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# EXISTENCE OF HIGH-ENERGY SOLUTIONS FOR SUPERCRITICAL FRACTIONAL SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$

LU GAN, WEIMING LIU

Abstract. In this article, we study supercritical fractional Schrödinger equations. Applying the finite-dimensional reduction method and the penalization method, we obtain the high-energy solutions for this equation.

## 1. Introduction and statement of main results

This article is devoted to the study of the problem

$$\begin{gathered}(-\Delta)^s u = V(x)u^{2^*(s)+\epsilon-1}, \quad u>0,\ x\in\mathbb{R}^N,\\ u\to 0 \quad \text{as } |x|\to+\infty,\end{gathered}\tag{1.1}$$

where $2^*(s)=\frac{2N}{N-2s}$, $N>2s$, $0<s<1$, $\epsilon>0$, $V$ is a positive continuous potential. Here, the fractional Laplacian of a function $f:\mathbb{R}^N\to\mathbb{R}$ is expressed by the formula

$$\begin{aligned}(-\Delta)^s f(x) &= C_{N,s}\,\text{p.v.}\int_{\mathbb{R}^N}\frac{f(x)-f(y)}{|x-y|^{N+2s}}dy\\ &= C_{N,s}\lim_{\delta\to0}\int_{\mathbb{R}^N\setminus B_\delta(x)}\frac{f(x)-f(y)}{|x-y|^{N+2s}}dy,\end{aligned}\tag{1.2}$$

where $C_{N,s}$ is some normalization constant.

The operator $(-\Delta)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion processes (see [1]). The Lévy processes occur widely in physics, biology, chemistry and finance (see [1, 3]). The stable Lévy processes that give rise to equations with fractional Laplacians have recently attracted much research interest, and there are a lot of results in the literature on the existence of such solutions. In [5], Barrios et al. studied the existence and multiplicity of solutions to the following critical problem with convex-concave nonlinearities

$$\begin{gathered}(-\Delta)^s u = \lambda u^q + u^{2^*(s)-1}, \quad u>0,\ x\in\Omega,\\ u=0, \quad x\in\mathbb{R}^N\setminus\Omega.\end{gathered}\tag{1.3}$$

As we know, the fractional power of the Laplacian can also be defined by using spectral decomposition. The same problem considered in [5] but for this spectral fractional Laplacian has been treated in [6]. As in [6] the purpose of this paper

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is to study the existence of weak solutions for (1.3). In [17], in order to construct solutions to the problem of the form $(-\Delta)^s u = \epsilon h u_+^q + u_+^p$, Dipierro et al. used the Lyapunov-Schmidt reduction, that takes advantage of the variational structure of the problem. For related results, we refer the reader to [7, 14, 19, 23, 24, 27, 29, 33, 34, 35].

Let us come back to equation (1.1). Recently, many results on the existence of solutions for problem (1.1) when $\epsilon = 0$ have been obtained. Liu [25] obtained infinitely many concentration solutions for (1.1) under certain conditions. Assume $V = 1 + \tau K$ and $K$ has at least two critical points satisfying some local conditions, Chen and Zheng [13] proved the existence of two-peak solutions when the positive number $\tau$ is small enough. When $V \equiv 1$, the existence of finite-energy sign-changing solutions to (1.1) has been established by Garrido and Musso [20]. In particular, DelaTorre et al. [16] constructed a class of Delaunay-type solutions for (1.1).

When $s = 1$, problem (1.1) reduces formally, to the classical Schrödinger equation

$$\begin{gathered} -\Delta u = V(x) u^{2^*+\epsilon-1}, \quad u > 0, \\ u \to 0 \quad \text{as } |x| \to +\infty. \end{gathered} \tag{1.4}$$

The study of problem (1.4) has attracted considerable attention in recent years, and there are several results in the literature on the existence of solutions. When $V$ is a perturbation of the constant, Ambrosetti et al. [2] and Cao et al. [9] proved the existence of two or many positive solutions. Li [25] proved that (1.4) has infinitely many positive solutions if $V$ is periodic, while similar result was obtained in [38] if $V$ has a sequence of strict local maximum points tending to infinity. Wei and Yan [36] obtained solutions with large number of bumps near infinity for (1.4) with $V$ being radial. Meanwhile, they proved that the energy of these solutions can be arbitrarily large. For related results, we refer the readers to [10, 11, 21, 22] and the references therein.

The aim of this article is to show the existence of high-energy solutions for the fractional Schrödinger equation with slightly supercritical exponent. We assume that the positive continuous potential $V$ satisfies the following conditions:

- (A1) There exist constants $q \in [0, 2s)$ and $C > 0$, such that $V(x) \le C(1+|x|)^q$ for all $x \in \mathbb{R}^N$;
- (A2) For some $\mu, r > 0$, $V \in C^{2,\mu}(B_r(y_0))$, and $\Delta V(y_0) > 0$, where $y_0 \in \mathbb{R}^N$ is a strict local minimum point of $V$.

Now, we recall the basic theory on fractional Laplacian operator. For $s \in (0,1)$, the nonlocal operator $(-\Delta)^s$ in $\mathbb{R}^N$ is defined on the Schwartz class through the Fourier transform

$$\widehat{(-\Delta)^s} f(\xi) = |\xi|^{2s} \widehat{f}(\xi),$$

or via the Riesz potential. And$\,\widehat{}\,$is the Fourier transform. When $f$ has some sufficiently regularity, the fractional Laplacian of a function $f : \mathbb{R}^N \to \mathbb{R}$ is expressed as (1.2). That integral makes sense directly when $s < 1/2$ and $f \in C^{0,\gamma}(\mathbb{R}^N)$ with $\gamma > 2s$, or if $f \in C^{1,\gamma}(\mathbb{R}^N)$ with $1 + 2\gamma > 2s$. It is well known that $(-\Delta)^s$ on $\mathbb{R}^N$ with $0 < s < 1$ is a nonlocal operator. In the remarkable work by Caffarelli and Silvestre [12], this nonlocal operator was expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with a local differential operator defined on the upper half-space $\mathbb{R}_+^{N+1} := \{(x, y) : x \in \mathbb{R}^N, y > 0\}$. That is, for a function $f : \mathbb{R}^N \to \mathbb{R}$, we consider the extension $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$

that satisfies the equations

$$u(x,0) = f(x), \tag{1.5}$$

$$\Delta_x u + \frac{1-2s}{y} u_y + u_{yy} = 0. \tag{1.6}$$

Equation (1.6) can also be written as

$$\operatorname{div}(y^{1-2s}\nabla u) = 0, \tag{1.7}$$

which is clearly the Euler-Lagrange equation for the functional

$$J(u) = \int_{y>0} |\nabla u|^2 y^{1-2s}\, dx\, dy.$$

From (1.5)-(1.7) it follows that

$$C(-\Delta)^s f = \lim_{y\to 0^+} -y^{1-2s} u_y = \frac{1}{2s} \lim_{y\to 0^+} \frac{u(x,y)-u(x,0)}{y^{2s}}.$$

In the rest of this article, the homogeneous fractional Sobolev space is given by

$$D^s(\mathbb{R}^N) = \Big\{ u \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 < +\infty \Big\}$$

with the norm

$$\|u\| = \Big( \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \Big)^{1/2},$$

which is induced by the inner product

$$\langle u, v\rangle_s = \Big( \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}(\xi)\widehat{v}(\xi) \Big)^{1/2}.$$

The so-called Gagliardo semi-norm of $u$ is defined as

$$[u]_{H^s(\mathbb{R}^N)} := \Big( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \Big)^{1/2}.$$

It can be proved [32, Proposition 3.4 and 3.6] that

$$[u]_{H^s(\mathbb{R}^N)} = C\Big( \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \Big)^{1/2} = C\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}$$

for a suitable positive constant $C$ depending only on $s$ and $N$.

We consider the equation

$$(-\Delta)^s u = u^{2^*(s)-1}, \ u > 0 \text{ on } \mathbb{R}^N. \tag{1.8}$$

It has been proved in [8, 26] that the following function, for $y \in \mathbb{R}^N$ and $\lambda > 0$,

$$U_{y,\lambda} = C_0 \Big( \frac{\lambda}{1+\lambda^2|x-y|^2} \Big)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N,$$

where $C_0 = C_0(N,s) > 0$, solves (1.8) on $\mathbb{R}^N$.

For any positive integer $m$, $\mathbf{y} = (y_1, y_2, \dots, y_m) \in \mathbb{R}^{mN}$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, and $\lambda_k > 0$, $k = 1, 2, \dots, m$, we define

$$\begin{aligned} E_{\mathbf{y},\lambda} = \Big\{ w \in D^s(\mathbb{R}^N) : \big\langle w, \frac{\partial U_{y_k,\lambda_k}}{\partial \lambda_k} \big\rangle_s = \big\langle w, \frac{\partial U_{y_k,\lambda_k}}{\partial y_{ki}} \big\rangle_s = 0, \\ k = 1, \dots, m, \ i = 1, \dots, N \Big\}. \end{aligned}$$

Our main result in this paper can be stated as follows.

**Theorem 1.1.** *Suppose that $N > 2s$, $0 < s < 1$. If $V$ satisfies* (A1) *and* (A2)*, then for any positive integer $m$, there exists an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, the problem* (1.1) *has a solution of the form $u_\varepsilon = \sum_{k=1}^m U_{y_{\varepsilon_k}, \lambda_{\varepsilon_k}} + \omega_\varepsilon$, where $\omega_\varepsilon \in E_{\mathbf{y},\lambda}$, $\mathbf{y} = (y_{\epsilon_1}, y_{\epsilon_2}, \dots, y_{\epsilon_m})$ and as $\varepsilon \to 0$, $y_{\varepsilon_k} \to y_0$, $\lambda_{\varepsilon_k} \to +\infty$, $k = 1, \dots, m$, $\|\omega_\varepsilon\| \to 0$,*

The proof of our results is inspired by the methods of [18, 37], we will combine a penalization argument and Lyapunov-Schmidt reduction scheme which are similar to [18, 37] to prove our main result.

This article is organized as follows. In section 2, we introduce the penalization problem, give some preliminary estimates and carry out the finite dimensional reduction. In section 3, we give the proof of Theorem 1.1. Some technical estimates are left in the appendix. Throughout this paper, we simply write $\int f$ to mean the Lebesgue integral of $f(x)$ in $\mathbb{R}^N$. The ordinary inner product between two vectors $a, b \in \mathbb{R}^N$ will be denoted by $a \cdot b$, and $C, \tilde{C}, c_i$ denote generic constants, which may vary inside a chain of inequalities. We use $O(t), o(t)$ to mean $|O(t)| \le C|t|$, $\frac{o(t)}{t} \to 0$ as $t \to 0$; $o(1)$ denotes quantities that tend to 0 as $|x| \to \infty$.

## 2. Preliminaries and finite dimensional reduction

In this section, we give some preliminary results, which are crucial in the proof of the main theorem and the finite-dimensional reduction. Problem (1.1) is the Euler-Lagrange equation of functional

$$I_1(u) = \frac{1}{2}\langle u, u\rangle_s - \frac{1}{2^*(s) + \epsilon}\int_{\mathbb{R}^N} V(x) u^{2^*(s)+\epsilon}.$$

As we know, under the conditions (A1) and (A2), the functional $I_1(u)$ will not be well defined and differentiable in $D^s(\mathbb{R}^N)$

Inspired by the idea introduced by Yan [37] and Deng et al. [18], we modify the nonlinearity as in [37]. To this end, we need to fix some notation. Choose $R > 0$ large enough. Define

$$f(x, u) = \chi_{B_R(0)}(x) f_1(u) + \chi_{B_R^C(0)}(x) f_2(u),$$

where $\chi_{B_R(0)}$ denotes the characteristic function of $B_R(0)$, and

$$f_1(u) = \begin{cases} u^{2^*(s)+\epsilon-1}, & \text{if } 0 \le u \le \epsilon^{-k_2 N}, \\ a_\epsilon u^{2^*(s)-1} + b_\epsilon, & \text{if } u \ge \epsilon^{-k_2 N}, \\ -f_1(-u), & \text{if } u < 0, \end{cases}$$

where

$$a_\epsilon = \Big(1 + \frac{\epsilon}{2^*(s) - 1}\Big)\epsilon^{-k_2 N \epsilon}, \quad b_\epsilon = -\frac{\epsilon}{2^*(s) - 1}\epsilon^{-k_2 N(2^*(s)+\epsilon-1)},$$

and $k_2 > 0$ is a constant to be determined in Proposition 3.1. Moreover,

$$f_2(x, u) = \frac{1}{|x|^{N+2s+\epsilon(N-2s)}} \bar{f}_2(|x|^{N-2s} u),$$

where the nonnegative $C^1$ function $\bar{f}_2$ satisfies:

$$\bar{f}_2(u) = \begin{cases} 0 & \text{if } u \ge 2, \\ u^{2^*(s)+\epsilon-1} & \text{if } u \in [0, 1], \\ -\bar{f}_2(-u) & \text{if } u < 0. \end{cases}$$

Let $F(x,u)=\int_0^u f(x,\tau)d\tau$, then we have the following Lemma.

**Lemma 2.1.** *Assume that $V(x)$ satisfies (A1). Then $\int_{\mathbb{R}^N} V(x)F(x,u)$ is well defined on $D^s(\mathbb{R}^N)$.*

*Proof.* Using (A1) and the definition of $f(x,u)$, we obtain

$$\begin{aligned}
&|\int_{\mathbb{R}^N} V(x)F(x,u)dx|\\
&\le \int_{\mathbb{R}^N\setminus B_R(0)} |V(x)F(x,u)|dx + \int_{B_R(0)} |V(x)F(x,u)|dx\\
&\le C\int_{\mathbb{R}^N\setminus B_R(0)} (1+|x|)^q\Big(\int_0^u f(x,\tau)d\tau\Big)dx + C\int_{B_R(0)} |u|^{2^*(s)}dx\\
&\le C\int_{\mathbb{R}^N\setminus B_R(0)} (1+|x|)^q\frac{1}{|x|^{N+2s+\varepsilon(N-2s)}}\Big(\int_0^u \bar f_2(|x|^{N-2s}\tau)d\tau\Big)dx\\
&\quad + C\int_{\mathbb{R}^N} |u|^{2^*(s)}dx\\
&\le C\int_{\mathbb{R}^N\setminus B_R(0)} (1+|x|)^q\frac{1}{|x|^{2N+\varepsilon(N-2s)}}\Big(\int_0^{|x|^{N-2s}u} \bar f_2(\tau)d\tau\Big)dx + C\int_{\mathbb{R}^N} |u|^{2^*(s)}dx\\
&\le C\int_{\mathbb{R}^N} \frac{1}{1+|x|^{2N-q+(N-2s)\varepsilon}}dx + C\int_{\mathbb{R}^N} |u|^{2^*(s)}dx<+\infty.
\end{aligned}$$

Consequently, the result follows from the above estimate. □

Now we consider the penalization problem

$$\begin{gathered}
(-\Delta)^s u = V(x)f(x,u),\quad x\in R^N,\\
u\in D^s(\mathbb{R}^N).
\end{gathered}\tag{2.1}$$

The functional associated with problem (2.1) is given by

$$I(u)=\frac12\langle u,u\rangle_s-\int_{\mathbb{R}^N} V(x)F(x,u),\quad u\in D^s(\mathbb{R}^N).\tag{2.2}$$

It follows from Lemma 2.1 that $I\in C^1$ is well defined in $E_{\mathbf{y},\lambda}$ and hence its critical points are solutions of problem (2.1).

Denote

$$U(\mathbf{y},\lambda)=\sum_{k=1}^m U_{y_k,\lambda_k}$$

and set

$$J(\mathbf{y},\lambda,w)=I\Big(\sum_{k=1}^m U_{y_k,\lambda_k}+w\Big),\quad \forall(\mathbf{y},\lambda,w)\in M_{\mathbf{y},\lambda},\tag{2.3}$$

where

$$M_{\mathbf{y},\lambda}=\{(\mathbf{y},\lambda,w): w\in E_{\mathbf{y},\lambda},(\mathbf{y},\lambda)\in D_{\mathbf{y},m},\|w\|\le\delta\},\tag{2.4}$$

$$\begin{aligned}
D_{\mathbf{y},m}=\Big\{&\mathbf{y}=(y_1,y_2,\dots,y_m)\in\mathbb{R}^{mN},\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m),y_k\in\overline{B_\delta(y_0)},\\
&\lambda_k\in[\epsilon^{-k_1},\epsilon^{-k_2}],\ k=1,\dots,m,\ \epsilon_{jk}\le\frac1L,\ j\ne k\Big\},
\end{aligned}$$

where small $k_1 > 0$ and large $k_2 > 0$ are constants to be determined in Proposition 3.1, $L > 0$ large enough, and

$$\epsilon_{jk} = \Big(\frac{\lambda_j}{\lambda_k} + \frac{\lambda_k}{\lambda_j} + \lambda_j\lambda_k|y_j - y_k|^2\Big)^{\frac{2s-N}{2}}.$$

**Lemma 2.2.** *There exists $\varepsilon_0 > 0$, such that, for $\varepsilon \in (0, \varepsilon_0]$, $L > 0$ large enough, $\delta > 0$ small enough, $(\mathbf{y}, \lambda, w) \in M_{\mathbf{y},\lambda}$ is a critical point of $J$ if and only if $u = \sum_{k=1}^m U_{y_k,\lambda_k} + w$ is a critical point of $I$ in $D^s(\mathbb{R}^N)$.*

The proof of Lemma 2.2 is standard, since can be complete it with the same arguments as those in [30, 13], we omit it. Without loss of generality, we assume that $y_0 = 0$ and $V(0) = 1$. Expanding $J(\mathbf{y}, \lambda, w)$, we obtain

$$J(\mathbf{y}, \lambda, w) = J(\mathbf{y}, \lambda, 0) + l_{\mathbf{y},\lambda}(w) + \frac{1}{2}\langle L_{\mathbf{y},\lambda}w, w\rangle_s + R_{\mathbf{y},\lambda}(w),$$

where

$$\begin{aligned}
l_{\mathbf{y},\lambda}(w) &= -\int_{\mathbb{R}^N} V(x)\Big(\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon-1} w + \langle U_{\mathbf{y},\lambda}, w\rangle_s,\\
\langle L_{\mathbf{y},\lambda}w, w\rangle_s &= \langle w, w\rangle_s - (2^*(s)+\varepsilon-1)\int_{\mathbb{R}^N} V(x)\Big(\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon-2} w^2,\\
R_{\mathbf{y},\lambda,w} &= -\int_{\mathbb{R}^N} V(x)F(x, U_{\mathbf{y},\lambda} + w) + \int_{\mathbb{R}^N} V(x)F(x, U_{\mathbf{y},\lambda})\\
&\quad + \int_{\mathbb{R}^N} V(x)\Big(\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon-1} w\\
&\quad + \frac{2^*(s)+\varepsilon-1}{2}\int_{\mathbb{R}^N} V(x)\Big(\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon-2} w^2.
\end{aligned}$$

Now, we state a lemma which is very important for our precise estimate on the functional energy and can be found in [31].

**Lemma 2.3.** *For $2 < q \le 3$ and $|a| > |b|$,*

$$\Big||a+b|^q - |a|^q - |b|^q - q|a|^{q-1}|b| - q|b|^{q-2}|a|\Big| \le C|b|^{q-1}|a|.$$

*For $q > 3$,*

$$\Big||a+b|^q - |a|^q - |b|^q - q|a|^{q-1}|b| - q|b|^{q-1}|a|\Big| \le C(|a|^{q-2}|b|^2 + |b|^{q-2}|a|^2).$$

Next, we show the invertibility of $L_{\mathbf{y},\lambda}$.

**Lemma 2.4.** *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for $(\mathbf{y}, \lambda) \in D_{\mathbf{y},m}$,*

$$\|L_{\mathbf{y},\lambda}w\| \ge C\|w\|, \quad \forall w \in E_{\mathbf{y},\lambda}.$$

*Proof.* We proceed by contradiction. Assume that there exist $\varepsilon_n \to 0, \delta_n \to 0, \lambda^n \to \infty$, $(\mathbf{y}^n, \lambda^n) = (y_1^n, \dots, y_m^n, \lambda_1^n, \dots, \lambda_m^n) \in D_{\mathbf{y},m}$ and $w_n \in E_{\mathbf{y},\lambda}$, such that

$$\langle L_{\mathbf{y}^n,\lambda^n} w_n, \varphi\rangle_s = o_n(1)\|w_n\|\|\varphi\|, \quad \forall \varphi \in E_{\mathbf{y},\lambda}. \tag{2.5}$$

Without loss of generality, we assume that $\|w_n\| = 1$. Let

$$\tilde{w}_{n,k}(x) = (\lambda_k^n)^{\frac{2s-N}{2}} w_n((\lambda_k^n)^{-1}x + y_k^n), \quad k = 1, \dots, m.$$

Assume that

$$\tilde{w}_{n,k} \rightharpoonup w_k^*, \quad k = 1,\dots,m, \quad \text{as } n \to \infty,$$
$$\tilde{w}_{n,k} \to w_k^*, \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N), \quad k = 1,\dots,m, \quad \text{as } \ n \to \infty.$$

From

$$\Big\langle \frac{\partial U_{y_k^n,\lambda_k^n}}{\partial \lambda_k^n}, w_n \Big\rangle_s = \Big\langle \frac{\partial U_{y_k^n,\lambda_k^n}}{\partial y_{k_i}^n}, w_n \Big\rangle_s = 0,$$

$k = 1,\dots,m, i = 1,\dots,N$, we obtain

$$\Big\langle \frac{\partial U_{0,1}}{\partial \lambda}\Big|_{\lambda=1}, \tilde{w}_{n,k} \Big\rangle_s = \Big\langle \frac{\partial U_{0,1}}{\partial x_i}\Big|_{x=0}, \tilde{w}_{n,k} \Big\rangle_s = 0,$$

for $k = 1,\dots,m$, $i = 1,\dots,N$. So $w_k^*$ satisfies

$$\Big\langle \frac{\partial U_{0,1}}{\partial \lambda}\Big|_{\lambda=1}, w_k^* \Big\rangle_s = \Big\langle \frac{\partial U_{0,1}}{\partial x_i}\Big|_{x=0}, w_k^* \Big\rangle_s = 0, \tag{2.6}$$

for $k = 1,\dots,m, i = 1,\dots,N$.

Define

$$\widetilde{E}_{\mathbf{y},\lambda} = \{\varphi \in D^s(\mathbb{R}^N), \Big\langle \frac{\partial U_{0,1}}{\partial \lambda}\Big|_{\lambda=1}, \varphi \Big\rangle_s = \Big\langle \frac{\partial U_{0,1}}{\partial x_i}\Big|_{x=0}, \varphi \Big\rangle_s = 0, i = 1,\dots,N\}.$$

Note that

$$o(1)\|\varphi\| = \langle w_n, \varphi\rangle_s - \int_{\mathbb{R}^N} V(x) f'(x, U_{\mathbf{y}^n,\lambda^n}) w_n \varphi. \tag{2.7}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N) \cap \widetilde{E}_{\mathbf{y},\lambda}$ and take $\varphi_n(x) := (\lambda_k^n)^{\frac{2s-N}{2}} \varphi[(x - y_k^n)\lambda_k^n]$. Letting $n \to \infty$, we obtain

$$\langle w_k^*, \varphi\rangle_s - (2^*(s) - 1) \int_{\mathbb{R}^N} U_{0,1}^{2^*(s)-2} w_k^* \varphi = 0.$$

It is easy to prove that

$$\langle w_k^*, \varphi\rangle_s - (2^*(s) - 1) \int_{\mathbb{R}^N} U_{0,1}^{2^*(s)-2} w_k^* \varphi = 0, \quad \forall \varphi \in \widetilde{E}_{\mathbf{y},\lambda}. \tag{2.8}$$

But (2.8) is true for $\varphi = c_0 \frac{\partial U_{0,1}}{\partial \lambda}\big|_{\lambda=1} + \sum_{i=1}^n c_i \frac{\partial U_{0,1}}{\partial x_i}\big|_{x=0}$. Thus, (2.8) is true for any $\varphi \in E_{\mathbf{y},\lambda}$, and hence $w_k^* = c_0 \frac{\partial U_{0,1}}{\partial \lambda}\Big|_{\lambda=1} + \sum_{i=1}^n c_i \frac{\partial U_{0,1}}{\partial x_i}\Big|_{x=0}$.

It follows from (2.6) that $c_i = 0$ $(i = 0, 1,\dots,N)$ and $w_k^* = 0$. Therefore, letting $\varepsilon_n, \delta_n > 0$ small enough, $\lambda^n$ big enough,

$$\begin{aligned} o(1) &= \langle w_n, w_n\rangle_s - \int_{\mathbb{R}^N} V(x) f'(x, U_{\mathbf{y}^n,\lambda^n}) w_n^2 \\ &\geq 1 - C \int_{\mathbb{R}^N} U_{0,1}^{2^*(s)-2} w_n^2 = 1 + O_R(1) + o(1). \end{aligned} \tag{2.9}$$

This contradicts (2.5). □

**Proposition 2.5.** *For $\varepsilon > 0$ sufficiently small and $(\mathbf{y}, \lambda) \in D_{\mathbf{y},m}$, there exists a $C^1$-map $w(\mathbf{y}, \lambda) : D_{\mathbf{y},m} \to M_{\mathbf{y},\lambda}$ such that $w(\mathbf{y}, \lambda)$ satisfies $\Big\langle \frac{\partial J(w)}{\partial w}, \varphi \Big\rangle = 0$ for all $\varphi \in M_{\mathbf{y},\lambda}$. Moreover,*

$$\|w\| \leq C\Big( \sum_{k=1}^m \Big( \frac{|DV(y_k)|}{\lambda_k} + \frac{1}{\lambda_k^2} + \frac{1}{\lambda_k^{\frac{N+2s}{2}}} |V(0) - V(y_k)| + \varepsilon \ln \lambda_k \Big) + \sum_{j \neq k} \varepsilon_{jk}^{\frac{1}{2}+\tau} \Big). \tag{2.10}$$

*Proof.* To find a critical point for $J(w)$, we only need to solve

$$l_{\mathbf{y},\lambda} + \langle L_{\mathbf{y},\lambda} w, w\rangle + \mathbb{R}^{'}(w) = 0. \tag{2.11}$$

From Lemma 2.4, we know that $L_{\mathbf{y},\lambda}$ is invertible. Therefore, (2.11) can be rewritten as

$$w = \mathcal{A}(w) =: -L_{\mathbf{y},\lambda}^{-1} l_{\mathbf{y},\lambda} - L_{\mathbf{y},\lambda}^{-1}\mathbb{R}^{'}(w).$$

Set

$$\begin{aligned}\mathcal{N} = \Big\{ w \in E_{\mathbf{y},\lambda} : \|w\| \le \sum_{k=1}^{m} \frac{|DV(y_k)|}{\lambda_k^{1-\delta}} + \frac{1}{\lambda_k^{2(1-\delta)}} + \frac{1}{\lambda_k^{\frac{N+2s}{2}(1-\delta)}} |V(0) - V(y_k)| \\ + \varepsilon^{1-\delta} \ln\lambda_k + \sum_{j\ne k} \varepsilon_{jk}^{(\frac{1}{2}+\tau)(1-\delta)} \Big\},\end{aligned}$$

where $\delta > 0$ is small enough.

As in [37], $R(w)$ is the higher order term satisfying

$$\mathbb{R}^{i}(w) = O(\|w\|^{2+\theta-i}), \quad i = 0, 1, 2,$$

where $\theta > 0$ is some constant.

Hence, Lemma 2.6 below implies

$$\begin{aligned}
&\|\mathcal{A}(w)\| \\
&\le C\|l_{y,\lambda}\| + C\|w\|^{1+\theta} \\
&\le C\Big(\sum_{k=1}^{m}\Big(\frac{|DV(y_k)|}{\lambda_k} + \frac{1}{\lambda_k^2} + \frac{1}{\lambda_k^{\frac{N+2s}{2}}}|V(0) - V(y_k)| + \varepsilon\ln\lambda_k\Big) + \sum_{j\ne k}\varepsilon_{jk}^{\frac{1}{2}+\tau}\Big) \\
&\quad + C\Big(\sum_{k=1}^{m}\frac{|DV(y_k)|}{\lambda_k^{1-\delta}} + \frac{1}{\lambda_k^{2(1-\delta)}} + \frac{1}{\lambda_k^{\frac{N+2s}{2}(1-\delta)}}|V(0) - V(y_k)| + \varepsilon^{1-\delta}\ln\lambda_k \\
&\quad + \sum_{j\ne k}\varepsilon_{jk}^{(\frac{1}{2}+\tau)(1-\delta)}\Big)^{1+\theta} \\
&\le \sum_{k=1}^{m}\frac{|DV(y_k)|}{\lambda_k^{1-\delta}} + \frac{1}{\lambda_k^{2(1-\delta)}} + \frac{1}{\lambda_k^{\frac{N+2s}{2}(1-\delta)}}|V(0) - V(y_k)| + \varepsilon^{1-\delta}\ln\lambda_k \\
&\quad + \sum_{j\ne k}\varepsilon_{jk}^{(\frac{1}{2}+\tau)(1-\delta)}.
\end{aligned} \tag{2.12}$$

Meanwhile,

$$\begin{aligned}
\|\mathcal{A}(w_1) - \mathcal{A}(w_2)\| &= \|L_{y,\lambda}^{-1}\mathbb{R}^{'}(w_1) - L_{y,\lambda}^{-1}\mathbb{R}^{'}(w_2)\| \\
&\le C\|\mathbb{R}^{'}(w_1) - \mathbb{R}^{'}(w_2)\| \\
&\le C\|\mathbb{R}^{''}(\varepsilon w_1 + (1-\varepsilon)w_2)\|\|w_1 - w_2\| \\
&\le C(\|w_1\|^{\theta} + \|w_2\|^{\theta})\|w_1 - w_2\| \le \frac{1}{2}\|w_1 - w_2\|,
\end{aligned}$$

where $\varepsilon \in (0, 1)$. Thus, $\mathcal{A}$ maps $\mathcal{N}$ to $\mathcal{N}$ and $\mathcal{A}$ is a contraction map.

By the contraction mapping theorem, we see that there is a unique $w$ such that (2.11) holds, and from (2.12) that (2.10) holds. □

**Lemma 2.6.**

$$\|l_{\mathbf{y},\lambda}\| \le C\Big(\sum_{k=1}^m\Big(\frac{|DV(y_k)|}{\lambda_k}+\frac{1}{\lambda_k^2}+\frac{1}{\lambda_k^{\frac{N+2s}{2}}}|V(0)-V(y_k)|+\varepsilon\ln\lambda_k\Big)+\sum_{j\neq k}\varepsilon_{jk}^{\frac12+\tau}\Big).$$

*Proof.* First, we know that

$$\langle U_{\mathbf{y},\lambda},w\rangle_s=\Big\langle\sum_{k=1}^m U_{y_k,\lambda_k},w\Big\rangle_s=\sum_{k=1}^m\int_{\mathbb{R}^N}U_{y_k,\lambda_k}^{\frac{N+2s}{N-2s}}w. \tag{2.13}$$

Next, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N}V(x)f(x,U_{\mathbf{y},\lambda})w\\
&=\int_{\mathbb{R}^N}V(x)\Big(\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon-1}w\\
&=\begin{cases}\sum_{k=1}^m\int_{\mathbb{R}^N}V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon-1}w\\ +O\Big(\sum_{j\neq k}\int_{\mathbb{R}^N}V(x)U_{y_j,\lambda_j}^{\frac{2^*(s)+\varepsilon-1}{2}}U_{y_k,\lambda_k}^{\frac{2^*(s)+\varepsilon-1}{2}}w\Big), & \text{if } 1<2^*(s)+\varepsilon-1\le 2,\\ \sum_{k=1}^m\int_{\mathbb{R}^N}V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon-1}w+O\Big(\sum_{j\neq k}\varepsilon_{jk}\Big)\|w\|, & \text{if } 2^*(s)+\varepsilon-1>2,\end{cases}\\
&=\begin{cases}\sum_{k=1}^m\int_{\mathbb{R}^N}V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon-1}w+O\Big(\sum_{j\neq k}\varepsilon_{jk}^{\frac12+\tau}\Big)\|w\|, & \text{if } 1<2^*(s)+\varepsilon-1\le 2,\\ \sum_{k=1}^m\int_{\mathbb{R}^N}V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon-1}w+O\Big(\sum_{j\neq k}\varepsilon_{jk}\Big)\|w\|, & \text{if } 2^*(s)+\varepsilon-1>2.\end{cases}
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_{\mathbb{R}^N}V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon-1}w\\
&=\int_{\mathbb{R}^N}V(x)U_{y_k,\lambda_k}^{2^*(s)-1}w+O(\varepsilon\ln\lambda_k)\|w\|\\
&=\int_{\mathbb{R}^N}V(y_k)U_{y_k,\lambda_k}^{2^*(s)-1}w+O\Big(\frac{|DV(y_k)|}{\lambda_k}+\frac{1}{\lambda_k^2}+\varepsilon\ln\lambda_k\Big)\|w\|.
\end{aligned} \tag{2.14}$$

Combining above estimates, we obtain

$$\begin{aligned}
|l_{y,\lambda}w| &= \sum_{k=1}^m\int_{\mathbb{R}^N}U_{y_k,\lambda_k}^{\frac{N+2s}{N-2s}}w-\sum_{k=1}^m\int_{\mathbb{R}^N}V(y_k)U_{y_k,\lambda_k}^{\frac{N+2s}{N-2s}}w\\
&\quad+O\Big(\sum_{k=1}^m\Big(\frac{|DV(y_k)|}{\lambda_k}+\frac{1}{\lambda_k^2}+\varepsilon\ln\lambda_k\Big)+\sum_{j\neq k}\varepsilon_{jk}^{\frac12+\tau}\Big)\|w\|\\
&=\sum_{k=1}^m\int_{\mathbb{R}^N}|V(0)-V(y_k)|U_{y_k,\lambda_k}^{\frac{N+2s}{N-2s}}w\\
&\quad+O\Big(\sum_{k=1}^m\Big(\frac{|DV(y_k)|}{\lambda_k}+\frac{1}{\lambda_k^2}+\varepsilon\ln\lambda_k\Big)+\sum_{j\neq k}\varepsilon_{jk}^{\frac12+\tau}\Big)\|w\|\\
&=O\Big(\sum_{k=1}^m\Big(\frac{|DV(y_k)|}{\lambda_k}+\frac{1}{\lambda_k^2}+\frac{1}{\lambda_k^{\frac{N+2s}{2}}}|V(0)-V(y_k)|+\varepsilon\ln\lambda_k\Big)\\
&\quad+\sum_{j\neq k}\varepsilon_{jk}^{\frac12+\tau}\Big)\|w\|.
\end{aligned}$$

□

## 3. Proof of main result

Let $w(\mathbf{y},\lambda)$ be the map obtained in Proposition 2.5. Define

$$\widetilde{I}(\mathbf{y},\lambda) := I(\mathbf{y},\lambda,w(\mathbf{y},\lambda)), \quad (\mathbf{y},\lambda)\in D_{\mathbf{y},m}.$$

Let $(\mathbf{y}_\varepsilon,\lambda_\varepsilon)\in D_{\mathbf{y},m}$ be any point for which

$$\widetilde{I}(\mathbf{y}_\varepsilon,\lambda_\varepsilon) = \sup\{\widetilde{I}(\mathbf{y},\lambda) : (\mathbf{y},\lambda)\in D_{\mathbf{y},m}\}. \tag{3.1}$$

The next Proposition shows that for small $\varepsilon>0$, $(\mathbf{y}_\varepsilon,\lambda_\varepsilon)$ is an interior point of $D_{\mathbf{y},m}$, and hence a critical point of $\widetilde{I}$.

**Proposition 3.1.** *Let $(\mathbf{y}_\varepsilon,\lambda_\varepsilon)$ satisfy (3.1). Then as $\varepsilon\to 0$,*

$$y_\varepsilon^k \to 0, \quad k=1,\dots,m,$$
$$\lambda_\varepsilon^k \in [\epsilon^{-k_1},\epsilon^{-k_2}], \quad k=1,\dots,m, \text{ for some positive constant } k_1<k_2,$$
$$\epsilon_{jk}\to 0, \quad j\neq k.$$

*Proof.* It follows from Lemma 4.1, Lemma 2.6 and Proposition 2.5 that

$$\begin{aligned}
&I(\mathbf{y},\lambda,w(\mathbf{y},\lambda))\\
&= I(\mathbf{y},\lambda,0) + O(\|l_{\mathbf{y},\lambda}\|\|w_{\mathbf{y},\lambda}\| + \|w_{\mathbf{y},\lambda}\|^2)\\
&= \Big(\frac{m}{2} - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^m V(y_k)\Big)\int_{\mathbb{R}^N} U^{2^*(s)} - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^m \frac{\Delta V(y_k)}{2\lambda_k^2}\int_{\mathbb{R}^N}|x|^2U^{2^*(s)}\\
&\quad - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^m\Big[\varepsilon V(y_k)\Big(\int_{\mathbb{R}^N}\ln\lambda_k^{\frac{N-2s}{2}}U^{2^*(s)} - \int_{\mathbb{R}^N}U^{2^*(s)}\ln U\Big)\Big]\\
&\quad - \sum_{k=1}^{m-1}\int_{\mathbb{R}^N} V(x)U_{y_k,\lambda_k}\Big(\sum_{j=k+1}^m U_{y_j,\lambda_j}\Big)^{2^*(s)+\varepsilon-1} + O\Big(\sum_{k=1}^m\frac{1}{\lambda_k^{2+\mu}}\Big)\\
&\quad + O\Big(\sum_{j\neq k}\varepsilon_{jk}^{1+\tau}\Big) + O\Big(\sum_{k=1}^m\frac{|DV(y_k)|^2}{\lambda_k^2}\Big) + O\Big(\sum_{k=1}^m(1-V(y_k))^{1+\tau}\Big)\\
&\quad + O\Big(\sum_{k=1}^m\Big(\frac{\varepsilon\ln\lambda_k}{\lambda_k} + \varepsilon^2\ln^2\lambda_k\Big)\Big).
\end{aligned}$$

Denote $z_\varepsilon^k = \varepsilon e_k, \overline{\lambda}_\varepsilon^k = \frac{1}{\varepsilon^2}$, $k=1,\dots,m$. Some unit vectors $e_1,\dots,e_m$ with $e_k\neq e_j$ $(k\neq j)$, for Then $|z_\varepsilon^j - z_\varepsilon^k|^2 = \varepsilon^2|e_j-e_k|^2\to 0$, $\overline{\lambda}_\varepsilon^k\to\infty$, as $\varepsilon\to 0$. Then

$$\begin{aligned}
I(\mathbf{y}_\varepsilon,\lambda_\varepsilon,w_\varepsilon(\mathbf{y}_\varepsilon,\lambda_\varepsilon)) &\geq I(z_\varepsilon,\overline{\lambda}_\varepsilon,w_\varepsilon(z_\varepsilon,\overline{\lambda}_\varepsilon))\\
&= m\Big(\frac{1}{2} - \frac{1}{2^*(s)+\varepsilon}\Big)\int_{\mathbb{R}^N}U^{2^*(s)} - mC\varepsilon\ln\frac{1}{\varepsilon} + O(\varepsilon).
\end{aligned}$$

Thus

$$
\begin{aligned}
&\Big(\frac{m}{2}-\frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^{m}V(y_\epsilon^k)\Big)\int_{\mathbb{R}^N}U^{2^*(s)}-\frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^{m}\frac{\Delta V(y_\epsilon^k)}{2(\lambda_\epsilon^k)^2}\int_{\mathbb{R}^N}|x|^2U^{2^*(s)}\\
&-\frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^{m}\Big[\varepsilon V(y_\epsilon^k)\Big(\int_{\mathbb{R}^N}\ln(\lambda_\epsilon^k)^{\frac{N-2s}{2}}U^{2^*(s)}-\int_{\mathbb{R}^N}U^{2^*(s)}\ln U\Big)\Big]\\
&-\sum_{k=1}^{m-1}\int_{\mathbb{R}^N}V(x)U_{y_\epsilon^k,\lambda_\epsilon^k}\Big(\sum_{j=k+1}^{m}U_{y_\epsilon^j,\lambda_\epsilon^j}\Big)^{2^*(s)+\varepsilon-1}+O\Big(\sum_{k=1}^{m}\frac{1}{(\lambda_\epsilon^k)^{2+\mu}}\Big)\\
&+O\Big(\sum_{j\neq k}\varepsilon_{jk}^{1+\tau}\Big)+O\Big(\sum_{k=1}^{m}\frac{|DV(y_\epsilon^k)|^2}{(\lambda_\epsilon^k)^2}\Big)+O\Big(\sum_{k=1}^{m}(1-V(y_\epsilon^k))^{1+\tau}\Big)\\
&\quad+O\Big(\sum_{k=1}^{m}\Big(\frac{\varepsilon\ln\lambda_\epsilon^k}{\lambda_\epsilon^k}+\varepsilon^2\ln^2\lambda_\epsilon^k\Big)\Big)\\
&\geq m\Big(\frac12-\frac{1}{2^*(s)+\varepsilon}\Big)\int_{\mathbb{R}^N}U^{2^*(s)}-mC\varepsilon\ln\frac{1}{\varepsilon}+O(\varepsilon).
\end{aligned}
\tag{3.2}
$$

Moreover,

$$
\sum_{k=1}^{m-1}\int_{\mathbb{R}^N}V(x)U_{y_\epsilon^k,\lambda_\epsilon^k}\Big(\sum_{j=k+1}^{m}U_{y_\epsilon^j,\lambda_\epsilon^j}\Big)^{2^*(s)+\varepsilon-1}\geq C\sum_{j\neq k}\epsilon_{jk}. \tag{3.3}
$$

This and (3.2) imply

$$
0\leq V(y_\epsilon^j)-1\leq C\epsilon\ln\frac{1}{\epsilon}+O(\epsilon), \tag{3.4}
$$

$$
\epsilon_{jk}\leq C\epsilon\ln\frac{1}{\epsilon}+O(\epsilon), j\neq k, \tag{3.5}
$$

$$
\begin{aligned}
&\varepsilon V(y_\epsilon^k)\int_{\mathbb{R}^N}\ln(\lambda_\epsilon^k)^{\frac{N-2s}{2}}U^{2^*(s)}+\frac{\Delta V(y_\epsilon^k)}{2(\lambda_\epsilon^k)^2}\int_{\mathbb{R}^N}|x|^2U^{2^*(s)}\\
&+O\Big(\sum_{k=1}^{m}\frac{|DV(y_\epsilon^k)|^2}{(\lambda_\epsilon^k)^2}\Big)+\sum_{k=1}^{m}O\Big(\frac{1}{(\lambda_\epsilon^k)^{2+\mu}}\Big)\\
&\leq(2^*(s)+\epsilon)mC\epsilon\ln\frac{1}{\epsilon}+O(\epsilon),
\end{aligned}
\tag{3.6}
$$

which implies $\lambda_\varepsilon^k\to+\infty$ for $k=1,\dots,m$, and $\epsilon_{jk}\to0$ for $j\neq k$, as $\epsilon\to0$, $k,j=1,2,\dots,m$.

If $\lambda_\varepsilon^k=\epsilon^{-k_1}$ for some $k$, then from (3.6), we obtain

$$
\epsilon^{2k_1}\frac{\Delta V(y_\epsilon^k)}{2}\int_{\mathbb{R}^N}|x|^2U^{2^*(s)}+O\Big(\sum_{k=1}^{m}|DV(y_\epsilon^k)|^2\Big)\epsilon^{2k_1}\leq C\epsilon\ln\frac{1}{\epsilon}.
$$

This is a contradiction if $k_1>0$ small enough.

If $\lambda_\varepsilon^k=\epsilon^{-k_2}$ for some $k$, then from (3.6), we obtain

$$
\frac{N-2s}{2}k_2\varepsilon\ln\frac{1}{\epsilon}V(y_\epsilon^k)\int_{\mathbb{R}^N}U^{2^*(s)}\leq(2^*(s)+\epsilon)mC\epsilon\ln\frac{1}{\epsilon}+O(\epsilon),
$$

which is impossible if we choose $k_2>\max\{\frac{2(2^*(s)+1)mC}{(N-2s)\int_{\mathbb{R}^N}U^{2^*(s)}},\frac12\}$. Consequently, the result $\lambda_\varepsilon^k\in[\epsilon^{-k_1},\epsilon^{-k_2}]$ $(k=1,\dots,m)$ follows from the above estimate. $\square$

*Proof of Theorem 1.1.* By Proposition 3.1, we can check that (3.1) is achieved by $(\mathbf{y}_\varepsilon, \lambda_\varepsilon)$ which is an interior point of $D_{\mathbf{y},m}$ for small $\epsilon$, It follows from Proposition 2.5 that $(\mathbf{y}_\varepsilon, \lambda_\varepsilon)$ is a critical point of $J$. Using Lemma 2.2, we can obtain $u = \sum_{k=1}^m U_{y_k,\lambda_k} + w$ is a critical point of $I$. Note that

$$(-\Delta)^s w = V(x) f\Big(x, \sum_{k=1}^m U_{y_k,\lambda_k} + w\Big) - \sum_{k=1}^m U_{y_k,\lambda_k}^{2^*(s)-1}.$$

On the other hand,

$$\Big|V(x) f\Big(x, \sum_{k=1}^m U_{y_k,\lambda_k} + w\Big) - \sum_{k=1}^m U_{y_k,\lambda_k}^{2^*(s)-1}\Big| \le C|w|^{2^*(s)-1} + C\epsilon^{-\frac{k_2(N+2s)}{2}}.$$

Since $\|w_\epsilon\| \to 0$ as $\epsilon \to 0$, by adapting the same approach explored in [4] and [15], we deduce that

$$|w_\epsilon| \le C\|w_\epsilon\|_{L^{2^*(s)}(B_2(0))} + C\epsilon^{-\frac{k_2(N+2s)}{2}}, \quad \forall x \in B_2(0),$$

which implies $|u_\epsilon| \le \epsilon^{-k_2 N}$ for all $|x| < R$.

Let $\bar{u}_\epsilon(x) = |x|^{-(N-2s)} u_\epsilon(x/|x|^2)$ which satisfies

$$(-\Delta)^s \bar{u}_\epsilon = g(x)\bar{u}_\epsilon := \begin{cases} |x|^{(N-2s)\epsilon} V\Big(\frac{x}{|x|^2}\Big)\bar{f}_2(\bar{u}_\epsilon), & \text{if } |x| \le \frac{1}{R}, \\ |x|^{-(N+2s)} V\Big(\frac{x}{|x|^2}\Big) f_1(|x|^{N-2s}\bar{u}_\epsilon), & \text{if } \frac{1}{R} \le |x| \le \frac{2}{R}. \end{cases}$$

We can choose $\gamma > 0$ small enough, and it also follows from the same approach explored in [4] and [15] that

$$|\bar{u}_\epsilon|_{L^\infty(B_\gamma(0))} \le C\|\bar{u}_\epsilon\|_{L^{2^*(s)}(B_{2\gamma}(0))} \to 0, \quad \text{as } \epsilon \to 0,$$

which implies $|x|^{N-2s} u_\epsilon \to 0$ as $\epsilon \to 0$. From above estimates, we can obtain that $u$ solves indeed the original problem (1.1). ☐

## 4. Appendix

In this section, we prove some estimates needed in the proof of our main results.

**Lemma 4.1.** *For any $(\mathbf{y}, \lambda) \in D_{\mathbf{y},m}$, we have*

$$\begin{aligned}
&I(U_{\mathbf{y},\lambda})\\
&= \Big(\frac{m}{2} - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^m V(y_k)\Big)\int_{\mathbb{R}^N} U^{2^*(s)} - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^m \frac{\Delta V(y_k)}{2\lambda_k^2}\int_{\mathbb{R}^N}|x|^2 U^{2^*(s)}\\
&\quad + O\Big(\sum_{k=1}^m \lambda_k^{-(2+\mu)}\Big) + O\Big(\sum_{k=1}^m\Big(\frac{\varepsilon\ln\lambda_k}{\lambda_k} + \varepsilon^2\ln^2\lambda_k\Big)\Big) + O\Big(\sum_{j\ne k}\Big(\varepsilon + \frac{1}{\lambda_j}\Big)\varepsilon_{jk}\Big)\\
&\quad - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^m\Big(\varepsilon V(y_k)\int_{\mathbb{R}^N}\ln\lambda_k^{\frac{N-2s}{2}} U^{2^*(s)} - \varepsilon V(y_k)\int_{\mathbb{R}^N} U^{2^*(s)}\ln U\Big)\\
&\quad + O\Big(\sum_{k=1}^m (1-V(y_k))^{1+\tau}\Big) - \sum_{k=1}^{m-1}\int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k}\Big(\sum_{j=k+1}^m U_{y_j,\lambda_j}\Big)^{2^*(s)+\varepsilon-1}\\
&\quad + O\Big(\sum_{j\ne k}\varepsilon_{j,k}^{1+\tau}\Big).
\end{aligned}$$

*Proof.* Firstly, for any $(\mathbf{y},\lambda)\in D_{\mathbf{y},m}$, we see that

$$\begin{aligned}
I(U_{\mathbf{y},\lambda}) &= \frac{1}{2}\langle U_{\mathbf{y},\lambda},U_{\mathbf{y},\lambda}\rangle_s - \int_{\mathbb{R}^N} V(x)F(x,U_{\mathbf{y},\lambda})\\
&= \frac{1}{2}\langle U_{\mathbf{y},\lambda},U_{\mathbf{y},\lambda}\rangle_s - \frac{1}{2^*(s)+\varepsilon}\int_{\mathbb{R}^N} V(x)U_{\mathbf{y},\lambda}^{2^*(s)+\varepsilon}\\
&= \frac{1}{2}\sum_{k=1}^m\langle U_{y_k,\lambda_k},U_{y_k,\lambda_k}\rangle_s + \sum_{j<k}\langle U_{y_j,\lambda_j},U_{y_k,\lambda_k}\rangle_s\\
&\quad - \frac{1}{2^*(s)+\varepsilon}\int_{\mathbb{R}^N} V(x)U_{\mathbf{y},\lambda}^{2^*(s)+\varepsilon}\\
&= \frac{1}{2}\sum_{k=1}^m\langle U_{y_k,\lambda_k},U_{y_k,\lambda_k}\rangle_s + \sum_{j<k}\int_{\mathbb{R}^N} U_{y_j,\lambda_j}^{2^*(s)-1}U_{y_k,\lambda_k}\\
&\quad - \frac{1}{2^*(s)+\varepsilon}\int_{\mathbb{R}^N} V(x)U_{\mathbf{y},\lambda}^{2^*(s)+\varepsilon}.
\end{aligned} \tag{4.1}$$

Next, we estimate $\int_{\mathbb{R}^N} V(x)U_{\mathbf{y},\lambda}^{2^*(s)+\varepsilon}$. Using Lemma 2.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)U_{\mathbf{y},\lambda}^{2^*(s)+\varepsilon}\\
&= \int_{\mathbb{R}^N} V(x)\Big(U_{y_1,\lambda_1}+\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon}\\
&= \int_{\mathbb{R}^N} V(x)U_{y_1,\lambda_1}^{2^*(s)+\varepsilon} + \int_{\mathbb{R}^N} V(x)\Big(\sum_{k=1}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon}\\
&\quad + (2^*(s)+\varepsilon)\int_{\mathbb{R}^N} V(x)U_{y_1,\lambda_1}^{2^*(s)+\varepsilon-1}\sum_{k=2}^m U_{y_k,\lambda_k}\\
&\quad + (2^*(s)+\varepsilon)\int_{\mathbb{R}^N} V(x)U_{y_1,\lambda_1}\Big(\sum_{k=2}^m U_{y_k,\lambda_k}\Big)^{2^*(s)+\varepsilon-1} + O\Big(\sum_{j\neq k}\varepsilon_{j,k}^{1+\tau}\Big).
\end{aligned}$$

By repeated applications of Lemma 2.3, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)U_{\mathbf{y},\lambda}^{2^*(s)+\varepsilon}\\
&= \sum_{k=1}^m\int_{\mathbb{R}^N} V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon} + (2^*(s)+\varepsilon)\int_{\mathbb{R}^N} V(x)U_{y_1,\lambda_1}^{2^*(s)+\varepsilon-1}\sum_{k=2}^m U_{y_k,\lambda_k}\\
&\quad + (2^*(s)+\varepsilon)\int_{\mathbb{R}^N} V(x)U_{y_1,\lambda_1}(\sum_{k=2}^m U_{y_k,\lambda_k})^{2^*(s)+\varepsilon-1} + O\Big(\sum_{j\neq k}\varepsilon_{j,k}^{1+\tau}\Big).
\end{aligned} \tag{4.2}$$

We also deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)U_{y_k,\lambda_k}^{2^*(s)+\varepsilon}\\
&= \int_{\mathbb{R}^N} V(x)U_{y_k,\lambda_k}^{2^*(s)} + \varepsilon\int_{\mathbb{R}^N} V(x)U_{y_k,\lambda_k}^{2^*(s)}\ln U_{y_k,\lambda_k} + O(\varepsilon^2\ln^2\lambda_k)\\
&= \int_{\mathbb{R}^N} V(x)U_{y_k,\lambda_k}^{2^*(s)} + \varepsilon\int_{\mathbb{R}^N} V(y_k)U_{y_k,\lambda_k}^{2^*(s)}\ln U_{y_k,\lambda_k} + O\Big(\frac{\varepsilon\ln\lambda_k}{\lambda_k}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ O(\varepsilon^2 \ln^2 \lambda_k) \\
&= \int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k}^{2^*(s)} + \varepsilon V(y_k)\Big(\int_{\mathbb{R}^N} \ln \lambda_k^{\frac{N-2s}{2}} U^{2^*(s)} - \int_{\mathbb{R}^N} U^{2^*(s)} \ln U\Big) \\
&+ O\Big(\frac{\varepsilon \ln \lambda_k}{\lambda_k}\Big) + O(\varepsilon^2 \ln^2 \lambda_k).
\end{aligned} \tag{4.3}
$$

From (4.2) and (4.3), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} V(x) U_{y,\lambda}^{2^*(s)+\varepsilon} \\
&= \sum_{k=1}^m \Big[\int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k}^{2^*(s)} + \varepsilon V(y_k)\Big(\int_{\mathbb{R}^N} \ln \lambda_k^{\frac{N-2s}{2}} U^{2^*(s)} - \int_{\mathbb{R}^N} U^{2^*(s)} \ln U\Big) \\
&+ O\Big(\frac{\varepsilon \ln \lambda_k}{\lambda_k}\Big) + O(\varepsilon^2 \ln^2 \lambda_k)\Big] + (2^*(s)+\varepsilon) \sum_{j<k} \int_{\mathbb{R}^N} V(x) U_{y_j,\lambda_j}^{2^*(s)+\varepsilon-1} U_{y_k,\lambda_k} \\
&+ (2^*(s)+\varepsilon) \sum_{k=1}^{m-1} \int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k} \Big(\sum_{j=k+1}^m U_{y_j,\lambda_j}\Big)^{2^*(s)+\varepsilon-1} + O\Big(\sum_{j\neq k} \varepsilon_{j,k}^{1+\tau}\Big).
\end{aligned}
$$

We also have

$$
\begin{aligned}
\int_{\mathbb{R}^N} V(x) U_{y_j,\lambda_j}^{2^*(s)+\varepsilon-1} U_{y_k,\lambda_k} &= \int_{\mathbb{R}^N} V(x) U_{y_j,\lambda_j}^{2^*(s)-1} U_{y_k,\lambda_k} + O(\varepsilon_{j,k}\varepsilon) \\
&= \int_{\mathbb{R}^N} V(y_j) U_{y_j,\lambda_j}^{2^*(s)-1} U_{y_k,\lambda_k} + O(\varepsilon_{j,k}\varepsilon) + O(\frac{1}{\lambda_i}\varepsilon_{j,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k}^{2^*(s)} \\
&= \int_{\mathbb{R}^N} V(y_k) U_{y_k,\lambda_k}^{2^*(s)} + \int_{\mathbb{R}^N} \langle DV(y_k), x-y_k\rangle U_{y_k,\lambda_k}^{2^*(s)} \\
&\quad + \int_{\mathbb{R}^N} \langle D^2V(y_k)(y_k-x), (y_k-x)\rangle U_{y_k,\lambda_k}^{2^*(s)} + O(\lambda_k^{-(2+\mu)}) \\
&= V(y_k) \int_{\mathbb{R}^N} U^{2^*(s)} + \frac{\Delta V(y_k)}{2\lambda_k^2} \int_{\mathbb{R}^N} |x|^2 U^{2^*(s)} + O(\lambda_k^{-(2+\mu)}).
\end{aligned}
$$

Combining above estimates, we obtain

$$
\begin{aligned}
I(U_{\mathbf{y},\lambda}) &= \frac{1}{2} \sum_{k=1}^m \langle U_{y_k,\lambda_k}, U_{y_k,\lambda_k}\rangle_s + \sum_{j<k} \int_{\mathbb{R}^N} U_{y_j,\lambda_j}^{2^*(s)-1} U_{y_k,\lambda_k} \\
&- \frac{1}{2^*(s)+\varepsilon} \sum_{k=1}^m \Big[\int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k}^{2^*(s)} + \varepsilon V(y_k)\Big(\int_{\mathbb{R}^N} \ln \lambda_k^{\frac{N-2s}{2}} U^{2^*(s)} \\
&- \int_{\mathbb{R}^N} U^{2^*(s)} \ln U\Big) + O\Big(\frac{\varepsilon \ln \lambda_k}{\lambda_k}\Big) + O(\varepsilon^2 \ln^2 \lambda_k)\Big] \\
&- \sum_{j<k} \int_{\mathbb{R}^N} V(x) U_{y_j,\lambda_j}^{2^*(s)+\varepsilon-1} U_{y_k,\lambda_k} \\
&- \sum_{k=1}^{m-1} \int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k} \Big(\sum_{j=k+1}^m U_{y_j,\lambda_j}\Big)^{2^*(s)+\varepsilon-1} + O\Big(\sum_{j\neq k} \varepsilon_{j,k}^{1+\tau}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\frac{m}{2} - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^{m} V(y_k)\Big)\int_{\mathbb{R}^N} U^{2^*(s)} \\
&\quad - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^{m} \frac{\Delta V(y_k)}{2\lambda_k^2}\int_{\mathbb{R}^N} |x|^2 U^{2^*(s)} + O\Big(\sum_{k=1}^{m} \lambda_k^{-(2+\mu)}\Big) \\
&\quad + O\Big(\sum_{k=1}^{m}\Big(\frac{\varepsilon \ln\lambda_k}{\lambda_k} + \varepsilon^2 \ln^2 \lambda_k\Big)\Big) + O\Big(\sum_{j\neq k}\Big(\varepsilon + \frac{1}{\lambda_j}\Big)\varepsilon_{jk}\Big) \\
&\quad + O\Big(\sum_{j\neq k} \varepsilon_{j,k}^{1+\tau}\Big) - \sum_{k=1}^{m-1}\int_{\mathbb{R}^N} V(x) U_{y_k,\lambda_k}\Big(\sum_{j=k+1}^{m} U_{y_j,\lambda_j}\Big)^{2^*(s)+\varepsilon-1} \\
&\quad - \frac{1}{2^*(s)+\varepsilon}\sum_{k=1}^{m}\Big(\varepsilon V(y_k)\int_{\mathbb{R}^N} \ln \lambda_k^{\frac{N-2s}{2}} U^{2^*(s)} \\
&\quad - \varepsilon V(y_k)\int_{\mathbb{R}^N} U^{2^*(s)} \ln U\Big) + O\Big(\sum_{k=1}^{m}(1-V(y_k))^{1+\tau}\Big).
\end{aligned}
$$

□

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## References


[1] D. Applebaum; *Lévy processes: From probability to finance and quantum groups*, Not. Am. Math. Soc., 51 (2004), 1336-1347.

[2] A. Ambrosetti, J. Garcia Azorero, I. Peral; *Perturbation of $-\Delta u = u^{\frac{N+2}{N-2}}$, the scalar curvature problem in $\mathbb{R}^N$, and related topics*, J. Funct. Anal., 165 (1999), 117-149.

[3] D. Applebaum; *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics vol 116, Cambridge University Press, (2009).

[4] C. O. Alves, O. H. Miyagaki; *Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method*, Calc. Var. Partial Differ. Equ., 55 (2016), 1-19.

[5] B. Barrios, E. Colorado, R. Servadei, F. Soria; *A critical fractional equation with concave-convex power nonlinearities*, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), no. 4, 875-900.

[6] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez; *On some critical problems for the fractional Laplacian operator*, J. Differ. Equ., 252 (2012), 6133-6162.

[7] L. Caffarelli, J. M. Roquejoffre, Y. Sire; *Variational problems with free boundaries for the fractional Laplacian*, J. Eur. Math. Soc., 12 (2010), 1151-1179.

[8] W. Chen, C. Li, B. Ou; *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., 59 (2006), 330-343.

[9] D. Cao, E. Noussair, S. Yan; *On the scalar curvature equation $-\Delta u = (1+\epsilon K)u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$*, Calc. Var. Partial Differ. Equ., 15 (2002), 403-419.

[10] D. Cao, S. Peng; *Concentration of solutions for the Yamabe problem on half-spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 143 (2013), 73-99.

[11] D. Cao, S. Peng, S. Yan; *On the Webster scalar curvature problem on the CR sphere with a cylindricaltype symmetry*, J. Geom. Anal., 23 (2013) 1674-1702.

[12] L. Caffarelli, L. Silvestre; *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32(2007), 1245-1260.

[13] G. Chen, Y. Zheng; *Peak solutions for the fractional Nirenberg problem*, Nonlinear Anal., 122 (2015), 100-124.

[14] J. Dávila, M. del Pino, Y. Sire; *Non-degeneracy of the bubble in the critical case for nonlocal equations*, Proc. Amer. Math. Soc., 141 (2013), 3865-3870.

[15] J. Dávila, M. del Pino, S. Dipierro, E. Valdinoci; *Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum*, Anal. PDE, 8 (2015), 1165-1235.

[16] A. DelaTorre, M. del Pino, M. del mar González, J. Wei; *Delaunay-type singular solutions for the fractional Yamabe problem*, Math. Ann., to appear

[17] S. Dipierro, M. Medina, I. Peral, E. Valdinoci; *Bifurcation results for a fractional elliptic equation with critical exponent in* $\mathbb{R}^n$. Manuscripta Math. (2016), DOI: 10.1007/s00229-016-0878-3

[18] Y. Deng, S. Peng, H. Pi; *Bound states with clustered peaks for nonlinear Schrödinger equations with compactly supported potentials*, Advanced Nonlinear Studies, 14 (2014), 463-481.

[19] S. Dipierro, G. Palatucci, E. Valdinoci; *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian*, Matematiche, 68 (2013), 201-216.

[20] D. Garrido, M. Musso; *Entire sign-changing solutions with finite energy to the fractional Yamabe equation*, Pacific J. Math., 283 (2016), 85-114.

[21] Z. Han; *The Yamabe problem on manifolds with boundary: existence and compactness results*, Duke Math. J., 99 (1999), 489-542.

[22] T. Jin, Y. Li, J. Xiong; *The Nirenberg problem and its generalizations: A unified approach*, Math. Ann., Doi:10.1007/s00208-016-1477-z

[23] T. Jin, Y. Li, J. Xiong; *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, J. Eur. Math. Soc., 16 (2014), 1111-1171.

[24] T. Jin, Y. Li, J. Xiong; *On a fractional Nirenberg problem, part II: existence of solutions*, Int. Math. Res. Not., 2015 (2015), 1555-1589.

[25] Y. Li; *On* $-\Delta u = K(x)u^5$ *in* $\mathbb{R}^3$, Comm. Pure Appl. Math., 46 (1993), 303-340.

[26] Y. Li; *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc., 6 (2004), 153-180.

[27] W. Liu; *Multi-peak positive solutions for nonlinear fractional Schrödinger systems in* $\mathbb{R}^N$, Adv. Nonlinear Stud., 16 (2016), 231-247.

[28] Z. Liu; *Concentration of solution for the fractional Nirenberg problem*, Commun. Pure Appl. Anal., 15(2016), 563-576.

[29] W. Liu, L. Gan; *Multi-bump positive solutions of a fractional nonlinear Schrödinger equation in* $\mathbb{R}^N$, Commu. Pure Appl. Anal.,15 (2016), 413-428.

[30] W. Long, J. Yang; *Positive or sign-changing solutions for a critical semilinear nonlocal equation*, Z. Angew. Math. Phys., 67 (2016), 1-30.

[31] E. S. Noussair, S. Yan; *On positive multipeak solutions of a nonlinear elliptic problem*, J. London Math. Soc., 62 (2002), 213-227.

[32] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136(2012), 521-573.

[33] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discret. Contin. Dyn. Syst., 33 (2013), 2105-2137.

[34] R. Servadei, E. Valdinoci; *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal., 12(2013) , 2445-2464.

[35] J. Tan; *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differ. Equ., 42(2012), 21-41.

[36] J. Wei, S. Yan; *Infinite many positive solutions for the prescribed scalar curvature problem on* $\mathbb{S}^N$. J. Funct. Anal., 258 (2010), 3048-3081.

[37] S. Yan; *High-energy solutions for a nonlinear elliptic problem with slightly supercritical exponent*, Nonlinear Anal. TMA, 38(1999), 527-546.

[38] S. Yan; *Concentration of solutions for the scalar curvature equation on* $\mathbb{R}^N$, J. Differ. Equ., 163 (2000), 239-264.



Lu Gan

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

*E-mail address*: ganlu0401@163.com

Weiming Liu (corresponding author)

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

*E-mail address*: whu.027@163.com