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BLOW-UP AND EXTINCTION OF SOLUTIONS TO A FAST DIFFUSION EQUATION WITH HOMOGENEOUS NEUMANN BOUNDARY CONDITIONS

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Abstract. In this article, we study blow-up and extinction properties of solutions to a fast diffusion p-Laplace equation with a nonlocal term under homogeneous Neumann boundary conditions. We first show that the solutions with positive initial energy will blow up in finite time, and then give some sufficient conditions for the solutions to vanish in finite time, using the method of integral estimates. Moreover, the decay rates near the extinction time are also derived.

1. Introduction

In this article, we consider the following p-Laplace equation under homogeneous Neumann boundary conditions,

$$\begin{gathered}
u_t=\mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})+|\mathrm{u}|^{\mathrm{q}-1}\mathrm{u}-\frac{1}{|\Omega|}\int_\Omega |\mathrm{u}|^{\mathrm{q}-1}\mathrm{u}\mathrm{dx},\quad \mathrm{x}\in\Omega,\ \mathrm{t}>0,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial n}=0,\quad x\in\partial\Omega,\ t>0,\\
u(x,0)=u_0(x),\quad x\in\Omega,
\end{gathered}\tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N\geq 2)$ with smooth boundary $\partial\Omega$, $1<p<2$, $q>0$, n is the unite outward normal on $\partial\Omega$ and the initial datum $u_0(x)$ satisfies

$$0\not\equiv u_0(x)\in L^\infty(\Omega)\cap W^{1,p}(\Omega),\quad \int_\Omega u_0(x)\mathrm{d}x=0.\tag{1.2}$$

It is immediately seen from the structure of the equation and the homogeneous boundary condition that the integral of the solution u to (1.1) is conserved, that is $\int_\Omega u(x,t)\mathrm{d}x=\int_\Omega u_0(x)\mathrm{d}x=0$ as long as $u(x,t)$ exists.

Problem (1.1) can be used to describe many physical models. For example, when $p=2$, it arises from the nuclear science where the growth of the temperature is known to be very fast, like u^q, but some absorption catalytic material is put into the system in such a way that the total mass is conserved. It can also be used to model other phenomena in population dynamics and biological sciences where the total mass is often conserved or known, but the growth of a certain cell is known to be of some form [11].

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In the past few years, much effort has been devoted to the study of global existence and blow-up of solutions to such kinds of problems. Among the huge amount of works, we only refer to [11], in which Hu et al established the blow-up result for (1.1) with $p = 2$ under the condition that the initial energy satisfies

$$E(0) = \int_\Omega \big[\frac{1}{2}|\nabla u_0|^2 - \frac{1}{q+1}|u_0|^{q+1}\big]\mathrm{d}x \le -C,$$

by using a convexity argument, where $C > 0$ is a constant depending on the measure of Ω. Later, Gao and Han [8] improved their results and showed that the solutions with small positive initial energy can also blow-up in finite time for $1 < q \le (N+2)/(N-2)$.

In 2007, Soufi et al [18] studied a slightly different model

$$u_t - \Delta u = |u|^q - \frac{1}{|\Omega|}\int_\Omega |u|^q \mathrm{d}x \tag{1.3}$$

with the same initial and boundary conditions as those given in problem (1.1). They established a new blow-up criterion for $1 < q \le 2$ based on partial Maximum Principles and on a Gamma-convergence argument, and proposed a conjecture that the conclusion might also be valid for all $q > 1$, a positive answer to which was given by Jazar et al in [12]. It is worth pointing out that (1.3) with $q = 2$ is also related to Navier-Stokes equations on an infinite slab for other reasons explained in [2]. Since mathematically we do not require that $u(x,t)$ is nonnegative, we use $|u|^{q-1}u$ instead of $|u|^q$ in our problem.

Extinction in finite time is another phenomenon shared by some evolution equations whereby the evolution of some nontrivial initial datum $u_0(x)$ produces a nontrivial solution $u(x,t)$ in a finite time interval $0 < t < T$, but $u(x,t) \equiv 0$ for almost every $(x,t) \in \Omega \times (T,\infty)$. In this case, T is called the extinction time. Extinction via fast diffusion was first observed by Sabinina [17], and since then, there has been increasing interest in this direction. Interested readers may refer to [3, 13, 14] for sufficient and necessary conditions for the solutions of general diffusion equations with or without reaction terms to vanish in finite time, to [1, 5, 6, 7] for the investigation of the asymptotic behaviors of solutions near the extinction time and to [10, 15, 19, 20] for the critical extinction exponents for fast diffusive equations with local or nonlocal sources. However, it is worth pointing out that most extinction results mentioned above concerns problems with Dirichlet boundary conditions and there are much fewer extinction results for Neumann problems, especially for problems with sign-changing solutions.

In [4], the authors considered the slow diffusion case, i.e. $p > 2$, and showed that the corresponding solutions blow up in finite time for positive but suitably small initial energy. In a recent paper [16], Qu et al considered a problem similar to (1.1), and proved that the sign-changing solutions blow up in finite time when the initial energy is non-positive and $q > 1$. As for the extinction results, they showed that if $p-1 < q < 1$, then all the weak solutions vanish in finite time for small initial data; if $q \ge 1$, then the bounded weak solutions vanish in finite time for small initial data. However, they did not show whether the problem admits extinction solutions or not for the case $0 < q \le p-1$. Later, Guo et al [9] showed that there will be non-extinction solutions provided that the initial energy is negative.

Motivated by the works mentioned above, we will consider both the blow-up and extinction properties of solutions to (1.1). As for the blow-up results we will improve those obtained in [8, 16] and show that the solutions will blow up in finite time for positive (but suitably small) initial energy. In the proof, some lower bound of $\|\nabla u\|_p$ (L^p norm $\|\cdot\|_{L^p(\Omega)}$ will be denoted by $\|\cdot\|_p$ throughout this paper) plays an essential role. When considering the extinction properties, we will show that the solutions behave in quite different ways depending on the parameters $p-1$ and q as well as the initial energy. A Sobolev-Poincaré type inequality for functions belonging to $W^{1,p}(\Omega)$ (not $W_0^{1,p}(\Omega)$) will be of great help.

The rest of this paper is organized as follows. We will show that the solutions will blow up in finite time for positive initial energy in Section 2, and the extinction properties of solutions will be investigated in Section 3.

2. Blow-up results

It is well known that the equation in (1.1) is singular at the points where $\nabla u = 0$, since $1 < p < 2$. Therefore, we have to work with its weak solutions.

Definition 2.1. We say that a function $u \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T; W^{1,p}(\Omega))$ with $u_t \in L^2(\Omega \times (0,T))$ is a weak solution to (1.1) if

$$\begin{aligned} &\int_0^t \int_\Omega \Big[u\varphi_s - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \Big(|u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u\mathrm{d}x\Big)\varphi\Big]\mathrm{dxds} \\ &= \int_\Omega u(x,t)\varphi(x,t)\mathrm{dx} - \int_\Omega \mathrm{u_0(x)\varphi(x,0)dx} \end{aligned} \tag{2.1}$$

holds for all $\varphi \in C^1(\overline{\Omega} \times [0,T])$.

The existence of local weak solutions can be obtained via the standard method of regularization [20, 21]. For convenience, we might as well assume that the weak solutions are appropriately smooth in what follows, or else, we can consider the corresponding regularized problem and the same result can also be obtained through an approximate process.

Denote by $W_*^{1,p}(\Omega)$ the subspace of $W^{1,p}(\Omega)$, the elements u that satisfy $\int_\Omega u\mathrm{d}x = 0$. We equip this subspace with the norm

$$\|u\|_{W_*^{1,p}(\Omega)} = \Big(\int_\Omega |\nabla u|^p \mathrm{d}x\Big)^{1/p}.$$

By using Poincaré's inequality, we see that this norm is equivalent to the classical norm equipped with $W^{1,p}(\Omega)$. Let $B > 0$ be the optimal constant of the embedding inequality

$$\|u\|_{q+1} \le B \parallel \nabla u \parallel_p, \quad u \in W_*^{1,p}(\Omega), \tag{2.2}$$

where $1 < q \le (Np - N + p)/(N - p)$, and set

$$\alpha_1 = B^{-\frac{q+1}{q-p+1}}, \quad E_1 = (\frac{1}{p} - \frac{1}{q+1})B^{-\frac{p(q+1)}{q-p+1}} > 0 \tag{2.3}$$

and the energy functional

$$E(t) = \int_\Omega \Big[\frac{1}{p}|\nabla u(x,t)|^p - \frac{1}{q+1}|u(x,t)|^{q+1}\Big]\mathrm{d}x. \tag{2.4}$$

Our main result in this section is as follows.

Theorem 2.2 (Blow-up with positive initial energy)**.** *Assume that* $\max\{1, \frac{2N}{N+2}\} < p < 2$, $1 < q \le (Np - N + p)/(N - p)$ *and that the initial datum* $u_0(x)$ *is chosen to satisfy* $E(0) < E_1$ *and* $\|\nabla u_0\|_p > \alpha_1$, *where* E_1 *and* α_1 *are given in* (2.3)*. Then the weak solutions* $u(x,t)$ *of* (1.1) *blow up in finite time.*

For the proof of the above theorem we need the following lemma.

Lemma 2.3. *The function* $E(t)$ *defined in* (2.4) *is nonincreasing in* t.

Proof. By direct computation, integration by parts and recalling the fact that $\int_\Omega u(x,t)\mathrm{dx} = 0$ we immediately obtain

$$
\begin{aligned}
\frac{d}{dt}E(t) &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u_t \mathrm{dx} - \int_\Omega |\mathrm{u}|^{\mathrm{q}-1}\mathrm{u}\mathrm{u}_\mathrm{t}\mathrm{dx} \\
&= -\int_\Omega u_t \Big[\mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}) + |\mathrm{u}|^{\mathrm{q}-1}\mathrm{u}\Big]\mathrm{dx} \\
&= -\int_\Omega u_t^2 \mathrm{d}x - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u\mathrm{d}x \cdot \int_\Omega u_t \mathrm{d}x \\
&= -\int_\Omega u_t^2 \mathrm{d}x \le 0.
\end{aligned}
$$

Thus, $E(t)$ is non-increasing in t. The proof is complete. □

The next lemma gives a uniform positive lower bound of $\|\nabla u(\cdot,t)\|_p$, which will play an essential role in the proof of Theorem 2.2.

Lemma 2.4. *Suppose that* $u(x,t)$ *is a weak solution of* (1.1)*,* $E(0) < E_1$ *and* $\|\nabla u_0\|_p > \alpha_1$*. Then there exists a positive constant* $\alpha_2 > \alpha_1$*, such that*

$$\|\nabla u(\cdot,t)\|_p \ge \alpha_2, \quad \forall t \ge 0, \tag{2.5}$$

$$\|u\|_{q+1} \ge B\alpha_2, \quad \forall t \ge 0. \tag{2.6}$$

Proof. It can be deduced from (2.2) and (2.4) that

$$
\begin{aligned}
E(t) &\ge \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q+1}B^{q+1}\|\nabla u\|_p^{q+1} \\
&= \frac{1}{p}\alpha^p - \frac{1}{q+1}B^{q+1}\alpha^{q+1} =: l(\alpha),
\end{aligned}
\tag{2.7}
$$

where $\alpha = \alpha(t) = \|\nabla u(\cdot,t)\|_p$. It is easy to see that $\alpha = \alpha_1$ is the only critical point of $l(\alpha)$, that l is strictly increasing for $0 < \alpha < \alpha_1$, strictly decreasing for $\alpha > \alpha_1$; $l(\alpha) \to -\infty$ as $\alpha \to +\infty$ and $l(\alpha_1) = E_1$, where α_1 and E_1 are defined in (2.3). Since $E(0) < E_1$, there exists an $\alpha_2 > \alpha_1$ such that $l(\alpha_2) = E(0)$.

Set $\alpha_0 = \|\nabla u_0\|_p$. From (2.7) we have $l(\alpha_0) \le E(0) = l(\alpha_2)$, which implies that $\alpha_0 \ge \alpha_2$ since $\alpha_0, \alpha_2 \ge \alpha_1$. To prove (2.5), we argue by contradiction. Suppose that $\|\nabla u(\cdot,t_0)\|_p < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|\nabla u(\cdot,t)\|_p$ with respect to t we may choose t_0 such that $\|\nabla u(\cdot,t_0)\|_p > \alpha_1$. Then it follows from (2.7) and the monotonicity of l that

$$E(0) = l(\alpha_2) < l(\|\nabla u(\cdot,t_0)\|_p) \le E(t_0),$$

which contradicts Lemma 2.3. Hence (2.5) is proved.

To prove (2.6), we see from (2.4) and Lemma 2.3 that

$$\frac{1}{p}\|\nabla u\|_p^p \le E(0) + \frac{1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x,$$

which implies that

$$\begin{aligned}\frac{1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x &\geq \frac{1}{p}\|\nabla u\|_p^p - E(0) \geq \frac{1}{p}\alpha_2^p - E(0)\\ &= \frac{1}{p}\alpha_2^p - g(\alpha_2)\\ &= \frac{1}{q+1}B^{q+1}\alpha_2^{q+1}.\end{aligned}$$

Therefore, (2.6) holds. The proof is complete. □

Let

$$H(t) = E_1 - E(t), \quad t \geq 0. \tag{2.8}$$

Lemma 2.5. *For all $t \geq 0$,*

$$0 < H(0) \leq H(t) \leq \frac{1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x. \tag{2.9}$$

Proof. It is easily seen from Lemma 2.3 that $H'(t) \geq 0$, which in turn implies $H(t) \geq H(0) > 0,\ t \geq 0$. On the other hand, by the definition of $E(t)$ and $H(t)$ we have

$$H(t) = E_1 - \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x.$$

Recalling (2.5) and (2.3) one obtains

$$E_1 - \frac{1}{p}\|\nabla u\|_p^p \leq E_1 - \frac{1}{p}\alpha_1^p = -\frac{1}{q+1}B^{q+1}\alpha_1^{q+1} \leq 0, \quad t \geq 0,$$

which completes the proof. □

We can now prove Theorem 2.2 on the basis of the above three lemmas.

Proof of Theorem 2.2. Define $G(t) = \frac{1}{2}\int_\Omega u^2(x,t)\mathrm{d}x$ and take derivative with respect to t to obtain

$$\begin{aligned}G'(t) &= \int_\Omega u u_t \mathrm{d}x\\ &= \int_\Omega u\Big[\mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}) + |\mathrm{u}|^{\mathrm{q}-1}\mathrm{u} - \frac{1}{|\Omega|}\int_\Omega |\mathrm{u}|^{\mathrm{q}-1}\mathrm{u}\mathrm{dx}\Big]\mathrm{dx}\\ &= \int_\Omega |u|^{q+1}\mathrm{d}x - \int_\Omega |\nabla u|^p \mathrm{d}x\\ &= \int_\Omega |u|^{q+1}\mathrm{d}x - pE(t) - \frac{p}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x\\ &= \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x - pE_1 + pH(t)\\ &\geq \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x - pE_1.\end{aligned} \tag{2.10}$$

Recalling (2.3) and (2.6) we have

$$\begin{aligned}pE_1 &= p(\frac{1}{p} - \frac{1}{q+1})B^{-\frac{p(q+1)}{q-p+1}}\\ &= \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}}\frac{q-p+1}{q+1}B^{q+1}\alpha_2^{q+1}\end{aligned}$$

$$\le \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}} \frac{q-p+1}{q+1} \int_\Omega |u|^{q+1} \mathrm{d}x.$$

Substituting the above inequality into (2.10) we obtain

$$G'(t) \ge \Big(1 - \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}}\Big) \frac{q-p+1}{q+1} \int_\Omega |u|^{q+1} \mathrm{d}x = C_0 \int_\Omega |u|^{p+1} \mathrm{d}x \ge 0, \tag{2.11}$$

where

$$C_0 = \Big(1 - \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}}\Big) \frac{q-p+1}{q+1} > 0.$$

On the other hand, by using Hölder's inequality we have

$$G^{\frac{q+1}{2}}(t) = \Big(\frac{1}{2} \int_\Omega u^2(x,t) \mathrm{d}x\Big)^{\frac{q+1}{2}} \le C \int_\Omega |u|^{q+1} \mathrm{d}x, \tag{2.12}$$

where $C > 0$ is a constant depending only on $|\Omega|$ and q. By combining (2.11) with (2.12) we have

$$G'(t) \ge \gamma G^{\frac{q+1}{2}}(t), \tag{2.13}$$

where $\gamma = C_0/C > 0$. A direct integration of (2.13) from 0 to t yields

$$G^{\frac{q-1}{2}}(t) \ge \frac{1}{G^{(1-q)/2}(0) - \frac{q-1}{2}\gamma t}.$$

Thus, $G(t)$ blows up at a finite time $T^* \le \frac{G^{(1-q)/2}(0)}{\frac{q-1}{2}\gamma}$, and so does $u(x,t)$. The proof is complete. $\square$

3. Extinction results

In this section, we confine ourselves to the study of the extinction properties of solutions to (1.1). More precisely, we will indicate whether or not the solutions will vanish in finite time, depending on the parameters p and q as well as on the initial energy $E(0)$. Before proving the main results, a Sobolev-Poincaré type inequality for functions belonging to $W^{1,p}(\Omega)$ (not $W_0^{1,p}(\Omega)$) will be established first. This inequality is a generalization of Sobolev-Poincaré type inequality under the assumption that $\int_\Omega v(x)\mathrm{dx} = 0$, and will play an critical role in the sequel.

Lemma 3.1. *Let $f(t)$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and $f(t) = 0$ implies $t = 0$. If $\int_\Omega f(v(x))\mathrm{dx} = 0$ and $v \in W^{1,p}(\Omega)$ $(p > 1)$, then*

$$\|v\|_q \le C\|\nabla v\|_p \tag{3.1}$$

for all $1 < q \le p^*$, where $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate of p, and $C > 0$ is a constant depending only on p, q and Ω.

Proof. It is easily seen from the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ that we need only to prove (3.1) for the case $q = p$. Assume on the contrary that there exists a sequence $\{v_n\}$ such that

$$\|v_n\|_p > n\|\nabla v_n\|_p. \tag{3.2}$$

Without loss of generality, we may assume that $\|v_n\|_p = 1$. Since $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$, there exist a subsequence of $\{v_n\}$, which we still denote by $\{v_n\}$ and a $v \in W^{1,p}(\Omega)$ such that v_n tends to v strongly in $L^p(\Omega)$, weakly in $W^{1,p}(\Omega)$ and almost everywhere in Ω. In particular, we have $\|v\|_p = 1$.

From (3.2) we know that ∇v_n tends to 0 as $n \to \infty$, and $\nabla v = 0$. Hence, v is a constant that not equals 0. On the other hand, we can deduce from $\int_\Omega f(v(x))\mathrm{dx} = |\Omega|\mathrm{f(v)} = 0$ that $v \equiv 0$, which is contradiction. The proof of this lemma is complete. □

Corollary 3.2. *Let $1 < p < 2$ and $s \geq \max\{0, \frac{1}{p^2}(2N-(N+2)p)\}$. If $\int_\Omega u(x)\mathrm{dx} = 0$ and $\nabla(|u|^s u) \in L^p(\Omega)$, then*

$$\|u\|_{ps+2}^{p(s+1)} \leq \gamma \|\nabla(|u|^s u)\|_p^p, \tag{3.3}$$

where $\gamma > 0$ is a constant depending only on p, s and Ω.

Proof. Taking $f(t) = |t|^{-\frac{s}{1+s}}t$, $v = |u|^s u$ and $q = \frac{ps+2}{s+1}$ in Lemma 3.1, we can easily prove this corollary. □

We are now in a position to prove the extinction properties of solutions to (1.1), by combining the method of energy estimates with the above corollary.

Theorem 3.3. *(I) If $p - 1 < q < 1$, then the weak solutions to* (1.1) *vanish in finite time provided that u_0 is suitably small;*

(II) If $q \geq 1$, then the bounded weak solutions to (1.1) *vanish in finite time provided that u_0 is suitably small;*

(III) If $q = p - 1$, then the weak solutions to (1.1) *vanish in finite time provided that $|\Omega|$ is suitably small.*

Proof. Multiplying the first equation in (1.1) by $|u|^{ps}u$ with $s \geq \max\Big\{0, \frac{1}{p^2}(2N - (N+2)p)\Big\}$ and integrating by parts over Ω, one obtains

$$\begin{aligned} &\frac{1}{ps+2}\frac{d}{dt}\int_\Omega |u|^{ps+2}\mathrm{dx} + \frac{\mathrm{ps}+1}{(\mathrm{s}+1)^{\mathrm{p}}}\int_\Omega |\nabla(|\mathrm{u}|^{\mathrm{s}}\mathrm{u})|^{\mathrm{p}}\mathrm{dx} \\ &= \int_\Omega |u|^{ps+q+1}\mathrm{dx} - \frac{1}{|\Omega|}\int_\Omega |\mathrm{u}|^{\mathrm{q}-1}\mathrm{u}\mathrm{dx}\int_\Omega |\mathrm{u}|^{\mathrm{ps}}\mathrm{u}\mathrm{dx}. \end{aligned} \tag{3.4}$$

(I) $p - 1 < q < 1$: Applying (3.3) to the second term on the left hand side of (3.4) and using Hölder's inequality on the right hand side, we arrive at

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |u|^{ps+2}\mathrm{dx} + \mathrm{C}\Big(\int_\Omega |\mathrm{u}|^{\mathrm{ps}+2}\mathrm{dx}\Big)^{\frac{\mathrm{ps}+\mathrm{p}}{\mathrm{ps}+2}} \\ &\leq 2(ps+2)|\Omega|^{\frac{1-q}{ps+2}}\Big(\int_\Omega |u|^{ps+2}\mathrm{dx}\Big)^{\frac{\mathrm{ps}+\mathrm{q}+1}{\mathrm{ps}+2}}, \end{aligned} \tag{3.5}$$

where $C = \frac{(ps+2)(ps+1)}{\gamma(s+1)^p}$. Set $J(t) = \int_\Omega |u|^{ps+2}\mathrm{dx}$, then the above inequality can be rewritten as

$$J'(t) \leq -J^{\frac{ps+p}{ps+2}}\Big[C - 2(ps+2)|\Omega|^{\frac{1-q}{ps+2}}J^{\frac{q+1-p}{ps+2}}(t)\Big]. \tag{3.6}$$

Choose u_0 sufficiently small such that

$$C - 2(ps+2)|\Omega|^{\frac{1-q}{ps+2}}J^{\frac{q+1-p}{ps+2}}(0) > 0,$$

then we have

$$J'(t) \leq -C_1 J^{\frac{ps+p}{ps+2}}, \tag{3.7}$$

where $C_1 = C - 2(ps+2)|\Omega|^{\frac{1-q}{ps+2}} J^{\frac{q+1-p}{ps+2}}(0)$. Noticing that $0 < \frac{ps+p}{ps+2} < 1$, by direct computation we have

$$J^{\frac{2-p}{ps+2}}(t) \le \Big[J^{\frac{2-p}{ps+2}}(0) - \frac{C_1(2-p)}{ps+2}t\Big]_+.$$

Thus, $J(t)$ vanishes in finite time and so does $u(x,t)$.

(II) $q \ge 1$: Suppose that $\|u\|_\infty \le M$. Then it can be deduced from (3.4) that

$$\frac{d}{dt}\int_\Omega |u|^{ps+2}\mathrm{dx} + \mathrm{C}\Big(\int_\Omega |\mathrm{u}|^{\mathrm{ps+2}}\mathrm{dx}\Big)^{\frac{\mathrm{ps+p}}{\mathrm{ps+2}}} \le 2(\mathrm{ps}+2)\mathrm{M}^{\mathrm{q-1}}\int_\Omega |\mathrm{u}|^{\mathrm{ps+2}}\mathrm{dx}. \tag{3.8}$$

Thus, by using the argument similar to Case (I) we can prove the finite time extinction of $u(x,t)$ provided that the initial datum $u_0(x)$ is suitably small.

(III) $q = p-1$: In this case, (3.6) becomes

$$J'(t) \le -J^{\frac{ps+p}{ps+2}}\big[C - 2(ps+2)|\Omega|^{\frac{2-p}{ps+2}}\big]. \tag{3.9}$$

Although the constant C in (3.9) depends on Ω, it does not tend to 0 as $|\Omega|$ tends to 0. Thus, we can choose $|\Omega|$ so small that $C_2 := C - 2(ps+2)|\Omega|^{\frac{2-p}{ps+2}} > 0$ since $p < 2$. The remaining argument is similar to that in Case (I) and therefore is omitted. The proof is complete. □

When $0 < q \le p-1$, problem (1.1) may admit non-extinction solutions. To prove this, we need the following lemma which gives a lower bound of the solutions to an ordinary differential inequality (see [9] for its proof).

Lemma 3.4. *Suppose that α,
beta, $\theta > 0$ and $h(t)$ is a non-negative and absolutely continuous function satisfying*

$$h'(t) + \alpha h^\theta(t) \ge \beta, \quad t \in (0,\infty).$$

Then $h(t) \ge \min\{h(0), (\frac{\beta}{\alpha})^{\frac{1}{\theta}}\}$.

Theorem 3.5. *If $0 < q < p-1$, then (1.1) admits no extinction solutions when $E(0) < 0$; If $q = p-1$, then (1.1) admits no extinction solutions when $E(0) \le 0$. Here $E(t)$ is defined in (2.4).*

Proof. We define $G(t) = \frac{1}{2}\int_\Omega u^2(x,t)\mathrm{d}x$ and take derivative with respect to t to obtain

$$\begin{aligned}
G'(t) &= \int_\Omega u u_t \mathrm{d}x \\
&= \int_\Omega u\Big[\mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p-2}}\nabla \mathrm{u}) + |\mathrm{u}|^{\mathrm{q-1}}\mathrm{u} - \frac{1}{|\Omega|}\int_\Omega |\mathrm{u}|^{\mathrm{q-1}}\mathrm{udx}\Big]\mathrm{dx} \\
&= \int_\Omega |u|^{q+1}\mathrm{d}x - \int_\Omega |\nabla u|^p \mathrm{d}x \\
&= \int_\Omega |u|^{q+1}\mathrm{d}x - pE(0) + p\int_0^t\int_\Omega u_s^2\mathrm{dxds} - \frac{\mathrm{p}}{\mathrm{q}+1}\int_\Omega |\mathrm{u}|^{\mathrm{q+1}}\mathrm{dx} \\
&\ge \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x - pE(0).
\end{aligned} \tag{3.10}$$

When $0 < q < p-1$, Hölder's inequality implies

$$\frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x \ge \frac{q-p+1}{q+1}|\Omega|^{\frac{1-q}{2}}\Big[\int_\Omega |u|^2\mathrm{d}x\Big]^{\frac{q+1}{2}}. \tag{3.11}$$

Substituting (3.11) into (3.10) and recalling $E(0) < 0$ and Lemma 3.4 we see that $G(t) > 0$ for all $t > 0$.

When $q = p - 1$, it follows from (3.10) and $E(0) \leq 0$ that $G'(t) \geq 0$, which then implies $G(t) \geq G(0) > 0$ since $u_0 \not\equiv 0$. Therefore, $u(x,t)$ can not vanish in finite time in each case. The proof is complete. □

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