Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 308, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

BOUNDARY BEHAVIOR OF THE UNIQUE SOLUTION OF A ONE-DIMENSIONAL PROBLEM

LING MI

Abstract. In this article, we analyze the blow-up rate of the unique solution to the singular boundary value problem

$$\begin{gathered} u''(t) = b(t)f(u(t)), \quad u(t) > 0,\ t > 0, \\ u(0) = \infty, \quad u(\infty) = 0, \end{gathered}$$

where $f(u)$ grows more slowly than u^p $(p > 1)$ at infinity, and $b \in C^1(0, \infty)$ which is positive and non-decreasing (it may vanish at zero).

1. Introduction and statement of main results

In this article, we consider the blow-up rate of the unique solution at zero of the singular boundary-value problem

$$\begin{gathered} u''(t) = b(t)f(u(t)), \quad u(t) > 0,\ t > 0, \\ u(0) = \infty, \quad u(\infty) = 0, \end{gathered} \tag{1.1}$$

under the following assumptions on the functions b and f:

(A1) $b \in C^1(0, \infty)$ is non-decreasing and $b(t) > 0$ for $t > 0$,
(A2) $f \in C^1[0, \infty)$, $f(0) = f'(0) = 0$, $f'(u) > 0$ for any $u > 0$,
(A3) the Keller-Osserman [13, 16] condition

$$\Theta(r) := \int_r^\infty \frac{ds}{\sqrt{2F(s)}} < \infty, \quad \forall r > 0, \quad F(s) = \int_0^s f(\tau)d\tau.$$

Boundary blow-up problems rise in many branches of mathematics and have been studied by many authors and in several contexts for a long time. Generally, solutions of boundary blow-up problems are said to be explosive solutions or large solutions. The pioneering research work on boundary blow-up problems goes back to Keller-Osserman [13, 16], who proved that the problem

$$\triangle u = f(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = \infty. \tag{1.2}$$

has one solution $u \in C^2(\Omega)$ if and only if (A3) holds.

2010 *Mathematics Subject Classification.* 35J25, 35J60, 35J65.
Key words and phrases. One-dimensional problems; uniqueness of the solution; boundary behavior.

Submitted June 30, 2015. Published November 30, 2016.

Loewner and Nirenberg [14] showed that if $f(u)=u^{p_0}$ with $p_0=\frac{N+2}{N-2}$, $N>2$, then problem (1.2) has a unique solution u satisfying

$$\lim_{d(x)\to 0} u(x)(d(x))^{(N-2)/2}=\Big(\frac{N(N-2)}{4}\Big)^{(N-2)/4}.$$

A function f is weakly superlinear when

$$f(s)=\beta_1 s(\ln s)^{\alpha}+\gamma_1 s(\ln s)^{\alpha-1}[1+o(1)] \quad \text{as } s\to\infty, \tag{1.3}$$

with $\beta_1>0$, $\alpha>2$ and $\gamma_1\in(-\infty,+\infty)$. This function grows more slowly at infinity than those variational functions with index $p>1$ or rapid ones. When f is weakly superlinear, Cîrstea and Du [9] consider the first order expansion of the blow-up solution of

$$\triangle u=b(x)f(u), \quad u>0, \quad x\in\Omega, \quad u|_{\partial\Omega}=\infty, \tag{1.4}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N\geq 2)$.

We point out that Cîrstea and Rădulescu [4]-[8], and Cîrstea and Du [9] introduced a new unified approach via the Karamata regular variation theory, to study the boundary behavior and uniqueness of solutions for boundary blow-up elliptic problems. For singular elliptic problems, we refer the reader to the papers [3, 12], [17]-[18], [21]-[22] and the references therein.

Now, let us return to problem (1.1). Cano-Casanova and López-Gómez [2] studied the existence, uniqueness and the blow-up rate of large solutions of

$$u''(t)=b(t)f(u(t)), \quad t>0, \quad u(0)=+\infty, \quad u(+\infty)=0, \tag{1.5}$$

where f satisfies (A2), (A3) and b satisfies

(A1') $b\in C[0,\infty)$ is non-decreasing and satisfies $b(t)>0$ for $t>0$,

Under the conditions (A2) and (A1'), problem (1.5) possesses a unique positive solution $\psi(t)$. Further, assuming that the following conditions are satisfied

(i) $f^*(u)=f(u)/u$ is non-decreasing on $(0,\infty)$ and, for some $\sigma>1$, $c_0:=\lim_{u\to\infty} f(u)/u^{\sigma}\in(0,\infty)$;

(ii) the limit

$$a_0:=\lim_{t\to 0^+}\frac{G(t)G''(t)}{[G'(t)]^2}\in(0,\infty)$$

is well defined for some $R>0$, where $G(t)$ stands for the function

$$G(t)=\int_t^R \frac{ds}{A(s)}, \quad A(t)=\Big(\int_0^t (b(\tau))^{1/(\sigma+1)}d\tau\Big)^{(\sigma+1)/(\sigma-1)}, \quad t\in(0,R],$$

the unique large solution $\psi(t)$ of (1.5) satisfies

$$\lim_{t\to 0^+}\frac{\psi(t)}{G(t)}=a_0^{-\sigma/(\sigma-1)}\Big(\frac{\sigma+1}{\sigma-1}\Big)^{(\sigma+1)/(\sigma-1)}c_0^{-1/(\sigma-1)}.$$

Later, using the Karamata regular variation theory, Zhang et al. [21] obtained the exact blow-up rate of the unique solution $\psi(t)$ of (1.5) for a more general nonlinear term f. Let b satisfy (A1) and $\sqrt{b}\in\Lambda$ (see the definition of Λ below), f satisfy (A2) and

(iii) $\int_0^1 \frac{d\nu}{f(\nu)}=\infty$;

(iv) $\lim_{s\to\infty} sf'(s)/f(s)=\sigma>1$.

Then, the unique solution $\psi(t)$ of (1.5) satisfies

$$\lim_{t\to 0^+}\frac{\psi(t)}{\varphi(K(t))}=\Big(\frac{2(C_k(\sigma-1)+2)}{\sigma-1}\Big)^{\sigma-1},$$

where $K(t)=\int_0^t\sqrt{b(s)}ds$ and φ is uniquely determined by the problem

$$\int_{\varphi(t)}^\infty\frac{d\nu}{f(\nu)}=t,\quad t>0.$$

However, there are fewer results for the exact blow-up rate of the unique solution to (1.1) at zero when $f(u)$ grows more slowly than u^p $(p>1)$ at infinity. This case is more difficult to handle than those foregoing cases, since the blow-up behavior of the solution depends more subtly on the behavior of $b(t)$ and $f(u)$.

Next we explain our assumption on $b(x)$. Let Λ denote the set of positive non-decreasing functions in $C^1(0,\delta_0)$ which satisfy

$$\lim_{t\to 0^+}\frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big):=C_k\in[0,\infty),\quad K(t)=\int_0^t k(s)ds.$$

We see that for each $k\in\Lambda$,

$$\lim_{t\to 0^+}\frac{K(t)}{k(t)}=0,\quad C_k\in[0,1]$$

and

$$\lim_{t\to 0^+}\frac{K(t)k'(t)}{k^2(t)}=1-\lim_{t\to 0^+}\frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big)=1-C_k. \tag{1.6}$$

The set Λ was first introduced by Cîrstea and Rădulescu [4] for studying the boundary behavior and uniqueness of solutions of problem (1.4).

Inspired by the above ideas, the main purpose of this article is to establish blow-up rate of the unique solution $l(t)$ at zero to (1.1) under appropriate conditions on the weight function b and the nonlinear term f. In this article, we assume that f growths more slowly than any u^p $(p>1)$ at infinity. In particular, we consider functions f which satisfy (A2) and (A3) and the following conditions hold:

- (A4) there exist two functions $f_1\in C^1[S_0,\infty)$ for some large $S_0>0$ and f_2 such that
$$f(s):=f_1(s)+f_2(s),\quad s\ge S_0;$$
- (A5)
$$\frac{f_1'(s)s}{f_1(s)}:=1+g(s),\quad s\ge S_0, \tag{1.7}$$
with $g\in C^1[S_0,\infty)$ satisfying
$$g(s)>0,\quad s\ge S_0,\quad \lim_{s\to\infty}g(s)=0, \tag{1.8}$$
$$\lim_{s\to\infty}\frac{sg'(s)}{g(s)}=0,\quad \lim_{s\to\infty}\frac{sg'(s)}{g^2(s)}=C_g\in\mathbb{R},\quad \lim_{s\to\infty}\frac{\sqrt{s/f_1(s)}}{g(s)}=0; \tag{1.9}$$
- (A6) either there exists a constant $E_1\neq 0$ such that
$$\lim_{s\to\infty}\frac{f_2(s)}{g(s)f_1(s)}=E_1 \tag{1.10}$$
or
$$\lim_{s\to\infty}\frac{f_2(s)}{g(s)f_1(s)}=0 \tag{1.11}$$

and there exists a constant $\mu \le 1$ such that

$$\lim_{s\to\infty} \frac{f_2(\xi s)}{f_2(s)} = \xi^{\mu}, \quad \forall \xi > 0. \tag{1.12}$$

Our main results are summarized as follows.

Theorem 1.1. *Assume* (A1)–(A6) *are satisfied. If* $b(t)$ *also satisfies*

(A7) *there exist* $k \in \Lambda$ *and a positive constant* b_0 *such that*

$$\lim_{t\to 0^+} \frac{b(t)}{k^2(t)} = b_0^2,$$

then the unique solution $l(t)$ *of* (1.1) *satisfies*

$$l(t) \sim \exp(\xi_0)\phi(b_0 K(t)), \tag{1.13}$$

where

$$\begin{gathered} \xi_0 = \frac{1}{2} - E_2 - (1 - C_k)\Big(\frac{1}{2} + C_g\Big), \\ E_2 = \begin{cases} E_1 & \text{if (1.10) holds;} \\ 0, & \text{if (1.11) and (1.12) hold,} \end{cases} \end{gathered} \tag{1.14}$$

and ϕ *is the unique solution of the problem*

$$\int_{\phi(t)}^{\infty} \frac{ds}{\sqrt{s f_1(s)}} = t, \quad \forall t > 0. \tag{1.15}$$

By $f_1(t) \sim f_1(t)$ as $t \to t_0 \in \bar{\mathbb{R}}$ we mean $\lim_{t\to t_0} \frac{f_1(t)}{f_2(t)} = c$, where c is a constant.

2. Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic process (see Bingham, Goldie and Teugels [1], Haan [10], Geluk and Haan [11], Maric [15], Resnick [19], Seneta [20] and the references therein.). In this section, we present some bases of Karamata regular variation theory which come from the Introductions and the Appendix in Maric [15], and Preliminaries in Resnick [19], Seneta [20].

Definition 2.1. *A positive measurable function* f *defined on* $[a, \infty)$*, for some* $a > 0$*, is called* regularly varying at infinity *with index* ρ*, written* $f \in RV_\rho$*, if for each* $\xi > 0$ *and some* $\rho \in \mathbb{R}$*,*

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^{\rho}. \tag{2.1}$$

In particular, when $\rho = 0$*,* f *is called* slowly varying at infinity*.*

Clearly, if $f \in RV_\rho$, then $L(t) := f(t)/t^\rho$ is slowly varying at infinity. Some basic examples of slowly varying functions at infinity are

(i) every measurable function on $[a, \infty)$ which has a positive limit at infinity;
(ii) $(\ln t)^q$ and $\big(\ln(\ln t)\big)^q$, $q \in \mathbb{R}$;
(iii) $e^{(\ln t)^q}$, $0 < q < 1$.

We also say that a positive measurable function h defined on $(0, a)$ for some $a > 0$, is *regularly varying at zero* with index ρ (written $h \in RVZ_\rho$) if $t \to h(1/t)$ belongs to $RV_{-\rho}$.

Proposition 2.2 (Uniform convergence). *If $f \in RV_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form (a_1, ∞) with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(0, a_1]$ provided f is bounded on $(0, a_1]$ for all $a_1 > 0$.*

Proposition 2.3 (Representation theorem). *A function L is slowly varying at infinity if and only if it can be written in the form*

$$L(t) = \varphi(t) \exp\Big(\int_{a_1}^t \frac{y(\tau)}{\tau} d\tau \Big), \quad t \geq a_1, \tag{2.2}$$

for some $a_1 \geq a$, where the functions φ and y are measurable and as $t \to \infty$, $y(t) \to 0$ and $\varphi(t) \to c_0$, with $c_0 > 0$.

We call

$$\hat{L}(t) = c_0 \exp\Big(\int_{a_1}^t \frac{y(\tau)}{\tau} d\tau \Big), \quad t \geq a_1, \tag{2.3}$$

its *normalized* slowly varying at infinity and

$$f(t) = t^\rho \hat{L}(t), \quad t \geq a_1, \tag{2.4}$$

its *normalized* regularly varying at infinity with index ρ (and write $f \in NRV_\rho$).

Similarly, h is called *normalized* regularly varying at zero with index ρ, written $h \in NRVZ_\rho$ if $t \to h(1/t)$ belongs to $NRV_{-\rho}$.

A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty), \quad \text{for some } a_1 > 0 \text{ and } \lim_{t\to\infty} \frac{tf'(t)}{f(t)} = \rho. \tag{2.5}$$

Proposition 2.4. *If functions L, L_1 are slowly varying at infinity, then*

(i) *L^ρ (for every $\rho \in \mathbb{R}$), $c_1 L + c_2 L_1$ ($c_1 \geq 0, c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(t) \to +\infty$ as $t \to \infty$), are also slowly varying at infinity.*

(ii) *For every $\rho > 0$ and $t \to \infty$,*

$$t^\rho L(t) \to +\infty, \quad t^{-\rho} L(t) \to 0.$$

(iii) *For $\rho \in \mathbb{R}$ and $t \to \infty$, $\ln(L(t))/\ln t \to 0$ and $\ln(t^\rho L(t))/\ln t \to \rho$.*

Proposition 2.5. *If $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$ with $\lim_{t\to\infty} f_2(t) = +\infty$, then $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$.*

Proposition 2.6 (Asymptotic behavior). *If a function L is slowly varying at infinity, then for $a \geq 0$ and $t \to \infty$,*

(i) *$\int_a^t s^\rho L(s) ds \cong (\rho+1)^{-1} t^{1+\rho} L(t)$, for $\rho > -1$'*

(ii) *$\int_t^\infty s^\rho L(s) ds \cong (-\beta-1)^{-1} t^{1+\rho} L(t)$, for $\rho < -1$.*

3. Auxiliary results

In this section, we give some results to be used in the proof of Theorem 1.1.

Lemma 3.1 ([22, Lemma 2.1]). *Let $k \in \Lambda$.*

(i) *When $C_k \in (0, 1)$, k is* normalized *regularly varying at zero with index $(1 - C_k)/C_k$;*

(ii) *when $C_k = 1$, k is normalized slowly varying at zero;*

(iii) *when $C_k = 0$, k grows faster than any t^p ($p > 1$) near zero.*

Denote

$$\Theta(r) = \int_r^\infty \frac{ds}{\sqrt{2F(s)}}, \quad \Theta_1(r) = \int_r^\infty \frac{ds}{\sqrt{sf_1(s)}}, \quad r > 0. \tag{3.1}$$

Then

$$\Theta'(r) = -\frac{1}{\sqrt{2F(r)}}, \quad \Theta_1'(r) = -\frac{1}{\sqrt{rf_1(r)}}, \quad r > 0. \tag{3.2}$$

Lemma 3.2. *Under the hypotheses of Theorem 1.1:*

(i)

$$\int_a^\infty \frac{ds}{\sqrt{sf_1(s)}} < \infty, \quad \forall a > 0;$$

(ii)

$$\lim_{r\to\infty} \frac{\Theta(\lambda r)}{\Theta(r)} = \lim_{r\to\infty} \frac{\Theta_1(\lambda r)}{\Theta_1(r)} = 1, \quad \forall \lambda \in (0,1);$$

(iii)

$$\lim_{r\to\infty} \frac{(r/f_1(r))^{1/2}}{\Theta_1(r)g(r)} = \frac{1}{2} + C_g;$$

(iv)

$$\lim_{r\to\infty} \frac{\frac{f_1(\xi r)}{\xi f_1(r)} - 1}{g(r)} = \ln \xi$$

uniformly for $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$*;*

(v)

$$\lim_{r\to\infty} \frac{f_2(\xi r)}{\xi g(r) f_1(r)} = E_2$$

uniformly for $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$*.*

Proof. By (1.7), (1.8) and (2.5), we see that $f_1 \in NRV_1$, hence,

$$\sqrt{sf_1(s)} \in NRV_1$$

Then, there exist $a_1 > 0$ and a function $\hat{L}$ which is normalized slowly varying at infinity such that

$$\sqrt{sf_1(s)} = c_0 s\hat{L}(s), \quad s \geq a_1. \tag{3.3}$$

(i) For arbitrary $\rho \in (1,\infty)$, it follows by Proposition 2.4 (ii) that

$$\lim_{s\to\infty} \frac{\sqrt{sf_1(s)}}{s^\rho} = c_0 \lim_{s\to\infty} s^{1-\rho}\hat{L}(s) = \infty.$$

Thus there exists $S_0 > 0$ such that

$$\sqrt{sf_1(s)} > s^\rho, \quad s \geq S_0,$$

i.e.

$$\frac{1}{\sqrt{sf_1(s)}} < \frac{1}{s^\rho}, \quad s \geq S_0,$$

and the results follow. The proof of (ii)–(v) can be found in [22, Lemma 2.6], we omit here. ☐

Lemma 3.3 ([22, Lemma 2.7]). *Assume hypotheses of Theorem 1.1, and let ϕ be the solution to the problem*

$$\int_{\phi(t)}^{\infty} \frac{ds}{\sqrt{sf_1(s)}} = t, \quad \forall t > 0.$$

Then

(i) $-\phi'(t) = \sqrt{\phi(t)f_1(\phi(t))}$, $\phi(t) > 0$, $t > 0$, $\phi(0) := \lim_{t\to 0^+}\phi(t) = \infty$, $\phi''(t) = \frac{1}{2}\big(f_1(\phi(t)) + \phi(t)f_1'(\phi(t))\big)$, $t > 0$;

(ii)
$$\lim_{t\to 0}\big(g(\phi(t))\big)^{-1}\Big(\frac{1}{2}\big(1 + \frac{\phi(t)f_1'(\phi(t))}{f_1(\phi(t))}\big) - \frac{f_1(\xi\phi(t))}{\xi f_1(\phi(t))}\Big) = \frac{1}{2} - \ln\xi;$$

(iii)
$$\lim_{t\to 0}\frac{\sqrt{\phi(t)f_1(\phi(t))}}{tg(\phi(t))f_1(\phi(t))} = \frac{1}{2} + C_g;$$

(iv)
$$\lim_{t\to 0}\frac{f_2(\xi\phi(t))}{\xi g(\phi(t))f_1(\phi(t))} = E_2$$

uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.

4. Proof of Theorem 1.1

Since the nonlinear term f satisfies (A2) and (A3), by [2, Theorem 2.1], we obtain under the assumptions on Theorem 1.1, that problem (1.1) has a unique positive solution.

Lemma 4.1. *Under the assumptions on Theorem 1.1, there are $\delta \in (0, \delta_0)$ and $0 < \varsigma_0 < \lambda_0$ such that for every $\varsigma \in (0, \varsigma_0]$ and $\lambda \in [\lambda_0, \infty)$, $\bar{u}(t) = \lambda\exp(\xi_0)\phi(b_0K(t))$ and $\underline{u}(t) = \varsigma\exp(\xi_0)\phi(b_0K(t))$ are a supersolution and a subsolution, respectively, of the problem*

$$u''(t) = b(t)f(u(t)), \quad u(t) > 0, \quad t > 0, \quad u(0) = \infty, \quad u(\delta) = l(\delta), \tag{4.1}$$

where $l(t)$ denotes the unique solution of (1.1).

Proof. Let

$$\begin{aligned}\Upsilon_0(t) = \big(g(\phi(b_0K(t)))\big)^{-1}\Big(\frac{1}{2}\big(1 + \frac{\phi(b_0K(t))f_1'(\phi(b_0K(t)))}{f_1(\phi(b_0K(t)))}\big) \\ - \frac{b(t)}{b_0^2k^2(t)}\frac{f_1(\omega\phi(b_0K(t)))}{\omega f_1(\phi(b_0K(t)))}\Big) - \frac{\sqrt{\phi(b_0K(t))f_1(\phi(b_0K(t)))}}{b_0K(t)g(\phi(b_0K(t)))f_1(\phi(b_0K(t)))}\frac{K(t)k'(t)}{k^2(t)},\end{aligned}$$

for $t \in (0, \delta_0)$, $\omega > 0$; and

$$\Upsilon_1(t) = \frac{b(t)}{b_0^2k^2(t)}\frac{f_2(\omega\phi(b_0K(t)))}{\omega g(\phi(b_0K(t)))f_1(\phi(b_0K(t)))}, \quad t \in (0, \delta_0),\ \omega > 0.$$

By (1.6), Lemma 3.3 and Proposition 2.2, we see that

$$\lim_{t\to 0^+}\Upsilon_0(t) = \theta_0 := \frac{1}{2} - \ln\omega - (\frac{1}{2} + C_g)(1 - C_k),$$

and

$$\lim_{t\to 0^+}\Upsilon_1(t) = E_2,$$

which has uniform convergence on intervals $(0, a_1]$ for all $a_1 > 0$ and $\omega \in (0, a_1]$.

Thus for each $m_0 \in (0,1), M_0 \in (1,\infty)$ and $\omega > 0$, there exists $\delta \in (0,\delta_0)$ such that

$$m_0\theta_0 < \Upsilon_0(t) < M_0\theta_0, \quad \forall t \in (0,\delta);$$
$$m_0 E_2 < \Upsilon_1(t) < M_0 E_2, \quad \forall t \in (0,\delta).$$

Let λ and ς be positive constants satisfying

$$\lambda \ge \lambda_0 := \max\Big\{\frac{l(\delta)\exp(-\xi_0)}{\phi(b_0K(\delta))}, \exp\Big(E_2 - \frac{m_0}{M_0}E_2\Big)\Big\},$$
$$\varsigma \le \varsigma_0 := \min\Big\{\frac{l(\delta)\exp(-\xi_0)}{\phi(b_0K(\delta))}, \exp\Big(E_2 - \frac{M_0}{m_0}E_2\Big)\Big\}.$$

By a direct computation, we have

$$\bar{u}''(t) \le b(t)f(\bar{u}(t)), \quad t \in (0,\delta), \quad \bar{u}(0) = \infty, \quad \bar{u}(\delta) \ge l(\delta);$$
$$\underline{u}''(t) \ge b(t)f(\underline{u}(t)), \quad t \in (0,\delta), \quad \underline{u}(0) = \infty, \quad \underline{u}(\delta) \le l(\delta).$$

i.e., $\bar{u}$ is a supersolution and $\underline{u}$ is a subsolution to (4.1). □

Lemma 4.2. *Let $\delta > 0$, $\varsigma_0 > 0$ and $\lambda_0 > 0$ be the positive constants given by Lemma 4.1. Then, for every $\varsigma \in (0,\varsigma_0]$ and $\lambda \in [\lambda_0,\infty)$,*

$$\varsigma\exp(\xi_0)\phi(b_0K(t)) \le l(t) \le \lambda\exp(\xi_0)\phi(b_0K(t)), \quad t \in (0,\delta),$$

where $l(t)$ denotes the unique solution of (1.1) *and ϕ is defined by* (1.15).

Proof. According to [2, Remark 1], $l(t)$ provides us with the unique positive solution of

$$u''(t) = b(t)f(u(t)), \quad t \in (0,\delta), \quad u(0) = \infty, \quad u(\delta) = l(\delta). \tag{4.2}$$

Subsequently, given $\varsigma \in (0,\varsigma_0]$ and $\lambda \in [\lambda_0,\infty)$, for each natural number $n > \delta^{-1}$ we consider the boundary value problem

$$\begin{gathered} u''(t) = b(t)f(u(t)), \quad t \in (n^{-1},\delta), \\ u(n^{-1}) = \frac{\varsigma+\lambda}{2}\exp(\xi_0)\phi(b_0K(n^{-1})), \quad u(\delta) = l(\delta). \end{gathered} \tag{4.3}$$

Set $\underline{u}(t) = \varsigma\exp(\xi_0)\phi(b_0K(t))$ and $\bar{u}(t) = \lambda\exp(\xi_0)\phi(b_0K(t))$. By Lemma 4.1, $(\underline{u},\bar{u})$ provides us with an ordered sub-supersolution pair of (4.3). Thus, this problem possesses a solution u_n such that

$$\underline{u}(t) \le u_n(t) \le \bar{u}(t), \quad t \in [n^{-1},\delta].$$

By a standard compactness argument, we can extract a subsequence of u_n, say u_{nm}, $m \ge 1$, approximating to a solution of (4.2); necessarily l, by uniqueness. Therefore, passing to the limit as $m \to \infty$ in the estimates

$$\underline{u}(t) \le u_{nm}(t) \le \bar{u}(t), \quad t \in [n_m^{-1},\delta],$$

we can get the result easily. □

Proof of Theorem 1.1. We consider the auxiliary function

$$h(t) = \frac{l(t)}{\exp(\xi_0)\phi(b_0K(t))}, \quad t \in (0,\delta].$$

By Lemma 4.2, $h(t)$ satisfies the estimate

$$\varsigma \le h(t) \le \lambda, \quad t \in (0,\delta],$$

and, hence,

$$0 < \varsigma \leq \underline{h} := \liminf_{t \to 0^+} h(t) \leq \bar{h} := \limsup_{t \to 0^+} h(t) \leq \lambda.$$

To show the existence of $\lim_{t \to 0^+} h(t)$, we argue by contradiction. Suppose $\underline{h} < \bar{h}$. Then, there exist two sequences $t_n, s_n, n \geq 1$, such that

$$\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = 0, \quad \lim_{n \to \infty} h(t_n) = \bar{h}, \quad \lim_{n \to \infty} h(s_n) = \underline{h},$$

and, for each $n \geq 1$,

$$h'(t_n) = h'(s_n) = 0, \quad h''(t_n) \leq 0, \quad h''(s_n) \geq 0. \tag{4.4}$$

Clearly,

$$l'(t) = \exp(\xi_0)\big(h'(t)\phi(b_0 K(t)) + b_0 h(t)\phi'(b_0 K(t))k(t)\big),$$

and

$$\begin{aligned} l''(t) = \exp(\xi_0)\big(&h''(t)\phi(b_0 K(t)) + 2b_0 h'(t)\phi'(b_0 K(t))k(t) \\ &+ b_0^2 h(t)\phi''(b_0 K(t))k^2(t) + b_0 h(t)\phi'(b_0 K(t))k'(t)\big). \end{aligned}$$

Since $l''(t) = b(t)f(l(t))$, we have

$$\begin{aligned} &\exp(\xi_0)\big(h''(t)\phi(b_0 K(t)) + 2b_0 h'(t)\phi'(b_0 K(t))k(t) \\ &+ b_0^2 h(t)\phi''(b_0 K(t))k^2(t) + b_0 h(t)\phi'(b_0 K(t))k'(t)\big) \\ &= b(t)f(l(t)), \quad t \in (0, \delta]. \end{aligned} \tag{4.5}$$

By (1.6), Proposition 2.2, Lemma 3.3 and Lemma 4.1, we have

$$\begin{aligned} \lim_{t \to 0} \big(g(\phi(b_0 K(t)))\big)^{-1} \Big(&\frac{b_0^2 \phi''(b_0 K(t))}{k^2(t) f_1(\phi(b_0 K(t)))} + \frac{b_0 \phi'(b_0 K(t))k'(t)}{k^2(t) f_1(\phi(b_0 K(t)))} \\ &- \frac{b(t) f_1(l(t))}{\exp(\xi_0) h(t) k^2(t) f_1(\phi(b_0 K(t)))}\Big) > 0. \end{aligned}$$

On the other hand, $k^2(t) > 0$, $g(\phi(b_0 K(t))) > 0$ and $f_1(\phi(b_0 K(t))) > 0$ for all $t \in (0, \delta]$. Hence, there exists $\delta_1 \in (0, \delta)$ such that

$$\begin{aligned} &b_0^2 \phi''(b_0 K(t))k^2(t) + b_0 \phi'(b_0 K(t))k'(t) - \frac{b(t) f_1(l(t))}{\exp(\xi_0) h(t)} \\ &= k^2(t) f_1(\phi(b_0 K(t))) \Big(\frac{b_0^2 \phi''(b_0 K(t))}{k^2(t) f_1(\phi(b_0 K(t)))} + \frac{b_0 \phi'(b_0 K(t))k'(t)}{k^2(t) f_1(\phi(b_0 K(t)))} \\ &\quad - \frac{\exp(-\xi_0) b(t) f_1(l(t))}{h(t) k^2(t) f_1(\phi(b_0 K(t)))}\Big) > 0, \quad t \in (0, \delta_1]. \end{aligned}$$

Thus, by (4.4) and (4.5), we obtain that, for any $n \geq 1$,

$$\begin{aligned} &h(t_n) \\ &\geq h(t_n) + h''(t_n) \frac{\phi(b_0 K(t_n))}{b_0^2 \phi''(b_0 K(t_n))k^2(t_n) + b_0 \phi'(b_0 K(t_n))k'(t_n) - \frac{b(t_n) f_1(l(t_n))}{\exp(\xi_0) h(t_n)}} \\ &= \frac{b(t_n) f_2(l(t_n))}{b_0^2 \phi''(b_0 K(t_n))k^2(t_n) + b_0 \phi'(b_0 K(t_n))k'(t_n) - \frac{b(t_n) f_1(l(t_n))}{\exp(\xi_0) h(t_n)}}, \end{aligned}$$

and

$$h(s_n)$$

$$\geq h(s_n)+h''(s_n)\frac{\phi(b_0K(s_n))}{b_0^2\phi''(b_0K(s_n))k^2(s_n)+b_0\phi'(b_0K(s_n))k'(s_n)-\frac{b(s_n)f_1(l(s_n))}{\exp(\xi_0)h(s_n)}}$$
$$=\frac{b(s_n)f_2(l(s_n))}{b_0^2\phi''(b_0K(s_n))k^2(s_n)+b_0\phi'(b_0K(s_n))k'(s_n)-\frac{b(s_n)f_1(l(s_n))}{\exp(\xi_0)h(s_n)}}.$$

Therefore, passing to the limit as $n\to\infty$ in these inequalities, it follows from (A7), (1.6), (1.14) and Lemma 3.3 that

$$\bar h\geq\frac{E_2\bar h}{E_2-\ln\bar h},\quad\text{and}\quad \underline{h}\leq\frac{E_2\underline{h}}{E_2-\ln\underline{h}}.$$

Consequently, $\bar h=\underline{h}=1$, which contradicts the assumption $\underline{h}<\bar h$. Therefore, the following limit exists

$$h_0:=\lim_{t\to0^+}\frac{l(t)}{\exp(\xi_0)\phi(b_0K(t))}\in[\varsigma,\lambda],$$

i.e. $l(t)\sim\exp(\xi_0)\phi(b_0K(t))$. The proof is complete. □

5. Examples

In this section, we shw some basic cases of the nonlinear term f, and apply our results to this examples.

Example 5.1. $f(s)=C_1^2s(\ln s)^{2\alpha}+f_2(s)$, where $\alpha>1$, $s>S_0$,

$$g(s)=2\alpha(\ln s)^{-1};\quad \lim_{s\to\infty}\frac{\sqrt{s/f_1(s)}}{g(s)}=\frac{1}{2\alpha C_1}\lim_{s\to\infty}(\ln s)^{-(\alpha-1)}=0;$$
$$\frac{sg'(s)}{g^2(s)}\equiv C_g=-\frac{1}{2\alpha};\quad \lim_{s\to\infty}\frac{f_2(s)}{g(s)f_1(s)}=\frac{1}{2\alpha C_1^2}\lim_{s\to\infty}\frac{f_2(s)}{s(\ln s)^{2\alpha-1}}=E_2;$$
$$\phi(t)=\exp\big(C_1(\alpha-1)t\big)^{-1/(\alpha-1)}.$$

Then

$$l(t)\sim\exp\Big(\frac12-E_2-\frac{(1-C_k)(\alpha-1)}{2\alpha}\Big)\exp\big(C_1(\alpha-1)b_0K(t)\big)^{-1/(\alpha-1)}$$

as $t\to0^+$.

In particular, when $f_2(s)=C_2s^{\mu}(\ln s)^{\beta}$ with $\beta\leq2\alpha-1$, $E_1=0$ for $\mu<1$ or $\mu=1$ and $\beta<2\alpha-1$, and $E_1=\frac{C_2}{2\alpha C_1^2}$ for $\mu=1$ and $\beta=2\alpha-1$.

Example 5.2. $f(s)=C_1^2se^{(\ln s)^q}+f_2(s)$, where $q\in(0,1)$, $s>S_0$,

$$g(s)=q(\ln s)^{-(1-q)};\quad \lim_{s\to\infty}\frac{\sqrt{s/f_1(s)}}{g(s)}=\frac{1}{qC_1}\lim_{s\to\infty}\frac{\exp(-\frac12(\ln s)^q)}{(\ln s)^{-(1-q)}}=0;$$
$$\lim_{s\to\infty}\frac{sg'(s)}{g^2(s)}=-\frac{1-q}{q}\lim_{s\to\infty}(\ln s)^{-q}=C_g=0;$$
$$\lim_{s\to\infty}\frac{f_2(s)}{g(s)f_1(s)}=\frac{1}{qC_1^2}\lim_{s\to\infty}\frac{f_2(s)}{s(\ln s)^{-(1-q)}\exp((\ln s)^q)}=E_2;$$

Then

$$l(t)\sim\exp\big(\frac{C_k}{2}-E_2\big)\phi(b_0K(t))\quad\text{as } t\to0^+,$$

where $\phi(t)$ is defined by

$$\int_{\ln(\phi(t))}^{\infty} \exp(-s^q/2)ds = C_1 t.$$

Example 5.3. $f(s) = C_1^2 s(\ln s)^2(\ln(\ln s))^{2\alpha} + f_2(s)$, where $\alpha > 1$, $s > S_0$,

$$g(s) = 2(\ln s)^{-1}\big(1 + \alpha(\ln(\ln s))^{-1}\big);$$

$$\lim_{s\to\infty} \frac{\sqrt{s/f_1(s)}}{g(s)} = \frac{1}{2C_1} \lim_{s\to\infty} \frac{(\ln(\ln s))^{-\alpha}}{1+\alpha(\ln(\ln s))^{-1}} = 0;$$

$$\lim_{s\to\infty} \frac{sg'(s)}{g^2(s)} = -\lim_{s\to\infty} \frac{1+\alpha(\ln(\ln s))^{-1} + \alpha(\ln(\ln s))^{-2}}{2(1+\alpha(\ln(\ln s))^{-1})^2} = C_g = -\frac{1}{2};$$

$$\lim_{s\to\infty} \frac{f_2(s)}{g(s)f_1(s)} = \frac{1}{2C_1^2} \lim_{s\to\infty} \frac{f_2(s)}{s\ln s(\ln(\ln s))^{2\alpha}(1+\alpha(\ln(\ln s))^{-1})} = E_2;$$

$$\phi(t) = \exp\big(\exp\big(C_1(\alpha-1)t\big)^{-1/(\alpha-1)}\big).$$

Then

$$l(t) \sim \exp\big(\frac{1}{2} - E_2\big) \exp\big(\exp\big(C_1(\alpha-1)b_0K(t)\big)^{-1/(\alpha-1)}\big).$$

Acknowledgments. This work was partially supported by the NSF of China (no. 11301250) and the NSF of Shandong Province (no. ZR2013AQ004).

References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels; *Regular Variation, Encyclopedia of Mathematics and its Applications 27,* Cambridge University Press, Cambridge, 1987.

[2] S. Cano-Casanova, J. López-Gómez; *Existence, uniqueness and blow-up rate of large solutions for a canonical class of one-dimension problems on the half-line,* J. Differential Equations **244** (2008), 3180-3203.

[3] M. Cencelj, D. Repovš, Ž.Virk; *Multiple perturbations of a singular eigenvalue problem,* Nonlinear Anal. **119** (2015), 37-45.

[4] F. Cîrstea, V. Rădulescu; *Uniqueness of the blow-up boundary solution of logistic equations with absorbtion,* C. R. Acad. Sci. Paris, Sér. I, **335** (2002), 447-452.

[5] F. Cîrstea, V. Rădulescu; *Blow-up solutions for semilinear elliptic problems,* Nonlinear Anal., **48** (2002), 541-554.

[6] F. Cîrstea, V. Rădulescu; *Asymptotics for the blow-up boundary solution of the logistic equation with absorption,* C. R. Acad. Sci. Paris, Sér. I, **336** (2003), 231-236.

[7] F. Cîrstea, V. Rădulescu; *Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach,* Asymptot. Anal. **46** (2006), 275-298.

[8] F. Cîrstea, V. Rădulescu; *Boundary blow-up in nonlinear elliptic equations of Bieberbach-Rademacher type,* Trans. Amer. Math. Soc., **359** (2007), 3275-3286.

[9] F. Cîrstea, Y. Du; *Large solutions of elliptic equations with a weakly superlinear nonlinearity,* J. Anal. Math., **103** (2007), 261-277.

[10] L. de Haan; *On Regular Variation and its Application to the weak Convergence of Sample Extremes,* University of Amsterdam / Maths. Centre Tract 32, Amsterdam, 1970.

[11] J. L. Geluk, L. de Haan; *Regular Variation, Extensions and Tauberian Theorems,* CWI Tract, Centrum Wisk. Inform., Amsterdam, 1987.

[12] M. Ghergu, V. Rădulescu; *Singular elliptic problems: bifurcation and asymptotic analysis.* Oxford Lecture Series in Mathematics and its Applications, 37. The Clarendon Press, Oxford University Press, Oxford, 2008.

[13] J. B. Keller; *On solutions of* $\triangle u = f(u)$, Comm. Pure Appl. Math. , **10** (1957), 503-510.

[14] C. Loewner, L. Nirenberg; Partial differential equations invariant under conformal or projective transformations, in: Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers), Academic Press, New York, 1974, 245-272.

[15] V. Maric; *Regular Variation and Differential Equations, Lecture Notes in Math.*, vol. 1726, Springer-Verlag, Berlin, 2000.

[16] R. Osserman; *On the inequality* $\triangle u \geq f(u)$, Pacific J. Math., **7** (1957), 1641-1647.

[17] V. Rădulescu; *Singular phenomena in nonlinear elliptic problems: from blow-up boundary solutions to equations with singular nonlinearities. Handbook of differential equations: stationary partial differential equations.* Vol. IV, 485-593, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007.

[18] D. Repovš; *Asymptotics for singular solutions of quasilinear elliptic equations with an absorption term,* J. Math. Anal. Appl., **395** (2012) no. 1, 78-85.

[19] S. I. Resnick; *Extreme Values, Regular Variation, and Point Processes,* Springer-Verlag, New York, Berlin, 1987.

[20] R. Seneta; *Regular Varying Functions,* Lecture Notes in Mathematics, vol. 508, Springer-Verlag, 1976.

[21] Z. Zhang, L. Mi, X. Yin; *Blow-up rate of the unique solution for a class of one-dimensional problems on the half-line,* J. Math. Anal. Appl., **348** (2008) 797-805.

[22] Z. Zhang; *Boundary behavior of large solutions for semilinear elliptic equations in borderline cases,* Electron. J. Differential Equations, **136** (2012) 1-11.

Ling Mi

School of Science, Linyi University, Linyi, Shandong 276005, China

E-mail address: `mi-ling@163.com`