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SHAPE DERIVATIVE OF AN ENERGY ERROR FUNCTIONAL FOR VOIDS DETECTION FROM SUB-CAUCHY DATA

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ABSTRACT. We study a new framework for a geometric inverse problem in linear elasticity. This problem concerns the recovery of cavities from the knowledge of partially overdetermined boundary data. The boundary data available for the reconstruction are given by the displacement field and the normal component of the normal stress, whereas there is lack of information about the shear stress. We propose an identification method based on a Kohn-Vogelius error functional combined with the shape gradient method. We put special focus on the identification of cavities and prove uniqueness for the case of monotonous cavities.

1. INTRODUCTION

This paper is devoted to the study of some geometric inverse problems related to the identification of cavities which arises in many areas of industry [1]. Indeed, flaws are introduced into materials during processing and in particular cavities can appear as small gas bubbles [4]. These defects have a strong influence on the lifetime of structural components [13, 14]. Nowadays, with the tremendous development of numerical techniques, the identification of cavities has become possible. Therefore, a major impetus has been given to this inverse problem and a lot of experimental, theoretical and numerical investigations have been carried out [5, 7, 8, 11] to improve damage resistance of mechanical components.

For the reconstruction of cavities, overdetermined boundary data are critically important. To the best of our knowledge, all geometric inverse problems in linear elasticity, investigated in the literature, are to be defined by complete overdetermined boundary data (see for example [5, 7, 8]) with the exception of a recent work [6] where data appear to be partial. Therefore, we focus our attention in this paper on this specific case where only the displacement field and the normal component of the normal stress are available. Indeed, it is the level of difficulty added when studying a related problem in a previous work [5].

This article is organized as follows: in the next section, we formulate the geometric inverse problem that will be investigated further in the following sections. In the third section, we discuss the identifiability of cavities and prove a uniqueness result only for monotonous cavities, highlighting the importance of the geometric inverse problem under consideration. In the fourth section, the inverse problem is

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transformed into a shape optimization one using a Kohn-Vogelius error functional. The fifth section is devoted to the shape sensitivity analysis. Some comments are drawn in the last section.

2. Formulation of the inverse problem

In this section, we first review the standard case of cavities identification problem combined with complete overdetermined boundary data; then we focus our discussion on the specific case of partially overdetermined boundary data.

We consider a linear elastic material which occupies an open bounded domain $B \subset \mathbb{R}^2$ with boundary Υ, the medium being assumed to be homogeneous and isotropic. We suppose that there is a cavity A namely a void inside B i.e. $\overline{A} \subset B$. For a given traction g acting on the boundary Υ, the displacement u satisfies the linear elasticity direct problem

$$\begin{gathered}
\operatorname{div} \sigma(u) = 0 \quad \text{in } \Omega, \\
\sigma(u) = \lambda \operatorname{tr} \varepsilon(u) \operatorname{Id} + 2\mu\varepsilon(u) \quad \text{in } \Omega, \\
\sigma(u) n = 0 \quad \text{on } \Gamma, \\
\sigma(u) n_\Upsilon = g \quad \text{on } \Upsilon,
\end{gathered}$$

where $\Omega = B \setminus \overline{A}$, Γ is the boundary of A and n_Υ and n are the outward unit normals to the boundary of Ω. ε is the strain tensor and σ is the Cauchy stress tensor related by the following Hooke constitutive law

$$\sigma(u) = \lambda \operatorname{tr} \varepsilon(u) \operatorname{Id} + 2\mu\varepsilon(u)$$

and conversely

$$\varepsilon(u) = \frac{1+\nu}{E}\sigma(u) - \frac{\nu}{E}(\operatorname{tr}\sigma(u)) \operatorname{Id}.$$

Above, tr denotes the trace of matrix and λ, μ are the Lamé coefficients related to Young's modulus E and Poisson's ratio ν via

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.$$

The inverse problem is then to recover the cavity A by applying some traction g on Υ and then measuring the displacement f induced by g i.e. $u = f$ on Υ.

However, in this work, we suppose that we have only access to the normal component of the normal stress g. In other words, the inverse problem investigated in this paper is the identification of a cavity A trapped in a material occupying the domain B where the displacement u satisfies

$$\begin{gathered}
\operatorname{div} \sigma(u) = 0 \quad \text{in } \Omega, \\
\sigma(u) = \lambda \operatorname{tr} \varepsilon(u) \operatorname{Id} + 2\mu\varepsilon(u) \quad \text{in } \Omega, \\
\sigma(u) n = 0 \quad \text{on } \Gamma, \\
(\sigma(u) n_\Upsilon) \cdot n_\Upsilon = g \cdot n_\Upsilon \quad \text{on } \Upsilon, \\
u = f \quad \text{on } \Upsilon.
\end{gathered} \tag{2.1}$$

We draw the reader's attention to the fact that it is not a standard situation since we have access to the displacement f and only to the normal component of the normal stress whereas no information on the shear stress, namely $(\sigma(u) n_\Upsilon) \cdot \tau$ is available on Υ.

3. Identifiability

From a theoretical point of view, there are several relevant questions about this geometric inverse problem because of, on the one hand, its ill-posedness and, on the other hand, the missing boundary measurements. Indeed, solving such a geometric inverse problem is a significant task since, to the best of our knowledge, the question of uniqueness is at present far from being solved. Hence, it poses a great challenge. In the following, we discuss the identifiability of cavities, i.e. the uniqueness question of the inverse problem in the case of monotonous cavities.

For $\Omega \subset \mathbb{R}^2$ an open and bounded domain with boundary Υ, let C_1 and C_2 be two connected domains such that $\overline{C_1} \subset C_2$ and $\overline{C_2} \subset \Omega$ (see Figure 1). For $i = 1, 2$, let u_i be the solution of the problem

$$
\begin{gathered}
-\operatorname{div} \sigma(u_i) = 0 \quad \text{in } \Omega \setminus \overline{C_i}, \\
\sigma(u_i)n = 0 \quad \text{on } \partial C_i, \\
\sigma(u_i)n_\Upsilon \cdot n_\Upsilon = g \quad \text{on } \Sigma, \\
u_i \cdot \tau = f \cdot \tau \quad \text{on } \Sigma, \\
\sigma(u_i)n_\Upsilon = 0 \quad \text{on } \Upsilon \setminus \Sigma,
\end{gathered}
\tag{3.1}
$$

where ∂C_i is the boundary of C_i, $\Sigma \subset \Upsilon$, n_Υ respectively n are the outward unit normals to the boundary of $\Omega \setminus \overline{C_i}$ on Υ respectively ∂C_i and τ is the tangent vector to the boundary Υ.

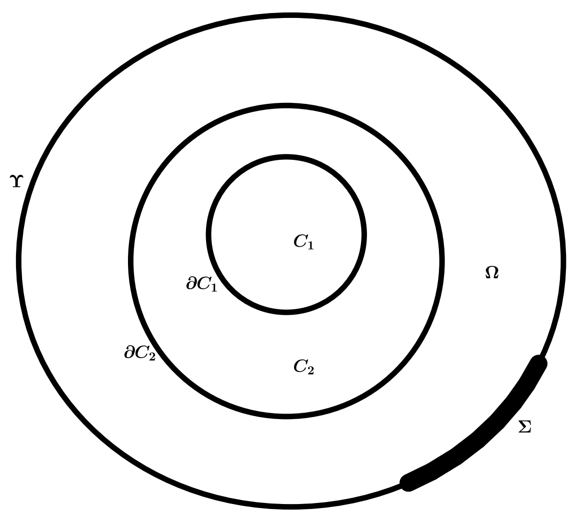

Figure 1. Domain with monotonous cavities.

Proposition 3.1. *Let C_1 and C_2 be two cavities such that $\overline{C_1} \subset C_2$ and for $i = 1, 2$, let u_i be the solution of the direct problem (3.1) defined in $\Omega \setminus \overline{C_i}$. Then, if C_1 and C_2 both lead to the same measured normal displacement on Σ, namely $u_1 \cdot n_\Upsilon = u_2 \cdot n_\Upsilon = f \cdot n_\Upsilon$ on Σ, we have $C_1 = C_2$.*

Proof. We suppose for simplicity that $f \cdot \tau = 0$ on Σ. u_2 is then the solution of the problem

$$\min_{v \in \mathcal{V}_2} \frac{1}{2} \int_{\Omega \setminus \overline{C2}} \sigma(v) : \varepsilon(v) dx - \int_{\Sigma} g(v \cdot n_{\Upsilon}) ds,$$

where

$$\mathcal{V}_2 = \{v \in [H^1(\Omega \setminus \overline{C_2})]^2 : \ v \cdot \tau = 0 \text{ on } \Sigma\}.$$

Hence, u_2 satisfies

$$\frac{1}{2} \int_{\Omega \setminus \overline{C_2}} \sigma(u_2) : \varepsilon(u_2) dx - \int_{\Sigma} g(u_2 \cdot n_{\Upsilon}) ds \leqslant \frac{1}{2} \int_{\Omega \setminus \overline{C_2}} \sigma(v) : \varepsilon(v) dx - \int_{\Sigma} g(v \cdot n_{\Upsilon}) ds,$$

for all $v \in \mathcal{V}_2$. In particular, since $\overline{C_1} \subset C_2$ and $u_1 \cdot \tau = f \cdot \tau = 0$ on Σ, we have

$$\begin{aligned} &\frac{1}{2} \int_{\Omega \setminus \overline{C_2}} \sigma(u_2) : \varepsilon(u_2) dx - \int_{\Sigma} g(u_2 \cdot n_{\Upsilon}) ds \\ &\leqslant \frac{1}{2} \int_{\Omega \setminus \overline{C_2}} \sigma(u_1) : \varepsilon(u_1) dx - \int_{\Sigma} g(u_1 \cdot n_{\Upsilon}) ds. \end{aligned} \tag{3.2}$$

Then, since $u_1 \cdot n_{\Upsilon} = u_2 \cdot n_{\Upsilon}$ on Σ, from (3.2) we obtain

$$\begin{aligned} \int_{\Omega \setminus \overline{C2}} \sigma(u_2) : \varepsilon(u_2) dx &\leqslant \int_{\Omega \setminus \overline{C_2}} \sigma(u_1) : \varepsilon(u_1) dx, \\ &\leqslant \int_{\Omega \setminus \overline{C1}} \sigma(u_1) : \varepsilon(u_1) dx - \int_{C2 \setminus \overline{C1}} \sigma(u_1) : \varepsilon(u_1) dx. \end{aligned} \tag{3.3}$$

Using the Green formula, on the one hand from the problem (3.1) related to u_2, we have

$$\int_{\Omega \setminus \overline{C2}} \sigma(u_2) : \varepsilon(v) dx = \int_{\Sigma} g(v \cdot n_{\Upsilon}) ds, \quad \forall v \in \mathcal{V}_2$$

and on the other hand from the problem (3.1) related to u_1,

$$\int_{\Omega \setminus \overline{C1}} \sigma(u_1) : \varepsilon(v) dx = \int_{\Sigma} g(v \cdot n_{\Upsilon}) ds, \quad \forall v \in \mathcal{V}_1,$$

where

$$\mathcal{V}_1 = \{v \in [H^1(\Omega \setminus \overline{C_1})]^2 ; \ v \cdot \tau = 0 \text{ on } \Sigma\}.$$

Then, from (3.3) we have

$$\int_{\Sigma} g(u_2 \cdot n_{\Upsilon}) ds \leqslant \int_{\Sigma} g(u_1 \cdot n_{\Upsilon}) ds - \int_{C2 \setminus \overline{C1}} \sigma(u_1) : \varepsilon(u_1) dx \,.$$

Hence, since $u_1 \cdot n_{\Upsilon} = u_2 \cdot n_{\Upsilon}$ on Σ, it follows that

$$0 \leqslant - \int_{C_2 \setminus \overline{C1}} \sigma(u_1) : \varepsilon(u_1) dx.$$

So, we obtain that $\text{meas}(C_2 \setminus \overline{C_1}) = 0$, that is $C_2 = C_1$. $\square$

So, one can distinguish two cavities C_1 and C_2 so that $\overline{C_1} \subset C_2$ from partially overdetermined boundary data on Σ.

4. Shape optimization problem

For a given Ω defined in the same way as in the second section, let (σ_D, u_D) and (σ_N, u_N) be the solutions of the Dirichlet problem

$$\begin{gathered} \operatorname{div} \sigma_D = 0 \quad \text{in } \Omega, \\ \varepsilon_D = \frac{1+\nu}{E}\sigma_D - \frac{\nu}{E}(\operatorname{tr}\sigma_D)\,\mathrm{Id} \quad \text{in } \Omega, \\ \sigma_D n = 0 \quad \text{on } \Gamma, \\ u_D = f \quad \text{on } \Upsilon, \end{gathered} \tag{4.1}$$

and the Neumann problem

$$\begin{gathered} \operatorname{div} \sigma_N = 0 \quad \text{in } \Omega, \\ \varepsilon_N = \frac{1+\nu}{E}\sigma_N - \frac{\nu}{E}(\operatorname{tr}\sigma_N)\,\mathrm{Id} \quad \text{in } \Omega, \\ \sigma_N n = 0 \quad \text{on } \Gamma, \\ (\sigma_N n_\Upsilon)\cdot n_\Upsilon = g \cdot n_\Upsilon \quad \text{on } \Upsilon, \\ u_N \cdot \tau = f \cdot \tau \quad \text{on } \Upsilon. \end{gathered} \tag{4.2}$$

Here we have used the Hellinger-Reissner principle [5, 17], namely the formulation in two fields. One can notice that the cavity to recover is reached when there is no misfit between both Dirichlet and Neumann solutions, that is, when $\sigma_D = \sigma_N$ and $u_D = u_N$. According to this observation, the cavities identification problem (2.1) can be transformed into a shape optimization one

$$\text{Find } \Omega \text{ such that } J(\Omega) = \min_{\tilde{\Omega}\subset B} J(\tilde{\Omega}), \tag{4.3}$$

by the minimization of the Kohn-Vogelius error functional, namely the energetic least-squares functional

$$J(\Omega) := \frac{1}{2}\int_\Omega (\sigma_D - \sigma_N) : (\varepsilon(u_D) - \varepsilon(u_N)) \tag{4.4}$$

over a class of admissible domains.

The functional (4.4) is called Kohn-Vogelius cost functional since Kohn and Vogelius were the first to use it in impedance computed tomography [12]. The variational formulation in two fields of the Dirichlet problem (4.1) is the following [5]: Find $(\sigma_D, u_D) \in L^2_s(\Omega) \times [H^1(\Omega)]^2; u_D = f$ on Υ such that

$$\begin{gathered} \forall \tau \in L^2_s(\Omega), \quad \int_\Omega \big[\frac{1+\nu}{E}\operatorname{tr}(\sigma_D\tau) - \frac{\nu}{E}\operatorname{tr}(\sigma_D)\operatorname{tr}(\tau)\big] - \int_\Omega \operatorname{tr}(\tau\nabla u_D) = 0, \\ \forall v \in \mathcal{V}_D, \quad \int_\Omega \operatorname{tr}(\sigma_D \nabla v) = 0, \end{gathered} \tag{4.5}$$

where

$$L^2_s(\Omega) = \big\{\tau = (\tau_{\alpha\beta}) \in [L^2(\Omega)]^4; \tau_{\alpha\beta} = \tau_{\beta\,\alpha}\big\}$$

and

$$\mathcal{V}_D = \big\{v \in [H^1(\Omega)]^2; v = 0 \text{ on } \Upsilon\big\}.$$

Let us define for $(\sigma,\tau) \in [L^2_s(\Omega)]^2$ and $v \in [H^1(\Omega)]^2$ the bilinear symmetric form $a(\cdot,\cdot)$ and the bilinear form $b(.,.)$, needed in the sequel, as follows :

$$a(\sigma,\tau) = \int_\Omega \big[\frac{1+\nu}{E}\operatorname{tr}(\sigma\tau) - \frac{\nu}{E}\operatorname{tr}(\sigma)\operatorname{tr}(\tau)\big]$$

and

$$b(\tau, v) = -\int_\Omega \operatorname{tr}(\tau\nabla v).$$

The formulation in two fields (4.5) can be rewritten as: Find $(\sigma_D, u_D) \in L^2_s(\Omega) \times [H^1(\Omega)]^2; u_D = f$ on Υ such that

$$\begin{gathered} \forall\tau \in L^2_s(\Omega), \quad a(\sigma_D, \tau) + b(\tau, u_D) = 0, \\ \forall v \in \mathcal{V}_D, \quad b(\sigma_D, v) = 0. \end{gathered} \tag{4.6}$$

The first equation reflects the constitutive law and the second the equilibrium equation. Proceeding in the same way as in the Dirichlet problem, the formulation in two fields of the Neumann problem (4.2) is: Find $(\sigma_N, u_N) \in L^2_s(\Omega) \times [H^1(\Omega)]^2; u_N \cdot \tau = f \cdot \tau$ on Υ such that

$$\begin{gathered} \forall\tau \in L^2_s(\Omega), \quad a(\sigma_N, \tau) + b(\tau, u_N) = 0, \\ \forall v \in \mathcal{V}_N, \quad b(\sigma_N, v) = -\int_\Upsilon (g \cdot n_\Upsilon)(v \cdot n_\Upsilon), \end{gathered} \tag{4.7}$$

where

$$\mathcal{V}_N = \left\{v \in [H^1(\Omega)]^2; v \cdot \tau = 0 \text{ on } \Upsilon\right\}.$$

The important point to note here is that in the optimization process, it is possible to deal with the topological gradient method [6]. However, we resort in this paper to the shape gradient method presented in the next section.

5. Shape sensitivity analysis

Nowadays, the shape optimization theory has achieved a high degree of success from theoretical and numerical points of view ever since the development of one of its famous tool: the shape gradient method. Due to the deep connection of the Kohn-Vogelius misfit functional with the shape gradient method [2, 3, 5, 10, 11], we focus in this paper on a shape sensitivity analysis of the Kohn-Vogelius functional (4.4) subject to partially overdetermined boundary data.

In the sequel, some basic tools related to the shape gradient method [16] are presented. Let us consider an open and bounded domain U and an initial domain Ω so that $\overline{\Omega} \subset U$. In order to define the shape gradient of the misfit functional (4.4), one needs to deform the so-called reference domain Ω using the perturbation of identity operator to the first order, that is the mapping

$$F_t : \overline{U} \longmapsto \mathbb{R}^2$$

defined by $F_t = id + th$ where id is the identity mapping. To make the exterior boundary Υ of Ω clamp during the shape reconstruction process, we consider the deformation field h belonging to the space

$$Q = \{h \in \mathcal{C}^{1,1}(\overline{U})^2;\ h = 0 \text{ on } \Upsilon\}.$$

We should note that for sufficiently small t, the mapping F_t is a diffeomorphism from Ω onto its image. Hence, the perturbed domains Ω_t and Γ_t are defined by

$$\Omega_t := F_t(\Omega) \quad \text{and} \quad \Gamma_t := F_t(\Gamma).$$

For $t = 0$, we have $\Omega_0 = \Omega$ (the reference domain). Once the diffeomorphism map between the reference domain Ω and the perturbed one is constructed, one can embed problems (4.1) and (4.2) into a family of perturbed problems defined in Ω_t.

More precisely, we consider the pairs (σ_{Dt}, u_{Dt}) and (σ_{Nt}, u_{Nt}) solutions for the following Dirichlet, respectively the Neumann problem

$$\begin{gathered}
\operatorname{div}\sigma_{Dt} = 0 \quad \text{in } \Omega_t,\\
\varepsilon_{Dt} = \frac{1+\nu}{E}\sigma_{Dt} - \frac{\nu}{E}(\operatorname{tr}\sigma_{Dt})\,\mathrm{Id} \quad \text{in } \Omega_t,\\
\sigma_{Dt}n_t = 0 \quad \text{on } \Gamma_t,\\
u_{Dt} = f \quad \text{on } \Upsilon,
\end{gathered} \tag{5.1}$$

respectively

$$\begin{gathered}
\operatorname{div}\sigma_{Nt} = 0 \quad \text{in } \Omega_t,\\
\varepsilon_{Nt} = \frac{1+\nu}{E}\sigma_{Nt} - \frac{\nu}{E}(\operatorname{tr}\sigma_{Nt})\,\mathrm{Id} \quad \text{in } \Omega_t,\\
\sigma_{Nt}n_t = 0 \quad \text{on } \Gamma_t,\\
(\sigma_{Nt}n_\Upsilon)\cdot n_\Upsilon = g\cdot n_\Upsilon \quad \text{on } \Upsilon,\\
u_{Nt}\cdot\tau = f\cdot\tau \quad \text{on } \Upsilon,
\end{gathered} \tag{5.2}$$

where n_t is the outward unit normal to Ω_t on Γ_t.

Definition 5.1. The first-order Eulerian derivative of a shape functional $J : \Omega \longmapsto \mathbb{R}$ at the domain Ω in the direction of the deformation field $h \in Q$ is given by

$$J'(\Omega, h) = \lim_{t\mapsto 0}\frac{J(\Omega_t) - J(\Omega)}{t}$$

if the limit exists.

Remark 5.2. J is called shape differentiable at Ω if $J'(\Omega, h)$ exists for all $h \in Q$ and if the mapping $h \mapsto J'(\Omega, h)$ is linear and continuous with respect to h.

Throughout the discussion, if φ_t is a function defined in the perturbed domain Ω_t, we denote by φ^t the function defined in the reference domain Ω by $\varphi^t = \varphi_t \circ F_t$. In particular, we consider the pairs (σ_D^t, u_D^t) and (σ_N^t, u_N^t) defined in Ω by

$$\begin{gathered}
\sigma_D^t = \sigma_{Dt}\circ F_t\\
u_D^t = u_{Dt}\circ F_t
\end{gathered}$$

and

$$\begin{gathered}
\sigma_N^t = \sigma_{Nt}\circ F_t\\
u_N^t = u_{Nt}\circ F_t.
\end{gathered}$$

For $t = 0$, (σ_D^0, u_D^0) respectively (σ_N^0, u_N^0) is the solution of (4.1) respectively (4.2).

Lemma 5.3 ([16]). *(i) If $\varphi_t \in L^1(\Omega_t)$, then $\varphi^t \in L^1(\Omega)$ and we have*

$$\int_{\Omega_t}\varphi_t = \int_\Omega \delta_t\,\varphi^t,$$

where $\delta_t = \det(DF_t) = \det(\mathrm{Id} + t\,\nabla h^{\mathsf{T}})$.

(ii) If $\varphi_t \in H^1(\Omega_t)$, then $\varphi^t \in H^1(\Omega)$ and we have

$$(\nabla\varphi_t)\circ F_t = M_t\nabla\varphi^t,$$

with $M_t = DF_t^{-T}$.

Above, DF_t is the Jacobian matrix of F_t and DF_t^T is the transpose of DF_t. It is easy to see that $(DF_t^T)^{-1} = (DF_t^{-1})^T$; so for the sake of simplicity, we shall write DF_t^{-T}.

5.1. **Asymptotic expansions.**

Dirichlet problem. Similarly to the variational formulation (4.5) of problem (4.1), we can get the formulation in two fields of the perturbed Dirichlet probelm (5.1), that is: Find $(\sigma_{Dt}, u_{Dt}) \in L^2_s(\Omega_t) \times [H^1(\Omega_t)]^2; u_{Dt} = f$ on Υ such that

$$\begin{gathered}
\forall \tau \in L^2_s(\Omega_t), \quad \int_{\Omega_t} \big[\frac{1+\nu}{E}\operatorname{tr}(\sigma_{Dt}\,\tau) - \frac{\nu}{E}\operatorname{tr}(\sigma_{Dt})\operatorname{tr}(\tau)\big] - \int_{\Omega_t}\operatorname{tr}(\tau\nabla u_{Dt}) = 0,\\
\forall v \in \mathcal{V}_{Dt}, \quad \int_\Omega \operatorname{tr}(\sigma_{Dt}\nabla v) = 0,
\end{gathered} \tag{5.3}$$

where

$$\begin{gathered}
L^2_s(\Omega_t) = \{\tau = (\tau_{\alpha\,\beta}) \in [L^2(\Omega_t)]^4; \tau_{\alpha\,\beta} = \tau_{\beta\,\alpha}\},\\
\mathcal{V}_{Dt} = \{v \in [H^1(\Omega_t)]^2; v = 0 \text{ on } \Upsilon\}.
\end{gathered}$$

Then, one needs to transfer the variational formulation (5.3) defined in the perturbed domain Ω_t to the reference domain Ω. Using Lemma 5.3 and that $u_{Dt} = u^t_D = u^0_D = f$ on Υ, the variational formulation in two fields of the perturbed Dirichlet problem, brought to the reference domain is then: Find $(\sigma^t_D, u^t_D) \in L^2_s(\Omega) \times [H^1(\Omega)]^2; u^t_D = f$ on Υ such that

$$\begin{aligned}
\forall \tau \in L^2_s(\Omega), \quad & \int_\Omega \big[\frac{1+\nu}{E}\operatorname{tr}(\sigma^t_D\tau) - \frac{\nu}{E}\operatorname{tr}(\sigma^t_D)\operatorname{tr}(\tau)\big]\det(DF_t)\\
&- \int_\Omega \operatorname{tr}[\tau(\nabla u^t_D(DF_t)^{-1})]\det(DF_t) = 0,\\
\forall v \in \mathcal{V}_D, \quad & \int_\Omega \operatorname{tr}[\sigma^t_D(\nabla v(DF_t)^{-1})]\det(DF_t) = 0.
\end{aligned} \tag{5.4}$$

Theorem 5.4 (Related to the Dirichlet problem)**.** *There exists $\eta_0 > 0$ such that, if $t < \eta_0$, we obtain*

$$(\sigma^t_D, u^t_D) = (\sigma^0_D, u^0_D) + t(\sigma^1_D, u^1_D) + to(t), \tag{5.5}$$

where $(\sigma^0_D, u^0_D), (\sigma^1_D, u^1_D)$ and $o(t)$ are elements of $L^2_s(\Omega) \times [H^1(\Omega)]^2$ satisfying:

- (i) *(σ^0_D, u^0_D) is the solution of the linear elasticity Dirichlet problem* (4.6) *in Ω.*
- (ii) $\lim_{t\mapsto 0} \|o(t)\|_{L^2_s(\Omega)\times\mathcal{V}_D} = 0$.
- (iii) *$(\sigma^1_D, u^1_D) \in L^2_s(\Omega) \times \mathcal{V}_D$ is the unique solution of the following problem*

$$\begin{gathered}
\forall \tau \in L^2_s(\Omega), \quad a(\sigma^1_D, \tau) + b(\tau, u^1_D) = -\int_\Omega \operatorname{tr}[\tau(\nabla u^0_D\nabla h)],\\
\forall v \in \mathcal{V}_D, \quad b(\sigma^1_D, v) = -\int_\Omega \operatorname{tr}[\sigma^0_D(\nabla v\,\nabla h)] + \int_\Omega \operatorname{tr}(\sigma^0_D\nabla v)\operatorname{div} h.
\end{gathered} \tag{5.6}$$

Proof. Let Φ be the function

$$\Phi : \mathbb{R} \times L^2_s(\Omega) \times [H^1(\Omega)]^2 \to L^2_s(\Omega) \times ([H^1(\Omega)]^2)'$$

defined as follows

$$\Phi(t, \sigma, u) = \begin{cases} A(t)\sigma + \overline{B(t)}u\\ B(t)\sigma \end{cases}$$

where we adopt the following notation:

$$A(t)\sigma = (\frac{1+\nu}{E}\,\sigma - \frac{\nu}{E}(\operatorname{tr}\sigma)\,\mathrm{Id})\det(DF_t),$$
$$\langle B(t)\,\sigma, v\rangle = -\int_\Omega [\operatorname{tr}(\sigma\nabla v(DF_t)^{-1})]\det(DF_t).$$

Then $\overline{B(t)}$ (which is the transpose of $B(t)$) will be defined by

$$\overline{B(t)}v = -\frac{1}{2}[(\mathrm{Id}+t\overline{\nabla h})^{-1}\,\overline{\nabla v} + \nabla v(\mathrm{Id}+t\nabla h)^{-1}]\det(DF_t).$$

So, the equations (5.4) can be written as follows

$$\begin{aligned}&\text{Find } (\sigma_D^t, u_D^t) \in L_s^2(\Omega)\times[H^1(\Omega)]^2 \text{ such that } u_D^t = f \text{ on } \Upsilon,\\ &\text{and } \Phi(t,\sigma_D^t,u_D^t) = 0.\end{aligned} \tag{5.7}$$

Here Φ is a linear application on (σ, u) and differentiable on t. We also have $\Phi(0,\sigma_D^0,u_D^0) = 0$, which reflects that (σ_D^0,u_D^0) is the solution of (4.6) defined in the reference domain Ω. The derivative of Φ, with respect to the variable (σ,u) is Φ itself which leads to

$$\begin{aligned}\frac{\partial\Phi}{\partial(\sigma,u)}(0,\sigma^0,u^0)(\sigma,u) &= \Phi(0,\sigma,u)\\ &= \begin{cases} A(0)\sigma + \overline{B(0)}u\\ B(0)\sigma\end{cases}\\ &= \begin{cases} (\frac{1+\nu}{E}\sigma - \frac{\nu}{E}(\operatorname{tr}\sigma)\,\mathrm{Id}) - \frac{1}{2}(\overline{\nabla u} + \nabla u)\\ B(0)\sigma.\end{cases}\end{aligned}$$

The partial derivative of Φ, with respect to (σ,u), is then, according to the theorem of Breziz, a bijection from $L_s^2(\Omega)\times[H^1(\Omega)]^2$ to $L_s^2(\Omega)\times([H^1(\Omega)]^2)'$. In addition, Φ is linear and continuous. The open mapping theorem states that Φ is an isomorphism from $L_s^2(\Omega)\times[H^1(\Omega)]^2$ to $L_s^2(\Omega)\times([H^1(\Omega)]^2)'$. It follows from the implicit function theorem that there exists a positive number η_0 and a neighborhood ϑ of (σ_D^0,u_D^0) in $L_s^2(\Omega)\times[H^1(\Omega)]^2$ such that for all $t\in]-\eta_0,\eta_0[$, there exists a unique pair (σ_D^t,u_D^t) in $L_s^2(\Omega)\times[H^1(\Omega)]^2$ such that

$$\Phi(t,\sigma_D^t,u_D^t) = 0.$$

Moreover, the application $t\mapsto(\sigma^t,u^t)$ is C^1, from $]-\eta_0,\eta_0[$ to ϑ. Then, we have

$$(\sigma_D^t,u_D^t) = (\sigma_D^0,u_D^0) + t(\sigma_D^1,u_D^1) + to(t),$$

where

$$\lim_{t\mapsto 0}\|o(t)\|_{L_s^2(\Omega)\times\mathcal{V}_D} = 0.$$

For (iii) by substituting equality (5.5) in (5.4), using the fact that (σ_D^0,u_D^0) is the solution of (4.6), and identifying the terms of the same order t, we show that (σ_D^1,u_D^1) is the solution of (5.6). □

Neumann problem. A similar result can be carried out for the pair (σ_N^t,u_N^t).

Theorem 5.5 (Related to the Neumann problem). *There exists $\delta_0>0$ such that, if $t<\delta_0$, we obtain*

$$(\sigma_N^t,u_N^t) = (\sigma_N^0,u_N^0) + t(\sigma_N^1,u_N^1) + to(t), \tag{5.8}$$

where $(\sigma_N^0,u_N^0),(\sigma_N^1,u_N^1)$ and $o(t)$ are elements of $L_s^2(\Omega)\times[H^1(\Omega)]^2$ satisfying:

(i) (σ_N^0, u_N^0) *is the solution of the linear elasticity Neumann problem* (4.7) *in* Ω.
(ii) $\lim_{t\mapsto 0} \|o(t)\|_{L^2_s(\Omega)\times\mathcal{V}_N} = 0$.
(iii) $(\sigma_N^1, u_N^1) \in L^2_s(\Omega)\times\mathcal{V}_N$ *is the unique solution of the following problem*

$$\begin{gathered}\forall\tau\in L^2_s(\Omega),\quad a(\sigma_N^1,\tau)+b(\tau,u_N^1) = -\int_\Omega \operatorname{tr}[\tau(\nabla u_N^0\nabla h)],\\ \forall v\in\mathcal{V}_N,\quad b(\sigma_N^1,v) = -\int_\Omega \operatorname{tr}[\sigma_N^0(\nabla v\nabla h)] + \int_\Omega \operatorname{tr}(\sigma_N^0\nabla v)\operatorname{div} h.\end{gathered} \tag{5.9}$$

5.2. **Main result.** Let us recall some useful lemmas.

Lemma 5.6 ([16]). *The mappings* $t\mapsto\delta_t$ *and* $t\mapsto M_t$ *with values in* $\mathcal{C}(\Omega)$ *and* $\mathcal{C}(\Omega)^{2\times 2}$ *respectively, are* $\mathcal{C}^1$ *in a neighborhood of* 0 *and we have*

$$\begin{gathered}\frac{d\delta_t}{dt}\Big|_{t=0} = \operatorname{div} h,\\ \frac{dM_t}{dt}\Big|_{t=0} = -\nabla h.\end{gathered}$$

Lemma 5.7 ([9]).

$$\begin{gathered}\overline{\operatorname{div}(\sigma_D^0\nabla u_D^0)} = \frac12\operatorname{grad}[\operatorname{tr}(\sigma_D^0\nabla u_D^0)],\\ \overline{\operatorname{div}(\sigma_N^0\nabla u_N^0)} = \frac12\operatorname{grad}[\operatorname{tr}(\sigma_N^0\nabla u_N^0)].\end{gathered}$$

Using the asymptotic expansions exposed in the previous subsection, one can express the shape gradient of J (4.4) with respect to the domain.

Theorem 5.8. *The functional*

$$J(\Omega_t) := \frac12\int_{\Omega_t}(\sigma_{Dt}-\sigma_{Nt}):(\varepsilon(u_{Dt})-\varepsilon(u_{Nt}))$$

is shape differentiable at Ω *and for* $h\in Q$ *the Eulerian derivative is*

$$J'(\Omega,h) = \int_\Gamma G(h\cdot n), \tag{5.10}$$

with

$$G = \frac12[(\sigma_D^0:\varepsilon(u_D^0)) - (\sigma_N^0:\varepsilon(u_N^0))]. \tag{5.11}$$

Proof. Using (5.5), (5.8), the transformation formulas (see Lemma 5.3) and the fact that

$$\det(DF_t) = 1+t\operatorname{div} h+o(t)\quad\text{and}\quad (DF_t)^{-1} = \operatorname{Id}-t\nabla h+o(t),$$

(see Lemma 5.6), we obtain

$$\begin{aligned}J'(\Omega,h) &= \int_\Omega[\frac{1+\nu}{E}(\sigma_D^0-\sigma_N^0):(\sigma_D^1-\sigma_N^1) - \frac{\nu}{E}\operatorname{tr}(\sigma_D^0-\sigma_N^0)\operatorname{tr}(\sigma_D^1-\sigma_N^1)]\\ &\quad+\frac12\int_\Omega\operatorname{div} h[\frac{1+\nu}{E}(\sigma_D^0-\sigma_N^0):(\sigma_D^0-\sigma_N^0) - \frac{\nu}{E}[\operatorname{tr}(\sigma_D^0-\sigma_N^0)]^2],\\ &= a(\sigma_D^0-\sigma_N^0,\sigma_D^1-\sigma_N^1)\\ &\quad+\frac12\int_\Omega\operatorname{div} h[\frac{1+\nu}{E}(\sigma_D^0-\sigma_N^0):(\sigma_D^0-\sigma_N^0) - \frac{\nu}{E}[\operatorname{tr}(\sigma_D^0-\sigma_N^0)]^2].\end{aligned}$$

Using (4.6), (4.7), (5.6) and (5.9), we obtain

$$\begin{aligned}a(\sigma_D^0-\sigma_N^0,\sigma_N^1)&=-b(\sigma_N^1,u_D^0-u_N^0)\\&=\int_\Omega \operatorname{tr}[\sigma_N^0(\nabla(u_D^0-u_N^0)\nabla h)]-\int_\Omega \operatorname{tr}[\sigma_N^0\nabla(u_D^0-u_N^0)]\operatorname{div} h,\end{aligned}$$

and

$$a(\sigma_D^1,\sigma_D^0-\sigma_N^0)=\int_\Omega \operatorname{tr}[(\sigma_D^0-\sigma_N^0)\nabla u_D^1]-\int_\Omega \operatorname{tr}[(\sigma_D^0-\sigma_N^0)(\nabla u_D^0\,\nabla h)].$$

One can prove that

$$\int_\Omega \operatorname{tr}[(\sigma_D^0-\sigma_N^0)\nabla u_D^1]=0.$$

Thus,

$$\begin{aligned}J'(\Omega,h)&=\int_\Omega \operatorname{tr}[\sigma_N^0(\nabla u_N^0\nabla h)]-\int_\Omega \operatorname{tr}[\sigma_D^0(\nabla u_D^0\nabla h)]\\&\quad-\frac{1}{2}\int_\Omega[\varepsilon(u_N^0):\sigma_N^0]\operatorname{div} h+\frac{1}{2}\int_\Omega[\varepsilon(u_D^0):\sigma_D^0]\operatorname{div} h.\end{aligned}$$

Using generalized Green's formula, we obtain

$$\begin{gathered}\frac{1}{2}\int_\Omega[\sigma_N^0:\varepsilon(u_N^0)]\operatorname{div} h=\frac{1}{2}\int_{\partial\Omega}[\sigma_N^0:\varepsilon(u_N^0)]h\cdot n-\frac{1}{2}\int_\Omega \operatorname{grad}[\operatorname{tr}(\sigma_N^0\nabla u_N^0)]\cdot h,\\ \frac{1}{2}\int_\Omega[\sigma_D^0:\varepsilon(u_D^0)]\operatorname{div} h=\frac{1}{2}\int_{\partial\Omega}[\sigma_D^0:\varepsilon(u_D^0)]h\cdot n-\frac{1}{2}\int_\Omega \operatorname{grad}[\operatorname{tr}(\sigma_D^0\nabla u_D^0)]\cdot h,\\ \int_\Omega \operatorname{tr}[\sigma_N^0(\nabla u_N^0\nabla h)]=\int_{\partial\Omega}(n^T\sigma_N^0)\cdot(\nabla u_N^0 h)-\int_\Omega \operatorname{div}(\sigma_N^0\nabla u_N^0)\cdot h,\\ -\int_\Omega \operatorname{tr}[\sigma_D^0(\nabla u_D^0\nabla h)]=-\int_{\partial\Omega}(n^T\sigma_D^0)\cdot(\nabla u_D^0\,h)+\int_\Omega \operatorname{div}(\sigma_D^0\nabla u_D^0)\cdot h.\end{gathered}$$

Then, using Lemma 5.7, we can deduce a simple formula for the derivative of J,

$$\begin{aligned}J'(\Omega,h)&=\frac{1}{2}\int_{\partial\Omega}[\sigma_D^0:\varepsilon(u_D^0)]h\cdot n-\frac{1}{2}\int_{\partial\Omega}[\sigma_N^0:\varepsilon(u_N^0)]h\cdot n\\&\quad+\int_{\partial\Omega}(n^T\sigma_N^0)\cdot(\nabla u_N^0\,h)-\int_{\partial\Omega}(n^T\sigma_D^0)\cdot(\nabla u_D^0 h).\end{aligned}$$

Hence, from the identities

$$\begin{gathered}h=(h\cdot n)n+(h\cdot\tau)\tau,\quad h=0 \text{ on } \Upsilon,\\ \sigma_D^0 n=\sigma_N^0 n=0 \text{ on } \Gamma,\end{gathered}$$

we obtain the desired result

$$J'(\Omega,h)=\int_\Gamma\Big[\frac{1}{2}[(\sigma_D^0:\varepsilon(u_D^0))-(\sigma_N^0:\varepsilon(u_N^0))]\Big]h\cdot n.$$

$\square$

Remark 5.9. The shape derivative (5.10) depends only on the normal component of the speed vector field h on the boundary of the cavity looking for. This property of the shape derivative concept is crucial for an iterative descent method if one aims to numerically solve the cavities identification problem. Indeed, in contrast to the classical shape optimization, a fruitful approach can be numerically designed to track domains changing the topology. The underlying technique behind this approach is to combine the shape gradient information (5.10) with the level set method [15]. We refer the reader to [3, 5, 11] for more details about this technique.

Comments. The geometric inverse problem investigated in this article tries to recover cavities from partially overdetermined boundary data. The problem is not in its usual form because the lack of overdetermined boundary data; it is rather the extension of a previous work [5] where data appeared to be complete. The problem has been addressed by means of the so-called Kohn-Vogelius formulation combined with the shape gradient method. The theoretical question related to the identifiability is still open since the uniqueness result was only derived for the case of monotonous cavities, which underlines the difficulty encountered when solving such an inverse problem. Moreover, an efficient optimization algorithm can be constructed. This algorithm can be seen as a descent method where the descent direction is determined by the shape derivative of the Kohn-Vogelius functional since it has been expressed in terms of a boundary integral. This will be a subject for a forthcoming publication.

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