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EXISTENCE OF SOLUTIONS FOR p-LAPLACIAN-LIKE DIFFERENTIAL EQUATION WITH MULTI-POINT NONLINEAR NEUMANN BOUNDARY CONDITIONS AT RESONANCE

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Abstract. This work concerns the multi-point nonlinear Neumann boundary-value problem involving a p-Laplacian-like operator

$$(\phi(u'))' = f(t,u,u'), \quad t \in (0,1),$$
$$u'(0) = u'(\eta), \quad \phi(u'(1)) = \sum_{i=1}^{m} \alpha_i \phi(u'(\xi_i)),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism with $\phi(\pm\infty) = \pm\infty$ such that

$$0 < \alpha(A) := \limsup_{s\to+\infty} \frac{\phi(A+s)}{\phi(s)} < \infty, \quad \text{for } A > 0.$$

By using an extension of Mawhin's continuation theorem, we establish sufficient conditions for the existence of at least one solution.

1. Introduction

In this article, by using an extension of Mawhin's continuation theorem, we obtain a solution for the p-Laplacian-like differential equation

$$(\phi(u'))' = f(t,u,u'), \quad t \in (0,1), \tag{1.1}$$

associated with the multi-point nonlinear Neumann type boundary conditions

$$u'(0) = u'(\eta), \quad \phi(u'(1)) = \sum_{i=1}^{m} \alpha_i \phi(u'(\xi_i)), \tag{1.2}$$

where $\eta \in (0,1)$, $\alpha_i \in \mathbb{R}$ and ξ_i, $i = 1,2,\dots,m$, are given numbers satisfying $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$; ϕ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory.

We notice that problem (1.1)-(1.2) is always at resonance in the sense that the associated boundary-value problem

$$(\phi(u'))' = 0, \quad t \in (0,1),$$
$$u'(0) = u'(\eta), \quad \phi(u'(1)) = \sum_{i=1}^{m} \alpha_i \phi(u'(\xi_i)),$$

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has the nontrivial solution $u(t) = c_1$ and $u(t) = c_1 + c_2t$, $c_1, c_2 \in \mathbb{R}$ (arbitrary constants) provided that $\sum_{i=1}^{m} \alpha_i \neq 1$ and $\sum_{i=1}^{m} \alpha_i = 1$, corresponding.

The study of multi-point boundary-value problems in the case $\phi = Id$ was initiated by Il'in and Moiseev in [11, 12] and has been studied extensively by many authors with different boundary conditions for both cases non-resonance and resonance [2, 3, 9, 10], [13] - [18].

Recently multi-point boundary-value problem involving p-Laplacian operator or p-Laplacian-like operator $(\phi(u'))'$ have been studied for both cases linear and nonlinear boundary conditions, see for example [5, 6, 7].

In [6, 7], by using topology degree arguments, Garcia-Huidobro, Gupta and Manasevich have studied the p-Laplacian-like differential equations (1.1) in (a, b) with nonlinear boundary conditions

$$u'(0) = 0, \quad \theta(u'(1)) = \sum_{i=1}^{m-2} a_i\theta(u'(\xi_i)) \quad \text{or}$$

$$u(0) = 0, \quad \theta(u'(1)) = \sum_{i=1}^{m-2} a_i\theta(u'(\xi_i))$$

where θ be two odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. In these setting, the set of nontrivial solutions of the associated homeogeneous problem is isomorphic to $\mathbb{R}$.

Ge and Ren [8] gave an extension of Mawhin's continuation theorem in order to solve the abstract equation $Mx = Nx$ when M is a noninvertible nonlinear operator. And then they used this result to study the existence of solutions for the boundary-value problem involving p-Laplacian operator at resonance of the form

$$\begin{gathered}(\phi(u'))' + f(t, u) = 0, \quad t \in (0, 1),\\ u(0) = 0 = G(u(\eta), u(1)),\end{gathered}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$ and $\eta \in (0, 1)$ is constant. By topology approach, the boundary-value problems with one dimension p-Laplacian or p-Laplacian like operator are usually reduced to fixed point problem. To avoid this reduction, the approach of Ge and Ren seems to be very useful. However, in [8], the definition of quasi-linear and M-compact operators in [8] have a little complicated and do not generalize the notations of Fredholm operator of index zero and L-compact operator [4, 17].

Motivated by these works, in this paper, we modify the Ge and Ren's result with some minor changes (e.g. Definition 2.1 and Definition 2.2) and then apply them to handle the problem (1.1)-(1.2). In our best of knowledge, most of the previous papers are only considered the cases $\dim\ker M = 0$ or $\dim\ker M = 1$. Complemented with these, in our setting, we deal with both cases $\dim\ker M = 1$ and $\dim\ker M = 2$ in which the most interesting occurs in the case $\dim\ker M = 2$ due to some technical difficulties like constructing the projector Q. In that case, we have to use some more delicate arguments (e.g. Lemma 2.7).

This article is organized as follows. In section 2, we first modify an extension of Mawhin's continuation Theorem which was introduced by Ge and Ren [8] and then present an abstract equation of the boundary-value problem (1.1)-(1.2) in which we can apply this Theorem. In section 3, we apply the modified Theorem to obtain several existence theorems for the boundary-value problem (1.1)-(1.2) and eventually illustrate some of our results by a simple example containing the p-Laplacian operator.

2. Preliminary results

We begin this section with slight modifications of an extension of Mawhin's continuation theorem which was given in [8].

2.1. An extension of Mawhin's continuation theorem.

Let X and Z be two real Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. We now introduce some definitions.

Definition 2.1. An operator $M : X \cap \operatorname{dom} M \to Z$ is said to be quasi-linear if

- (i) $\ker M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$, where $\operatorname{dom} M$ denotes the domain of the operator M;
- (ii) there exists a subspace Z_2 of Z possessing finite codimension such that $\operatorname{Im} M$ is a closed subset of Z_2 and

$$\dim \ker M = \operatorname{codim} Z_2.$$

It follows from (i) and (ii) that there exist the continuous projectors $P : X \to X$, $Q : Z \to Z$ such that

$$\operatorname{Im} P = \ker M \quad \text{and} \quad \ker Q = Z_2.$$

And hence we have the decompositions $X = \ker M \oplus \ker P$ and $Z = \operatorname{Im} Q \oplus Z_2$.

We now let Ω be an open bounded subset of X and let $N : X \to Z$. Then for each $\lambda \in [0, 1]$, we put

$$\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = \lambda Nx\}.$$

Definition 2.2. The operator N is said to be M-compact in $\overline{\Omega}$ if there exists $R : \overline{\Omega} \times [0, 1] \to \ker P$ being completely continuous such that

- (a) the map $QN : \overline{\Omega} \to Z$ is continuous and $QN(\overline{\Omega})$ is bounded in Z,
- (b) $R(\cdot, 0)$ is the zero operator and $R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$,
- (c) $M[P + R(\cdot, \lambda)] = \lambda(I - Q)N$.

Let $J : \operatorname{Im} Q \to \ker M$ be an isomorphism. We define $S_\lambda : \overline{\Omega} \cap \operatorname{dom} M \to X$, $\lambda \in [0, 1]$ by

$$S_\lambda = P + JQN + R(\cdot, \lambda).$$

Then S_λ is a completely continuous mapping.

Remark 2.3. In the Definition 2.1, if M is a linear operator, then M is a Fredholm operator of index zero by taking $Z_2 = \operatorname{Im} L$. On the other hand, we notice that the assumption on continuity of the operator M (in [8]) is unnecessary.

Moreover, the continuity assumption on N_λ in [8] is not enough to ensure that S_λ is a completely continuous operator. To overcome this situation, we need the assumption (a) in Definition 2.2.

Lemma 2.4. *Let X and Z be Banach spaces, $\Omega \subset X$ an nonempty open and bounded set, M be a quasi-linear operator and N be a M-compact operator in $\overline{\Omega}$. Then the abstract equation $Mx = \lambda Nx$ is equivalent to the fixed point equation $x = S_\lambda x$, for $\lambda \in (0, 1]$ and $x \in \overline{\Omega}$.*

Theorem 2.5. *Let X and Z be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and $\Omega \subset X$ an nonempty open and bounded set. Suppose that $M : X \cap \operatorname{dom} M \to Z$ is a quasi-linear operator and $N : \overline{\Omega} \to Z$ is M-compact. In addition, if the following conditions hold:*

1. *$Mx \neq \lambda Nx$ for every $(x, \lambda) \in (\partial\Omega \cap \operatorname{dom} M) \times (0, 1)$;*
2. *$\deg(JQN; \Omega \cap \ker M, 0) \neq 0$, where $J : \operatorname{Im} Q \to \ker M$ is an isomorphism, and $Q : Z \to Z$ is a projector given as above.*

Then the equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$.

The proof of Lemma 2.4 and Theorem 2.5 are similar to the proof of Ge and Ren [8] with some minor changes. However, for the sake of completeness, we present the proofs here.

Proof of Lemma 2.4. Let $x \in \overline{\Omega}$ and $\lambda \in (0, 1]$ such that $Mx = \lambda Nx$, then we have $Nx \in \operatorname{Im} M \subset Z_2 = \ker Q$, that is, $QNx = 0$. And therefore, we obtain

$$JQNx = 0, \tag{2.1}$$

where $J : \operatorname{Im} Q \to \ker M$ is an isomorphism.

On the other hand, since N is M-compact in $\overline{\Omega}$, we deduce from (b) of Definition 2.2 that

$$R(x, \lambda) = (I - P)x. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$x = Px + R(x, \lambda) = Px + R(x, \lambda) + JQNx.$$

And hence, x is a fixed point of S_λ in $\overline{\Omega}$; that is,

$$x = S_\lambda x, \quad x \in \overline{\Omega}, \lambda \in (0, 1].$$

Conversely, we assume that $x \in \overline{\Omega}$ satisfies

$$x = S_\lambda x, \quad \lambda \in (0, 1]. \tag{2.3}$$

Since N is M-compact on $\overline{\Omega}$, we have $PR(x, \lambda) = 0$. And therefore, we deduce from (2.3) and the Definition of operator S_λ that

$$Px = PS_\lambda x = Px + P(JQNx),$$

which implies $JQNx = 0$ and $QNx = 0$. Hence, from (2.3), we obtain

$$x = Px + R(x, \lambda).$$

From (c) of Definition 2.2, we obtain

$$\begin{aligned} Mx &= M[Px + R(x, \lambda)] \\ &= \lambda(I - Q)Nx \\ &= \lambda Nx - \lambda QNx \\ &= \lambda Nx. \end{aligned}$$

The proof is complete. □

Proof of Theorem 2.5. By Lemma 2.4, the equation $Mx = \lambda Nx$ is equivalent to the fixed point equation

$$x = S_\lambda x,$$

for all $x \in \overline{\Omega}$, $\lambda \in (0, 1]$. Furthermore, it is obviously that S_λ is a completely continuous mapping for $(x, \lambda) \in \overline{\Omega} \times [0, 1]$ due to the M-compactness of N in $\overline{\Omega}$.

To apply the Leray-Schauder degree, we need to prove that S_λ does not possess any fixed point on $\partial\Omega$. In fact, by Lemma 2.4 and condition (1) of Theorem 2.5, we obtain

$$x \neq S_\lambda x, \quad \lambda \in (0, 1), x \in \partial\Omega.$$

Furthermore, without loss of generality, we can assume that $x \neq S_1 x$ for $x \in \partial\Omega$. Since if it is not valid, there exists $x_0 \in \partial\Omega$ such that $x_0 = S_1 x_0$. By Lemma 2.4, we obtain $Mx_0 = Nx_0$ for $x_0 \in \partial\Omega \subset \overline{\Omega}$. So the Theorem 2.5 is verified for this case.

For $\lambda = 0$, assumption (2) of Theorem 2.5 implies $x \neq S_0 x$ for $x \in \partial\Omega$. In fact, if there exists $x \in \partial\Omega$ satisfying $x = S_0 x$, then $x = Px + JQNx \in \ker M$. So we obtain $Px = Px + P(JQNx)$ which implies $JQN = 0$ for $x \in \partial\Omega \cap \ker M$. This contradicts to the condition (2) of Theorem 2.5. Thus, we gain

$$x \neq S_\lambda x, \quad \lambda \in [0, 1], x \in \partial\Omega.$$

By the invariant property of homotopy and condition (2), one has

$$\begin{aligned} \deg(I - S_1, \Omega \cap \operatorname{dom} M, 0) &= \deg(I - S_0, \Omega \cap \operatorname{dom} M, 0) \\ &= \deg(I - P - JQN, \Omega \cap \operatorname{dom} M, 0) \\ &= \deg(I - P - JQN, \Omega \cap \ker M, 0) \\ &= \deg(-JQN, \Omega \cap \ker M, 0) \neq 0. \end{aligned}$$

Hence, S_1 has a fixed point $x_0 \in \Omega$, that is, $Mx_0 = Nx_0$. This completes the proof of Theorem 2.5. □

2.2. **Abstract equation of the boundary-value problem** (1.1)**-**(1.2)**.** To apply the Theorem 2.5, we shall rewrite the boundary-value problem (1.1)-(1.2) as an abstract operator equation in the form of

$$Mu = Nu,$$

where M is a quasi-linear operator and N is a M-compact operator.

Let us introduce the spaces $X = C^1[0, 1]$ with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\},$$

and $Z = L^1[0, 1]$ with its usual norm $\|u\|_1 = \int_0^1 |u(s)|ds$. Let $\mathcal{B}_1 : Z \to \mathbb{R}$ and $\mathcal{B}_2 : Z \to \mathbb{R}$ defined by

$$\mathcal{B}_1(z) = \int_0^\eta z(s)ds, \quad \text{and} \quad \mathcal{B}_2(z) = \int_0^1 z(s)ds - \sum_{i=1}^m \alpha_i \int_0^{\xi_i} z(s)ds. \tag{2.4}$$

Then it is not difficult to show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are linearly continuous operators. We now consider two cases:

Case 1: $\sum_{i=1}^m \alpha_i = \alpha \neq 1$**.** We define the operator $M_1 : X \cap \operatorname{dom} M_1 \to Z$ by $M_1 u := (\phi(u'))'$, where

$$\operatorname{dom} M_1 = \Big\{u \in X : \phi(u') \in AC[0,1],\, u'(0) = u'(\eta),\, \phi(u'(1)) = \sum_{i=1}^m \alpha_i \phi(u'(\xi_i)),\\ \sum_{i=1}^m \alpha_i = \alpha \neq 1\Big\}.$$

Then it is not difficult to see that

$$\ker M_1 = \{u \in X : u(t) = c_1, t \in [0,1], c_1 \in \mathbb{R}\},$$

and

$$\operatorname{Im} M_1 = \{z \in Z : \mathcal{B}_1(z) = 0\}. \tag{2.5}$$

Indeed, let $z \in \operatorname{Im} M_1$, then there exists $u \in \operatorname{dom} M_1$ such that $M_1 u = z$. It follows that

$$\phi(u'(t)) = \phi(u'(0)) + \int_0^t z(s)ds,\ t \in [0,1].$$

Since $u \in \operatorname{dom} M_1$, we have $u'(0) = u'(\eta)$, $\phi(u'(1)) = \sum_{i=1}^m \alpha_i \phi(u'(\xi_i))$, and $\sum_{i=1}^m \alpha_i = \alpha \neq 1$. And therefore, we obtain

$$\mathcal{B}_1 z = \int_0^\eta z(s)ds = 0. \tag{2.6}$$

Conversely, if $z \in Z$ satisfies (2.6), then it is not difficult to see that $z = M_1 u$, where $u \in \operatorname{dom} M_1$ defined by

$$u(t) = a + \int_0^t \Big[\phi^{-1}(\phi(b) + \int_0^s z(\tau)d\tau)\Big]ds,$$

with $a \in \mathbb{R}$, b satisfying $(\alpha - 1)\phi(b) = \mathcal{B}_2(z)$. This shows that $z \in \operatorname{Im} M_1$. Thus, (2.5) is valid.

Case 2: $\sum_{i=1}^m \alpha_i = 1$**.** We define the operator $M_2 : X \cap \operatorname{dom} M_2 \to Z$ by $M_2 u := (\phi(u'))'$, where

$$\operatorname{dom} M_2 = \Big\{u \in X : \phi(u') \in AC[0,1],\, u'(0) = u'(\eta),\, \phi(u'(1)) = \sum_{i=1}^m \alpha_i \phi(u'(\xi_i)),\\ \sum_{i=1}^m \alpha_i = 1\Big\}.$$

By using similar argument, it is not difficult to show that

$$\ker M_2 = \{u \in X : u(t) = c_1 + c_2 t, t \in [0,1], c_1, c_2 \in \mathbb{R}\}.$$

and

$$\operatorname{Im} M_2 = \{z \in Z : \mathcal{B}_1(z) = 0 \text{ and } \mathcal{B}_2(z) = 0\}. \tag{2.7}$$

Next, we have the following useful lemmas.

Lemma 2.6. *Let $\alpha_i \in \mathbb{R}$ satisfy $\sum_{i=1}^m \alpha_i = \alpha \neq 1$ and $u \in \operatorname{dom} M_1$. Then we have*

$$\phi(\|u'\|_\infty) \le C\|M_1 u\|_1,$$

where $C = 1 + \frac{1}{|\alpha-1|}\big(1 + \sum_{i=1}^m |\alpha_i|\big)$.

Proof. Let $u \in \operatorname{dom} M_1$. Then we have

$$\phi(u'(\xi_i)) = \phi(u'(0)) + \int_0^{\xi_i} M_1 u(s)ds, \quad i = 1, 2, \dots, m,$$
$$\phi(u'(1)) = \phi(u'(0)) + \int_0^1 M_1 u(s)ds.$$

Because u holds the condition $\phi(u'(1)) = \sum_{i=1}^m \alpha_i \phi(u'(\xi_i))$ with $\alpha_i \in \mathbb{R}$ satisfying $\sum_{i=1}^m \alpha_i = \alpha \neq 1$, we obtain

$$(\alpha - 1)\phi(u'(0)) = \mathcal{B}_2(M_1 u).$$

It follows from the definition of the operator $\mathcal{B}_2$ that

$$|\phi(u'(0))| \leq \frac{1}{|\alpha - 1|}\Big(1 + \sum_{i=1}^m |\alpha_i|\Big)\|M_1 u\|_1.$$

On the other hand, from the identity

$$\phi(u'(t)) = \phi(u'(0)) + \int_0^t M_1 u(s)ds,$$

we obtain

$$\begin{aligned} |\phi(u'(t))| &\leq |\phi(u'(0))| + \int_0^t |M_1 u(s)|ds \\ &\leq \Big[1 + \frac{1}{|\alpha - 1|}\Big(1 + \sum_{i=1}^m |\alpha_i|\Big)\Big]\|M_1 u\|_1, \end{aligned}$$

for all $t \in [0, 1]$. Since ϕ is an odd increasing homeomorphism, we obtain $\phi(\|u'\|_\infty) \leq C\|M_1 u\|_1$, where $C = 1 + \frac{1}{|\alpha-1|}(1 + \sum_{i=1}^m |\alpha_i|)$. □

Lemma 2.7. *Let $\alpha_i \in \mathbb{R}$, $i = 1, \dots, m$ satisfy $\sum_{i=1}^m \alpha_i = 1$. Then the set*

$$S = \Big\{n \in \mathbb{N} : \eta\Big(1 - \sum_{i=1}^m \alpha_i \xi_i^{n+1}\Big) - \eta^{n+1}\Big(1 - \sum_{i=1}^m \alpha_i \xi_i\Big) = 0\Big\},$$

is finite.

Proof. Suppose that S is an infinite set. Then there exists a sequence $\{n_j\}$ such that $n_j < n_{j+1}$ and

$$\eta\Big(1 - \sum_{i=1}^m \alpha_i \xi_i^{n_j+1}\Big) - \eta^{n_j+1}\Big(1 - \sum_{i=1}^m \alpha_i \xi_i\Big) = 0.$$

Let $n_j \to +\infty$ with noting that $\eta \in (0, 1)$, $\xi_i \in (0, 1)$, for all $i \in \{1, 2, \dots, m\}$ and $\sum_{i=1}^m \alpha_i = 1$, we obtain a contradiction $\eta = 0$. This completes the proof. □

In the case $\sum_{i=1}^m \alpha_i = 1$, by setting $\varphi_1(t) = 1$ and $\varphi_2(t) = t^k$, $t \in [0, 1]$, with $k > \max\{n : n \in S\}$, then straightforward calculation gives us

$$\mathcal{B}_1(\varphi_1) = \eta, \quad \mathcal{B}_2(\varphi_1) = 1 - \sum_{i=1}^m \alpha_i \xi_i,$$
$$\mathcal{B}_1(\varphi_2) = \frac{1}{k+1}\eta^{k+1}, \quad \mathcal{B}_2(\varphi_2) = \frac{1}{k+1}(1 - \sum_{i=1}^m \alpha_i \xi_i^{k+1}).$$

It follows from Lemma 2.7 that

$$\kappa = \mathcal{B}_1(\varphi_1)\mathcal{B}_2(\varphi_2) - \mathcal{B}_1(\varphi_2)\mathcal{B}_2(\varphi_1) \neq 0.$$

Next, we define the operators $Q_2^1 : Z \to \mathbb{R}$ and $Q_2^2 : Z \to \mathbb{R}$ as follows

$$Q_2^1(z) = \kappa^{-1}[\mathcal{B}_2(\varphi_2)\mathcal{B}_1(z) - \mathcal{B}_1(\varphi_2)\mathcal{B}_2(z)], \tag{2.8}$$

$$Q_2^2(z) = \kappa^{-1}[\mathcal{B}_1(\varphi_1)\mathcal{B}_2(z) - \mathcal{B}_2(\varphi_1)\mathcal{B}_1(z)]. \tag{2.9}$$

Then Q_2^1 and Q_2^2 are continuous mappings by the continuity of the operators $\mathcal{B}_1$, $\mathcal{B}_2$. Furthermore, from the linearity of the operators $\mathcal{B}_1$ and $\mathcal{B}_2$, it is not difficult to see that

$$\begin{gathered} Q_2^1(Q_2^1(z)\varphi_1) = Q_2^1(z), \quad Q_2^1(Q_2^2(z)\varphi_2) = 0, \\ Q_2^2(Q_2^1(z)\varphi_1) = 0, \quad Q_2^2(Q_2^2(z)\varphi_2) = Q_2^2(z). \end{gathered} \tag{2.10}$$

Lemma 2.8. *The mappings $M_j : X \cap \operatorname{dom} M_j \to Z$, $j = 1, 2$ are quasi-linear operators.*

Proof. It is clear that $\ker M_j$ is linearly homeomorphic to $\mathbb{R}^j$, $j = 1, 2$ and $\operatorname{Im} M_j \subset Z$. Furthermore, since $\mathcal{B}_1$ and $\mathcal{B}_2$ are linearly continuous operators, we gain $\operatorname{Im} M_j$, $j = 1, 2$ are closed subspaces of Z. We consider two following cases

Case 1: $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = \alpha \neq 1$. We now define the operators $P_1 : X \to X$ and $Q_1 : Z \to Z$ as follows

$$P_1 u(t) = u(0), \quad Q_1 z(t) = \frac{1}{\eta}\int_0^\eta z(s)ds.$$

Then, it is not difficult to show that P_1 and Q_1 are linearly continuous projectors and

$$\operatorname{Im} P_1 = \ker M_1 \quad \text{and} \quad \ker Q_1 = \operatorname{Im} M_1.$$

Therefore, we have $X = \ker M_1 \oplus \ker P_1$ and $Z = \operatorname{Im} Q_1 \oplus \operatorname{Im} M_1$. Furthermore, it is obviously that $\dim \ker M_1 = \dim \operatorname{Im} Q_1 = 1$. Hence, there exists a closed subspace $\operatorname{Im} M_1$ of Z and $\dim \ker M_1 = \operatorname{codim} \operatorname{Im} M_1 = 1$. Thus M_1 is a quasi-linear operator.

Case 2: $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$. We define the operators $P_2 : X \to X$ and $Q_2 : Z \to Z$ as follows

$$P_2 u(t) = u(0) + u'(0)t, \quad Q_2 z(t) = Q_2^1(z)\varphi_1(t) + Q_2^2(z)\varphi_2(t),$$

where $Q_2^1(z)$ and $Q_2^2(z)$ are defined by (2.8) and (2.9). Then it is clear that P_2 is a linearly continuous projector satisfying $\operatorname{Im} P_2 = \ker M_2$. Furthermore, it follows form (2.10) that Q_2 is also a linearly continuous projector and $\ker Q_2 = \operatorname{Im} M_2$. Hence, we have $X = \ker M_2 \oplus \ker P_2$ and $Z = \operatorname{Im} Q_2 \oplus \operatorname{Im} M_2$ and we also have $\dim \ker M_2 = \dim \operatorname{Im} Q_2 = 2$. As a result, we can find a closed subspace $\operatorname{Im} M_2$ of Z satisfying $\dim \ker M_2 = \operatorname{codim} \operatorname{Im} M_2 = 2$. Thus, M_2 is also a quasi-linear operator. □

In the sequel, we assume that $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory condition; that is,

(a) $f(\cdot, u, v)$ is measurable for $(u, v) \in \mathbb{R}^2$,
(b) $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for almost every where $t \in [0, 1]$,
(c) For each compact set $K \subset \mathbb{R}^2$, the function $m_K(t) = \sup\{|f(t, u, v)| : (u, v) \in K\}$ defined on $[0, 1]$ satisfies $m_K \in L^1[0, 1]$.

With each function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfying conditions above, we associate its *Nemytskii operator* $N : X \to Z$ defined by

$$N(u)(t) = f(t, u(t), u'(t)).$$

Then problem (1.1)-(1.2) can be written as the operator equation

$$M_j u = Nu,$$

where $j = 1, 2$ provided that $\sum_{i=1}^m \alpha_i \neq 1$ and $\sum_{i=1}^m \alpha_i = 1$, respectively.

By using the assumption on f and dominated convergence theorem, it is not difficult to see that N is continuous mapping and takes bounded sets into bounded sets.

Next, we define the operator $R_1 : X \times [0,1] \to \ker P_1$ as follows

$$R_1(u,\lambda) = \int_0^t \phi^{-1}\Big[c + \int_0^s \lambda(N(u)(\tau) - Q_1 \circ N(u)(\tau))d\tau\Big]ds,$$

where c is a constant depending on (u,λ) and satisfying

$$(\alpha - 1)c = \lambda \mathcal{B}_2 \circ N(u) - \lambda \mathcal{B}_2 \circ Q_1 \circ N(u).$$

And define the operator $R_2 : X \times [0,1] \to \ker P_2$ as follows

$$\begin{aligned} R_2(u,\lambda) = \int_0^t \phi^{-1}\Big[\int_0^s \lambda(N(u)(\tau) - \sum_{i=1}^2 Q_2^i \circ N(u)\varphi_i(\tau))d\tau + \phi(u'(0))\Big]ds \\ - u'(0)t. \end{aligned}$$

Then, it is not difficult to show that $R_1(u,\lambda) \in C^1[0,1]$, $R_2(u,\lambda) \in C^1[0,1]$ and $R_1(u,\lambda)(0) = 0$ and $R_2(u,\lambda)(0) = R_2(u,\lambda)'(0) = 0$ hold by using $\phi^{-1}(0) = 0$. Hence R_1 and R_2 are well defined. Furthermore, by the continuity of operators composing R_1, R_2, we deduce that R_1 and R_2 are continuous.

Lemma 2.9. $R_j : X \times [0,1] \to \ker P_j$, $j = 1, 2$ *are completely continuous operators.*

Proof. We first prove that R_1 is completely continuous operator. By the arguments above, it suffices to prove that R_1 takes bounded sets into relatively compact sets. Let $\Omega \subset X$ be a nonempty and bounded set. Then there exists a positive constant r such that $\|u\| \leq r$. From the hypotheses of the function f we deduce that there exists a positive function $m_r \in Z$ such that, for all $u \in \overline{\Omega}$,

$$|Nu(t)| = |f(t, u(t), u'(t))| \leq m_r(t), \quad \forall t \in [0,1]. \tag{2.11}$$

Let

$$g(u)(t) = c + \int_0^t \lambda[N(u)(s) - Q_1 \circ N(u)(s)]ds, \quad t \in [0,1],$$

where c is a constant depending on (u,λ) and satisfying

$$(\alpha - 1)c = \lambda \mathcal{B}_2 \circ N(u) - \lambda \mathcal{B}_2 \circ Q_1 \circ N(u).$$

It follows from (2.11) and the definition of the operator $\mathcal{B}_2$ that

$$|g(u)(t)| \leq \big(1 + \frac{1}{\eta}\big)\Big[1 + \frac{1}{|\alpha - 1|}(1 + \sum_{i=1}^m |\alpha_i|)\Big]\|m_r\|_1 := G, \tag{2.12}$$

for all $t \in [0,1], u \in \overline{\Omega}$. Hence, we can find a positive constant C_1 such that

$$|R_1(u,\lambda)(t)| \leq C_1, \quad \text{and} \quad |R_1(u,\lambda)'(t)| \leq C_1, \quad \forall t \in [0,1],\ u \in \overline{\Omega}.$$

Thus, $R_1(\overline{\Omega} \times [0,1])$ is bounded in X. On the other hand, for any $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$, we infer from (2.12), the increasing property of ϕ^{-1}, and the definition of the operator R_1 that

$$|R_1(u,\lambda)(t_1) - R_1(u,\lambda)(t_2)| \le \int_{t_1}^{t_2} |\phi^{-1}(G)| ds,$$

which implies $\{R_1(u,\lambda) : u \in \overline{\Omega}\}$ are equicontinuous on $[0,1]$. Further, we also have

$$|R_1(u,\lambda)'(t_1) - R_1(u,\lambda)'(t_2)| \le |\phi^{-1} \circ g(u)(t_1) - \phi^{-1} \circ g(u)(t_2)|.$$

For $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in \overline{\Omega}$, we have

$$\begin{aligned} |g(u)(t_1) - g(u)(t_2)| &= \Big| \int_{t_1}^{t_2} \lambda[N_f(u)(s) - Q_1 \circ N_f(u)(s)] ds \Big| \\ &\le \int_{t_1}^{t_2} \big(|m_r(t)| + \frac{1}{\eta}\|m_r\|_1\big) ds. \end{aligned}$$

It follows from $m_r \in L^1[0,1]$ that $\{g(u) : u \in \overline{\Omega}\}$ are equicontinuous on $[0,1]$. Since ϕ^{-1} is uniformly continuous on $[-G, G]$, we obtain that $\{R_1(u,\lambda)' : u \in \overline{\Omega}\}$ are equicontinuous on $[0,1]$. Thus, R_1 is a completely continuous operator by Arzela-Ascoli's theorem. By similar arguments, we can be able to prove that R_2 is a completely continuous operator. □

Lemma 2.10. *Let Ω be a nonempty, open and bounded subset of X. Then N is M_j-compact in $\overline{\Omega}$, $j = 1, 2$.*

Proof. Since N is a continuous operator and takes the bounded sets into bounded sets, so do $Q_i N$, $i = 1, 2$. By Lemma 2.9, the operators $R_j : \overline{\Omega} \times [0,1] \to \ker P_j$, $j = 1, 2$ are completely continuous. It follows from the definitions of R_j that $R_j(u, 0) = 0$ for all $u \in X$, $j = 1, 2$. Let $u \in \sum_\lambda^1 := \{u \in \overline{\Omega} : M_1 u = \lambda N u\}$. Then we have $u \in \operatorname{dom} M_1$, $\lambda N u \in \operatorname{Im} M_1 = \ker Q_1$ and $(\phi(u'))' = \lambda N(u)$. It follows that

$$\begin{aligned} R_1(u,\lambda)(t) &= \int_0^t \phi^{-1}\Big[c + \int_0^s (\phi(u'(\tau)))' d\tau\Big] ds \\ &= \int_0^t \phi^{-1}[c + \phi(u'(s)) - \phi(u'(0))] ds. \end{aligned}$$

On the other hand, since $u \in \operatorname{dom} M_1$, we have $\phi(u'(1)) = \sum_{i=1}^m \alpha_i \phi(u'(\xi_i))$ and therefore c satisfies

$$\begin{aligned} (\alpha - 1)c &= \lambda \mathcal{B}_2 \circ N(u) \\ &= \int_0^1 [\phi(u'(s))]' ds - \sum_{i=1}^m \alpha_i \int_0^{\xi_i} [\phi(u'(s))]' ds \\ &= (\alpha - 1)\phi(u'(0)). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} R_1(u,\lambda)(t) &= \int_0^t \phi^{-1}[c + \phi(u'(s)) - \phi(u'(0))] ds \\ &= u(t) - u(0) \\ &= (I - P_1)u(t). \end{aligned}$$

Further, for $u \in X$, we have

$$M_1[P_1u + R_1(u,\lambda)](t) = \lambda N(u)(t) - \lambda Q_1 \circ N(u)(t) = \lambda(I - Q_1)Nu(t).$$

Thus, by Definition 2.2, N is M_1-compact in $\overline{\Omega}$. Similarly, let $u \in \sum_\lambda^2 := \{u \in \overline{\Omega} : M_2u = \lambda Nu\}$. Then we have $\lambda Nu \in \operatorname{Im} M_2 = \ker Q_2$ and $(\phi(u'))' = \lambda N(u)$. It follows that

$$\begin{aligned} R_2(u,\lambda)(t) &= \int_0^t \phi^{-1}[\int_0^s (\phi(u'(\tau)))'d\tau + \phi(u'(0))]ds - u'(0)t \\ &= u(t) - u(0) - u'(0)t \\ &= (I - P_2)u(t). \end{aligned}$$

And, for $u \in X$, we have

$$M_2[P_2u + R_2(u,\lambda)](t) = \lambda N(u)(t) - \lambda \sum_{i=1}^{2} Q_2^i \circ N(u)\varphi_i(t) = \lambda(I - Q_2)Nu(t).$$

Thus, N is also M_2-compact in $\overline{\Omega}$. This completes the proof. □

3. Existenceof solutions

In this section we use Theorem 2.5 to prove the existence of solutions for problem (1.1)-(1.2) in both cases $\sum_{i=1}^m \alpha_i = \alpha \neq 1$ and $\sum_{i=1}^m \alpha_i = 1$.

We first prove the existence of solutions in the case $\sum_{i=1}^m \alpha_i = \alpha \neq 1$. For this purpose, we assume that the following conditions hold:

- (A1) there exists a positive constant A such that for each $u \in C^1[0,1]$ with $\min_{t\in[0,1]} |u(t)| > A$, we have
$$\int_0^\eta f(s,u(s),u'(s))ds \neq 0;$$
- (A2) there exist non-negative functions $a, b, c \in Z$ satisfying $\|a\|_1\alpha(A) + \|b\|_1 < \frac{1}{C}$, with C is constant defined by Lemma 2.6 such that
$$|f(t,u,v)| \le a(t)\phi(|u|) + b(t)\phi(|v|) + c(t),$$
for a.e. $t \in [0,1]$ and for all $u, v \in \mathbb{R}$;
- (A3) there exists a constant $\rho_1 > 0$ such that for all $c_1 \in \mathbb{R}$ with $|c_1| > \rho_1$, then either
$$c_1 \int_0^\eta f(s,c_1,0)ds < 0, \tag{3.1}$$
or
$$c_1 \int_0^\eta f(s,c_1,0)ds > 0. \tag{3.2}$$

Then we have the following lemmas.

Lemma 3.1. *Let $\Omega_1^1 = \{u \in \operatorname{dom} M_1 : M_1u = \lambda Nu, \lambda \in (0,1)\}$. Then Ω_1^1 is bounded in X.*

Proof. Let $u \in \Omega_1^1$. Then there exists $\lambda \in (0,1)$ such that $\lambda Q_1 Nu = 0$. This implies $Q_1Nu(t) = 0$ for all $t \in [0,1]$, that is,

$$\int_0^\eta f(s,u(s),u'(s))ds = 0.$$

It follows from the assumption (A1) that there exists $t_0 \in [0,1]$ such that

$$|u(t_0)| \leq A.$$

On the other hand, since $u(t) = u(t_0) + \int_{t_0}^{t} u'(s)ds$, we obtain

$$|u(t)| \leq A + \|u'\|_\infty, \ \forall t \in [0,1]. \tag{3.3}$$

It follows from (3.3), the increasing property of ϕ, the assumption (A1) and Lemma 2.6 that

$$\begin{aligned} \phi(\|u'\|_\infty) &\leq C\|M_1u\|_1 \leq C\|Nu\|_1 \\ &\leq C[\|a\|_1\phi(\|u\|_\infty) + \|b\|_1\phi(\|u'\|_\infty) + \|c\|_1] \\ &\leq C[\|a\|_1\phi(A + \|u'\|_\infty) + \|b\|_1\phi(\|u'\|_\infty) + \|c\|_1]. \end{aligned} \tag{3.4}$$

Furthermore, because $\|a\|_1\alpha(A) + \|b\|_1 < \frac{1}{C}$, we deduce from (3.4) that there exists a positive constant K_1 such that

$$\|u'\|_\infty \leq K_1. \tag{3.5}$$

Hence, it follows from (3.3) and (3.5) that Ω_1^1 is bounded in X. This completes the proof. □

Lemma 3.2. *The set $\Omega_2^1 = \{u \in \ker M_1 : Nu \in \operatorname{Im} M_1\}$ is a bounded subset in X.*

Proof. Let $u \in \Omega_2^2$. Since $u \in \ker M_1$, we can assume that $u(t) = c_1$, where $c_1 \in \mathbb{R}$. Further it is clear that $Q_1Nu = 0$ because of $Nu \in \operatorname{Im} M_1 = \ker Q_1$. By the same arguments as in the proof of Lemma 3.1, we can find a positive constant k_1 such that $\|u\| \leq k_1$. Thus, Ω_2 is bounded in X. □

Lemma 3.3. *Assume that $\Omega_{3^-}^1 = \{u \in \ker M_1 : -\lambda u + (1-\lambda)J_1Q_1Nu = 0, \lambda \in [0,1]\}$ and*

$$\Omega_{3^+}^1 = \{u \in \ker M_1 : \lambda u + (1-\lambda)J_1Q_1Nu = 0, \lambda \in [0,1]\},$$

where $J_1 : \operatorname{Im} Q_1 \to \ker M_1$ is the linear isomorphism defined by $J_1^{-1}(c_1) = c_1$, $c_1 \in \mathbb{R}$. Then $\Omega_{3^-}^1$ and $\Omega_{3^+}^1$ are bounded subsets in X provided that (3.1) *and* (3.2)*, respectively.*

Proof. First we assume that (3.1) holds. Let $u \in \Omega_3^-$. Then, since $u \in \ker M_1$, there exists $c_1 \in \mathbb{R}$ such that $u(t) = c_1$, for all $t \in [0,1]$. Further, we have

$$\lambda J_1^{-1}(c_1) = (1-\lambda)Q_1N(c_1), \quad \forall t \in [0,1],$$

which is equivalent to

$$\lambda c_1 = (1-\lambda)\frac{1}{\eta}\int_0^\eta f(s, c_1, 0)ds.$$

If $\lambda = 1$ then $c_1 = 0$. And therefore, Ω_3^- is bounded. On the other hand, if $\lambda \in [0,1)$ and $|c_1| > \rho_1$ then, by assumption (3.1), we obtain a contradiction

$$0 \leq \eta\lambda c_1^2 = (1-\lambda)c_1\int_0^\eta f(s, c_1, 0)ds < 0.$$

Therefore, $\|u\| = |c_1| \leq \rho_1$. Thus, $\Omega_{3^-}^1$ is bounded in X. If (3.2) holds then by using the same arguments as above we are able to prove that $\Omega_{3^+}^1$ is also bounded in X. □

Theorem 3.4. *We assume that the assumptions* (A1)-(A3) *hold and* $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ *with* $\sum_{i=1}^m \alpha_1 = \alpha \neq 1$. *Then problem* (1.1)-(1.2) *has at least one solution in* X.

Proof. We shall prove that all conditions of the Theorem 2.5 are satisfied, where Ω^1 is an open and bounded such that $\cup_{i=1}^3 \overline{\Omega_i^1} \subset \Omega^1$. Then we have M_1 is a quasi-linear operator by Lemma 2.8 and N is M_1-compact on $\overline{\Omega^1}$ by Lemma 2.10. It is clear that the condition (1) of Theorem 2.5 hold by using Lemma 3.1. And therefore, it remains to verify that the second condition of Theorem 2.5 holds. For this purpose, we apply the degree property of invariance under a homotopy. Let us define

$$H_1(u, \lambda) = \pm\lambda u + (1 - \lambda) J_1 Q_1 N u.$$

By Lemma 3.2 and Lemma 3.3, we obtain that H_1 is a homotopy and $H_1(u, \lambda) \neq 0$ for all $(u, \lambda) \in (\ker M_1 \cap \partial\Omega^1) \times [0, 1]$. So

$$\begin{aligned} \deg(J_1 Q_1 N; \Omega^1 \cap \ker M_1, 0) &= \deg(H_1(\cdot, 0); \Omega^1 \cap \ker M_1, 0) \\ &= \deg(H_1(\cdot, 1); \Omega^1 \cap \ker M_1, 0) \\ &= \deg(\pm Id; \Omega^1 \cap \ker M_1, 0) = \pm 1 \neq 0. \end{aligned}$$

Thus, Theorem 3.4 is proved. □

Next, we establish the existence result for (1.1)-(1.2) in the case $\sum_{i=1}^m \alpha_i = 1$, with $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. To gain this, we assume the following conditions:

- (A4) there exist a positive constant B such that for each $u \in C^1[0, 1]$ satisfying $|u(t)| + |u'(t)| > B$, for all $t \in [0, 1]$, we have $Q_2 N u(t) \neq 0$;
- (A5) there exist positive functions $a, b, c \in Z$ with $\|a\|_1 \alpha(B) + \|b\|_1 < 1$ such that
$$|f(t, u, v)| \leq a(t)\phi(|u|) + b(t)\phi(|v|) + c(t),$$
for a.e. $t \in [0, 1]$ and for all $u, v \in \mathbb{R}$.
- (A6) there exists a positive constant ρ_2 such that if $c_1, c_2 \in \mathbb{R}$ with $\sum_{i=1}^2 |c_i| > \rho_2$, then there exists $i \in \{1, 2\}$ such that either
$$c_i Q_2^i N(c_1 + c_2 t) < 0 \tag{3.6}$$
or
$$c_i Q_2^i N(c_1 + c_2 t) > 0. \tag{3.7}$$

Then we have the following lemmas.

Lemma 3.5. *Let* $\Omega_1^2 = \{u \in \operatorname{dom} M_2 : M_2 u = \lambda N u, \lambda \in (0, 1)\}$. *Then* Ω_1^2 *is bounded in* X.

Proof. Let $u \in \Omega_1^2$. Then there exists $\lambda \in (0, 1)$ such that $\lambda Q_2 N u = 0$. This implies $Q_2 N u(t) = 0$ for all $t \in [0, 1]$. By using the assumption (A4), there exist $t_0 \in [0, 1]$ such that

$$|u(t_0)| + |u'(t_0)| \leq B.$$

Then, from ϕ begin increasing homeomorphism and the identity

$$\phi(u'(t)) = \phi(u'(t_0)) + \int_{t_0}^t M_2 u(s) ds,$$

we infer that

$$\phi(|u'(t)|) \leq \phi(B) + \|M_2 u\|_1 \leq \phi(B) + \|N u\|_1, \quad \forall t \in [0, 1]. \tag{3.8}$$

On the other hand, since $u(t) = u(t_0) + \int_{t_0}^t u'(s)ds$, we obtain

$$|u(t)| \le B + \|u'\|_\infty, \quad \forall t \in [0,1]. \tag{3.9}$$

Combining (3.8), (3.9) and the assumption (A5), it follows that

$$\begin{aligned} \phi(|u'(t)|) &\le \phi(B) + \|a\|_1\phi(\|u\|_\infty) + \|b\|_1\phi(\|u'\|_\infty) + \|c\|_1 \\ &\le \|a\|_1\phi(B + \|u'\|_\infty) + \|b\|_1\phi(\|u'\|_\infty) + \|c\|_1 + \phi(B), \quad \forall t \in [0,1]. \end{aligned}$$

This implies

$$\phi(\|u'\|_\infty) \le \|a\|_1\phi(B + \|u'\|_\infty) + \|b\|_1\phi(\|u'\|_\infty) + \|c\|_1 + \phi(B). \tag{3.10}$$

Because $\|a\|_1\alpha(B) + \|b\|_1 < 1$, we deduce from (3.10) that there exists a positive constant K_2 such that

$$\|u'\|_\infty \le K_2. \tag{3.11}$$

Thus, it follows from (3.9) and (3.11) that Ω_1^2 is bounded in X. □

Lemma 3.6. *The set $\Omega_2^2 = \{u \in \ker M_2 : Nu \in \operatorname{Im} M_2\}$ is a bounded subset in X.*

Proof. Let $u \in \Omega_2^2$. Since $u \in \ker M_2$ we can assume that $u(t) = c_1 + c_2t$, where $c_1, c_2 \in \mathbb{R}$. Further it is clear that $Q_2Nu = 0$ because of $Nu \in \operatorname{Im} M_2$. By the same arguments as in the proof of Lemma 3.5, we can find a positive constant k_2 such that $\|u\| \le k_2$. Thus, Ω_2^2 is bounded in X. □

Lemma 3.7. *Assume that $\Omega_{3^-}^2 = \{u \in \ker M_2 : -\lambda u + (1-\lambda)J_2Q_2Nu = 0, \lambda \in [0,1]\}$ and*

$$\Omega_{3^+}^2 = \{u \in \ker M_2 : \lambda u + (1-\lambda)J_2Q_2Nu = 0, \lambda \in [0,1]\},$$

where $J_2 : \operatorname{Im} Q_2 \to \ker M_2$ is the linear isomorphism which is defined by

$$J^{-1}(c_1 + c_2t) = c_1\varphi_1(t) + c_2\varphi_2(t), \quad c_1, c_2 \in \mathbb{R},$$

where $(\varphi_1(t), \varphi_2(t)) = (1, t^k)$, with k defined in the previous arguments. Then $\Omega_{3^-}^2$ and $\Omega_{3^+}^2$ are bounded subsets in X provided that $c_iQ_iN(c_1 + c_2t)$ are negative for some $i \in \{1,2\}$ and that $c_iQ_iN(c_1 + c_2t)$ are positive for some $i \in \{1,2\}$, respectively.

Proof. First we assume that (3.6) holds. Let $u \in \Omega_{3^-}^2$. Then, since $u \in \ker M_2$, there exists $c_1, c_2 \in \mathbb{R}$ such that $u(t) = c_1 + c_2t$, for all $t \in [0,1]$. Further, we have

$$\lambda J^{-1}(c_1 + c_2t) = (1-\lambda)Q_2N(c_1 + c_2t), \quad \forall t \in [0,1],$$

which is equivalent to

$$\lambda \sum_{i=1}^{2} c_i\varphi_i(t) = (1-\lambda)\sum_{i=1}^{2} Q_2^iN(c_1 + c_2t)\varphi_i(t), \quad \forall t \in [0,1].$$

Hence, from the independence of system of vectors $\{\varphi_1, \varphi_2\}$ in Z, we deduce that

$$\lambda c_i = (1-\lambda)Q_iN(c_1 + c_2t), \quad \forall i \in \{1,2\}.$$

If $\lambda = 1$, then $c_i = 0$ for all $i \in \{1,2\}$. And therefore $\Omega_{3^-}^2$ is bounded. On the other hand, if $\lambda \in [0,1)$ and $\sum_{i=1}^2 |c_i| > \rho_2$ then, by the assumption (3.6), we obtain a contradiction

$$0 \le \lambda c_i^2 = (1-\lambda)c_iQ_iN(c_1 + c_2t) < 0, \quad \forall i \in \{1,2\}.$$

Therefore, we have $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\} \le \rho_2$. Thus, $\Omega^2_{3^-}$ is bounded in X. If $(A_6) - (3.7)$ holds then by using the same arguments as above we can be able to prove that $\Omega^2_{3^+}$ is also bounded in X. $\square$

Theorem 3.8. *We assume that the assumptions* (A4)–(A6) *hold and* $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ *with* $\sum_{i=1}^m \alpha_i = 1$. *Then* (1.1)-(1.2) *has at least one solution in* X.

Proof. We shall verify all conditions of the Theorem 2.5 are satisfied, where Ω^2 is an open and bounded such that $\cup_{i=1}^3 \overline{\Omega_i^2} \subset \Omega^2$. Then we have M_2 is a quasi-linear operator by Lemma 2.8 and N is M_2-compact on $\overline{\Omega^2}$ by Lemma 2.10. It is clear that condition (1) of Theorem 2.5 holds by using Lemma 3.5. And therefore, it remains to verify that the second condition of Theorem 2.5 holds. To gain this, we apply the degree property of invariance under a homotopy. Let us define

$$H_2(u, \lambda) = \pm\lambda u + (1 - \lambda) J_2 Q_2 N u.$$

By Lemma 3.6 and Lemma 3.7, we obtain that H_2 is a homotopy and $H_2(u, \lambda) \neq 0$ for all $(u, \lambda) \in (\ker M_2 \cap \partial\Omega^2) \times [0, 1]$. So

$$\begin{aligned}
\deg(J_2 Q_2 N; \Omega^2 \cap \ker M_2, 0) &= \deg(H_2(\cdot, 0); \Omega^2 \cap \ker M_2, 0) \\
&= \deg(H_2(\cdot, 1); \Omega^2 \cap \ker M_2, 0) \\
&= \deg(\pm Id; \Omega^2 \cap \ker M_2, 0) = \pm 1 \neq 0.
\end{aligned}$$

Thus, Theorem 3.8 is proved. $\square$

We now give an example to illustrate our results.

Example. Consider the one dimension p-Laplacian differential equation

$$\big(|u'(t)|^{p-2}u'(t)\big)' = f(t, u(t), u'(t)), \quad t \in (0, 1), \tag{3.12}$$

subjected to the multi-point nonlinear Neumann type boundary condition

$$\begin{gathered}
u'(0) = u'(\frac{1}{2}), \\
|u'(1)|^{p-2}u'(1) = -\frac{2}{3}|u'(\frac{1}{3})|^{p-2}u'(\frac{1}{3}) + \frac{1}{2}|u'(\frac{2}{3})|^{p-2}u'(\frac{2}{3}),
\end{gathered} \tag{3.13}$$

where $f(t, u, v) = \frac{1}{27}(1 + t)|u|^{p-1} + \frac{t}{11}\sin(|v|^{p-2}v) + t^2 + 1$ and $p > 1$.

By setting $\phi(t) = \varphi_p(t) = |t|^{p-2}t$, $p > 1$, $\eta = \frac{1}{2}$, $\alpha_1 = -\frac{2}{3}$, $\alpha_2 = \frac{1}{2}$, $\xi_1 = \frac{1}{3}$ and $\xi_2 = \frac{2}{3}$. Then the problem (3.12)-(3.13) is a particular case of the problem (1.1)-(1.2). Because of $\alpha = \sum_{i=1}^2 \alpha_i = -\frac{1}{6} \neq 1$, to show that (3.12)-(3.13) has one solution, it suffices to verify the conditions of Theorem 3.4.

First, we note that $f(t, u, v) > 0$ provided that $|u| > \varphi_q(54)$, with $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Hence, choosing $A = \varphi_q(54) > 0$, then we have

$$\int_0^{1/2} f(s, u(s), u'(s))ds \neq 0$$

as $\min_{t\in[0,1]} |u(t)| > A$. So we obtain the condition (A1).

Next, by the definition of f, we obtain that $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory condition and

$$|f(t, u, v)| \le a(t)\phi(|u|) + b(t)\phi(|v|) + c(t),$$

for a.e. $t \in [0,1]$ and for all $u, v \in \mathbb{R}$, where

$$a(t) = \frac{1}{27}(1+t), \quad b(t) = \frac{t}{11}, \quad c(t) = 1 + t^2.$$

It is not difficult to calculate

$$C = 1 + \frac{1}{|\alpha - 1|}(1 + \sum_{i=1}^{2} |\alpha_i|) = \frac{20}{7}$$

and to see $a, b, c \in L^1[0,1]$ satisfying $C(\|a\|_1 \alpha(A) + \|b\|_1) = C(\|a\|_1 + \|b\|_1) = \frac{20}{7}(\frac{1}{18} + \frac{1}{22}) < 1$. Therefore, the condition (A2) holds.

Finally, it is not difficult to see that $f(t,c,0) > 0$ and $f(t,c,0) < 0$ provided that $c > \varphi_q(54)$ and $c < \varphi_q(-54)$, corresponding. Therefore, by choosing $\rho_1 = \varphi_q(54)$, we obtain

$$c \int_0^{1/2} f(s,c,0)ds > 0.$$

Hence, the condition (A3) holds. Thus the problem (3.12)-(3.13) has one solution.

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