\lambda e^{\mu\psi(x,t)}}dt\Big)dx\\
&\le\int_I\Big(\int_{-T}^{T}|g_0(t)G(x,0)|^2e^{2\lambda e^{\mu[(x-x_0)^2-\beta t^2]}}dt\Big)dx\\
&=\int_I\Big(\int_{-T}^{T}|g_0(t)|^2e^{2\lambda e^{\mu[(x-x_0)^2-\beta t^2]}}e^{-2\lambda\varphi(x,0)}dt|G(x,0)|^2e^{2\lambda\varphi(x,0)}\Big)dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_I \Big(\int_{-T}^{T} |g_0(t)|^2 e^{2\lambda e^{\mu[(x-x_0)^2-\beta t^2]}} e^{-2\lambda e^{\mu(x-x_0)^2}} dt |G(x,0)|^2 e^{2\lambda\varphi(x,0)} \Big) dx \\
&= \int_I \Big(\int_{-T}^{T} |g_0(t)|^2 e^{2\lambda e^{\mu(x-x_0)^2}(e^{-\beta t^2}-1)} dt |G(x,0)|^2 e^{2\lambda\varphi(x,0)} \Big) dx \\
&\le \int_I \Big(\int_{-T}^{T} |g_0(t)|^2 e^{2\lambda(e^{-\beta t^2}-1)} dt |G(x,0)|^2 e^{2\lambda\varphi(x,0)} \Big) dx \\
&=: \int_I (\int_{-T}^{T} h_\lambda(t) dt |G(x,0)|^2 e^{2\lambda\varphi(x,0)}) dx.
\end{aligned}
$$

We have that $h_\lambda(t)$ is in $L^1(-T,T)$, and $\lim_{\lambda\to\infty} h_\lambda(t) = 0$ for $t \neq 0$ and $|h_\lambda(t)| \le |g_0|^2 \in L^1(-T,T)$. Hence the Lebesgue theorem implies

$$
\int_{-T}^{T} |g_0(t)|^2 e^{2\lambda(e^{-\beta t^2}-1)} dt = o(1)
$$

as $\lambda \to \infty$, so that

$$
\int_{-T}^{T} \int_I G_t^2 \theta^2 \, dx \, dt = o(1) \int_I G(x,0)^2 e^{2\lambda\varphi(x,0)} dx. \tag{3.7}
$$

By the assumption in (3.2), we have

$$
\|G\|^2_{H^1(0,T;L^2(I))} \le C \int_I G(x,0)^2 dx. \tag{3.8}
$$

Therefore (3.2), (3.6) and (3.8) yield $\|G\|^2_{H^1(0,T;L^2(I))} \le CD(y)$. □

Now we can prove Theorem 1.1. In Proposition 3.1, we set $y = u(p) - u(q)$ and $G(x,t) = (p-q)u(q)$. Recalling that $H^1(0,T;L^2(I)) \subset C([0,T];L^2(I))$, it is easy to see that assumption (3.2) holds, and that (3.3) implies

$$
\begin{aligned}
\|p-q\|_{L^2(I)} &\le C\|(p-q)(x)u(q)(0,x)\|_{L^2(I)} \\
&\le C\|(p-q)u(q)\|_{C([0,T];L^2(I))} \\
&\le C\|(p-q)u(q)\|_{H^1(0,T;L^2(I))} \\
&= C\|G\|_{H^1(0,T;L^2(I))} \\
&\le C \int_0^T (y_{xxt}^2 + y_{xxxt}^2)(1,t) dt \\
&\le C \int_0^T [(u(p)-u(q))_{xxt}^2 + (u(p)-u(q))_{xxxt}^2](1,t) dt,
\end{aligned}
$$

this completes the proof.

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References

[1] Lucie Baudouin, Eduardo Cerpa, Emmanuelle Crépeau, Alberto Mercado; *On the determination of the principal coefficient from boundary measurements in a KdV equation*, Journal of Inverse and Ill-Posed Problems, 22 (2014), 819-845.

[2] Lucie Baudouin, Alberto Mercado, Axel Osses; *A global Carleman estimate in a transmission wave equation and application to a one-measurement inverse problem*, Inverse Problems, 23 (2007), 257-278.

[3] L. Baudouin, J.-P. Puel; *Uniqueness and stability in an inverse problem for the Schrödinger equation*. Inverse Problems, 18 (2002), 1537-1554.

[4] Mourad Bellassoued; *Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation*, Inverse Problems, 20 (2004), 1033-1052.

[5] L. Cardoulis, M. Cristofol, P. Gaitan; *Inverse problem for the Schrödinger operator in an unbounded strip*. J. Inverse Ill-Posed Probl., 16 (2008), 127-146.

[6] T. Carleman; *Sur un problème d'unicité pour les systèmes d'équations aux derivées partielles à deux variables independentes*, Ark. Mat. Astr. Fys., 2B (1939), 1-9.

[7] M. Chen, P. Gao; *A new unique continuation property for the Korteweg-de Vries equation*. Bulletin of the Australian Mathematical Society, 90(01) (2014), 90-98.

[8] A. Doubova, A. Osses; *Rotated weights in global Carleman estimates applied to an inverse problem for the wave equation*, Inverse Problems, 22 (2006), 265-296.

[9] Y. V. Egorov; *Linear Differential Equations of Principal Type*, New York: Consultants Bureau, 1986.

[10] X. Fu, J. Yong, X. Zhang; *Exact controllability for multidimensional semilinear hyperbolic equations*, SIAM J. Control Optim., 46(5) (2007), 1578-1614.

[11] A. V. Fursikov, O. Y. Imanuvilov; *Controllability of evolution equations*, volume 34 of Lecture Notes Series. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[12] P. Gao; *Insensitizing controls for the Cahn-Hilliard type equation*, Electron. J. Qual. Theory Differ. Equ., 35 (2014), 1-22.

[13] P. Gao; *A new global Carleman estimate for the one-dimensional Kuramoto-Sivashinsky equation and applications to exact controllability to the trajectories and an inverse problem*, Nonlinear Analysis: Theory, Methods & Applications, 117 (2015), 133-147.

[14] P. Gao; *A new global Carleman estimate for Cahn-Hilliard type equation and its applications*, J. Differential Equations 260 (2016), 427-444.

[15] P. Gao; *Local exact controllability to the trajectories of the Swift-Hohenberg equation*. Nonlinear Analysis: Theory, Methods & Applications, 139 2016, 169-195.

[16] P. Gao, M. Chen, Y. Li; *Observability estimates and null controllability for forward and backward linear stochastic Kuramoto-Sivashinsky equations*, SIAM J. Control Optim., 53 (1) (2015), 475-500.

[17] O. Glass, S. Guerrero; *Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit*, Asymptot. Anal., 60 (1-2) (2008), 61-100.

[18] P. Guzmán; *Local exact controllability to the trajectories of the Cahn-Hilliard equation*, prepint.

[19] L. Hörmander; *Linear Partial Differential Operators*, Berlin: Springer, 1963.

[20] L. Hörmander; *The Analysis of Linear Partial Differential Operators*, ICIV, Berlin: Springer, 1985.

[21] O. Y. Imanuvilov; *On Carleman estimates for hyperbolic equations*, Asymptot. Anal., 32(3-4) (2002), 185-220.

[22] O. Y. Imanuvilov, M. Yamamoto; *Global Lipschitz stability in an inverse hyperbolic problem by interior observations*, Inverse Problems, (2001), 717-728.

[23] O. Y. Imanuvilov, M. Yamamoto; *Carleman estimate for a parabolic equation in a Sobolev space of negative order and its applications*, Control of Nonlinear Distributed Parameter Systems (New York: Dekker), (2001) 113-37 .

[24] V. Isakov; *Inverse Source Problems*, Providence, RI: American Mathematical Society, 1990.

[25] V. Isakov; *Carleman type estimates in an anisotropic case and applications*, J. Diff. Eqns 105 (1993), 217-238.

[26] V. Isakov; *Inverse Problems for Partial Differential Equations*, Berlin: Springer, 1998.

[27] J. L. Lions, E. Magenes; *Non-homogeneous BoundaryValue Problems and Applications*, Berlin: Springer (1972).

[28] P. G. Meléndez; *Lipschitz stability in an inverse problem for the mian coefficient of a Kuramoto-Sivashinsky type equation*, J. Math. Anal. Appl., 408 (2013), 275-290.

[29] A. Mercado, A. Osses, L. Rosier; *Lionel Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights*. Inverse Problems, 24 (2008), 015017.

[30] L. Rosier, B.-Y. Zhang; *Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain*, SIAM Journal on Control and Optimization, 45(3) (2006), 927-956.
[31] D. Tataru; *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures Appl., 75 (1996), 367-408.
[32] M. Taylor; *Pseudodifferential Operators, Princeton*, NJ: Princeton University Press, 1981.
[33] F. Trèves; *Linear Partial Differential Equations*, New York: Gordon and Breach, 1970.
[34] B. Wu; *Carleman estimate for a strongly damped wave equation and applications to an inverse problem*, Math. Methods Appl. Sci., 35 (2012), 427-437.
[35] M. Yamamoto; *Carleman estimates for parabolic equations and applications*. Inverse Problems, 25(12) (209), 123013.
[36] Z. C. Zhou; *Observability estimate and null controllability for one-dimensional fourth order parabolic equation*, Taiwanese J. Math., 16 (6) (2012), 1991-2017.

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