Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 213, pp. 1–33.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DIMENSION OF THE SET OF POSITIVE SOLUTIONS TO NONLINEAR EQUATIONS AND APPLICATIONS

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Abstract. We study the covering dimension of the set of (positive) solutions to various classes of nonlinear equations involving condensing and A-proper maps. It is based on the nontriviality of the fixed point index of a certain condensing map or on oddness of a nonlinear map. Applications to nonlinear singular integral equations and to semilinear ordinary and elliptic partial differential equations are given with finite or infinite dimensional null space of the linear part.

1. Introduction and statements of the basic results

Let K be a retract of a Banach space X (e.g., K is a closed and convex subset of X, say a cone). Then K is closed. Let $D \subset \mathbb{R}^m \times K$ be an open bounded subset and $F : \overline{D} \subset \mathbb{R}^m \times K \to X$ be continuous and ϕ-condensing. Our first objective is to study the set of positive solutions of the equation

$$x - F(\lambda, x) = f \tag{1.1}$$

This equation is undetermined and under suitable assumptions on F we shall not only prove the existence of its solutions but also that its set of positive solutions has a covering dimension at least m. Unless otherwise specified, we shall assume that m is a positive integer throughout the paper.

Our dimension results for (1.1) will be used to study semilinear operator equations of the form

$$Lx + Nx = f(x \in \bar{D}, f \in Y) \tag{1.2}$$

where $L : X \to Y$ is a continuous linear surjective map with dimension of the null space $m \le \infty$, $N : X \to Y$ is a suitable continuous nonlinear map, and X and Y are Banach spaces. We prove that the dimension of the solution set of (1.2) is at least m. The previous studies [10, 11, 12, 13, 14, 18, 22, 23, 29, 30, 31, 32, 33, 43, 44] and the references therein) dealt with dimension results for compact perturbations of the identity map, or with approximation-proper maps of the form $L + N$ with L a Fredholm linear map of positive index. The study of the latter class of maps requires that spaces (X, Y) posses projectionally complete schemes and, in particular, be separable. Our study of (1.1)-(1.2) involves perturbations $F(\lambda, x)$ and N that are

2010 *Mathematics Subject Classification.* 47H08, 47H09, 47J05, 45G05, 35L61.
Key words and phrases. Covering dimension, semilinear equations; fixed point index; condensing and A-proper maps; singular integral equations; exponential dichotomy; ordinary differential equations; elliptic PDE's.
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Submitted May 7, 2016. Published August 10, 2016.

ϕ-condensing relative to a general measure of noncompactness ϕ and is done in general, not necessarily separable, Banach spaces. Applications to singular integral equations and partial differential equations in Hölder spaces require such results. Various existing generalized first Fredholm theorems are applicable to (1.2) with L a linear homeomorphism or a surjective positive-homogeneous map. The solvability of (1.2) with L a Fredholm linear map of index zero has been done under various Landesman-Lazer type conditions. There is a vast literature on such study of (1.2) (see [32, 38] and the references therein).

Beginning with a detailed study by Fitzpatrick-Massabo-Pejsachowicz [10], when $K = X$ the dimension of the solution set of equation (1.1) with $F(\lambda, x)$ compact and of equation (1.2) with L a Fredholm map of index $m > 0$ has been studied by many authors using algebraic topology arguments (see [6, 10, 18] for extensive expositions on the subject). In [10], the authors studied (1.2) with L a Fredholm map of index $m > 0$ and $L + N$ approximation-proper by reducing it to the form (1.1). In our works [30, 31], we have studied (1.2) directly and proved dimension results for approximation proper maps L+N with the index$(L) > 0$ under conditions on N that are extensions of the corresponding Landesman-Lazer type conditions for (1.2) when the index $L = 0$. No surjectivity of L is required in any of these works.

Zorn's lemma argument in the study of dimensions of solution sets of nonlinear equations has been used by Ize-Massabo-Pejsachowisz-Vignoli [22, 23]. Still other approaches to studying the dimension of the solution sets of (1.1) and (1.2) based on the selection theorems of Michael [30] and Saint-Raymond [46] can be found in Ricceri [43, 44, 45], on degree theory in Gelman [12, 13, 14] with $K = X$, and on equivariant essential maps by the author [29, 30, 31], Gorniewisz [18] and the references therein. When the nonlinear perturbation is a k-Lipschitzian, Ricceri [45] has shown that the solution set is an absolute extensor for paracompact spaces, but no dimension assertion of it is given.

In this work, we shall prove our results using the fixed point index method for multivalued ϕ-condensing maps developed in Fitzpatrick-Petryshyn [11], in conjunction with the selection results of Michael [28] and Saint-Raymond [46]. In this approach, we introduce a notion of a complementing map by a continuous multivalued compact map. It differs from the notion of complementing maps by a finite dimensional single valued map introduced in [10]. But, in either case, the existence of a complementing map implies a dimension result. We prove that if the restriction of F to $\bar{D}_0 = \bar{D} \cap (0 \times K)$ has a nonzero fixed point index, then F is a complementing map. To the best of our knowledge, no prior dimension results for positive solutions of nonlinear equations exist. When N is a k-Lipschitzian, using Ricceri's result [45], we prove that the solution set is also an absolute extensor for paracompact spaces.

As we will see below, the main assumption on the linear part in our dimension results for semilinear equations (1.2) with non-odd N is that it is surjective and has a continuous linear right inverse. We know that the existence of such an inverse is equivalent to the existence of a complement of the null space X_0 of L in X. Such complements always exist if X_0 is finite dimensional, or if X_0 is closed and either the domain space is a Hilbert space or if the codimension of X_0 is finite. In general, it is known that there are continuous linear surjections between Banach spaces which do not possess any continuous linear right inverse. We note that, beginning with a negative result of Grothendieck, existence and nonexistence of continuous linear right inverses for various classes of partial differential operators have been extensively studied and we refer to the survey paper by Vogt [47]. In Sections 5-9, we prove the existence of a linear continuous right inverse for various ordinary and elliptic partial differential operators L that also have infinite dimensional null space. Michael [28] established that each continuous surjective linear map L between Banach spaces X and Y has a continuous right inverse $K : Y \to X$ such that $LK(y) = y$ for each $y \in Y$ and $K(ty) = tK(y)$ for all t and $\|K(y)\| \leq k\|y\|$ for some k and all y. No other properties of K are known except that it is linear if and only if the null space of L has a complement in X. Its existence is suitable for studying nonlinear compact perturbations of L as was done by Gelman [13]. In view of this, we require the existence of a continuous linear right inverse in order to study various general classes of noncompact nonlinear perturbations. Another approach to obtain dimension results for semilinear problems (1.2) with N compact is given by Ricceri [45, 46] and is based on a really deep selection theorem by Saint Raymond [46] conjectured by Ricceri. This approach does not require the existence of a continuous right inverse of L.

To state our basic results, we need to introduce a notion of a complementing map. Let $D \subset \mathbb{R}^m \times K$ be an open subset (in the relative topology) and $F : \overline{D} \to K$ be a continuous condensing map, i.e. $\phi(F(Q)) < \phi(Q)$ for $Q \subset \mathbb{R}^m \times K$ with $\phi(Q) \neq 0$, where ϕ is a measure of noncompactness. We say that F is complemented by a continuous compact multivalued map $G : \overline{D} \to CV(\mathbb{R}^m)$ if the fixed point index $i(H, D, \mathbb{R}^m \times K) \neq 0$ for the multivalued condensing map $H : \overline{D} \subset (\mathbb{R}^m \times K) \to CV(\mathbb{R}^m \times K)$ given by $H(\lambda, x) = (G(\lambda, x), F(\lambda, x))$. Note that $(\lambda, x) - (G(\lambda, x), F(\lambda, x)) = (I - H)(\lambda, x)$ is a condensing perturbation of the identity and $\operatorname{Fix}(H, D) = \{(\lambda, x) : (\lambda, x) \in H(\lambda, x)\} \subset S(F, D) = \{(\lambda, x) : F(\lambda, x) = x\}$. Our definition of a complementing map differs from the notion of a complementing by finite dimensional single valued maps in Fitzpatrick-Massabo-Pejsachowicz [10]. A basic assumption that implies that F has a complement is that the fixed point index for the condensing map $i(F(0, .), D \cap (0 \times K), K) \neq 0$. In that sense, our results are of a continuation type involving an m-dimensional parameter space $\mathbb{R}^m$

Recall that if D is a topological space, and m is a positive integer, then D has the covering dimension equal to m provided that m is the smallest integer with the property that whenever U is a family of open subsets of D whose union covers D, there exists a refinement, U', of U whose union also covers D and no subfamily of U' consisting of more than $m+1$ members has nonempty intersection. If D fails to have this refinement property for each positive integer, then D is said to have infinite dimension. Recall that when D is a convex set in a Banach space, the covering dimension of D coincides with the algebraic dimension of D, the latter being understood as ∞ if it is not finite. A covering dimension is a topological invariant, i.e., if B and D are metric spaces and $F : B \to D$ is a homeomorphism, then $\dim(B) = \dim(D)$. Moreover, if D is a locally compact metric space, then $\dim(D) = 0$ if and only if D is hereditarily disconnected, i.e, the connected components of D are singletons. If $\dim D > 0$, it is known that the cardinality $\operatorname{card}(D) \geq c$, where c denotes the cardinality of the continuum. The converse is false as the set of irrational numbers shows. In the absence of a manifold structure on D, the concept of dimension is a natural way in which to describe its size.

Unless otherwise stated, X and Y will be Banach spaces. Some of our basic results for (1.1) and (1.2) are stated next.

Theorem 1.1. *Let m be a positive integer and $F : \overline{D} \subset \mathbb{R}^m \times K \to K$ be a continuous condensing map complemented by a continuous multivalued compact map $G : \overline{D} \to CV(\mathbb{R}^m)$ with $\dim G(\lambda, x) = m$ for each $(\lambda, x) \in \overline{D}$. Then dim $S(F, D) \geq m$, and $S(F, D)$ contains a nondegenerate (nonsingleton) connected component.*

The next result shows that F is complemented if its restriction to $\bar{D}_0 = \bar{D} \cap (0 \times K)$ has a nonzero index.

Corollary 1.2. *Let m be a positive integer, $F : \overline{D} \subset \mathbb{R}^m \times K \to K$ be continuous and condensing, $\overline{D}_0 = \overline{D} \cap (0 \times K)$ and $F_0(x) = F(0, x) : \overline{D}_0 \subset K \to K$ be such that its index $i(F_0, D_0, K) \neq 0$. Then F is complemented and $\dim S(F, D) \geq m$. Moreover, $S(F, D)$ contains a nondegenerate connected component.*

Recall that a map $T : X \to Y$ satisfies condition (+) if $\{x_n\}$ is bounded whenever $Tx_n \to y$ in Y. A nonlinear mapping T is quasibounded with the quasinorm $|T|$ if

$$|T| = \limsup_{\|x\| \to \infty} \|Tx\| / \|x\| < \infty$$

For a map $T : X \to Y$, let Σ be the set of all points $x \in X$ where T is not locally invertible, and let card $T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$. Define $S(f) = \{x : Lx - Nx = f\}$.

Theorem 1.3. *Let $L : X \to Y$ be a not injective continuous linear surjection, $L^+ : Y \to X$ be a continuous linear right inverse of L and $N : X \to Y$ be a k-ϕ contraction with $k\|L^+\| < 1$ such that $I - tNL^+ : Y \to Y$ satisfies condition (+), $t \in [0, 1]$. Then $L - N : X \to Y$ is surjective and, for each $f \in Y$,*

$$\dim S(f) \geq \dim \ker (L).$$

Moreover, $S(f)$ contains a nondegenerate connected component and $S(f)$ is unbounded if $\|Nx\| \leq a\|x\| + b$ for some positive a and b with $a\|L^+\| < 1$. It is an absolute extensor for paracompact spaces if N is k-Lipschitzian. If L is a homeomorphism, then $S(f) \neq \emptyset$ compact set for each $f \in Y$ and the cardinal number of $S(f)$ is constant and finite on each connected component of $Y \setminus (L - N)(\Sigma)$.

In dealing with some semilinear equations, like singular integral equations in Hölder spaces, a nonlinear map can not be globally k-Lipschitzian unless it is affine (see Section 5). For studying such problems we have the following result for locally ϕ-contractive nonlinearities.

Theorem 1.4. *Let $L : X \to Y$ be a not injective continuous linear surjection, $L^+ : Y \to X$ be a continuous linear right inverse of L with $\|L^+\| \leq 1$ and $N : X \to Y$ be such that for some $r > 0$, $N : \bar{B}(0, r) \subset X \to Y$ is a $k(r)$-ϕ- contraction with $k(r)\|L^+\| < min\,\{1, r\}$ and $\|Nx\| \leq k(r)\|x\|$ on $\bar{B}(0, r)$. Then* (1.2) *is solvable for each $f \in Y$ satisfying*

$$\|f\| < r - \|L^+\| k(r)$$

and $\dim(S(f) \cap \bar{B}(0, r)) \geq \dim \ker(L)$.

Next, to study wider classes of nonlinearities N, we need that spaces are separable and $L - N$ is approximation-proper relative to a suitable projection scheme.

The following basic result for such maps with infinite dimensional null space of L is an easy extension of of Fitzpatrick-Massabo-Pejsachwisz [10, Theorem 1.2]. No A-properness of $L - N$ on the whole space X is needed.

Theorem 1.5. *Let X and Y be separable Banach spaces, $L : X \to Y$ be a not injective continuous linear surjection with a continuous linear right inverse, $X_0 = \ker(L)$ be infinite dimensional, and $\tilde{X}$ be a complement of X_0 in X. Let $\Gamma = \{X_n, Y_n, Q_n\}$ be a projectionally complete scheme for $(\tilde{X}, Y)$ and $N : X \to Y$ be a continuous map such that for each m-dimensional subspace $U_m \subset \ker L$, the map $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ for $(U_m \oplus \tilde{X}, Y)$ with $\dim X_n = \dim Y_n$ and the degree $\deg((Q_n(L-N)|X_n, X_n, 0) \neq 0$ for all large n. Assume that a projection P_m of $U_m \oplus \tilde{X}$ onto U_m is proper on $\{x \in U_m \oplus \tilde{X}; | \, Lx - Nx = f\}$ for each $f \in Y$. Then*

$$\dim\{x : Lx - Nx = f\} = \infty.$$

Moreover, for each $m > 0$, there is a connected subset of the solution set whose dimension at each point is at least m.

Corollary 1.6. *Let $L : X \to Y$ be a not injective continuous surjection with a continuous linear right inverse and $N : X \to Y$ be a nonlinear map such that, for each finite dimensional subspace U_m of X_0=ker(L), the restriction $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to Γ_m. Let*

$$\|Nx\| \le a\|x\| + b \quad \text{for all } x \in X$$

and $a\|L^+\| < 1$. Then, for each $f \in Y$, $S(f)$ is unbounded and

$$\dim S(f) \ge \dim \ker L.$$

Moreover, for each $m > 0$, there is a connected subset of the solution set whose dimension at each point is at least m. If L is a homeomorphism, then $S(f) \neq \emptyset$ compact set for each $f \in Y$ and the cardinal number of $S(f)$ is constant and finite on each connected component of $Y \setminus (L - N)(\Sigma)$.

Next, the study of the dimension of the solution set of semilinear equations of the form $Lx - Nx = 0$ when L and N are equivariant relative to some group of symmetries and L is a Fredholm linear map of positive index has been done by many authors and we refer to the (survey) articles and books [6, 18, 22, 23]. Rabinowitz [42] estimated the genus (and therefore the dimension) of the solution set for compact perturbations of continuous Fredholm maps of positive index. His result has been extended by the author to the case when $L - N$ is A-proper in [29, 30, 31] and by Gelman [14] for compact perturbations of linear surjective maps. In Section 4, we shall extend these results to odd perturbations of linear maps with infinite dimensional null space. No surjectivity of L is required in this case. A basic result is as follows.

Theorem 1.7. *Let $L : X \to Y$ be a continuous linear map with $X_0 = \ker L$, $\dim \ker L = \infty$, $\tilde{X}$ be a complement of X_0 in X and $N : X \to Y$ be an odd nonlinear map such that, for each finite dimensional subspace U_m of X_0, the restriction $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ at 0, where $\{X_n, Y_n, Q_n\}$ is a projectionally complete scheme for $(\tilde{X}, Y)$. Then, for each open, bounded and symmetric relative to 0 subset D of X*

$$\dim\{x \in \partial D : Lx - Nx = 0\} = \infty.$$

In Sections 5-9, we give applications of the above results to semilinear singular integral equations and to ordinary and elliptic partial differential equations. In Section 5, we establish a dimension result for semilinear one-dimensional singular integral equations with a Cauchy kernel

$$a(s)x(s) + \frac{b(s)}{\pi i} \int_c^d \frac{x(t)}{t-s} dt + \int_c^d \frac{k(s,t)}{t-s} f(t, x(t)) dt = h(s) \quad (c \le s \le d).$$

in the classical Hölder space $H^\alpha([c,d])$ $(0 < \alpha < 1)$ where $a(s)$, $b(s)$, $h(s)$, $k : [c,d] \times [c,d] \to C$ and $f : [c,d] \times R \to C$ are given functions. Here the induced nonlinear Nemitskii map is locally k-Lipschitzian. It is known ([26, 27]) that the Nemitskii map in $H^\alpha([c,d])$ is globally k-Lipschitzian if and only if $f(t,x(t))$ is affine, i.e., $f(t,x(t)) = a(t)x + b(t)$ for some functions $a(t)$ and $b(t)$. Such equations arise in a variety of applications in physics, aerodynamics, elasticity and other fields of engineering. We do not know of any dimension results for these equations. An interested reader is referred to [33] for dimension results for semilinear Wiener-Hopf integral equations.

Next, in Sections 6 and 7, we establish dimension results for ODE's defined on finite as well as infinite dimensional spaces

$$u'(t) + A(t)(u(t)) - F(t, u(t), u'(t)) = f(t) \quad \text{for all } t \in I.$$

Here, the surjectivity of the linear part in various Banach spaces of functions follows naturally from its ordinary or exponential dichotomy that have been studied extensively (see [25, 34] and the references therein). In [34], some results have also been proven about the surjectivity of $Lu = u'(t) + A(t)u(t)$ when it doesn't have any dichotomy but satisfies a certain Riccati differential inequality. In Sections 8 and 9, we apply our results to semilinear partial differential equations with finite and infinite dimensional null space of the linear operator in Hölder and Sobolev spaces

$$Lu - F(x, u, Du, D^2u) = f$$

that have continuous right inverses. We remark that some applications of the dimension results of Ricceri [43, 44] to semilinear elliptic equations on bounded domains involving nonlocal terms can be found in [43, 9]. Next, if $L : X \to Z$ is a not injective continuous linear map with closed range $Y = L(X)$ in Z and if a nonlinear map $N : X \to Y$, then our results apply to $L + N : X \to Y$. A particular case of this setting was given in Ricceri [43]. The closedness of the range may be avoided sometimes (see [48]). In Section 9.2, we prove a unique solvability result for convolution perturbations of elliptic differential maps L that have infinite dimensional null space with the range $R(L)$ of L not closed and the range of N is contained in $R(L)$.

2. Proofs of Theorems 1.1–1.4

Let X be a Banach space, and $K(X)$ be closed convex subset of X. We need the following continuous selection results of Michael [28] and Saint-Raymond [46].

Theorem 2.1. *(a) ([28]) Let Y be a paracompact topological space, X be a Banach space and $G : Y \to K(X)$ be a lower semicontinuous multivalued map. Then, for each closed subset A of Y and each continuous selection ψ of $G|_A$, there is a continuous selection ϕ of G such that $\phi_A = \psi$.*

(b) ([46]) Let Y be a compact metrisable subspace of dimension at most $m-1$ of a Banach space X, $H : Y \to K(X)$ be a multivalued lower semicontinuous map such that $0 \in H(x)$ and dim $H(x) \geq m$ for each $x \in Y$. Then there is a continuous selection h of H such that $h(x) \neq 0$ for all $x \in Y$.

Recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0 : D$ has a finite covering by sets of diameter less than d$\}$. The *ball-measure of noncompactness* of D is defined as $\chi(D) = \inf\{r > 0 : D \subset \cup_{i=1}^{n} B(x_i, r),\ x_i \in X, n \in N\}$. Let ϕ denote either the set or the ball measure of noncompactness. Then a mapping $F : D \subset X \to Y$ is said to be k-ϕ-*contractive (ϕ-condensing)* if $\phi(F(Q)) \leq k\phi(Q)$ (respectively, $\phi(F(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$). Next, let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of X and Y, respectively, with $\cup_{n=1}^{\infty} X_n$ dense in X, $m = \dim X_n - \dim Y_n \geq 0$ for each n and $Q_n : Y \to Y_n$ be a projection onto Y_n for each n. Recall also that a map $F : D \subset X \to Y$ is A-proper (at f) with respect to a projection scheme $\Gamma_m = \{X_n, P_n, Y_n, Q_n\}$ for (X,Y) if $Q_n F : D \cap X_n \to Y_n$ is continuous for each large n and whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $Q_{n_k} F x_{n_k} \to f$, a subsequence $x_{n_{k(i)}} \to x$ with $Fx = f$. This is a customary definition when $\dim X_n = \ dimY_n$ (see [38]). In dealing with dimension results, we need that $m > 0$ and such schemes were first used in [10, 29]. The class of A-proper maps is rather large (see [10, 32, 38] and also Proposition 3.1 below).

Recall that a closed subset K of a Banach space X is called a retract of X if there is a continuous map, called a retraction, $r : X \to K$ such that $r(x) = x$ for all $x \in K$. For example, any closed convex subset K, say a cone, is a retract of X. Let $CV(K)$ be compact and convex subsets of K, $D \subset K$ be an open subset of K (in the relative topology on K). When dealing with multivalued positive condensing maps F, we use the fixed point index $i(F, D, K)$ of Fitzpatrick and Petryshyn [11]. Let $\mathrm{Fix}(F, D) = \{x \in D : x \in F(x)\}$.

We begin by first proving a more general version of Theorem 1.1.

Theorem 2.2. *Let $F : \overline{D} \subset K \to CV(K)$ be an upper semicontinuous condensing map, $x \notin F(x)$ for each $x \in \partial D$ and the fixed point index $i(F, D, K) \neq 0$. Suppose that there is an open neighborhood U in K with $\mathrm{Fix}(F, D) \subset U \subset D$ and a lower semicontinuous map $G : U \to CV(X)$ such that $G(x) \subset F(x)$, dim $G(x) \geq m$ for each $x \in U$ and $x \in G(x)$ for each $x \in \mathrm{Fix}(F, D)$. Then $\dim \mathrm{Fix}(F, D) \geq m$ and $\mathrm{Fix}(F, D)$ contains a nondegenerate connected component.*

Proof. Suppose that the claim is false, i.e., dim $\mathrm{Fix}(F, D) \leq m - 1$. Since F is upper semicontinuous and condensing, it is easy to show that $\mathrm{Fix}(F, D)$ is a compact metric subspace of X. Let $H : U \to CK(X)$ be given by $H = I - G$ and $H_1 = I - G|_{\mathrm{Fix}(F,D)}$. Since G is lower semicontinuous from K to $CV(X)$, it follows that $H_1 : \mathrm{Fix}(F, D) \to CV(X)$ is lower semicontinuous from K to $CV(X)$, $0 \in H_1(x)$ and dim $H_1(x) \geq m$ for each $x \in \mathrm{Fix}(F, D)$. Then, by Saint Raymond's Theorem 2.1-(b) there is a continuous selection $h_1 : \mathrm{Fix}(F, D) \to X$ of H_1, with $h_1 = I - f_1 : \mathrm{Fix}(F, D) \to X$, $0 \neq h_1(x) \in H_1(x)$ for each $x \in \mathrm{Fix}(F, D)$. Since U is paracompact and $H : U \to CV(X)$ is lower semicontinuous, by Michael's Theorem 2.1-(a) there is a continuous selection $h : U \to X$, $h(x) \in H(x)$ for each $x \in U$, such that $h|_{\mathrm{Fix}(F,D)} = h_1$ and $h(x) \neq 0$ for each $x \in U$ since $0 \notin H(x)$ if $x \in U \backslash \mathrm{Fix}(F, D)$. Moreover, $h(x) = x - f(x) \in H(x) \subset x - F(x)$ with $f(x) \in G(x) \subset K$ for each $x \in U$.

Define a new multivalued map $F_1 : \overline{D} \subset K \to CV(K)$ by $F_1(x) = f(x)$ for $x \in U$ and $F_1(x) = F(x)$ for $x \notin U$. It is easy to see that F_1 is an upper semicontinuous condensing multivalued map with $x \notin F_1(x) \subset F(x)$ for all $x \in \overline{D}$. Since F_1 and F coincide on the boundary of D, we have that $i(F_1, D, K) = i(F, D, K) \neq 0$. Hence, $x \in F_1(x)$ for some $x \in D$, in contradiction to the definition of F_1. Thus, $\dim \operatorname{Fix}(F, D) \geq m$. Since $\operatorname{Fix}(F, D)$ is a compact metric space and $m \geq 1$, it contains a nondegenerate connected component. □

Remark 2.3. If F in Theorem 2.2 is also lower semicontinuous, and therefore continuous, then we can take $G = F$ and $U = D$ in Theorem 2.2. When $K = X$, Theorem 2.2 was proved by Gelman [12] using the degree theory for the multivalued map $I - F$.

Proof of Theorem 1.1. Since F is complemented by G, the map $H : \overline{D} \to CV(\mathbb{R}^m \times K)$, given by $H(\lambda, x) = (G(\lambda, x), F(\lambda, x))$, is a multivalued continuous condensing map with compact convex values, $\dim(\lambda, x) = \dim G(\lambda, x) \geq m$ for each $(\lambda, x) \in \overline{D}$ and has a nonzero fixed point index $i(H, D, \mathbb{R}^m \times K)$. Hence, $\dim \operatorname{Fix}(H, D) \geq m$ by Theorem 2.2. Since $\operatorname{Fix}(H, D) \subset S(F, D)$, we get that dim $S(F, D) \geq m$ by the monotonicity property of dimension. Moreover, since $\operatorname{Fix}(H, D)$ is a compact metric space, as in Theorem 2.2, we get a nondegenerate connected component of $S(F, D)$. □

Proposition 2.4. *Let $F : \overline{D} \subset \mathbb{R}^m \times K \to K$ be continuous and condensing, $D_0 = \overline{D} \cap (0 \times K)$ and $F_0(x) = F(0, x) : D_0 \subset K \to K$ be such that $i(F_0, D_0, K) \neq 0$. Then F is complemented by the continuous compact multivalued map $G(\lambda, x) = \overline{B}(0, r) \subset \mathbb{R}^m$ for all $(\lambda, x) \in \overline{D}$ and some fixed $r > 0$.*

Proof. Define $H_r : \overline{D} \to CV(\mathbb{R}^m \times K)$ by $H_r(\lambda, x) = \overline{B}(0, r) \times F(\lambda, x)$. We claim that H_r has no fixed points in ∂D for some $r > 0$. If not, then there would exist $(\lambda_k, x_k) \in \partial D$ such that $(\lambda_k, x_k) \in H_k(\lambda_k, x_k) = \overline{B}(0, 1/k) \times F(\lambda_k, x_k)$ for each positive integer k. Hence $\lambda_k \in \overline{B}(0, r)$ and $x_k \in F(\lambda_k, x_k)$. Since F is condensing, we have that $x_k \to x_0 \in \partial D_0$ and therefore $(\lambda_k, x_k) \to (0, x_0)$ and $x_0 = F(0, x_0)$ in contradiction to our assumption on F. Thus, for some $r > 0$, $(\lambda, x) \notin H_r(\lambda, x)$ for all $(\lambda, x) \in \partial D$. Since $h(\lambda, x) = (0, F(\lambda, x))$ is a continuous selection of $H_r(\lambda, x)$, we have that the fixed point index

$$i(H_r, D, B(0, r) \times K) = i(h, D, B(0, r) \times K).$$

Since $h : D \subset \mathbb{R}^m \times K \to K$ and K is a retract of $\mathbb{R}^m \times K$, the permanence property of the index implies that $i(h, D, B(0, r) \times K) = i(h, D_0, 0 \times K)$. Hence, $i(H_r, D, B(0, r) \times K) = i(F_0, D_0, K) \neq 0$, proving that F is complemented by the (constant) compact multivalued map $G(\lambda, x) = \overline{B}_m(0, r) \subset \mathbb{R}^m$ for some $r > 0$. □

Proof of Corollary 1.2. By Proposition 2.4, F is complemented by a continuous compact multivalued map $G(\lambda, x) = \overline{B}_m(0, r) \subset \mathbb{R}^m$ for all $(\lambda, x) \in \overline{D}$ and some $r > 0$. Hence, $H : \overline{D} \to \mathbb{R}^m \times K$ given by $H(\lambda, x) = \overline{B}_m(0, r) \times F(\lambda, x)$ is a continuous multivalued condensing map with compact convex values and $\dim H(\lambda, x) = \dim \overline{B}_m(0, r) \geq m$ for each $(\lambda, x) \in \overline{D}$. By Theorem 1.1, dim $S(F, D) \geq m$ and the other conclusion also holds. □

We need the following result to study the unboundedness of the solution set. If X is a Banach space, define a norm of the Banach space $X_1 = R \times X$ by

$\|(t,x)\| = (|t|^2 + \|x\|^2)^{1/2}$. Let $\overline{B}_1(0,r)$ be the closed ball of radius r in X_1 and $S_r = \partial B_1(0,r)$ be its boundary.

Lemma 2.5. *Let K be a closed convex subset of a Banach space X containing zero, $F : K_1 = \mathbb{R}^1 \times K \to K$, $\|F(t,x)\| \le r$ for all $(t,x) \in S_r \cap K_1$ and satisfy either one of the following conditions:*

- (a) *F is continuous and condensing on $[0,r] \cap K$, i.e., $\phi(F([0,r] \times Q)) < \phi(Q)$ for all $Q \subset K$ with $\phi(Q) > 0$.*
- (b) *The map $H : K_1 \to K$ given by $H(t,x) = x - F(t,x)$ is A-proper on $\overline{B_1}(0,r) \cap K_1$ with respect to $\Gamma = \{R \times X_n, X_n, P_n\}$ with $P_n(K) \subset K$.*

Then $F(t,x) = x$ has a solution in $S_r \cap K_1$, and in case (a) $\dim S(F, B_r) = \dim\{(t,x) \in B_1(0,r) : F(t,x) = x\} \ge 1$ provided that $\|F(0,x)\| < r$ for $\|x\| = r$.

Proof. (a) Let $\overline{B}$ be the closed ball of radius r in X. Define the map $G : \overline{B} \cap K \subset X \to K$ by $G(x) = F((r^2 - \|x\|^2)^{1/2}, x)$. For $x \in \overline{B}$, $\|(r^2 - \|x\|^2)^{1/2}, x)\| = r$ and therefore G maps $\overline{B} \cap K$ into itself and is ϕ-condensing. Indeed, let $Q \subset \overline{B} \cap K$ with $\phi(Q) > 0$. Then $\phi(G(Q)) \le \phi(F([0,r] \times Q)) < \phi(Q)$. Hence, $G(x) = F((r^2 - \|x\|^2)^{1/2}, x) = x$ for some $x \in \overline{B} \cap K$ by Sadovski's fixed point theorem and therefore $F(t,x) = x$ with $t^2 = r^2 - \|x\|^2$ and $\|(t,x)\| = r$. Moreover, if $\|F(0,x)\| < r$ for $\|x\| = r$ in $\overline{B}$, then the fixed point index of $F_0 = F_{|\overline{B} \cap K} : \overline{B} \cap K \to K$, $i(F_0, B \cap K, K) = 1$ since the homotopy $H(t,x) = x - tF(0,x) \neq 0$ for $\|x\| = r$ and $t \in [0,1]$. Hence, dim $S(F, B_1(0,r)) \ge 1$ by Corollary 1.2.

(b) Define G as in (a). Then $I - G$ is A-proper on $\overline{B} \cap K \subset X$ with respect to $\Gamma = \{X_n, P_n\}$ for X. Indeed, let $x_{n_k} \in \overline{B}(0,r) \cap X_n \cap K$ and $x_{n_k} - P_{n_k} G x_{n_k} \to f$, i.e., $x_{n_k} - P_{n_k} F((r^2 - \|x_{n_k}\|^2)^{1/2}, x_{n_k}) \to f$. Then a subsequence of $\{(r^2 - \|x_{n_k}\|^2)^{1/2}, x_{n_k}\}$ converges to $(r^2 - \|x\|^2)^{1/2}, x)$ with $x - G(x) = x - F((r^2 - \|x\|^2)^{1/2}, x) = f$ by the A-properness of $H(t,x)$. Since $P_n G(\overline{B}(0,r) \cap X_n \cap K) \subset \overline{B}(0,r) \cap X_n \cap K$ and $P_n G : \overline{B}(0,r) \cap X_n \cap K \to \overline{B}(0,r) \cap X_n \cap K$ is compact, by Brouwer's fixed point theorem $P_n G(x_n) = P_n F((r^2 - \|x_n\|^2)^{1/2}, x_n) = x_n$ for some $x_n \in B(0,r) \cap X_n \cap K$ and all large n. Hence, $P_n F(t_n, x_n) = x_n$ with $t_n^2 = r^2 - \|x_n\|^2$ and $\|(t_n, x_n)\| = r$. By the A-properness of H on K_1, a subsequence of $\{(t_n, x_n)\}$ converges to $(t,x) \in K_1$ with $F(t,x) = x$ and $\|(t,x)\| = r$. □

For the space $\mathbb{R}^m \times X$, we use the norm $\|(\lambda, x)\| = \sqrt{\|\lambda\|^2 + \|x\|^2}$.

Theorem 2.6. *Let $m > 0$ be a positive integer, K be a closed unbounded subset of a Banach space X containing zero and $F : \mathbb{R}^m \times K \to K$ be continuous, condensing and quasibounded, i.e.*

$$|F| = \limsup_{\|(\lambda,x)\| \to \infty} \|F(\lambda,x)\| / \|(\lambda,x)\| < 1.$$

Then $S(F, \mathbb{R}^m \times K) = \{(\lambda,x) : F(\lambda,x) = x\}$ is unbounded and, for each r sufficiently large, dim $S(F, B(0,r) \cap (\mathbb{R}^m \times K)) \ge m$ and $S(F, B(0,r) \cap (\mathbb{R}^m \times K)$ contains a nondegenerate connected component. If $K = X$, the same conclusions hold for $S(F - f, \mathbb{R}^m \times X)$ for each $f \in X$. If $m = 0$, then $S(F,K) \neq \emptyset$ and compact. If $m = 0$ and $K = X$, then the cardinality of $S(F - f, X)$ is positive, finite and constant for each f in connected components of $X \setminus (I - F)(\Sigma)$.

Proof. Let $m > 0$ and $\epsilon > 0$ be such that $|F| + \epsilon < 1$ and $r_\epsilon > 0$ be such that

$$\|F(\lambda,x)\| \le (|F| + \epsilon)\|(\lambda,x)\| < \|(\lambda,x)\| \text{ for all } \|(\lambda,x)\| \ge r_\epsilon.$$

Moreover, there is an $r_0 > r_\epsilon$ such that for each $r > r_0$, $H(t,x) = x - tF(0,x) \neq 0$ for all $t \in [0,1]$ and $\|x\| = r$ in K. If not, then there would exist $t_n \to t$ and x_n with $\|x_n\| \to \infty$ such that $H(t_n, x_n) = 0$ for all n. Hence, $\|x_n\| \leq \|F(0,x_n)\| \leq (|F| + \epsilon)\|x_n\| < \|x_n\|$, which is a contradiction. Thus such an r_0 exists and by the homotopy theorem for condensing maps, $i(F(0,.), B(0,r) \cap K, K) = 1$ for each $r > r_0$. Hence, $\dim S(F, B(0,r) \cap (\mathbb{R}^m \times K)) \geq m$ and $S(F, B(0,r) \cap (\mathbb{R}^m \times K))$ contains a nondegenerate connected component by Corollary 1.2.

Next, let us prove that $S(F, \mathbb{R}^m \times K)$ is unbounded. For a fixed $e \in \mathbb{R}^m$ with $\|e\| = 1$, define $F_e : \mathbb{R}^1 \times K \to K$ by $F_e(t,x) = F(te,x)$. Note that if $(t,x) \in \partial B(0,r) \subset \mathbb{R}^1 \times K$, then $(te,x) \in \partial B(0,r) \subset \mathbb{R}^m \times K$. Then for each $r > r_0$, $\|F_e(t,x)\| \leq r$ for $(t,x) \in \overline{B}(0,r) \subset \mathbb{R}^1 \times K$ and by Lemma 2.5, $F_e(t,x) = x$ for some $(t,x) \in \partial B(0,r) \subset \mathbb{R}^1 \times K$. Hence, $x = F(te,x)$ with $(te,x) \in \partial B(0,r) \subset \mathbb{R}^m \times K$ and therefore $S(F, \mathbb{R}^m \times K)$ is unbounded. If $m = 0$, then the above proof shows that $S(F,K) \neq \emptyset$. Moreover, if $x \in S(F,K)$ is such that $\|x\| \geq r$, then as above

$$\|x\| \leq \|F(x)\| < \|x\|$$

which is a contradiction. Hence, $S(F,K)$ is bounded and therefore compact by the properness of $I - F$ on bounded closed subsets. If $K = X$, then $F_f = F - f$ satisfies all conditions of F for each $f \in X$ and the conclusions of the theorem hold for F_f. If $m = 0$ and $K = X$, then $I - F$ is locally proper and satisfies condition (+), i.e., $\{x_n\}$ is bounded whenever $x_n - Fx_n \to y$ in X. Hence, the cardinality of $(I-F)^{-1}(f)$ is positive, finite and constant for each $f \in X \setminus (I-F)(\Sigma)$ by [31, Theorem 3.5]. □

Proof of Theorem 1.3. Let $X_0 = \ker L$, $m = \dim(X_0)$ if X_0 is finite dimensional and $m < \dim(X_0)$ be any positive integer otherwise. Let U_m be an m-dimensional subspace of X_0. Since $N_f x = Nx - f$ has the same properties as N, we may assume $f = 0$ and study the equation $Lx - Nx = 0$. Define a map $F : U_m \times Y \to Y$ by $F(u,y) = N(u + L^+ y)$ with $\|(u,y)\| = \max\{\|u\|, \|y\|\}$. We claim that F is $k\|L^+\|$-set contractive. Let $Q \subset U_m \times Y$ be bounded. Then, without loss of generality, we can assume that $Q = Q_1 \times Q_2$ with both $Q_1 \subset U_m$ and $Q_2 \subset Y$ bounded. Moreover, $Q_3 = \{u + L^+(y) : (u,y) \in Q\}$ is also bounded. Hence

$$\begin{aligned}
\phi(F(Q)) = \phi(N(Q_3)) &\leq k\phi(Q_3) \leq k\phi(Q_1 + L^+(Q_2)) \\
&\leq k(\phi(Q_1) + \phi(L^+(Q_2))) = k\phi(L^+(Q_2)) \\
&\leq k\|L^+\|\,\phi(Q_2) = k\|L^+\|\,\phi(Q)
\end{aligned}$$

since Q_1 is compact. Then $(u,y) \in U_m \times Y$ is a solution of $N(u + L^+ y) = y$ if and only if $x = u + L^+ y \in U_m \oplus L^+(Y)$ is a solution of $Lx - Nx = 0$. Since $X = X_0 \oplus L^+(Y)$, the map $A : U_m \times Y \to U_m \oplus L^+(Y)$ defined by $A(u,y) = u + L^+ y$ is a continuous bijection. Its surjectivity is clear. It is injective since $(u_1,y_1) \neq (u_2,y_2)$ implies that $A(u_1,y_1) \neq A(u_2,y_2)$ by the injectivity of $L^+ : Y \to L^+(Y)$. Next, we claim that there is an $r > 0$ such that $H(t,(0,y)) = (0,y) - tF(0,y) \neq 0$ for all $t \in [0,1]$ and $(0,y) \in \{0\} \times \partial B_Y(0,r)$. If not, then there would exist $t_k \in [0,1]$, $y_k \in Y$ such that $\|y_k\| \to \infty$ and $H(t_k,(0,y_k)) = 0$ for each k. This contradicts condition (+) for $I - tF(0,.) = I - tNL^+$. Hence, the homotopy $H : [0,1] \times (0 \times Y) \to Y$ given by $H(t,(0,y)) = y - tF(0,y)$ is not zero for $t \in [0,1]$ and $y \in \partial B_Y(0,r)$ for some $r > 0$. Thus, the degree $\deg(I - F(0,.), 0 \times B_Y(0,r), 0) = 1$ and $\dim S(F, U_m \times Y) \geq \dim S(F, B_m(0,r) \times Y) \geq m$ by Corollary 1.2.

Since $S(F, B_m(0,r)\times Y)$ is compact, the map $A(u,y) = u + L^+y$ is a homeomorphism from $S(F, B_m(0,r)\times Y)$ onto its range in $S(0)$. There is a nondegenerate connected component C_m of $S(F, B_m(0,r)\times Y)$ for each m and therefore $A(C_m)$ is a connected component of $S(0)$. Moreover, by the monotonicity of the dimension

$$\dim S(0) \geq \dim S(F, B_m(0,r)\times Y) \geq m.$$

Since m was arbitrary, we have

$$\dim S(0) \geq \dim \ker(L).$$

Next, let N have a sublinear growth with $a\|L^+\| < 1$ and show that $S(0)$ is unbounded. Observe that $x \in S(0)$ if and only if $x = u + L^+y$ for a solution (u,y) of $y - N(u+L^+y) = 0$, where $F(u,y) = N(u+L^+y)$ is $k\|L^+\|$-set contractive as shown above. Suppose that $S(0)$ is bounded. Since N is bounded, the set $NS(0)$ is also bounded and so $\|Nx\| \leq C$ for all $x \in S(0)$ and some $C > 0$. For a fixed $e \in X_0$ with $\|e\| < (1 - a\|L^+\|)/\|a\|$, define $F_e : R\times Y \to Y$ by $F_e(t,y) = N(te + L^+y)$. Let $r \geq b/(1 - a\|L^+\| - a\|e\|)$. Then for $(t,y) \in \partial B(0,r) \subset \mathbb{R}^1 \times Y$, we get that $|t|, \|y\| \leq r$ and

$$\|N(te + L^+y)\| \leq a|t|\,\|e\| + a\|L^+\|y\| + b \leq ar\|e\| + ar\|L^+\| + b \leq r.$$

Hence, $F_e(t,y) = y$ for some $(t,y) \in \partial B(0,r)$ and therefore $x = te + L^+y \in S(0)$. Let $t_n \to \infty$ as $n \to \infty$ and note that $r_n = t_n\|e\|\,| \geq b/(1 - a\|L^+\| - a\|e\|)$ for large n. Then, again by Lemma 2.5, there is (t_n, y_n) in the sphere $S_{r_n} \subset R\times Y$ such that $y_n - F(t_n e, y_n) = 0$ and therefore $x_n = t_n e + L^+y_n \in S(0)$. Hence,

$$\|y_n\| = \|N(t_n e + L^+y_n)\| \leq C \text{ for all } n.$$

Then

$$\|t_n e\| = |t_n|\,\|e\| \leq \|x_n\| + \|L^+\|\,\|y_n\| \leq C_1 \text{ for all } n$$

for some constant $C_1 > 0$. This contradicts the fact that $|t_n| \to \infty$ as $n \to \infty$. Thus $S(0)$ is unbounded. If L is a homeomorphism, then $S(f) \neq \emptyset$ by the above proof, bounded and compact by the properness of $I - F$ on bounded closed subsets. The finite solvability on connected components follows from [31, Theorem 3.5]. □

In case of a k-Lipschitzian map N, we can say more. Recall that a topological space V is an *absolute extensor* for paracompact (respectively, normal) spaces if for each paracompact (respectively, normal) topological space U, each closed subset A of U and each continuous function $\psi : A \to V$, there exists a continuous function $\phi : U \to V$ such that $\phi_{|A} = \psi$. Note that an absolute extensor for paracompact (respectively, normal) spaces is an absolute retract and is arcwise connected.

Theorem 2.7. *Let $L : X \to Y$ be a not injective continuous linear surjection with a continuous linear right inverse L^+ and $N : X \to Y$ be a k-Lipschitzian map with $k\|L^+\| < 1$. Then $S(f)$ is unbounded and dim $S(f) \geq \dim\ker L$. Moreover, $S(f)$ is a nonempty absolute extensor for paracompact spaces.*

Proof. Since a k-Lipschitzian map is a k-ϕ-contraction, the dimension assertion follows from Theorem 1.3. The absolute extensor property of $S(f)$ was proved in Ricceri [45]. □

In the case of compact nonlinearities, we do not need the linearity of a continuous right inverse of L. As mentioned before, if $L : X \to Y$ is a linear continuous surjection, then by Michael's result [28], there is a continuous map $K : Y \to X$ such that $LK(y) = y$ and $\|K(y)\| \le k\|y\|$ for all $y \in Y$ for any $k > c$ and $K(ty) = tK(y)$ for all t, where

$$c = \sup\{\inf\{\|x\| : x \in L^{-1}(y)\} : \ y \in Y, \|y\| \le 1\}.$$

We say that a continuous map $N : X \to Y$ is L-compact if $\overline{N(B \cap L^{-1}(A))}$ is compact for each bounded subsets $B \subset X$ and $A \subset Y$. The dimension part of the next result is an extension of a theorem by Gelman [13], who assumed that nonlinearities have a linear growth and used different arguments. This result also extends a result of Ricceri [43, 44] for a compact map N with bounded range proven by a completely different method based on a deep result by Sain-Raymond [46] on fixed points of convex-valued multifunctions that was conjectured by Ricceri. Existence of a continuous right inverse of L is not required in [43, 44].

Theorem 2.8. *Let $L : X \to Y$ be a not injective continuous linear surjection, and $N : X \to Y$ be L-compact such that $I - tNK : Y \to Y$ satisfies condition (+), $t \in [0, 1]$. Then $L - N : X \to Y$ is surjective and, for each $f \in Y$,*

$$\dim S(f) \ge \dim \ker(L).$$

Moreover, $S(f)$ contains a nondegenerate connected component and $S(f)$ is unbounded if $\|Nx\| \le a\|x\| + b$ for all $x \in X$ with $a\|L^+\| < 1$. If L is a homeomorphism, then $S(f) \neq \emptyset$ compact set for each $f \in Y$ and the cardinal number of $S(f)$ is constant and finite on each connected component of $Y \setminus (L - N)(\Sigma)$.

Proof. Let U_m be an m-dimensional subspace of $\ker(L)$. Define the map $F : U_m \times Y \to Y$ by $F(u, y) = N(u + K(y))$. We shall prove that F is a compact map. Let $Q \subset U_m \times Y$ be bounded. Then, without loss of generality, we can assume that $Q = Q_1 \times Q_2$ with both $Q_1 \subset U_m$ and $Q_2 \subset Y$ bounded. Moreover, $Q_3 = \{u + K(y) : (x, y) \in Q\}$ is also bounded since $\|K(y)\| \le k\|y\|$ and $Q_3 \subset L^{-1}(Q_2)$. Hence $F(Q) = N(Q_3)$ is compact by the L-compactness of N and so F is a compact map. Continuing as in Theorem 1.3, we get the conclusions. □

Finally, we conclude this section by proving Theorem 1.4 for locally ϕ-contractive nonlinearities which cannot be globally ϕ-contractive.

Proof of Theorem 1.4. If the $\ker(L)$ is finite dimensional, let$m = \dim \ker(L)$. If $\ker(L)$ is infinite dimensional, let $m = \dim U_m$ for a finite dimensional subspace U_m of $U = \ker(L)$. Let $F : U_m \times Y$ be given by $F(u, y) = N(u + L^+y)$. For each $f \in Y$ such that $\|f\| < r - \|L^+\|k(r)$ and $\|y\| \le r$ we have that

$$\|NL^+y + f\| \le \|L^+\|k(r) + \|f\| < r.$$

Thus, $NL^+ + f : \bar{B}_Y(0, r) \to B_Y(0, r)$. Let $D = \bar{B}(0, r) \subset X$ and $Q \subset \{(u, y) : \|(u, y)\| \le r\} \subset U_m \times Y$ be bounded, where $\|(u, y)\| = \max\{\|u\|, \|y\|\}$. Then $Q = Q_1 \times Q_2$ with $Q_1 \subset B_{U_m}(0, r) = \{u \in U_m : \|u\| \le r\}$ and $Q_2 \subset B_Y(0, r) = \{y \in Y : \|y\| \le r\}$. As in the proof of Theorem 1.3, we see that $F(u, y) = N(u + L^+y) + f$ is $k(r)\|L^+\|$ - set-contractive on Q with $k(r)\|L^+\| < 1$. Now, $H(t, (0, y)) = (0, y) - tF(0, y) = (0, y) - tNL^+y - tf \neq 0$ for all $t \in [0, 1]$ and $(0, y) \in \{0\} \times \partial B_Y(0, r)$ since $NL^+ + f : \bar{B}_Y(0, r) \to B_Y(0, r)$. Continuing as in Theorem 1.3, we get the conclusion. □

Corollary 2.9. *Let $L : X \to Y$ be a non injective continuous linear surjection, $L^+ : Y \to X$ be a continuous linear right inverse of L with $\|L^+\| \le 1$ and $N : X \to Y$ be locally Lipschitzian, i.e, for some $r > 0$, there is a $k(r) > 0$ such that*

$$\|Nx - Ny\| \le k(r)\|x - y\| \quad \text{(for all } \|x\|, \|y\| \le r\text{)} \tag{2.1}$$

with $k(r)\|L^+\| < \min\{1, r\}$ and $N(0) = 0$. Then (1.2) is solvable for each $f \in Y$ satisfying

$$\|f\| < r - \|L^+\|k(r) \tag{2.2}$$

and $\dim(S(f) \cap \bar{B}(0, r)) \ge \dim \ker(L)$.

Proof. Since N is defined on the whole space X and N is $k(r)$-Lipschitzian on $\bar{B}(0, r)$, it follows that N is k(r)-set contractive on $\bar{B}(0, r)$. Since $N(0) = 0$, we get that $\|Nx\| \le k(r)\|x\|$ for each $\|x\| \le r$. Then the result follows from Theorem 1.4. □

3. Dimension results for semilinear equations involving A-proper maps

The continuation theorem of Leray-Schauder on $[0, 1]$ has been extended to the whole line $\mathbb{R}$ by Rabinowitz [42] and to $\mathbb{R}^m$, $m > 1$, by Fitzpatrick-Masabo-Pejsashowitz [10]. Theorem 1.5 extends the continuation theorem to infinite dimensional parameter spaces. Let $L : X \to Y$ be a linear continuous surjection, $X_0 = \text{KerL}$ with $\dim X_0 = \infty$ and $X = X_0 \oplus \tilde{X}$ for some closed subspace $\tilde{X}$ of X. Take an increasing sequence of finite dimensional subspaces of X_0: $U_1 \subset U_2 \subset \dots \subset U_m \subset \dots$. whose union is dense in X_0. Then $L : U_m \oplus \tilde{X} \to Y$ is a surjective Fredholm map of index equal to $\dim U_m$. Let $P_m : U_m \oplus \tilde{X} \to U_m$ be the projection onto U_m.

Proof of Theorem 1.5. Let $f \in Y$ be fixed and let $U_1 \subset U_2 \subset \dots \subset U_m \subset \dots$ be a sequence of finite dimensional subspaces of X_0 whose union is dense in X_0. Then the restriction $L : U_m \oplus \tilde{X} :\to Y$ is a Fredholm map of index dim U_m. Moreover, the restriction $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to the scheme $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ for $\{U_m \oplus \tilde{X}, Y\}$. The degree assumption implies that $L - N : U_m \oplus \tilde{X} \to Y$ is complemented by the projection P_m of $U_m \oplus \tilde{X}$ onto U_m in the sense of [10]. Since P_m is proper on $\{x \in U_m \oplus \tilde{X} : Lx - Nx = f\}$, by Fitzpatrick-Massabo-Pejsachowisz [10, Theorem 1.2] applied to the restriction $L - N : U_m \oplus \tilde{X} \to Y$ we get that

$$\dim\{x : Lx - Nx = f, x \in U_m \oplus \tilde{X}\} \ge \dim U_m.$$

Letting $m \to \infty$, this implies the conclusion of the theorem. The existence of a connected subset of the solution set in the theorem follows from [10, theorem 1.2]. □

Proof of Corollary 1.6. Let $X_0 = kerL \neq \{0\}$ and $X = X_0 \oplus \tilde{X}$ for some closed subspace $\tilde{X}$ of X. Set $U_m = X_0$ if $\dim X_0 < \infty$ or let $\{U_m\}$ be an increasing sequence of finite dimensional subspaces of X_0 whose union is dense in X_0. For a given $f \in Y$, let $Bx = Nx - f$. We need to show that $\deg(Q_n(L - B)|X_n, X_n, 0) \neq 0$ for all large n. Consider the restriction of $L - B$ to $\tilde{X}$. Define the homotopy $H : [0, 1] \times \tilde{X} \to Y$

by $H(t,x) = Lx - tBx$. Since L restricted to $\tilde{X}$ is a bijection from $\tilde{X}$ onto Y, it follows that for some $c > 0$

$$\|Lx\| \geq c\|x\|, \ x \in \tilde{X}.$$

Since the quasinorm $|B|$ is sufficiently small, let $\epsilon > 0$ be such that $|B| + \epsilon < c$ and $R = R(\epsilon) > 0$ be such that

$$\|Bx\| \leq (|B| + \epsilon)\|x\| \quad \text{for all } \|x\| \geq R.$$

Then, for $x \in \tilde{X} \setminus B(0,R)$, we get that

$$\|Lx - tBx\| \geq (c - |B| - \epsilon)\|x\|$$

and therefore $\|H(t,x)\| = \|Lx - tBx\| \to \infty$ as $\|x\| \to \infty$ in $\tilde{X}$ independent of t. Hence, arguing by contradiction, we see that there are an $r > R$ and $\gamma > 0$ such that

$$\|H(t,x)\| \geq \gamma \quad \text{for all } t \in [0,1], \ x \in \partial B(0,r) \subset \tilde{X}.$$

Since H is an A-proper homotopy relative to $\Gamma_0 = \{X_n, Y_n, Q_n\}$, this implies that there is an $n_0 \geq 1$ such that

$$Q_n H(t,x) \neq 0 \quad \text{for all } t \in [0,1], x \in \partial B(0,r) \cap X_n, \ n \geq n_0.$$

By the properties of the Brouwer degree we see that $\deg(Q_n(L-B)|X_n, X_n, 0) \neq 0$ for each $n \geq n_0$.

Next, we need to show that the projection $P_m : U_m \oplus \tilde{X} \to U_m$ is proper on $(L-B)^{-1}(0) \cap (U_m \oplus \tilde{X})$. To see this, it suffices to show that if $\{x_n\} \subset U_m \times \tilde{X}$ is such that $y_n = Lx_n - Bx_n \to 0$ and $\{P_m x_n\}$ is bounded, then $\{x_n\}$ is bounded since the A-proper map $L - B$ is proper when restricted to bounded closed subsets. We have that $x_n = x_{0n} + x_{1n}$ with $x_{0n} \in X_m$ and $x_{1n} \in \tilde{X}$ and $c\|x_{1n}\| \leq \|Lx_{1n}\| \leq (|N| + \epsilon)\|x_{1n}\| + \|y_n\|$ for some $\epsilon > 0$ with $|N| + \epsilon < c$ if $\|x_{1n}\| \geq R$. This implies that $\{x_{1n}\}$ is bounded as before. Since $\{x_{0n}\} = \{P_m x_n\}$ is bounded, it follows that $\{x_n\}$ is also bounded. Hence, for each $f \in Y$, the conclusions about the dimension and a connected subset of the corollary follows from Theorem 1.5.

Next, let us show that $S(f)$ is unbounded. This can be done as in the proof of Theorem 1.3. Or, as in that proof, by Lemma 2.5, the equation $N(te + L^+ y) = y$ has a solution $(te, y) \in \partial B(0,r)$ for any unit vector $e \in U_m$. Then $x = te + L^+ y$ is a solution of $Lx - Nx = 0$. Since $t^2 + \|y\|^2 = r^2$, then either $|t| > r/\sqrt{2}$ or $\|y\| > r/\sqrt{2}$. If $\|y\| > r/\sqrt{2}$, then $\|y\| = \|L(x)\| \leq \|L\|\|x\|$, or $\|x\| \geq r/(\sqrt{2}\|L\|)$. If $|t| > r/\sqrt{2}$, then, since $\|L^+\| \leq c$ for some positive c,

$$\|x\| \geq \|t\| - \|L^+ y\| \geq r/\sqrt{2} - c\|y\| \geq r/\sqrt{2} - c\|L\|\|x\|$$

and so

$$\|x\| \geq r/(\sqrt{2}(1 + c\|L\|)).$$

Hence, in ether case $\|x\| \to \infty$ as $r \to \infty$ and so $S(f)$ is unbounded. If L is a homeomorphism, then $S(f) \neq \emptyset$ by the above proof. Moreover, $a\|L^+\| < 1$ implies that $\|x\| \leq \|L^+ f\| + a\|L^+\| \ \|x\| + b$ for each $x \in S(f)$ and therefore $S(f)$ is bounded. The other assertions follow from [33, Theorem 3.5]. □

Theorem 1.5 and Corollary 1.6 apply to many types of nonlinearities N. One class of them is given in Proposition 3.1 below (see also [10]). It involves Fredholm maps $L : D(L) \subset X \to Y$ of index $i(L) = m \geq 0$ and a scheme $\Gamma_m = \{X_n, P_n, Y_n, Q_n\}$ for (X,Y) such that $Q_n Lx = Lx$ for each $x \in X_n$ and any n. If $i(L) = 0$, such a scheme always exist for separable Banach spaces X and Y.

Namely, since $i(L) = 0$, there is a compact linear map from X to Y such that $K = L + C : D(L) \subset X \to Y$ is bijective. Let $\{Y_n\}$ be a sequence of finite dimensional subspaces of Y and $Q_n : Y \to Y_n$ be projections such that $Q_n y \to y$ for each $y \in Y$. Define $X_n = K^{-1}(Y_n) \subset D(L)$. Then $\Gamma = \{X_n, Y_n, Q_n\}$ is a projection scheme for (X, Y) with $Q_n Lx = Lx$ for each $x \in X_n$ and all n. Such a scheme can also be constructed when $i(L) = m > 0$. Let X_0=null space of L and $\tilde{X}$ be its complement so that $X = X_0 \oplus \tilde{X}$. Since $\tilde{Y}$ = the range of L of finite codimension, there is a finite dimensional subspace Y_0 of Y such that $Y = Y_0 \oplus \tilde{Y}$. Let $P : X \to X_0$ and $Q : Y \to Y_0$ be projections onto X_0 and Y_0, respectively. The restriction of L to $D(L) \cap \tilde{X}$ has a bounded inverse L^+ on $\tilde{Y}$ so that $LL^+ y = y$ for each $y \in \tilde{Y}$. Let $\{X_n\}$ be a monotonically increasing sequence of finite dimensional subspaces of X and $P_n : X \to X_n$ be continuous linear projections onto X_n for each n such that $P_n x \to x$ for each $x \in x$, $X_0 \subset X_n$ and $PP_n = P$ for each n. Then $P_n \tilde{X} \subset \tilde{X}$ and $(I - P_n)(X) \to \tilde{X}$ for each n. Define $Q_n = Q + LP_n L^+(I - Q)$. Then $Q_n : Y \to Y_n = Q_n(Y)$ is a continuous projection with $\{Y_n\}$ being an increasing sequence of finite dimensional subspaces of Y with $Y_0 \subset Y_n$, $QQ_n = Q_n Q$, $Q_n(\tilde{Y}) \subset \tilde{Y}$, $(I - Q)(Y) \subset \tilde{Y}$ and $Q_n Lx = LP_n x$ for all $x \in D(L)$ and $\dim X_n - \dim Y_n = m$ for each n. Moreover, $Q_n y \to y$ for each $y \in Y$ if $LP_n x \to Lx$ for each $x \in D(L)$, and, in particular when L is continuous. The required approximation scheme for (X, Y) is $\Gamma_m = \{X_n, P_n, Y_n, Q_n\}$ (cf. [32, 38]).

Let us construct such a scheme for any separable Banach spaces X and Y and a Fredholm map $L : X \to Y$ of index $i(L) = m > 0$. Using the above notation, select a sequence $\{X_n\}$ of increasing finite dimensional subspaces of $\tilde{X}$, as well as a sequence $\{Y_n = L(X_n)\}$ of finite dimensional subspaces of $\tilde{Y}$. Let $\tilde{Q}_n : \tilde{Y} \to Y_n = L(X_n)$ be projections onto Y_n. Define $Q_n : Y \to Y_0 \oplus Y_n$ by $Q_n(y_0 + y_1) = y_0 + \tilde{Q}_n y_1$ for $y_0 \in Y_0$ and $y_1 \in \tilde{Y}$. Then $\Gamma_m = \{X_0 \oplus X_n, Y_0 \oplus Y_n, Q_n\}$ is a projection scheme for (X, Y) with $Q_n Lx = Lx$ for all $x \in X_n$. When L is continuous and surjective, we get a scheme $\Gamma_m = \{X_0 \oplus X_n, Y_n, Q_n\}$ with $Q_n Lx = Lx$ for all $x \in X_n$.

Proposition 3.1 ([31, 32]). *Let $L : X \to Y$ be a not injective linear continuous surjective map, $X_0 = \ker(L)$, $\dim X_0 \leq \infty$, and X_0 have a complement $\tilde{X}$ in X. Let $N : X \to Y$ be a continuous k-ball contractive map with $k\delta < 1$, where $\delta = sup_n \|Q_n\| < \infty$. Then, for each finite dimensional subspace U_m of X_0, $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ with $Q_n Lx_n = Lx_n$ for $x_n \in X_n$.*

Proof. Since $X = X_0 \oplus \tilde{X}$, the restriction $L : \tilde{X} \to Y$ is continuous and bijective, and therefore $\|Lx\| \geq c\|x\|$ for some $c > 0$ and all $x \in \tilde{X}$. As in [31, 32], we can show that for any bounded sequence $\{x_n\} \subset \tilde{X}$, the ball measure of noncompactness $\chi(\{Lx_n\}) \geq c\chi(\{x_n\})$. Let U_m be a finite dimensional subspace of X_0 and note that the restriction $L : U_m \oplus \tilde{X} \to Y$ is Fredholm of index m. Let $u_n + x_n \in U_m \oplus X_n$ be such that $\{u_n + x_n\}$ is bounded and $y_n = L(u_n + x_n) - Q_n N(u_n + x_n) \to y$. Then $\{u_n\}$ is precompact and

$$c\chi(\{u_n + x_n\}) = c\chi(\{x_n\}) \leq \chi(\{L(x_n)\}) \leq \delta\chi(\{N(u_n + x_n)\}) \leq k\delta\chi(\{x_n\}).$$

Hence $\{x_n\}$ is precompact and a subsequence of $\{u_n + x_n\}$ converges to $u + x$ with $L(u + x) - N(u + x) = y$. This proves that the map $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to Γ_m. □

The above example has an interesting feature. It shows that $L-N : U_m \oplus \tilde{X} \to Y$ is A-proper for each finite dimensional subspace U_m of the null space of X_0 of L, but it can not be A-proper from $X = X_0 \oplus \tilde{X} \to Y$ if $\dim X_0 = \infty$. However, this is sufficient to prove that the dimension of the solution set is infinite. To show that $L - N : X \to Y$ is not A-proper, take a bounded sequence $\{u_n + x_n\}$ in $X_0 + \tilde{X}$ with $y_n = L(u_n + x_n) - Q_n N(u_n + x_n) \to y$. Then $c\chi(\{u_n + x_n\}) = c\chi(\{x_n\}) \leq \chi(\{L(x_n)\})$ only if $\{u_n\} \subset X_0$ is compact, which implies that X_0 must be finite dimensional. If $\dim\ker(L)$ is finite, then no surjectivity of L is needed.

4. Semilinear equations with odd nonlinearities

Let X, Y be Banach spaces and look now at odd perturbations of linear maps $L : X \to Y$ with infinite dimensional null space. Here no surjectivity of L is needed. We begin with nonlinear perturbations $N : X \to Y$ of a closed densely defined Fredholm map of positive index $L : D(L) \subset X \to Y$. Then $V = D(L)$ is a Banach space with the graph norm $|x| = \|x\| + \|Lx\|$. The following result with L continuous on X was proved by Rabinowitz [42]. It extends easily to closed densely defined maps.

Theorem 4.1. *Let $L : D(L) \subset X \to Y$ be a closed densely defined Fredholm map of positive index $i(L)$ and $N : X \to Y$ be a compact odd nonlinear map. Then for each closed bounded symmetric neighborhood Ω of 0 in V the solution set $Z = \{x \in \partial\Omega : Lx - Nx = 0\} \neq \emptyset$ and its genus $\gamma(Z) \geq i(L)$. In particular, the dim$(Z) \geq i(L) - 1$.*

Proof. We have that $L : V \to Y$ is continuous and Fredholm of index $i(L)$. Since $\|x\| \leq |x|$, a bounded set in V is also bounded in X and therefore $N : V \to Y$ is also compact. The result follows from the Rabinowitz's theorem [42]. □

Rabinowitz proved his result by constructing finite dimensional odd approximations of N of Schauder type. In [29], we have extended Rabinowitz's result to noncompact perturbations N assuming that $L-N$ is A-proper. Later, Gelman [14] proved the dimension assertion of solutions of $Lx - Nx = 0$ with L a surjective linear map and N an odd compact map on the boundary of the ball $B(0, r)$ using an odd selection theorem of Michael's type. The next result, Theorem 1.7, extends the above results to semilinear equations with infinite dimensional null space of the linear map L that need not be surjective.

Proof of Theorem 1.7. Let $U_1 \subset U_2 \subset \dots \subset U_m \subset \dots$ be a sequence of finite dimensional subspaces of X_0 whose union is dense in X_0, $\dim U_m = m$. Let D be an open, bounded and symmetric relative to 0 subset of X. Then $L - N : \bar{D} \cap (U_m \oplus V) \to Y$ is A-proper with respect to $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ at 0. Hence, by [29, Theorem 2.1], $Z = \{x \in \partial(D \cap (U_m \oplus V)) : Lx - Nx = 0\} \neq \emptyset$, its genus $\gamma(Z) \geq m$ and $\dim Z \geq \gamma(Z) - 1 \geq m - 1$. Letting $m \to \infty$, we get the conclusion. □

In view of Proposition 3.1, we have the following corollary of Theorem 1.7. When $\dim\ker(L)$ is finite, no surjectivity of L is needed.

Corollary 4.2. *Let $L : X \to Y$ be a linear continuous surjective map with $X_0 = kerL$, $\dim X_0 = \infty$, $\tilde{X}$ be a complement of X_0 in X and $N : X \to Y$ be an odd k-ball contractive map with $k < 1$ and $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ such that $Q_n Lx = Lx$*

for $x \in X_n$. Then, for each open, bounded and symmetric relative to 0 subset D of X

$$\dim\{x \in \partial D : Lx - Nx = 0\} = \infty.$$

For densely defined linear maps L we have the following corollary.

Corollary 4.3. *Let $L : D(L) \subset X \to Y$ be a closed linear surjective map with $X_0 = \ker L$, $\dim X_0 = \infty$, $\tilde{X}$ be a complement of X_0 in X and $N : X \to Y$ be an odd k-Lipschitzian map with $k < 1$ and $\Gamma_m = \{U_m \oplus X_n, Y_n, Q_n\}$ such that $Q_n Lx = Lx$ for $x \in X_n$. Then, for each open, bounded and symmetric relative to 0 subset D of V,*

$$\dim\{x \in \partial D : Lx - Nx = 0\} = \infty.$$

Proof. Let $V = D(L)$ be the Banach space with the graph norm. Then $N : V \to Y$ is again k-Lipschitzian and, for each finite dimensional subspace U_m of X_0, the restriction $L - N : U_m \oplus \tilde{X} \to Y$ is A-proper with respect to Γ_m at 0 by Proposition 3.1. Hence, the conclusion follows from Theorem 1.7. □

Remark 4.4. In Theorem 1.7, $L - N : X \to Y$ need not be A-proper (see Proposition 3.1). It remains valid for G-equivariant A-proper maps at 0 for any index theory related to the G-representation on X and having the d-dimension property as discussed in [30, 31], where G is a compact Lie group.

5. Nonlinear singular integral equations

Consider a nonlinear one-dimensional singular integral equation with a Cauchy kernel

$$a(s)x(s) + \frac{b(s)}{\pi i}\int_c^d \frac{x(t)}{t-s}dt + \int_c^d \frac{k(s,t)}{t-s} f(t,x(t))dt = h(s) \dim(c \le s \le d) \quad (5.1)$$

where $a(s)$, $b(s)$, $h(s)$, $k : [c,d] \times [c,d] \to C$ and $f : [c,d] \times R \to C$ are given functions. We will study this equation in the classical Hölder space $H^\alpha([c,d])$ $(0 < \alpha < 1)$, equipped with the usual norm $\|x\|_\alpha = \|x\|_C + [x]_\alpha$, where

$$[x]_\alpha = \sup_{s \neq t} \frac{|x(s) - x(t)|}{|s-t|^\alpha}$$

and $\|x\|_C$ is the sup norm. Write this equation in the operator form in $H^\alpha([c,d])$ as

$$Lx + Nx = h \quad (5.2)$$

where

$$Lx(s) = a(s)x(s) + \frac{b(s)}{\pi i}\int_c^d \frac{x(t)}{t-s}dt$$

and $N = SF$ with

$$Sy(s) = \int_c^d \frac{k(s,t)}{t-s} y(t)dt$$

and $Fx(t) = f(t, x(t))$. Suppose that a(s)+b(s) and a(s)-b(s) do not vanish anywhere on $[c,d]$. It is known that (Muskhlishvili [35]) if the index of L, $i(L) \ge 0$, then $L : H^\alpha([c,d]) \to H^\alpha([c,d])$ is surjective and the dimension of the null space of L is equal to the ind(L). We assume that $S : H^\alpha([c,d]) \to H^\alpha([c,d])$ is linear and continuous. Some sufficient condition for the continuity of S are given in Gusejnov and Mukhtarov [19] with an upper estimate for $\|S\|$ in $H^\alpha([c,d])$. To apply Theorem 1.4 with $D = B(0,R)$, we need a good upper estimate in terms of $f(t,s)$ for the local Lipschitz constant $k(r)$ with

$$\|Fx - Fy\|_\alpha \le k(r)\|x-y\|_\alpha \quad (x, y \in \bar{B}(0,r), r \le R), \tag{5.3}$$

where $k(r)$ denotes the minimal Lipschitz constant for F on the ball $\bar{B}(0,R)$, i.e.

$$k(r) = \sup\{\frac{\|Fx-Fy\|_\alpha}{\|x-y\|_\alpha} : \|x\|_\alpha, \|y\|_\alpha \le r\ ; x \ne y\}.$$

It was shown in [26, 27] that F could satisfy the global Lipschitz condition on $H^\alpha([c,d])$, i.e. $k(r)$ is a constant independent of r, only if the function f is affine in the second variable, i.e. $f(t,u) = a(t)+b(t)u$ with fixed coefficients $a, b \in H^\alpha([c,d])$. For simplicity, we assume that $F(0) = 0$.

Suppose $g(t,u) = \partial f(t,u)/\partial u$ exists and defines a superposition map $Gz(t) = g(t,z(t))$ in $H^\alpha([c,d])$. Since

$$f(t,x(t)) - f(t,y(t)) = [x(y) - y(t)] \int_0^1 g[t,(1-\lambda)x(t) + \lambda y(t)]d\lambda$$

and $H^\alpha([c,d])$ is a normed algebra, we get

$$\|Fx - Fy\|_\alpha \le \|x-y\|_\alpha \|\int_0^1 g[t,(1-\lambda)x(t) + \lambda y(t)]d\lambda\|_\alpha.$$

Thus, $k(r) \le \sup\{\|Gz\|_\alpha\ ;\ \|z\|_\alpha \le r\}$. It was shown in [4] that

$$\sup\{\|Gz\|_\alpha : \|z\|_\alpha \le r\} = \max\{\gamma_C(r), \gamma_\alpha(r)\},$$

where

$$\gamma_C(r) = \sup\{|g(t,u)| : a \le t \le b\, |u| \le r\}$$
$$\gamma_\alpha(r) = \sup\{\frac{|g(t,u)-g(s,v)|}{|t-s|^\alpha};\ a \le t,s \le b;\ |u|,|v| \le r;\ |u-v| \le |t-s|^\alpha\}.$$

Theorem 5.1. *Let the index of $L : H^\alpha \to H^\alpha$ be positive, $\|L^+\| \le 1$, $F(0) = 0$, S, F and G act in H^α and be bounded, and $r > 0$ be such that $k_1(r)\|L^+\| \le \min\{1,r\}$, where*

$$k_1(r) = \|S\| \max\{\gamma_C(r), \gamma_\alpha(r)\}).$$

Then (5.1) *is solvable for each $h \in H^\alpha$ satisfying*

$$\|h\|_\alpha < r - k_1(r)\|L^+\|$$

and the dimension of the solution set is at least $\operatorname{ind}(L)$.

Proof. Since the index of L is positive, it is surjective and has a finite dimensional null space [35]. Hence, it has a continuous right inverse L^+. Since $N = SF$, by the above discussion we have that $\|Nx - Ny\|_\alpha \le k_1(r)\|x-y\|_\alpha$ for each $x, y \in \bar{B}(0,r) \subset H^\alpha$. Since N is defined on all of H^α, it follows that it is $k_1(r)\|L^+\| - \phi$-contractive with $k_1(r)\|L^+\| < 1$. Moreover, we have that N=SF:$\bar{B}(0,r) \to B(0,r)$ since $\|Nx\|_\alpha \le \|S\|\,\|Fu\| \le k_1(r)\|x\|_\alpha < \|x\|_\alpha$. Hence, Theorem 5.1 follows from Theorem 1.4. □

Example 5.2. Let $f(u) = u^2 + pu + q$ for some $p > 0$ and q. Then $\gamma_C(r) = 2r + p$ and $\gamma_\alpha = 2r$ and therefore

$$k(r) \le \max\{\gamma_C(r), \gamma_\alpha(r)\} = 2r + p.$$

Then $k_1(r) = \|S\|(2r+p)$ and we need that $\|L^+\|k_1(r) = \|L^+\|\ \|S\|(2r+p) < \min\{1,r\}$. Since $\|L^+\|$ and $\|S\|$ do not depend on r, the above inequality holds for suitably chosen r and p depending on the sizes of $\|L^+\|$ and $\|S\|$. Observe that $k(r) \to \infty$ as $r \to \infty$ and F is not globally Lipschitzian. Note that if we would work in L_p, then $F(L_p) \subset L_p$ is known to imply that $|f(t,u)| \leq a(t) + b|u|$ for some $a \in L_p$ and $b \geq 0$. Hence, in this case we have to restrict ourselves to sublinear nonlinearities, unlike in the Hölder space setting.

By the above remarks, using a local Lipschitz condition allow us to study superlinear nonlinearities. Actually, it was proven in [4], that if the derivative $f'(u)$ of $f(u)$ satisfies the local Lipschitz condition

$$|f'(u) - f'(v)| \leq k_1(r)|u-v|\ (|u|,|v| \leq r),$$

then

$$k_1(r) \leq \frac{2k(2r)+1}{r}$$

with $k(r)$ being the local Lipschitz constant for $F(u)$ in the Hölder space. So, if $k(r)$ can be chosen independent of r, then $k_1(r) \to 0$ as $r \to \infty$, showing that $f'(u)$ is actually a constant, which means that $f(u)$ must be an affine function. So, if $f(u)$ is not affine, then $k(r)$ must depend on r and $f(u)$ must have a superlinear growth for large values of r since

$$\liminf_{r\to\infty} \frac{k(r)}{r} > 0.$$

Let us now make some more remarks about the Nemitskii map. If F is induced by an autonomous f, i.e. Fx=f(x(t)), then it is known that $F : H^\alpha([c,d]) \to H^\alpha([c,d])$ and is bounded if and only if $f \in Lip_{\rm loc}(R)$ and F is locally Lipschitz if and only if $f \in Lip^1_{\rm loc}(R)$ [8, 17]. In the non autonomous case, F(x)=f(t,x(t)) maps $H^\alpha([c,d])$ into itself and is bounded if and only if (cf. [14]) for each $r > 0$ there is a constant $M(r) > 0$ such that

$$|f(t,u) - f(s,u)| \leq M(r)|t-s|^\alpha \quad \text{for } t,s \in [c,d], |u| \leq R. \tag{5.4}$$

Moreover, F is locally Lipschitz in $H^\alpha([c,d])$ if and only if (cf. [2, 16]) for each $r > 0$ there is a constant $M(r) > 0$ such that

$$|f(t,u) - f(s,v)| \leq M(r)(|t-s|^\alpha + |u-v|/r \quad \text{for } t,s \in [c,d], |u|,|v| \leq r. \tag{5.5}$$

and f'_u satisfies this condition too. Clearly, condition (5.5) implies condition (5.4) and $f \in C([c,d],R)$. Moreover, if f satisfies (5.4) and $f'_u \in C([c,d],R)$, then f satisfies (5.5) too.

Remark 5.3. The study of (5.1) with a superlinear nonlinearity f has to be done in a Hölder space. Since these spaces are nonseparable and therefore have no approximation schemes, (5.1) can not be studied using A-proper mapping theory. The condition $\|L^+\| \leq 1$ in Theorem 5.1 is satisfied for any L in a reformulated equation when N is replaced by λN in (5.2) with sufficiently small λ (see the proof of Theorem 8.1 below).

6. Semilinear ODE systems on the half-line

Let $|\cdot|$ be the norm in $\mathbb{R}^M$ induced by a given inner product $(\cdot,\cdot)$ in $\mathbb{R}^M$. Denote by $|\cdot|_p$ the norm of $L_p = L_p((0,\infty),\mathbb{R}^M)$, $1 \leq p \leq \infty$. Then the norm on

$W_p^1 = W_p^1((0,\infty), \mathbb{R}^M)$ with $p < \infty$ is

$$\|u\|_{1,p} = \{|u|_p^p + |\dot{u}|_p^p\}^{1/p}.$$

Let $A : \bar{R}_+ \to L(\mathbb{R}^M)$ be a locally bounded family of linear maps. Recall that the problem

$$Lu = \dot{u} + Au = 0 \tag{6.1}$$

is said to have an exponential dichotomy (on $\mathbb{R}_+$) if there are a projection Π and positive constants K, α and β such that

$$|\Phi(t)\Pi\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for all } t \ge s \ge 0 \tag{6.2}$$

and

$$|\Phi(t)(I-\Pi)\Phi^{-1}(s)| \le Ke^{-\beta(s-t)} \quad \text{for all } s \ge t \ge 0, \tag{6.3}$$

where $\Phi(t)$ denotes the fundamental matrix of the system (6.1) satisfying $\Phi(0) = I$. In this case, we say that L has an exponential dichotomy with projection Π. It is well known that the range of Π (but not Π itself) is uniquely determined, i.e. if L has also an exponential dichotomy with projection Π', then $\text{rge}(\Pi') = \text{rgn}(\Pi)$. The exponential dichotomy of L is a basic assumption that implies its surjectivity needed to study its nonlinear perturbations below. Some sufficient conditions using a Riccati type inequality for exponential dichotomy of L in W_2^1 can be found in [34, Corollary 3.4] as well as in [25]. The surjectivity of L can be also obtained when it has no exponential dichotomy (see [34] and remarks at the end of the section). Detailed study of the surjectivity of L between two suitable function spaces linked to various definitions of dichotomy can be found in Massera and Schaffer [25]. Their study is used in Section 7 for ordinary differential equations in Banach spaces of the form (6.4).

For nonlinear perturbations of (6.1)

$$\dot{u} + Au - F(t,u) = f \tag{6.4}$$

we have the following result.

Theorem 6.1. *Let $A \in L_\infty$ and L have an exponential dichotomy with projection Π and $F : [0,\infty) \times \mathbb{R}^M \to \mathbb{R}^M$ be a Caratheodory function such that for some $a(t) \in L_p$ and $b \ge 0$,*

$$|F(t,x)| \le a(t)|x| + b \quad \text{for all } t \in [0,\infty),\ x \in \mathbb{R}^M \tag{6.5}$$

$$|F(t,x) - F(t,y)| \le k|x-y| \quad \text{for all } t \in (0,\infty),\ x,y \in \mathbb{R}^M \tag{6.6}$$

with k sufficiently small. Then, for $1 \le p \le \infty$ and each $f \in L_p$

$$\dim\{u \in W_p^1 : \dot{u} + Au - F(t,u) = f\} \ge \dim\ker L = \text{rank}(\Pi)$$

and the solution set is an absolute extensor for paracompact spaces.

Proof. As shown in [34], the map $L : X = W_p^1 \to Y = L_p$ defined by $Lu = \dot{u} + Au$ is surjective and $\dim\ker L = \text{rank}\,\Pi$. Since the null space of L is finite dimensional, it has a complement $\tilde{X}$ in X. Thus L has a continuous right inverse denoted by L^+ from Y onto $\tilde{X}$. Set $N(u) = F(t,u)$. Then, $N : X \to Y$ is a k-Lipschitzian by condition (6.6). Since the quasinorm $|N| \le k$, we get that $I - tNL^+$ satisfies condition (+) for $t \in [0,1]$ for k sufficiently small. Hence, the conclusion of the theorem follows from Theorem 1.3. The solution set is an absolute extensor for paracompact spaces by a theorem of Ricceri [45] (see Theorem 2.7). $\square$

Let us give a corollary to Theorem 6.1 when L has constant coefficients. Let $A \in L(\mathbb{R}^M, \mathbb{R}^M)$, $\sigma(A)$ be its spectrum and $\sigma_0(A) = \{\lambda \in \sigma(A) : Re\lambda = 0\} = \sigma(A) \cap iR = \emptyset$. Decompose $\mathbb{R}^M$ as in Amann [1]

$$\mathbb{R}^M = X_0 + X_+ + X_- \text{ such that } AX_0 \subset X_0,\ AX_+ \subset X_+,\ AX_- \subset X_-.$$

X_+ is called the positive (generalized) eigenspace of A. If $\sigma(A) = \emptyset$, then $\mathbb{R}^M = X_+ + X_-$.

Corollary 6.2. *Let $A \in L(\mathbb{R}^M, \mathbb{R}^M)$ with $\sigma_0(A) = \emptyset$ and $F : [0, \infty) \times \mathbb{R}^M \to \mathbb{R}^M$ be a Caratheodory function satisfying* (6.1)-(6.2)*. Then the conclusions of Theorem 6.1 hold.*

Proof. As shown in [41], the bounded linear map $L : X = W_p^1 \to Y = L_p$ defined by $Lu = \dot{u} + Au$ is a Fredholm map if and only if $\sigma_0(A) = \emptyset$. Its null space is $\ker L = \{e^{-tA}\psi : \psi \in X_+\}$, its range is $R(L) = Y$ so that its index $i(L) = \dim X_+$. Here, L has an exponential dichotomy (see the observation below). Since the null space of L is finite dimensional, it has a complement $\tilde{X}$ in X. Thus L has a continuous right inverse given by L^+ from Y onto $\tilde{X}$. Hence, the conclusions follow from Theorem 1.3 or Theorem 6.1. □

Next, we look at the boundary value problem

$$\dot{u} + Au - F(t, u) = 0 \tag{6.7}$$

$$P_1 u(0) = 0 \tag{6.8}$$

associated with the splitting $\mathbb{R}^M = X_1 \oplus X_2$ and $P_1 : \mathbb{R}^M \to X_1$ is a projection. Define the linear map $\Lambda : W_p^1 \to L_p \times X_1$ by $\Lambda u = (Lu, P_1 u(0))$ with $Lu = u'(t) + A(t)u(t)$. The solutions of (6.7)-(6.8) are solutions of $\Lambda u + Nu = (0, 0)$. Hence, the following theorem follows from Corollary 4.2. As remarked before it, no surjectivity of L needed when $\dim \ker(L)$ is finite.

Theorem 6.3. *Let $A \in L_\infty$ and L have an exponential dichotomy with projection Π and $F : [0, \infty) \times \mathbb{R}^M \to \mathbb{R}^M$ be a Caratheodory function that satisfies conditions* (6.5) *and* (6.6) *and F is odd, i.e.,$F(t, -u) = -F(t, u)$ for each $(t, u) \in [0, \infty) \times \mathbb{R}^M$. Assume that rank $\Pi >$dim X_1. Then, for $1 \le p \le \infty$ and each $r > 0$*

$$\dim\{u \in \partial B(0, r) \subset W_p^1 : \Lambda u + (F(t, u(t)), 0) = (0, 0)\ \} \ge \mathit{rank}\, \Pi - \dim X_1 - 1.$$

Proof. The linear map $u \in W_p^1 \to (0, P_1 u(0)) \in L_p \times X_1$ has finite rank and therefore the index of the map $\Lambda : W_p^1 \to L_p \times X_1$ is rank Π-dim $X_1 > 0$ (see [32]). The map $N : W_p^1 \to L_p \times X_1$ given by $Nu = (F(t, u(t)), 0)$ is odd and k-Lipschitzian with k sufficiently small. Hence, the theorem follows from Corollary 4.2. □

In view of Theorem 6.1, it is useful to have easily verifiable conditions that imply an exponential dichotomy. Exponential dichotomy and the characterization of rank Π can be obtained through various available criteria (see [25, 34]). For example, if $\lim_{t\to\infty} A(t) = A^\infty$ exists (which includes the constant case) and the spectrum $\sigma(A^\infty) \cap iR = \emptyset$, then L has an exponential dichotomy and $\operatorname{rank} \Pi$ is the number of eigenvalues of A^∞ with positive real part. More generally, if A is bounded and continuous, then L has an exponential dichotomy and $\operatorname{rank} \Pi$ coincides with the number of eigenvalues of $A(t)$ with positive real part for large enough t, provided that these eigenvalues are bounded away from the imaginary axis and A is "slowly" varying (see [34] for a more detailed discussion). Detailed study of the surjectivity of L (among other things) from a "natural" space $W_A^{1,2} = \{u : \dot{u} \in L_2, Au \in L_2\}$ onto L_2 can be found in [34] without assuming that A is bounded or that L has any dichotomy. It is based on Riccati differential inequalities. If A is bounded and L has exponential dichotomy, then $W_A^{1,2} = W_2^1$ [34]. If A is bounded and $W_A^{1,2} \subset L_2$ (so that $W_A^{1,2} = L_2$), additional conditions are needed for L to have an exponential dichotomy (see [34]). When L has no exponential dichotomy, then there are conditions in [34] that ensure that $W_A^{1,2} \subset L_2$ continuously and therefore $W_A^{1,2}$ is continuously embedded in $W^{1,2}$, (or just require that $W_A^{1,2} \subset L_2$) and to study (6.2) from $W_A^{1,2}$ into L_2, it is enough to have that N is k-Lipschitzian N from W_2^1 to L_2. For other criteria for exponential dichotomy of L we refer to [34] and the references therein.

7. Semilinear ordinary differential equations in Banach spaces

We need the following result about the existence of continuous linear right inverses of surjective linear maps.

Proposition 7.1. *Let X and Y be Banach spaces, $L : D(L) \subset X \to Y$ be a closed surjective linear map and $P : X \to Y$ be linear and continuous. Suppose that the abstract boundary value problem: $Lx = y$ with $Px = 0$ has a unique solution $x \in D(L)$ such that $\|x\| \le k\|y\|$ for all $y \in Y$ and some constant k. Then L has a continuous linear right inverse $L^+ : Y \to X$.*

Proof. For a given $y \in Y$, define $L^+y = x$ where $x \in D(L)$ is the unique solution of the BVP in the theorem. It is clear that L^+ is linear and continuous since $\|L^+y\| \le k\|y\|$ for each $y \in Y$. Moreover, $LL^+ = I$. □

Let E be an (infinite dimensional) Banach, $L(E)$ be the space of all continuous linear maps from E into E with the usual norm and I be a (nondegenerate) compact real interval. Let $X = C^1(I, E)$ and $Y = C(I, E)$ be the Banach spaces of E-valued continuously differential and continuous function with the usual norms $\|\cdot\|_1$ and $\|\cdot\|$, respectively.

Proposition 7.2. *Let $A : I \to L(E)$ be a continuous function and $L : X \to Y$ be the linear map given by $Lu = u' + A(t)u$. Then L has a continuous linear right inverse.*

Proof. For any $f \in Y$, the Cauchy problem $Lu = f$, $u(t_0) = 0$ has a unique solution for a fixed $t_0 \in I$. It is the unique solution of the integral equation

$$u(t) = \int_{t_0}^t f(s)ds - \int_{t_0}^t A(s)u(s)ds.$$

Applying Gronwall's lemma, we obtain

$$\|u(t)\| \le \int_{t_0}^t \|f(s)\|ds \exp \int_{t_0}^t \|A(s)\|ds \le e^C \int_{t_0}^t \|f(s)\|ds$$

since $\int_{t_0}^t \|A(s)\|ds \le C$ for some positive constant C. Hence, $\|u\| \le K\|f\|$ for some K. Since $u'(t) = f(t) - A(t)u(t)$, for some $K_1 > 0$, we have

$$\begin{aligned}\|u'(t)\| &\le \|f\| + \|A(t)\|\,\|u(t)\| \le \|f\| + K_1\|u(t)\| \\ &\le \|f\| + K_1K\|f\| = (1 + KK_1)\|f\|.\end{aligned}$$

Thus

$$\|u\|_1 \le (K + KK_1)\|f\|.$$

Define a right inverse $L^+ f = u$, where U is the unique solution of the above Cauchy problem. It is linear and continuous since $\|L^+ f\|_1 \le (K + KK_1)\|f\|$. □

We have the following extension of [43, Theorem 2], where it is assumed that the nonlinearity depends only on $(t, u(t))$.

Theorem 7.3. *Let E be a Banach space, $A : I \to L(E)$ be a continuous function and $F : I \times E \times E \to E$ be such that*

(i) *For each $y \in E$ fixed, the function $F(.,y) : I \times E \to E$ is uniformly continuous with relatively compact range.*
(ii) *$\|F(t,x,y) - F(t,x,z)\| \le k\|y - z\|$ for all $t \in I$, $x, y, z \in E$ for k sufficiently small.*

Then for each $f \in C(E)$,

$$\begin{aligned} &\dim\{u \in C^1(I,E) : u'(t) + A(t)(u(t)) - F(t, u(t), u'(t)) = f(t) \text{ for all } t \in I\} \\ &\ge \dim E. \end{aligned}$$

The solution set is an absolute extensor for paracompact spaces if $F(t,.)$ is k-Lipschitzian.

Proof. Set $X = C^1(I,E)$, $Y = C(I,E)$ and $Lu = u' + A(.)(u(.))$ for all $u \in X$. It is well known that $L : X \to Y$ is a linear continuous surjection with $\dim \ker(L) = X_0 = \infty$. It has a linear continuous right inverse L^+ by Proposition 7.2. Hence, X_0 has a complement $\tilde{X}$ in X. Define $N : X \to Y$ by $Nu = F(\cdot, u(\cdot), u'(.))$. Let $U(u,v) = F(t,u,v')$. Then, for each fixed $v \in X$, the map $U(\cdot, v) : X \to Y$ is completely continuous by condition (i) and the Ascoli-Arzela theorem. Moreover, for each fixed $u \in X$, the map $U(u,\cdot) : X \to Y$ is a k-Lipschitzian by condition (ii). Hence, the map $Nu = U(u,u)$ is k-ball-contraction (see Webb [49]) with $k\|L^+\| < 1$. Moreover, the quasinorm $|N| < k$ and so $I - tNL^+$ satisfies condition (+) for $t \in [0,1]$ since k is sufficiently small. Hence, the conclusion of the theorem follow from Theorems 1.3 and 2.7. □

Next, we shall look at the surjectivity question of the linear map $Lu = u' + A(t)u$ in various Banach space valued function spaces defined on an interval $J \subset R$ and the existence of its right continuous inverse. It is based on ordinary or exponential dichotomy of L and we refer to [25] for a detailed discussion. Let W denote the space of real valued functions on J with the topology of convergence in the mean L_1 on compact intervals of J. Then W is a Frechet (complete, linear metric) space. Let $L_p = L_p(J,R)$, $1 \le p \le \infty$, denote the usual Banach spaces of real-valued functions with the norm $\|\cdot\|_p$. For other Banach spaces B of real-valued, measurable functions $\phi(t)$, the notation $|\phi|_B$ will be used for the norm of $\phi(t)$ in B. For a Banach space Z, $L(Z)$, $L_p(Z)$, $B(Z)$, …will represent the spaces of measurable vector valued functions $y(t)$ on J with values in Z such that $\phi(t) = \|y(t)\|$ is in W, L_p, B,… With L_p or B, the norm $|\phi|_p$ or $|\phi|_B$ will be abbreviated to $|y|_p$ or $|y|_B$. A Banach space U will be said to be stronger than $L(Z)$ if U is contained in $L(Z)$ and the convergence in U implies the convergence in $L(Z)$. Each one of the following spaces is stronger than $L(Z) : L_p(Z)$, $1 \le p \le \infty$, $C_b(Z)$ - the space of continuous bounded functions on J with the sup norm, $A(Z)$ - the space of continuous bounded almost periodic functions, etc. (see [25]).

If U is a Banach space stronger than $L(Z)$, a U-solution $u(t)$ of $u' + A(t)u = 0$ or $u' + A(t)u = y(t)$ means a solution $u(t) \in U$. The pair (U, V) of Banach spaces is said to be admissible for $A(t)$ if each is stronger than $L(Z)$, and, for every $f(t) \in V$, the differential equation $u' + A(t)u = f(t)$ has a U-solution. Hence, the map $Lu = u' + A(t)u$ maps $D(L) \subset U$ onto V. It is known [18, 23] that the map L is closed and $\dim\ker(L) = \dim(Z)$ and the null space of L is isomorphic to Z. Detailed discussion of various pairs (U, V) of (strongly) admissible spaces for L can be found in Massera-Schaffer [25], Corduneanu [2].

Let $Z_0 = Z_{0D}$ denote the set of all initial values $u(0) \in Z$ of U-solutions $u(t)$ of $Lu = u' + A(t)u = 0$. The space Z_0 may not be closed even if Z is a Hilbert space, nor be complemented in Z if it is closed (cf. Massera-Schaffer [25]). If Z_0 has a complement Z_1 in Z, let P_0 be the projection from Z onto Z_0 that annihilates Z_1. The following lemma gives conditions under which L has a continuous linear right inverse.

Lemma 7.4. *Let (U, V) be admissible for $A(t)$ and Z_0 be complemented by Z_1 in Z. Then $L : D(L) \subset U \to V$ has a continuous linear right inverse from V into U and from V into the Banach space $U_1 = D(L)$ endowed with the graph norm induced by L.*

Proof. By assumption, the linear map $L : D(L) \subset U \to V$ is surjective. Since U and V are stronger than $L(X)$, [25, Theorem 31.D] implies that the graph of L is closed in $U \times V$ and so L is closed and for each $f \in V$ there is a solution $y(t) \in L^{-1}(f)$ such that $\|y\|_U \le K\|f\|_V$ by the Open mapping Theorem (see [20]) with K independent of f. By [25, Theorem 51.E], for each $f(t) \in V$ there is a unique solution $u(t) \in U$ such that $u(0) \in Z_1$ and satisfies $\|u\|_U \le \max\{1, \|P\|\}K'\|f\|_V$, where P is the projection along Z_0 onto Z_1, and $K' = K + K_1$ is independent of f, with the constant K_1 explicitly determined in [23]. Define the linear map L^+ by $L^+ f = u$, where U is this unique solution. Hence, $L^+ : V \to U$ is a continuous right inverse of L.

Next, as above, for each $f(t) \in V$ there is a unique solution $u(t) \in U$ such that $u(0) \in Z_1$. Then there is a one-to-one linear correspondence between $f \in V$ and the solutions u(t) of $u' + A(t)u = 0$ with $u(0) \in Z_1$. The proof of the fact that L is closed from $D(L) \subset U \to V$ (in Hartman [20, Lemma 6.2]) shows that if L_1 is the restriction of L with domain consisting of $u(t) \in D(L)$ such that $u(0) \in Z_1$, then L_1 is closed. Hence, $L_1 : D(L_1) \subset U_1 \subset U \to V$ is a closed linear one-to-one surjection. Therefore, by the Open mapping Theorem [20], there is a constant $K > 0$ such that if $L_1 u = f$, then $\|u\|_{U_1} \le K\|f\|_V$ for each $f \in V$. Define the linear map L^+ by $L^+ f = u$ where $L_1 u = f$. Then $L^+ : V \to U_1$ is a continuous right inverse of L with $\|L^+ f\|_{U_1} \le K\|f\|_V$. □

Theorem 7.5. *Let (U, V) be admissible for $A(t)$ and Z_0 be complemented by Z_1. Let $U_1 = D(L)$ be the Banach space with the graph norm and $F : J \times X \to X$ be a k-Lipschitzian map, i.e., there is a sufficiently small k such that for each $u_1, u_2 \in U$,*

$$\|F(t, u_1(t)) - F(t, u_2(t))\|_V \le k\|u_1(t) - u_2(t)\|_U \tag{7.1}$$

Then, for each $f \in V$,

$$\dim\{u \in U : u' + A(t)u - F(t, u) = f\} \ge \dim\ker(L).$$

and the solution set is an absolute extensor for paracompact spaces.

Proof. Let U_1 be the Banach space $D(L)$ endowed with the graph norm induced by L. Then the map $L : U_1 \to V$ is continuous and surjective and, by Lemma 7.4, it has a continuous linear right inverse $L^+ : V \to U_1$. Set $Nu = F(t, u(t))$. Then the map $N : U \to V$ is a k-Lipschitzian with $k\|L^+\| < 1$ as well as from U_1 into V. Since the quasinorm $|NL^+| < 1$, it follows that $I - tNL^+$ satisfies condition (+) in V. Hence, the proof follows from Theorem 2.7. □

The following corollary is a consequence of Lemma 7.4 for the pair (C_b, V), and extends in different ways a result of Perron [37] for a finite dimensional system of the form $u' + A(t)u = F(t, u)$ where it is assumed the existence of bounded solutions of the linear part.

Corollary 7.6. *Let (C_b, V) be admissible for $A(t)$ and Z_0 be complemented by Z_1. Let U_1=$D(L)$ be the Banach space with the graph norm and $F : J \times X \to X$ be a k-Lipschitzian, i.e., there is a sufficiently small k such that for each $u_1, u_2 \in C_b$,*

$$\|F(t, u_1(t)) - F(t, u_2(t))\|_V \le k\|u_1(t) - u_2(t)\|_{C_b}.$$

Then, for each $f \in V$,

$$\dim\{u \in C_b : u' + A(t)u - F(t, u) = f\} \ge \dim\ker(L).$$

and the solution set is an absolute extensor for paracompact spaces.

As V in this corollary we can take any of the spaces: $L_p(Z)$, $1 \le p \le \infty$, $C_b(Z)$ - the space of continuous bounded functions on J with the sup norm, $A(Z)$ - the space of continuous bounded almost periodic functions, etc. (see [25]). Moreover, for these choices of V, the pair (C_b, V) is admissible if and only if there is a bounded solution for each f in V such that $\|f(t)\| = 1$ for all $t \ge 0$ [25]. Let M be the space of functions $f \in V$ for which $\int_t^{t+1} \|f(s)\|ds$ is bounded for $t \in J$ with the norm $\|f\|_M = \sup_{t\in J} \int_t^{t+1} \|f(s)\|ds$. Let V be either M or L_p, $1 \le p \le \infty$ and $F(t, u)$ be a function defined for $t \in J$, $u \in Z$, $\|u\| < a$ $(0 < a \le \infty)$ such that $F(t, u)$ is a measurable function in t for each $\|u\| < a$, $F(t, 0) \in B$ with $\|F(t, 0)\|_V = \beta$, and, for each $u_1, u_2 \in Z$ with norms less than a

$$\|F(t, u_1) - F(t, u_2)\| \le \gamma(t)\|u_1 - u_2\| \tag{7.2}$$

holds for all $t \ge 0$ and some function $\gamma(t) \in B(R)$. If β and $\gamma = \|\gamma(t)\|_V$ are sufficiently small, then $F(t, u(t)) \in V$ and condition (7.2) holds (see [25]). Similarly, if $V = C_b$, (7.2) holds and $F(t, u)$ is a continuous function with $F(t, 0) \in C_b$ with $\|F(0, t)\| = \beta$ that satisfies (7.2) with $\gamma(t) = \gamma$, a constant, then (7.2) holds if β and γ are sufficiently small (see [25]).

Remark 7.7. Since U and V need not be separable spaces (e.g., $V = L_\infty(Z)$), therefore have no approximation schemes, Corollary 1.6 for A-proper maps cannot be used in Theorem 7.5 and Corollary 7.6.

8. Semilinear elliptic equations on bounded domains

Let $Q \subset \mathbb{R}^n$ have smooth boundary and $F = F(x, t, p, q)$ be a real valued function defined on $\bar{Q} \times R \times \mathbb{R}^n \times \mathbb{R}^{n^2} = \bar{Q} \times \mathbb{R}^m$, where $m = 1 + n + n^2$. Consider the equation

$$\Delta u - \lambda F(x, u(x), Du, D^2u) = h \quad (u \in H^{2,\alpha}(\bar{Q}, R)), h \in H^\alpha(\bar{Q}, R)) \tag{8.1}$$

where Du and D^2u are shorthand notations for the first, respectively second order derivatives of u and $H^{2,\alpha}(\bar{Q}, R)$, $0 < \alpha < 1$, is the Hölder space of real functions defined on $\bar{Q}$ with derivatives up to second order in $H^{\alpha}(\bar{Q}, R)$ equipped with the norm

$$\|u\|_{2,\alpha} = \Sigma_{|k|\le 2}\|D^k u\|_{\alpha}$$

where $k = (k_1, \dots, k_n)$ is a multi-index, $|k| = k_1 + \dots + k_n$ and

$$D^k u = \frac{\partial^{|k|}u}{\partial^{k_1}x_1 \dots \partial^{k_n}x_n}.$$

We will also need the Hölder space $H^{\alpha}(\overline{Q}, \mathbb{R}^m)$ with the norm

$$\|u\|_{\alpha} = \Sigma_{i=1}^m \|u_i\|_{\alpha}\ (u = (u_1, \dots, u_m)).$$

Let I denote a bounded interval in $\mathbb{R}^m$:

$$I = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : a_i < x_i < b_i, i = 1, 2, \dots, m\}$$

with a_i and b_i real numbers, $a_i < b_i$, $i = 1, \dots, m$, and $\bar{I}$ is the closure of I. Let $N_1 u = F(x, u(x))$, $F_s' = (F_{s_1}, \dots, F_{s_m})$ denote the gradient of $F(x, s)$ with respect to the variables $s \in \mathbb{R}^m$.

Theorem 8.1. *Let $F : \overline{Q}\times\mathbb{R}^m \to R$ be a continuous function of class $H^{0,1}(\bar{Q}\times\bar{I}, R)$ for any bounded interval $I \subset \mathbb{R}^m$, be differentiable with respect to the $\mathbb{R}^m$ variable, $F_s \in H^{0,1}(\bar{Q} \times \bar{I}, \mathbb{R}^m)$ for any bounded interval $I \subset \mathbb{R}^m$, and $\lambda > 0$ be sufficiently small, $F(0) = 0$. Then* (8.1) *is solvable for each $h \in H^{\alpha}$ of sufficiently small norm and*

$$\dim\{u \in H^{2,\alpha}(\overline{Q}, R) : \Delta u - \lambda F(x, u(x), Du, D^2u) = h\} = \infty.$$

Proof. Set $X = H^{2,\alpha}(\overline{Q}, R)$ and $Y = H^{\alpha}(\overline{Q}, R)$. Define $L : X \to Y$ by $Lu = \Delta u$. As shown in [43], $\dim\ker(L) = \infty$. By the classical PDE theory (see [15, Theorem 6.14 and page 123]), there is a positive constant C such that for every $f \in Y$ there is a unique solution $u \in X$ of $Lu = f$, $u_{|\partial Q} = 0$ with $\|u\| \le C\|f\|$. Hence, $L : X \to Y$ is surjective and has a continuous linear right inverse L^+, and therefore the null space of L has a complement in X. Since λ is sufficiently small, the equation $\Delta u = \lambda F(x, u(x), Du, D^2u) + h$ can be written as $\lambda_1^{-1}\Delta u = \lambda_2 F(x, u(x), Du, D^2u) + \lambda_1^{-1}h$ with $\lambda = \lambda_1\lambda_2$ such that $\lambda_1^{-1}\|L^+\| < 1$ and $\lambda_1^{-1}L$ has the same properties as L. Let $N : X \to Y$ be a map defined by $Nu = \lambda_2 F(x, u, Du, D^2u)$. Define the Nemitskii map $N_1 : Z = H^{\alpha}(\overline{Q}, \mathbb{R}^m) \to Y$ by $N_1 u = F(x, u(x))$. It was shown in [34] that N_1 maps Z into Y and is locally Lipschitz.

Next, the map $Ju = (u, Du, D^u)$ is an isometry from $H^{2,\alpha}(\overline{Q}, R)$ onto $H^{\alpha}(\overline{Q}, \mathbb{R}^m)$. Since the map $N_1 : H^{\alpha}(\overline{Q}, \mathbb{R}^m) \to Y$ is locally Lipschitz, there is an $r > 0$ such that $\|N_1 u - N_1 v\|_Z \le k(r)\|u - v\|_Y$ for some constant $k(r)$ and all $\|u\|_Z, \|v\|_Z \le r$. Set $N_2 = N_1 J$. Since J is an isometry with $J(0) = 0$, for each $u, v \in \bar{B}(0, r) \subset X$, $Ju, Jv \in \bar{B}(0, r) \subset Z$ and therefore

$$\|N_2 u - N_2 v\|_Y = \|N_1 Ju - N_1 Jv\|_Y \le k(r)\|Ju - Jv\|_Z = k(r)\|u - v\|_Y.$$

Hence, $N_2 : \bar{B}(0, r) \subset X \to Y$ is locally Lipshitzian. Since λ_2 is sufficiently small, we have that $N = \lambda_2 N_2 : \bar{B}(0, r) \subset X \to Y$ is locally Lipschitzian with the Lipschitz constant $\lambda_2 k(r)$. The equation $\Delta u = \lambda F(x, u(x), Du, D^2u) + h$ is equivalent to $\lambda_1 L = Nu + \lambda_1^{-1}h$ and the conclusion follows from Theorem 1.4 since λ_2 is sufficiently small. ☐

Remark 8.2. Since X and Y are not separable spaces, A-proper mapping results like Corollary 1.6 cannot be used in the above proof. Dimension results for nonlocal perturbations of the Laplacian are given in Ricceri [43] and Faraci and Iannizzotto [9].

Next, we shall study (8.1) in Sobolev spaces in which case we can allow a much wider class of nonlinearities. Here, the induced nonlinear map can be globally k-Lipschitzian. More generally, our result requires the k-contractivity only in variables that correspond only to the highest derivatives in the equation. Let $Q \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded set with a smooth boundary and $\mathring{W}_2^2(Q)$ be the Sobolev space of functions that are zero on the boundary of Q with the usual norm.

Theorem 8.3. *Let $F : \overline{Q} \times \mathbb{R}^m \to R$, $m = 1 + n + n^2$, be a continuous function such that*

(1) *There is a sufficiently small constant $k > 0$ such that*

$$|F(x, y, z_1) - F(x, y, z_2)| \leq k|z_1 - z_2| \quad \text{for all } x \in \overline{Q}, y \in \mathbb{R}^{n+1}, z_1, z_2 \in \mathbb{R}^{n^2}$$

(2) *For some $a > 0$ sufficiently small and $b(x) \in L_2(Q)$*

$$|F(x, y)| \leq a|y| + b(x) \quad \text{for all } x \in \overline{Q}, y \in \mathbb{R}^m.$$

Then

$$\dim\{u \in \mathring{W}_2^2(Q) : \Delta u = F(x, u(x), Du, D^2u)\} = \infty.$$

The solution set is an absolute extensor for paracompact spaces if $F(t, .)$ is k-Lipschitzian.

Proof. Set $X = \mathring{W}_2^2(Q)$ and $Y = L_2(Q)$. Define $L : X \to Y$ by $Lu = \Delta u$. As shown in [43], $\dim\ker(L) = \infty$ in the Hölder space $C^{2,\alpha}(\overline{Q}))$. But, a C^2 function that satisfies Lu=0 in the classical sense satisfies also $Lu = 0$ in the generalized sense by the divergence theorem. Hence, $\dim\ker(L) = \infty$ in X. By the classical PDE theory, (see [15, Theorem 8.12]), there is a positive constant C such that for every $f \in Y$ there is a unique solution $u \in X$ of $Lu = f$, $u_{|\partial Q} = 0$ with $\|u\| \leq C\|f\|$. Hence, $L : X \to Y$ is surjective and has a continuous linear right inverse and therefore the null space of L has a complement in X. Let $N : X \to Y$ be a map defined by $Nu = F(x, u, Du, D^2u)$. Define the map $U(\cdot, \cdot)$ by $U(u, v) = F(x, u, Du, D^2v)$. The continuity and boundedness of $N : X \to Y$ and the Rellich compactness embedding theorem imply that for each fixed v, if $\{u_n\} \subset X$ converges weakly to u in X, then $U(u_n, v)$ converges to $U(u, v)$ in Y. Moreover, the map $U(u, \cdot) : X \to Y$ is k-Lipschitzian by condition (1). Hence, the map $Nu = U(u, u)$ is k-ball-contractive (see [49]) and $I - tNL^+$ satisfies condition (+). Thus, the conclusions follow from Theorems 1.3 and 2.7. □

Let us now look at the two dimensional problem with oblique derivative boundary conditions

$$\Delta u - F(x, y, u, u_x, u_y, D^2u) = 0, \quad \text{for all } (x, y) \in Q \tag{8.2}$$

$$a(x, y)\partial u/\partial x - b(x, y)\partial u/\partial y = 0, \quad \text{for all } (x, y) \in \partial Q. \tag{8.3}$$

with $Q \subset \mathbb{R}^2$ a bounded domain with smooth boundary, $a(x, y)$ and $b(x, y)$ are smooth with $a^2 + b^2 = 1$.

Suppose that the following limits exist

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x).$$

Let $b_+ > 0$, $b_- > 0$ or $b_+ < 0, b_- < 0$ and I be the interval connecting the point (a_+, b_+) with the point (a_-, b_-). Let C be the curve $(a(x), b(x))$, x$\in \mathbb{R}^1$, completed by the interval I and considered from (a_-, b_-) in the direction of growing values of x. The rotation r of the vector $(a(x), b(x))$ is the number of rotations of the curve C around the origin in the counterclockwise direction. Assume that $r > 0$.

Theorem 8.4. *Suppose that the above assumptions on Q, a and b hold so that $r > 0$ and that $F : Q \times \mathbb{R}^4 \to R$ is a Caratheodory function such that*

$$|F(x, y, z)| \le d|z| + c(x, y), \quad (x, y) \in Q,\ z \in \mathbb{R}^7 \tag{8.4}$$

$$\begin{gathered} |F(x, y, z, w_1) - F(x, y, z, w_2)| \le k|w_1 - w_2|, \\ (x, y) \in Q, z \in \mathbb{R}^3,\ w_1, w_2 \in \mathbb{R}^4 \end{gathered} \tag{8.5}$$

for some $d > 0$ and k sufficiently small and a function $c(x, y) \in L_1(Q)$. Then the dimension of the solutions of the BVP (8.2)-(8.3) *is at least the index of the associated linear map. The solution set is an absolute extensor for paracompact spaces if $F(x, .)$ is k-Lipschitzian.*

Proof. Set $X = W_2^2(Q)$, $Y = L_2(Q) \times W_2^{1/2}(\partial Q)$ and $L : X \to Y$,

$$Lu = (\Delta u, a\partial u/\partial x - b(x, y)\partial u/\partial y)$$

be the linear map corresponding to BVP (8.2)-(8.3). The index of L is $i(L) = 2r+2$ (see [48]), where r is the number of counterclockwise rotations of the vector (a, b). Then L is surjective with dimension of the null space equals $i(L)$. Define the map $N : X \to Y$ by $Nu = (F(x, y, u, Du, D^2u), 0)$ and $U(u, v) = F(x, y, u, Du, D^2v)$. By the compactness of the embedding of $W_2^2(Q)$ into $L_2(Q)$, for each fixed v, the map $U(., v) : W_2^2(Q) \to L_2(Q)$ is compact. For each u, the map $U(u, .) : W_2^2(Q) \to L_2(Q)$ is k-Lipschitzian by condition (8.5). Hence, $N_1u = U(u, u)$ is k-ball contractive with $k\|L^+\| < 1$ as is $Nu = (N_1u, 0)$ from X to Y. Moreover, $\|Nu\|_Y \le d\|u\|_X + c$ for each $u \in W_2^2(Q)$ and some positive constants d and c. Since d is sufficiently small, $I - tNL^+$ satisfies condition (+) and the conclusions follow from Theorems 1.3 and 2.7. ☐

Remark 8.5. If $F(x, \cdot)$ is odd, i.e., $F(x, -u) = -F(x, u)$ for all x and u, then the solution sets of equations in Theorems 8.3 and 8.4 have infinite, respectively $i(L)$ dimension on the boundary of the ball $B(0, r)$ for each $r > 0$ by Corollary 4.2.

Next, we give more examples of surjective Fredholm maps of positive index defined on bounded and unbounded domains to which our results can apply.

Example 8.6. Let $Q \subset \mathbb{R}^2$ have a C^∞ boundary. Then the map $L : W_2^2(Q) \to L_2(Q) \times W_2^{1/2}(\partial Q)$ given by $Lu = (\Delta u, \partial u/\partial x|_{\partial Q})$ is a surjective Fredholm map of index 2 (see Hörmander [21]). Its null space is $\{u = ay + b\}$.

Example 8.7. Let

$$Lu = (a(x) - 1)u'' + (b(x) - b_1(x))u' + (c(x) - 2)u,$$

where $b_1(x)$ is a smooth function such that $b_1(x) = 2$ for $x \ge 1$ and $b_1(x) = -2$ for $x \le -1$ and a(x), $b(x)$ and c(x) are continuous on $\mathbb{R}$ and such that the map $B : W_p^2(\mathbb{R}^1) \to L_p(\mathbb{R}^1)$ given by $Bu = a(x)u'' + b(x)u' + c(x)$ is continuous and has a sufficiently small norm. Define $L_1u = -u'' - b_1(x)u' - 2u$. It is shown by Rabier [39] that $L_1 : W_p^2(\mathbb{R}^1) \to L_p(\mathbb{R}^1)$ is surjective and $\dim\ker L_1 = 2$. Hence, the map $L = L_1 + B : W_p^2(\mathbb{R}^1) \to L_p(\mathbb{R}^1)$ is surjective and of index 2 since B has a sufficiently small norm (see Jorgen [24, page 94]).

Example 8.8. Let $H^{2,\alpha}(R, \mathbb{R}^n)$ and $H^\alpha(R, \mathbb{R}^n)$ be Hölder spaces and let $L : H^{2,\alpha}(R, \mathbb{R}^n) \to H^\alpha(R, \mathbb{R}^n)$ be defined by

$$Lu = a(x)u'' + b(x)u' + c(x)u$$

where $a(x)$, $b(x)$ and $c(x)$ are smooth $n \times n$ matrices having, respectively, the limits $a^\pm$, $b^\pm$ and $c^\pm$ as $x \to \pm\infty$. Then it was shown in [48] that if

$$T^\pm(\lambda) = -a^\pm \lambda^2 + b^\pm i\lambda + c^\pm$$

are invertible matrices for each $\lambda \in \mathbb{R}$, then L is a Fredholm map of index $k^+ - k^-$, where $k^\pm$ are the number of solutions to the equation

$$\det(a^\pm \lambda^2 - b^\pm \lambda + c^\pm) = 0$$

which have positive real part.

9. Semilinear elliptic equations on $\mathbb{R}^M$ with infinite dimensional null space

In this section we shall study semilinear elliptic equations with infinite dimensional null space defined on $\mathbb{R}^M$.

9.1. **Linearities with a continuous right inverse.** In this subsection we assume that the null space of a linear map is infinite dimensional and has a continuous right inverse. The following result provides some linear elliptic operators with infinite dimensional null space when $M > 1$.

Lemma 9.1 (Rabier-Stuart [40]). *Let $L : W_p^2(\mathbb{R}^M) \to L_p(\mathbb{R}^M)$, $p \in (1, \infty)$, be a second order linear elliptic differential operator with continuous M-periodic coefficients. Then*

(1) $\dim\ker L = 0$ *or* ∞ *and* $\dim\ker L^* = 0$ *or* ∞.
(2) *If $M = 1$ then $\dim\ker L = 0$ and, if in addition the range of L is closed, then L is a homeomorphism.*
(3) *If $M > 1$, $p \geq 2$, L has constant coefficients and is semi-Fredholm (i.e., has a finite dimensional null space and a closed range), then it is a homeomorphism.*

Theorem 9.2. *Let $L : W_p^2(\mathbb{R}^M) \to L_p(\mathbb{R}^M)$, $p \in (1, \infty)$, be a second order linear elliptic differential operator with continuous M-periodic coefficients and have a closed range. Let $F : \mathbb{R}^M \times \mathbb{R}^{s_2} \to \mathbb{R}^1$ be a Caratheodory function such that*

$$|F(x, \xi)| \leq a|\xi| + b(x) \quad \text{for } x \in \mathbb{R}^M, \ \xi \in \mathbb{R}^{s_2}$$

and $F(x, \xi)$ is such that $F(., 0) \in L_\infty(\mathbb{R}^M)$ and for some $k > 0$ sufficiently small,

$$|F(x, \xi) - F(x, \xi')| \leq k \sum_{|\alpha| \leq 2} |\xi_\alpha - \xi'_\alpha|.$$

Let $Nu = F(x, u, Du, D^2u)$.

(a) *If either $M = 1$, or $M > 1$, $p \geq 2$, L has constant coefficients and $\dim\ker L = 0$, then either*
    (i) *$L - N$ is locally injective, in which case $L - N$ is a homeomorphism, or*

(ii) *$L-N$ is not locally injective, in which case $(L-N)^{-1}(f)$ is compact for each $f \in L_p(\mathbb{R}^M)$ and the cardinal number* $\operatorname{card}(L-N)^{-1}(f)$ *is positive, finite on each connected component of the set $L_p(\mathbb{R}^M) \setminus (L-N)(\Sigma)$.*

(b) *If $M > 1$, $p \geq 2$, L has constant coefficients,* $\dim\ker L = \infty$ *and the* $\ker L$ *has a complement also when $p \neq 2$, then*

$$\dim\{u \,| Lu - F(x,u,Du,D^2u) = f\} = \infty$$

for each $f \in L_p(\mathbb{R}^M)$ and the solution set is an absolute extensor for paracompact spaces.

Proof. (a) By Lemma 9.1, L is an isomorphism if $M = 1$ and then parts (i) and (ii) follow from [33, Theorem 3.5].

(b) Since $p \geq 2$, the conjugate $p' \leq 2$ and the adjoint L^* of L also has constant coefficients. Since $p' \leq 2$, the Fourier transform maps $L_{p'}(\mathbb{R}^M)$ to $L_p(\mathbb{R}^M)$ (see [7]). It follows at once that $\ker L^* = 0$ and, since the range of L is closed, L is surjective onto $L_p(\mathbb{R}^M)$. Since $Nu = F(x,u,Du,D^2u)$ is k-Lipschitzian from $W_p^2(\mathbb{R}^M)$ to $L_p(\mathbb{R}^M)$, the result now follows from Theorem 2.7. □

Let us now give some examples of linear elliptic PDE's with infinite dimensional null space. As noted above, Rabier and Stuart [40] proved that a second order linear elliptic partial differential operator with continuous M-periodic coefficients $L : W_p^2(\mathbb{R}^M) \to L_p(\mathbb{R}^M)$, $p \in (1,\infty)$, has either a trivial null space or an infinite dimensional null space. Moreover, it is known that an elliptic partial differential operator with constant coefficients $L = -\sum_{i,k=1}^M A_{ik}\partial_{ik}^2 + \sum_{i=1}^M B_i\partial_i + C$ is semi-Fredholm (i.e., it has a finite dimensional null space and a closed range) from $W_p^2(\mathbb{R}^M)$ to $L_p(\mathbb{R}^M)$, $1 < p < \infty$, if and only if it is an isomorphism (see [40]). This amounts to $C > 0$ if either $M \geq 2$ or $M = 1$ and $B_1 = 0$. If $M = 1$ and $B_1 \neq 0$, then we need to assume $C \neq 0$. Hence, if these conditions on the coefficients are not satisfied, then the null space of L is infinite dimensional in $W_p^2(\mathbb{R}^M)$ by Lemma 9.1, but its range may not be closed as shown below by the Helmholtz operator. Here, $C < 0$.

9.2. **Convolution perturbations of elliptic PDE with nonclosed range.** The closedness of the range of L, and in particular its surjectivity, is a crucial assumption in our results. Here, we consider some linear elliptic maps with infinite dimensional null space, a nonclosed range and yet whose perturbations by nonlinear maps of convolution type have a unique solution. The Helmholtz map $-\Delta - 1$ is not Fredholm. It has an infinite dimensional null space in $W_p^2(\mathbb{R}^2)$ for $p > 4$ since $u(x) = J_0(|x|)$, where J_0 is the Bessel function of the first kind and index 0, and its translates $u(x+a)$ for $a \in \mathbb{R}^2$, are solutions to $-\Delta u - u = 0$ in $\mathbb{R}^2$ (Dautry and Lions, [5, p. 642]). The range of $L = -\Delta - k^2$, $k > 0$, is not closed in $L_2(\mathbb{R}^M)$. Indeed, let $f_n \in \mathbb{R}(L)$ be such that $f_n \to f$ in $L_2(\mathbb{R}^M)$ and that its Fourier transform $\hat{f}_n(\xi)$ vanishes at $|\xi|^2 = k$. These functions can converge in $L_2(\mathbb{R}^M)$ to $\hat{f}(\xi)$ which does not vanish at $|\xi|^2 = k$ (see [48]). Hence, $f \notin R(L)$ and we can not apply our results to perturbations of the Helmholtz operator. There is no solvability theory of such non Fredholm maps nor of their perturbations (see Volpert [48]). Here, we present a special nonlinear perturbation result. Since it has constant coefficients, we can use the Fourier transform to obtain the following unique solvability result for k-Lipschitz convolution perturbations. Consider the general linear differential elliptic operator with constant coefficients $L : W_2^2(\mathbb{R}^M) \to L_2(\mathbb{R}^M)$ with an infinite dimensional null space whose range is not closed and look at its perturbation by a convolution operator

$$Lu - \int_{\mathbb{R}^M} s(x-y)F(y,u(y))dy = 0 \tag{9.1}$$

with $s \in L_2(\mathbb{R}^M)$, and F satisfying the following conditions

$$|F(x,y_1) - F(x,y_2)| \le k|y_1 - y_2| \quad \text{for all } x \in \mathbb{R}^M,\ y_1, y_2 \in \mathbb{R} \tag{9.2}$$

$$|F(x,y)| \le K|y| + h(x) \quad \text{for all } x \in \mathbb{R}^M,\ y \in \mathbb{R} \tag{9.3}$$

for some positive constants k, K and $h(x) \in L_2(\mathbb{R}^M)$. Applying the Fourier transform, we see that $Lu = f$ has a unique solution $u \in L_2(\mathbb{R}^M)$ if and only if $\hat{f}(\xi)/\phi(\xi) \in L_2(\mathbb{R}^M)$, where $\hat{Lu} = \phi(\xi)\hat{u}$ is the Fourier transform of Lu. Assume that for some $C > 0$,

$$|\hat{s}(\xi)/\phi(\xi)| \le C \quad \text{for all } \xi \in \mathbb{R}^M \tag{9.4}$$

Theorem 9.3. *Let conditions* (9.2)-(9.4) *hold and* $kC < 1$. *Then* (9.1) *has a unique solution in* $L_2(\mathbb{R}^M)$.

Proof. For each $v \in L_2(\mathbb{R}^M)$, the equation

$$Lu - \int_{\mathbb{R}^M} s(x-y)F(y,v(y))dy = 0 \tag{9.5}$$

has a unique solution $u \in L_2(\mathbb{R}^M)$. Define the map $N : L_2(\mathbb{R}^M) \to L_2(\mathbb{R}^M)$ by $Nv = u$. N is a k-Lipschitzian. Indeed, for each v_1, v_2 in $L_2(\mathbb{R}^M)$, let u_1 and u_2 be the unique solutions of (9.5). Then

$$\hat{u}_1(\xi) - \hat{u}_2(\xi) = \hat{s}(\xi)/\phi(\xi)(\hat{f}_1(\xi) - \hat{f}_2(\xi)),$$

where $\hat{f}_i(\xi)$ is the Fourier transform of $F(y, v_i(y))$. This implies

$$\begin{aligned} \|u_1 - u_2\| &\le C\|\hat{f}_1 - \hat{f}_2\| \\ &= C\Big(\int_{\mathbb{R}^M} |F(y,v_1(y)) - F(y,v_2(y))|^2 dy\Big)^{1/2} \\ &\le kC\|v_1 - v_2\|. \end{aligned}$$

Hence, the conclusion follows from the contraction principle. □

For $Lu = -\Delta u - k^2 u$ with $k > 0$, Theorem 9.3 was proved in [48]. Theorem 9.3 provides an example of a nonlinear map whose range is contained in the range of a non surjective linear map that even has no closed range. Nonunique solvability of (9.1) can be obtained in a similar way if condition (9.2) is replaced by conditions on F that imply that N is compact in $L_2(\mathbb{R}^M)$ and maps a closed convex set B into itself. Our dimension results do not apply here since the range of L is not closed.

Acknowledgments. The author expresses his warm gratitude to the reviewer for giving valuable suggestions for improving the paper.

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