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GLOBAL AND LOCAL BEHAVIOR OF THE BIFURCATION DIAGRAMS FOR SEMILINEAR PROBLEMS

TETSUTARO SHIBATA

Abstract. We consider the nonlinear eigenvalue problem

$$u''(t) + \lambda(u(t)^p - u(t)^q) = 0, \quad u(t) > 0, \quad -1 < t < 1,$$
$$u(1) = u(-1) = 0,$$

where $1 < p < q$ are constants and $\lambda > 0$ is a parameter. It is known in [13] that the bifurcation curve $\lambda(\alpha)$ consists of two branches, which are denoted by $\lambda_\pm(\alpha)$. Here, $\alpha = \|u_\lambda\|_\infty$. We establish the asymptotic behavior of the turning point α_p of $\lambda(\alpha)$, namely, the point which satisfies $d\lambda(\alpha_p)/d\alpha = 0$ as $p \to q$ and $p \to 1$. We also establish the asymptotic formulas for $\lambda_+(\alpha)$ and $\lambda_-(\alpha)$ as $\alpha \to 1$ and $\alpha \to 0$, respectively.

1. Introduction

We consider the nonlinear eigenvalue problem

$$u''(t) + \lambda(u(t)^p - u(t)^q) = 0, \quad t \in I := (-1, 1), \tag{1.1}$$
$$u(t) > 0, \quad t \in I, \tag{1.2}$$
$$u(1) = u(-1) = 0, \tag{1.3}$$

where $1 < p < q$ are constants and $\lambda > 0$ is a parameter.

Nonlinear elliptic eigenvalue problems have been studied by many authors from the standpoint of bifurcation analysis . We refer to [1, 2, 3, 4, 5, 6, 7, 8, 12, 14] and the references therein. In particular, it is quite significant to study the equations which have fine structures of the bifurcation diagrams. Among other things, (1.1)–(1.3) is well known as the model equation which has the parabola-like bifurcation curve (cf. Figure 1 below). More precisely, it has been proved in [9, Theorem 2.15] and [13, Theorem 6.19] the following basic properties of the structure of bifurcation diagram for (1.1)–(1.3).

Theorem 1.1 ([9, 13]). *Assume that $1 < p < q$. Then there exists a critical λ_0 such that (1.1)–(1.3) has no positive solution for $0 < \lambda < \lambda_0$, exactly one positive solution at $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda > \lambda_0$. Furthermore, all solutions lie on a smooth solution curve, and λ is parameterized by $\alpha := \|u_\lambda\|_\infty$ as $\lambda = \lambda(\alpha)$. Further, $\lambda(\alpha)$ consists of two branches $\lambda_\pm(\alpha)$ and is a parabola-like*

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curve with exactly one turn to the right at $\alpha = \alpha_p \in (0,1)$*, namely,* α_p *is the unique point which satisfies* $\lambda'(\alpha_p) = 0$*. Furthermore,*

$$\lambda_-(\alpha) \to \infty \quad \text{as } \alpha \to 0, \tag{1.4}$$

$$u_-(\lambda, 0) \to 0, \tag{1.5}$$

$$\lambda_+(\alpha) \to \infty \quad \text{as } \alpha \to 1, \tag{1.6}$$

$$u_+(\lambda, t) \to 1 \quad (t \in I) \quad \text{as } \lambda \to \infty, \tag{1.7}$$

where $u_\pm(\lambda, t)$ *is a solution of* (1.1)–(1.3) *corresponding to* $\lambda = \lambda_\pm(\alpha)$*.*

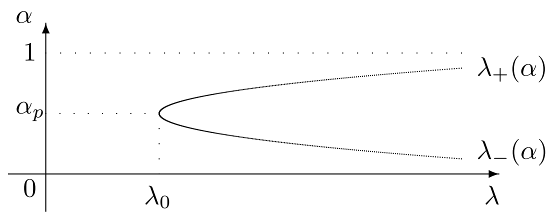

FIGURE 1. Bifurcation curve $\lambda_\pm(\alpha)$

On the other hand, if $p = 1$, then we know from [1] that the shape of the bifurcation curve looks as in Figure 2.

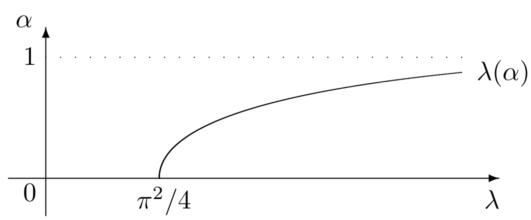

FIGURE 2. Bifurcation curve for $p = 1$

Consider now the relationship between Figures 1 and 2. It is quite natural to expect that if $p \to 1$, then the shape of the graph in Figure 1 will approach to the graph in Figure 2 in some sense. To obtain the evidence of this expectation, it is important to investigate the asymptotic behavior of the graph in Figure 1 as $p \to 1$. Related to the observation above, it is worth studying the asymptotic behavior of the graph in Figure 1 as $p \to q$.

Now we study the global behavior of $\lambda_\pm(\alpha)$.

Theorem 1.2. *Let* $1 < p < q$ *be fixed constants. Let an arbitrary* $0 < \delta \ll 1$ *be fixed. (i) As* $\alpha \to 1$*,*

$$\sqrt{\lambda_+(\alpha)} = \sqrt{\frac{1}{q-p}(1 + O(\delta))\alpha^{(1-p)/2}|\log(1 - \alpha^{q-p})| + O(\delta^{-1})}. \tag{1.8}$$

(ii) As $\alpha \to 0$*,*

$$\sqrt{\lambda_-(\alpha)} = \sqrt{\frac{p+1}{2}}\alpha^{(1-p)/2}(b_0 + b_1\alpha^{q-p} + O(\alpha^{2(q-p)}), \tag{1.9}$$

where

$$b_0 = \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} ds, \tag{1.10}$$

$$b_1 = \frac{p+1}{2(q+1)} \int_0^1 \frac{1-s^{q+1}}{(1-s^{p+1})^{3/2}} ds. \tag{1.11}$$

Our main purpose is to study the local behavior of $0 < \alpha_p < 1$, which is the turning point of $\lambda(\alpha)$. We show how Figure 1 tends to Figure 2 as $p \to 1$. To do this, we establish the asymptotic formula for α_p as $p \to 1$.

Theorem 1.3. *Let $q > 1$ be a fixed constant. Then as $p \to 1$,*

$$\alpha_p^{q-p} = \frac{b_0(q+1)}{k_0(p+1)(q-p)}(p-1) + O((p-1)^2), \tag{1.12}$$

where

$$k_0 = \int_0^1 \frac{1-s^{q+1}}{(1-s^{p+1})^{3/2}} ds. \tag{1.13}$$

Furthermore,

$$\sqrt{\lambda(\alpha_p)} = D_p(p-1)^{(1-p)/(2(q-p))}(b_0 + O(p-1)), \tag{1.14}$$

where

$$D_p := \sqrt{\frac{p+1}{2}} \Big[\frac{b_0(q+1)}{k_0(p+1)(q-p)}\Big]^{(1-p)/(2(q-p))}. \tag{1.15}$$

Clearly, as $p \to 1$,

$$D_p \to 1, \quad (p-1)^{(1-p)/(2(q-p))} \to 1, \quad b_0 \to \frac{\pi}{2}. \tag{1.16}$$

Therefore, we see from (1.14) that $\lambda(\alpha_p) \to \pi^2/4$ as $p \to 1$, and the shape of the bifurcation curve when $0 < p - 1 \ll 1$ is as shown in Figure 3.

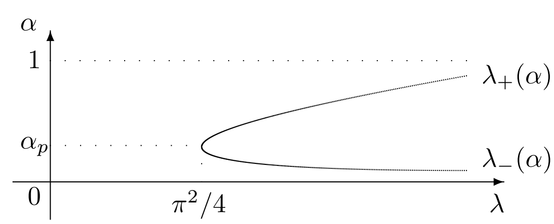

FIGURE 3. Bifurcation curve for $0 < p - 1 \ll 1$

Finally, we establish the asymptotic formula for α_p as $p \to q$.

Theorem 1.4. *Let $q > 1$ be fixed. Then*

$$\alpha_p^{q-p} = 1 - O\Big(\frac{q-p}{|\log(q-p)|^{2/3}}\Big) \quad \text{as } p \to q. \tag{1.17}$$

It should be mentioned that as far as the author knows, the results such as Theorems 1.3 and 1.4 seem to be new.

Our methods to prove Theorems 1.2–1.4 are based on the precise and complicated calculation of the time map.

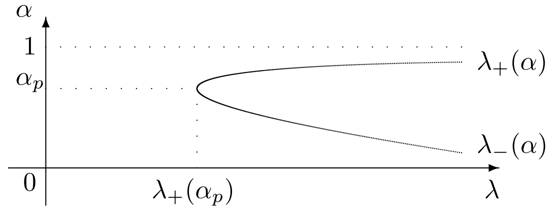

FIGURE 4. Bifurcation curve for $0 < q - p \ll 1$

2. Proof of Theorem 1.2

In this section, we let $0 < p - 1 \ll 1$. We know that if $(\lambda, u) \in \mathbb{R}_+ \times C^2(\bar{I})$ satisfies (1.1)–(1.3), then

$$u(t) = u(-t), \quad 0 \le t \le 1, \tag{2.1}$$

$$u(0) = \max_{-1\le t\le 1} u(t), \tag{2.2}$$

$$u'(t) < 0, \quad 0 < t \le 1. \tag{2.3}$$

We parameterize the solution pair by using the L^∞ norm of the solution $\alpha = \|u_\lambda\|_\infty$ such as $(\lambda, u) = (\lambda(\alpha), u_\alpha)$ $(0 < \alpha < 1)$. By (1.1), for $t \in \bar{I}$,

$$[u_\alpha''(t) + \lambda(u_\alpha(t)^p - u_\alpha(t)^q)]u_\alpha'(t) = 0.$$

This implies that for $t \in \bar{I}$,

$$\frac{d}{dt}\Big[\frac{1}{2}u_\alpha'(t)^2 + \lambda\Big(\frac{1}{p+1}u_\alpha(t)^{p+1} - \frac{1}{q+1}u_\alpha(t)^{q+1}\Big)\Big] = 0.$$

By this, (2.2) and putting $t = 0$, for $-1 \le t \le 1$, we obtain

$$\begin{aligned} &u_\alpha'(t)^2 + 2\lambda\Big(\frac{1}{p+1}u_\alpha(t)^{p+1} - \frac{1}{q+1}u_\alpha(t)^{q+1}\Big) \\ &= \text{constant} = 2\lambda\Big(\frac{1}{p+1}\alpha^{p+1} - \frac{1}{q+1}\alpha^{q+1}\Big). \end{aligned}$$

By (2.3), for $-1 \le t \le 0$, we obtain

$$u_\alpha'(t) = \sqrt{2\lambda}\sqrt{\frac{1}{p+1}(\alpha^{p+1} - u_\alpha(t)^{p+1}) - \frac{1}{q+1}(\alpha^{q+1} - u_\alpha(t)^{q+1})}. \tag{2.4}$$

By this and putting $\alpha s = u_\alpha(t)$, we obtain

$$\begin{aligned} \sqrt{\lambda} &= \frac{1}{\sqrt{2}}\int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{(\alpha^{p+1} - u_\alpha(t)^{p+1})/(p+1) - (q+1)(\alpha^{q+1} - u_\alpha(t))/(q+1)}}dt \\ &= \sqrt{\frac{p+1}{2}}\alpha^{(1-p)/2}\int_0^1 \frac{1}{\sqrt{1 - s^{p+1} - (p+1)\alpha^{q-p}(1 - s^{q+1})/(q+1)}}ds. \end{aligned} \tag{2.5}$$

We put

$$A_p(\alpha) := \int_0^1 \frac{1}{\sqrt{1 - s^{p+1} - (p+1)\alpha^{q-p}(1 - s^{q+1})/(q+1)}}ds. \tag{2.6}$$

Furthermore, we put for $0 \le s \le 1$,

$$h(s) := 1 - s^{p+1} - \frac{p+1}{q+1}\alpha^{q-p}(1 - s^{q+1}). \tag{2.7}$$

It is clear that

$$h'(s) = -(p+1)s^p + (p+1)s^q\alpha^{q-p}, \tag{2.8}$$

$$h''(s) = (p+1)s^{p-1}(qs^{q-p}\alpha^{q-p} - p). \tag{2.9}$$

Lemma 2.1. *Let an arbitrary $0 < \delta \ll 1$ be fixed. Then as $\alpha \to 1$,*

$$A_p(\alpha) = \sqrt{\frac{2}{(p+1)(q-p)}}(1 + O(\delta))|\log(1 - \alpha^{q-p})| + O(\delta^{-1}). \tag{2.10}$$

Proof. In what follows, C denotes various positive constants independent of α and δ. We apply the argument of [15, Theorem 1.1] to the proof of (2.10). By (2.6), we put

$$A_p(\alpha) = K_1 + K_2 := \int_0^{1-\delta} \frac{1}{\sqrt{h(s)}} ds + \int_{1-\delta}^1 \frac{1}{\sqrt{h(s)}} ds. \tag{2.11}$$

We first calculate K_1. For $0 \le s \le 1$, we have

$$h(s) > h_1(s) := 1 - s^{p+1} - \frac{p+1}{q+1}(1 - s^{q+1}). \tag{2.12}$$

Then for $0 \le s \le 1$,

$$h_1'(s) = (p+1)s^p(-1 + s^{q-p}) \le 0. \tag{2.13}$$

This implies that for $0 \le s < 1 - \delta$, by Taylor expansion,

$$h(s) > h_1(s) > h_1(1-\delta) \ge C\delta^2. \tag{2.14}$$

By this and (2.11), we obtain

$$K_1 \le C \int_0^{1-\delta} \frac{1}{\sqrt{\delta^2}} ds = O(\delta^{-1}). \tag{2.15}$$

We next calculate K_2. By (2.8), (2.9) and Taylor expansion, for $1 - \delta < s < 1$, there exists $s < \xi_s < 1$ such that

$$\begin{aligned} h(s) &= h(1) + h'(1)(s-1) + \frac{1}{2}h''(\xi_s)(s-1)^2 \\ &= (p+1)(1 - \alpha^{q-p})(1-s) + \frac{1}{2}(p+1)\xi_s^{p-1}(q\xi_s^{q-p}\alpha^{q-p} - p)(1-s)^2 \\ &= d_1(1-s)^2 + d_2(1-s), \end{aligned} \tag{2.16}$$

where

$$d_1 := \frac{1}{2}(p+1)\xi_s^{p-1}(q\xi_s^{q-p}\alpha^{q-p} - p), \quad d_2 := (p+1)(1 - \alpha^{q-p}). \tag{2.17}$$

By this, (2.16) and putting $x = 1 - s$, we obtain

$$
\begin{aligned}
K_2 &= \int_0^\delta \frac{1}{\sqrt{d_1 x^2 + d_2 x}} dx \\
&= \frac{1}{\sqrt{d_1}} \big[\log |2d_1 x + d_2 + 2\sqrt{d_1(d_1 x^2 + d_2 x)}| \big]_0^\delta \\
&= \frac{1}{\sqrt{d_1}} \Big(\log(2d_1\delta + d_2 + 2\sqrt{d_1(d_1\delta^2 + d_2\delta)}) - \log |d_2| \Big) \\
&= \frac{1}{\sqrt{d_1}} (|\log(C\delta + C(1-\alpha^{q-p}))| - \log(p+1) + |\log(1-\alpha^{q-p})|) \\
&= \frac{1}{\sqrt{d_1}} (O(|\log \delta|) + |\log(1-\alpha^{q-p})|).
\end{aligned} \tag{2.18}
$$

We have

$$
\frac{1}{\sqrt{d_1}} = \sqrt{\frac{2}{(p+1)(q-p)}} + \sqrt{\frac{2}{p+1}} L_1, \tag{2.19}
$$

where

$$
L_1 := \sqrt{\frac{1}{\xi_s^{p-1}(q\xi_s^{q-p}\alpha_p^{q-p} - p)}} - \sqrt{\frac{1}{q-p}}. \tag{2.20}
$$

Then by direct calculation, as $\alpha \to 1$, we obtain

$$
L_1 \le C(1 - \alpha^{q-p} + \delta) \le C\delta. \tag{2.21}
$$

By this, (2.11), (2.15), (2.18) and (2.19), we obtain (2.10). The proof is complete $\square$

Proof of Theorem 1.2. By (2.5), (2.6) and Lemma 2.1, we obtain (1.8). We next prove (1.9). By (2.6) and Taylor expansion theorem, for $0 < \alpha \ll 1$, we obtain

$$
\begin{aligned}
A_p(\alpha) &= \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}\sqrt{1 - \frac{p+1}{q+1}\alpha^{q-p}\frac{1-s^{q+1}}{1-s^{p+1}}}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} \big\{ 1 + \frac{p+1}{2(q+1)} \alpha^{q-p} \frac{1-s^{q+1}}{1-s^{p+1}} + O(\alpha^{2(q-p)}) \big\} ds \\
&= b_0 + b_1 \alpha^{q-p} + O(\alpha^{2(q-p)}).
\end{aligned} \tag{2.22}
$$

By this and (2.5), we obtain (1.9). Thus the proof is complete. $\square$

3. Proof of Theorem 1.3

In this section, we study the asymptotic behavior of α_p as $p \to 1$. We put

$$
B_p(\alpha) := \int_0^1 \frac{1 - s^{q+1}}{\{1 - s^{p+1} - (p+1)\alpha^{q-p}(1-s^{q+1})/(q+1)\}^{3/2}} ds. \tag{3.1}
$$

By this, (2.5) and (2.6), we have

$$
(\sqrt{\lambda(\alpha)})' = \sqrt{\frac{p+1}{2}} \alpha^{-(1+p)/2} \big\{ -\frac{p-1}{2} A_p(\alpha) + \frac{(p+1)(q-p)}{2(q+1)} B_p(\alpha) \alpha^{q-p} \big\}. \tag{3.2}
$$

Lemma 3.1. $\alpha_p \to 0$ *as* $p \to 1$.

Proof. Since $\lambda'(\alpha) = 2\sqrt{\lambda(\alpha)}(\sqrt{\lambda(\alpha)})'$, by (3.2), we consider the equation

$$-\frac{p-1}{2}A_p(\alpha_p) + \frac{(p+1)(q-p)}{2(q+1)}B_p(\alpha_p)\alpha_p^{q-p} = 0. \tag{3.3}$$

Then there are three cases to consider. Let an arbitrary $0 < \delta \ll 1$ be fixed.

Case 1. Assume that there exists a subsequence of $\{\alpha_p\}$, denoted by $\{\alpha_p\}$ again, and constants c_0 and c_1 such that $0 < c_0 < \alpha_p < c_1 < 1$ for $\lambda \gg 1$. Then for $0 < p - 1 \ll 1$,

$$\begin{aligned} B_p(\alpha_p) &= \int_0^{1-\delta} \frac{1-s^{q+1}}{h(s)^{3/2}}ds + \int_{1-\delta}^1 \frac{1-s^{q+1}}{h(s)^{3/2}}ds \\ &=: I_1 + I_2 \\ &> I_1 > C\int_0^{1-\delta} \frac{(q+1)\delta}{[1-(p+1)c_0^{q-p}/(q+1)]^{3/2}}ds \\ &= C_{1,\delta} > 0. \end{aligned} \tag{3.4}$$

By (2.11), we also obtain

$$\begin{aligned} A_p(\alpha_p) = K_1 + K_2 &:= \int_0^{1-\delta} \frac{1}{\sqrt{h(s)}}ds + \int_{1-\delta}^1 \frac{1}{\sqrt{h(s)}}ds \\ &< C_{2,\delta} + K_2. \end{aligned} \tag{3.5}$$

By Taylor expansion and (2.8), for $1 - \delta < s < 1$, we have

$$\begin{aligned} h(s) &= h(1) + h'(\tilde{s})(s-1) = (p+1)\tilde{s}^p(1 - \alpha_p^{q-p}\tilde{s}^{q-p})(1-s) \\ &> (p+1)(1-\delta)^p(1-\alpha_p^{q-p})(1-s) \\ &> (p+1)(1-\delta)^p(1-c_1^{q-p})(1-s), \end{aligned} \tag{3.6}$$

where $s < \tilde{s} < 1$. By this and (3.5), we obtain

$$\begin{aligned} A_p &< C_{2,\delta} + K_2 \\ &< C_{2,\delta} + \int_{1-\delta}^1 \frac{1}{\sqrt{(p+1)(1-\delta)^p(1-c_1^{q-p})(1-s)}}ds = C_{3,\delta}. \end{aligned} \tag{3.7}$$

By this and (3.3), we obtain

$$\frac{p-1}{2}C_{3,\delta} > \frac{p-1}{2}A_p = \frac{(p+1)(q-p)}{2(q+1)}B_p\alpha_p^{q-p} > \frac{(p+1)(q-p)}{2(q+1)}C_{1,\delta}c_0^{q-p}. \tag{3.8}$$

This is a contradiction, since $0 < p - 1 \ll 1$.

Case 2. We assume that $\alpha_p \to 1$ and derive a contradiction. By (2.18), (3.4), Taylor expansion and putting $x = 1 - s$, we obtain

$$\begin{aligned} I_2 &\ge C\int_0^\delta \frac{(q+1)x}{(d_1x^2 + d_2x)^{3/2}}dx \\ &= C\Big[\frac{2d_2(q+1)x}{d_2^2\sqrt{d_1x^2 + d_2x}}\Big]_0^\delta \\ &= \frac{2C(q+1)\delta}{d_2\sqrt{d_1\delta^2 + d_2\delta}}. \end{aligned} \tag{3.9}$$

By this, (2.17) and (3.2), as $\alpha_p \to 1$, we obtain

$$0 = (\sqrt{\lambda(\alpha_p)})' \ge -C|\log(1-\alpha_p^{q-p})| + C_\delta \frac{1}{1-\alpha_p^{q-p}} \to \infty. \tag{3.10}$$

This is a contradiction. Thus the proof is complete. □

Proof of Theorem 1.3. By (2.7), Lemma 3.1 and Taylor expansion, we obtain

$$\begin{aligned}
&B_p(\alpha_p)\\
&= \int_0^1 \frac{1-s^{q+1}}{(1-s^{p+1})^{3/2}(1-\frac{p+1}{q+1}\alpha_p^{q-p}\frac{1-s^{q+1}}{1-s^{p+1}})^{3/2}}ds\\
&= \int_0^1 \frac{1-s^{q+1}}{(1-s^{p+1})^{3/2}}\Big\{1+\frac{3(p+1)}{2(q+1)}\alpha_p^{q-p}\frac{1-s^{q+1}}{1-s^{p+1}} + O(\alpha^{2(q-p)})\Big\}ds\\
&= k_0 + k_1\alpha^{q-p} + O(\alpha^{2(q-p)}),
\end{aligned} \tag{3.11}$$

where k_0 is a constant defined in (1.13) and

$$k_1 = \frac{3(p+1)}{2(q+1)}\int_0^1 \frac{(1-s^{q+1})^2}{(1-s^{p+1})^{5/2}}ds. \tag{3.12}$$

By this, (2.22) and (3.3), for $0 < p-1 \ll 1$, we obtain

$$\alpha_p^{q-p} = \frac{b_0(q+1)}{k_0(p+1)(q-p)}(p-1) + O((p-1)^2). \tag{3.13}$$

This implies (1.12). Finally, we show (1.14). By (3.13), as $p \to 1$, we obtain

$$\begin{aligned}
\log\alpha_p^{(1-p)/2} &= \frac{1-p}{2}\log\alpha_p\\
&= \frac{1-p}{2(q-p)}\Big(\log(p-1) + \log\Big(\frac{b_0(q+1)(1+o(1))}{k_0(p+1)(q-p)}\Big)\Big) \to 0.
\end{aligned} \tag{3.14}$$

This implies that $\alpha_p^{(1-p)/2} \to 1$ as $p \to 1$. By this, (1.9), (2.5) and (3.12), we obtain

$$\begin{aligned}
\sqrt{\lambda(\alpha_p)} &= \sqrt{\lambda_-(\alpha_p)}\\
&= \sqrt{\frac{p+1}{2}}\Big[\frac{b_0(q+1)}{k_0(p+1)(q-p)}(p-1) + O((p-1)^2)\Big]^{(1-p)/(2(q-p))}\\
&\quad \times (b_0 + b_1\alpha_p^{q-p} + O(\alpha_p^{2(q-p)})\\
&= \sqrt{\frac{p+1}{2}}\Big[\frac{b_0(q+1)}{k_0(p+1)(q-p)}\Big]^{(1-p)/(2(q-p))}\\
&\quad \times (p-1)^{(1-p)/(2(q-p))}(b_0 + O(p-1)).
\end{aligned} \tag{3.15}$$

The proof is complete. □

4. Proof of Theorem 1.4

In this section, we prove Theorems 1.4 by following the strategy of the proof of Theorems 1.2 and 1.3.

Lemma 4.1. *$\alpha_p \to 1$ as $p \to q$.*

Proof. As in the proof of Lemma 3.1, we consider (3.3). There are two cases to be considered.

Case 1. Assume that there exists a subsequence of $\{a_p\}$, denoted by $\{a_p\}$ again, and constants c_0 and c_1 such that $0 < c_0 < \alpha_p < c_1 < 1$ for $\lambda \gg 1$. Let $0 < \delta \ll 1$ be a fixed constant. By (3.1), we obtain

$$B_p(\alpha_p) < \int_0^1 \frac{1 - s^{q+1}}{(1 - s^{p+1} - (p+1)c_1^{q-p}(1 - s^{q+1})/(q+1))^{3/2}} ds < C. \tag{4.1}$$

Since $A_p(\alpha_p) > C_{4,\delta}$ by (3.5), by (3.3) and (4.1), we have

$$\begin{aligned} &-\frac{p-1}{2}A_p(\alpha_p) + \frac{(p+1)(q-p)}{2(q+1)}B_p(\alpha_p)\alpha_p^{q-p} \\ &< -\frac{p-1}{2}C_{4,\delta} + \frac{(p+1)(q-p)}{2(q+1)}C < 0. \end{aligned} \tag{4.2}$$

This is a contradiction.

Case 2. Assume that there exists a subsequence of $\{\alpha_p\}$, denoted by $\{\alpha_p\}$ again, such that $\alpha_p \to 0$. Then clearly, we have

$$A_p \to \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} ds, \quad B_p \to \int_0^1 \frac{1 - s^{q+1}}{(1 - s^{p+1})^{3/2}} ds. \tag{4.3}$$

This implies

$$-\frac{p-1}{2}A_p(\alpha_p) + \frac{(p+1)(q-p)}{2(q+1)}B_p(\alpha_p)\alpha_p^{q-p} < 0. \tag{4.4}$$

This is a contradiction that completes the proof. □

Proof of Theorem 1.4. We calculate $B_p(\alpha_p)$ by using (3.4). Let $0 < \delta_0 \ll 1$ be a fixed constant. Then for $0 < \delta < \delta_0$, we put

$$I_1 := I_{1,1} + I_{1,2} = \int_0^{1-\delta_0} \frac{1 - s^{q+1}}{h(s)^{3/2}} ds + \int_{1-\delta_0}^{1-\delta} \frac{1 - s^{q+1}}{h(s)^{3/2}} ds. \tag{4.5}$$

It is cleat that $I_{1,1} = O(1)$. By (2.16), (2.17), (2.18) and Taylor expansion, we have

$$\begin{aligned} I_{1,2} &\le C\int_\delta^{\delta_0} \frac{(q+1)x}{(d_1x^2 + d_2x)^{3/2}} dx \\ &\le C\int_\delta^{\delta_0} \frac{(q+1)x}{d_1x^2 + d_2x} \frac{1}{\sqrt{d_1x^2 + d_2x}} dx \le C\delta^{-1}\int_\delta^{\delta_0} \frac{1}{\sqrt{d_1x^2 + d_2x}} dx \\ &\le C\delta^{-1}|\log\delta|. \end{aligned} \tag{4.6}$$

By mean value theorem, for $1 - \delta < s < 1$, we have

$$s^{q+1} = 1 + (q+1)\eta_s^q(s-1), \tag{4.7}$$

where $1-\delta < \eta_s < 1$. By this, (2.16) and (2.17), we obtain

$$\begin{aligned} I_2 &= \int_{1-\delta}^{1} \frac{(q+1)\eta_s^q(1-s)}{(d_1(1-s)^2+d_2(1-s))^{3/2}}ds \\ &= \int_0^{\delta} \frac{(q+1)\eta_s^q x}{(d_1x^2+d_2x)^{3/2}}dx \\ &= (q+1)(1+O(\delta))\Big[\frac{2d_2x}{d_2^2\sqrt{d_1x^2+d_2x}}\Big]_0^{\delta} \\ &= 2(q+1)(1+O(\delta))\frac{\delta}{d_2\sqrt{d_1\delta^2+d_2\delta}} \\ &= \frac{2(q+1)}{(p+1)(1-\alpha_p^{q-p})}(1+O(\delta))M_\delta, \end{aligned} \tag{4.8}$$

where

$$M_\delta := \frac{\delta}{\sqrt{d_1\delta^2+d_2\delta}}. \tag{4.9}$$

By this and (3.3), we obtain

$$\frac{p-1}{2}A_p(\alpha_p) = (q-p)\Big(M_\delta\frac{\alpha_p^{q-p}}{1-\alpha_p^{q-p}} + O(\delta^{-1}|\log\delta|)\Big). \tag{4.10}$$

This implies

$$\alpha_p^{q-p} = \frac{1}{1+O((q-p)M_\delta/A_p)} = 1 - O\Big(\frac{(q-p)M_\delta}{A_p}\Big). \tag{4.11}$$

By (2.17) and (4.9), we have

$$M_\delta \le \sqrt{\frac{\delta}{d_2}} \le C\frac{1}{\sqrt{1-\alpha_p^{q-p}}}. \tag{4.12}$$

By this, Lemma 2.1, (2.17) and (4.11), we obtain

$$1-\alpha_p^{q-p} \le C\frac{(q-p)^{3/2}}{|\log(1-\alpha_p^{q-p})|\sqrt{1-\alpha_p^{q-p}}}. \tag{4.13}$$

This implies $1-\alpha_p^{q-p} \le C(q-p) \ll 1$, namely,

$$\frac{1}{|\log(1-\alpha_p^{q-p})|} \le C\frac{1}{|\log(q-p)|}. \tag{4.14}$$

By this and (4.13), we obtain Theorem 1.4. Thus the proof is complete. $\square$

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Tetsutaro Shibata
Laboratory of Mathematics, Institute of Engineering, Hiroshima University, Higashi-Hiroshima, 739-8527, Japan
E-mail address: shibata@amath.hiroshima-u.ac.jp