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DETERMINATION OF THE ORDER OF FRACTIONAL DERIVATIVE AND A KERNEL IN AN INVERSE PROBLEM FOR A GENERALIZED TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. A generalized time fractional diffusion equation containing a lower order term of a convolutional form is considered. Inverse problem to determine the order of a fractional derivative and a kernel of the lower order term from measurements of states over the time is posed. Existence, uniqueness and stability of the solution of the inverse problem are proved.

1. Introduction

Subdiffusion processes in porous, fractal, biological etc. media are described by differential equations containing fractional time (time and space) derivatives [1, 2, 13, 14, 27].

In many practical situations parameters of media or model are unknown or scarcely known. They can be determined solving inverse problems for governing differential equations.

Analytical and numerical study of inverse problems for fractional diffusion equations is undergoing an intensive development during the present decade. Series of papers are devoted to problems to determine unknown source terms [4, 19, 22, 25, 28], boundary conditions [8], initial conditions [12], coefficients [3, 15, 11], orders of derivatives [3, 7, 15, 18] and nonlinear terms [9, 20, 21, 23].

Fractional time derivatives in diffusion models result from postulating the power law waiting time density of a stochastic processes going on in micro-level. However, there are no convincing arguments that the waiting time density has to be exactly of the power law. In the present paper we consider a more general model that is governed by an equation that involves "almost" fractional time derivative. Namely, we replace the power function $t^{\beta-1}$ occurring in the fractional derivative by the sum of $t^{\beta-1}$ and a convolution of $t^{\beta-1}$ with an arbitrary kernel m.

We pose an inverse problem to reconstruct β and m from measurements of the states over the time. We prove the existence and uniqueness of the solution of the inverse problem and establish a stability estimate for m with respect to the data. Results are global in time. Moreover, we deduce an explicit formula for β and present a numerical example. The analysis is implemented in the Fourier domain.

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2. Formulation of direct and inverse problems

Continuous time random walk models of subdiffusion with power law waiting time densities yield in macro-level differential equations that contain fractional derivatives of order between 0 and 1. The simplest equation of such kind is [1, 2, 14, 27]

$$u_t(x,t) = D^{1-\beta} u_{xx}(x,t),$$

where u is the state variable, x is the space variable, t is the time and

$$D^{1-\beta} u(x,t) = \frac{d}{dt}\int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} u(x,\tau)d\tau$$

is the Riemann-Liouville fractional derivative of the order $1-\beta$ with $0<\beta<1$.

An equation that corresponds to general waiting time densities is [2, Eq. (10)]

$$u_t = \frac{d}{dt}\int_0^t M(t-\tau) u_{xx}(x,\tau)d\tau, \tag{2.1}$$

where M is an arbitrary function. Because of the physical background, M is positive, decreasing and has a weak singularity at $t=0$. Let us suppose that the function M has the form

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \frac{t^{\beta-1}}{\Gamma(\beta)} * m(t) \tag{2.2}$$

with some kernel m, where $*$ denotes the time convolution; i.e.,

$$v_1 * v_2(t) = \int_0^t v_1(t-\tau) v_2(\tau) d\tau.$$

Then the equation (2.1) reads $u_t = D^{1-\beta}(u_{xx} + m * u_{xx})$.

Our aim is to pose and study an inverse problem to determine the order of the fractional derivative β and the kernel m in this equation. But before we proceed, we generalize this equation a bit:

$$u_t = D^{1-\beta}(u_{xx} + m * u_{xx} + m^0 * u_{xx}) + g. \tag{2.3}$$

The function g is a source term. The inclusion of the addend with m^0 has a mathematical reason. Namely, the study of stability in Section 7 requires a previously proved existence result for an inverse problem that contains the additional term with m^0. Therefore, it makes sense to incorporate this term already from the beginning. On the other hand, m^0 can be interpreted as an initial guess for an unknown kernel of the form $m^0 + m$. In this case, the perturbation part m of the kernel is to be determined in the inverse problem.

Next we transform the equation under consideration to a more common in the mathematical literature form. To this end we introduce the operator of fractional integration I^α defined by the formula

$$I^\alpha v(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * v(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} v(\tau) d\tau.$$

Applying the operator $I^{1-\beta}$ to the equation (2.3), we reach the equivalent equation

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * [u_t - g] = u_{xx} + m * u_{xx} + m^0 * u_{xx}.$$

We mention that the left hand side of (2.7) contains the Caputo derivative of the order β of $\mathcal{u}$, i.e. $\frac{t^{-\beta}}{\Gamma(1-\beta)} * \mathcal{u}_t$.

Let us formulate the following initial-boundary value problem for this equation in a bounded domain $(x_1, x_2) \times (0, T)$:

$$\begin{gathered}\frac{t^{-\beta}}{\Gamma(1-\beta)} * [\mathcal{u}_t(x,t) - \mathcal{g}(x,t)] = \mathcal{u}_{xx}(x,t) + m * \mathcal{u}_{xx}(x,t) + m^0 * \mathcal{u}_{xx}(x,t), \\ (x,t) \in (x_1, x_2) \times (0,T), \\ \mathcal{u}(x,0) = \mathcal{u}_0(x), \quad x \in (x_1, x_2), \\ \mathcal{B}_1 \mathcal{u}(\cdot, t) = b_1(t), \quad \mathcal{B}_2 \mathcal{u}(\cdot,t) = b_2(t), \quad t \in (0,T),\end{gathered} \tag{2.4}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are boundary operators at $x = x_1$ and $x = x_2$, respectively. More precisely,

$$\begin{gathered}\text{for any } j \in \{1;2\} \text{ either } \mathcal{B}_j v = v(x_j) \text{ or} \\ \mathcal{B}_j v = v'(x_j) + \theta_j v(x_j) \text{ with } \theta_j \in \mathbb{R},\ (-1)^j \theta_j \ge 0.\end{gathered} \tag{2.5}$$

Here and in the sequel we use for x- and t-dependent functions $v(x,t)$ the alternative notation $v(\cdot, t)$ that means a function of t with values as functions of x.

To formulate an inverse problem, let us introduce an observation functional Φ that maps functions defined on the interval $[x_1, x_2]$ onto $\mathbb{R}$. For instance, Φ can be defined as follows:

$$\Phi[v] = v(x_0) \quad \text{or} \quad \Phi[v] = v'(x_0) + \vartheta v(x_0) \quad \text{or} \quad \Phi[v] = \int_{x_1}^{x_2} \kappa(x) v(x) dx,$$

where $x_0 \in [x_1, x_2]$, $\vartheta \in \mathbb{R}$, $\kappa : (x_1, x_2) \to \mathbb{R}$ are given. It is natural to assume that Φ does not coincide with any of the boundary operators, i.e. $\Phi \neq \mathcal{B}_1$ and $\Phi \neq \mathcal{B}_2$.

Now we are in a situation to formulate the *inverse problem*. Given $\mathcal{g}, m^0, \mathcal{u}_0, b_1$ and b_2, find the pair (β, m) such that the solution $\mathcal{u}$ of the (direct) problem (2.4) satisfies the additional condition

$$\Phi[\mathcal{u}(\cdot, t)] = \mathcal{h}(t), \quad t \in (0,T), \tag{2.6}$$

where $\mathcal{h}$ is a prescribed function (observation of the physical state $\mathcal{u}$).

It is more convenient to deal with a problem with homogeneous boundary conditions. Then it is possible to interpret the second order space derivative in the equation (2.4) as a linear operator in some functional space. Let $\widehat{u}$ be a function satisfying the nonhomogeneous boundary conditions, i.e. $\mathcal{B}_1 \widehat{u}(\cdot,t) = b_1(t)$ and $\mathcal{B}_2 \widehat{u}(\cdot, t) = b_2(t)$ for $t \in (0,T)$. Performing the change of variables $\mathcal{u} = \widehat{u} + u$, we obtain the following equation and conditions for u:

$$\begin{gathered}\frac{t^{-\beta}}{\Gamma(1-\beta)} * [u_t(x,t) - g(x,t)] \\ = u_{xx}(x,t) + f(x,t) + m * [u_{xx}(x,t) + \psi(x,t)] + m^0 * u_{xx}(x,t), \\ (x,t) \in (x_1, x_2) \times (0,T), \\ u(x,0) = \varphi(x), \quad x \in (x_1, x_2), \\ \mathcal{B}_1 u(\cdot, t) = 0, \quad \mathcal{B}_2 u(\cdot, t) = 0, \quad t \in (0,T),\end{gathered} \tag{2.7}$$

and

$$\Phi[u(\cdot, t)] = h(t), \quad t \in (0,T), \tag{2.8}$$

where $g = \mathcal{g} - \widehat{u}_t$, $\psi = \widehat{u}_{xx}$, $f = \widehat{u}_{xx} + m^0 * \widehat{u}_{xx}$, $\varphi = \mathcal{u}_0 - \widehat{u}(\cdot, 0)$ and $h = \mathcal{h} - \Phi[\widehat{u}]$.

The relations (2.7) form a direct problem for u. The *inverse problem* consists in determining (β, m) such that the solution u of (2.7) satisfies the additional condition (2.8).

In this article we prove well-posedness results for the inverse problem with the component m in spaces $L_p(0,T)$, where $p \in [1,\infty)$. This covers as particular cases functions M of the form $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{i=1}^n c_i t^{s_i-1}$, where $s_i > \beta$ (for with such M, see [7]). Then $m(t) = \sum_{i=1}^n \frac{c_i\Gamma(s_i)}{\Gamma(s_i-\beta)} t^{s_i-\beta-1}$. Another example of m is the exponentially decreasing flux relaxation (memory) kernel $m(t) = \sum_{i=1}^n c_i e^{-\alpha_i t}$, where $\alpha_i > 0$ [24].

3. Abstraction and reformulation in Fourier domain

Let X be a Hilbert space and $A : \mathcal{D}(A) \to X$ be a linear operator with the domain $\mathcal{D}(A) \subseteq X$. Moreover, let $g, f, \psi : (0,T) \to X$, $m^0, h : (0,T) \to \mathbb{R}$, be given functions, $\varphi \in X$ a given element and $\Phi : \mathcal{D}(A) \to \mathbb{R}$ a given linear functional.

In the *abstract inverse problem* we seek for a number β and a function $m : (0,T) \to \mathbb{R}$ such that a solution $u : [0,T] \to X$ of the (forward) problem

$$\begin{gathered}\frac{t^{-\beta}}{\Gamma(1-\beta)} * [u'(t) - g(t)] \\ = Au(t) + f(t) + m * [Au(t) + \psi(t)] + m^0 * Au(t), \quad t \in (0,T), \\ u(0) = \varphi\end{gathered} \tag{3.1}$$

satisfies the additional condition

$$\Phi[u(t)] = h(t), \quad t \in (0,T). \tag{3.2}$$

Firstly, let us formulate a theorem that gives sufficient conditions for the well-posedness of the abstract direct problem (3.1).

Theorem 3.1. *Assume that A is closed and densely defined in X and satisfies the following property:*

$$\rho(A) \supset \Sigma(\beta\pi/2)\,, \;\; \exists M > 0 : \|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda|} \;\forall \lambda \in \Sigma(\beta\pi/2), \tag{3.3}$$

where $\rho(A)$ is the resolvent set of A and $\Sigma(\theta) = \{\lambda \in \mathbb{C} : |\arg\lambda| < \theta\}$. Let X_A be the domain of A endowed with the graph norm $\|z\|_{X_A} = \|z\| + \|Az\|$. Moreover, assume $\varphi \in X_A$, $f, \psi, g \in W_1^1((0,T); X)$ and $m, m^0 \in L^1(0,T)$. Then (3.1) has a unique solution in the space $C([0,T]; X_A)$ and $\frac{t^{-\beta}}{\Gamma(1-\beta)} * u' \in C([0,T]; X)$. The solution continuously depends on φ, f, ψ, g, m and m^0 in norms of the mentioned spaces.

The above theorem follows from [17, Theorem 2.3 and Proposition 1.2].

Remark 3.2. Define $X = L_2(x_1, x_2)$. Then the operator $A = \frac{d^2}{dx^2}$ with the domain

$$D(A) = \{w : z \in W_2^2(x_1,x_2),\, \mathcal{B}_1 w = 0,\, \mathcal{B}_2 w = 0\} \tag{3.4}$$

satisfies the assumptions of Theorem 3.1 (see [10, Theorem 3.1.3]). Consequently, Theorem 3.1 applies to the problem (2.7).

Our next step is to reformulate the abstract inverse problem (3.1), (3.2) in the Fourier domain. Let us further assume that

the spectrum of A is discrete, the eigenvalues λ_i, $i = 1, 2, \dots$ of the operator $-A$ are nonnegative, ordered in the usual manner, i.e. $0 \le \lambda_1 \le \lambda_2 \le \dots$ and the corresponding eigenvectors v_i, $i = 1, 2, \dots$, form an orthonormal basis in X. (3.5)

Remark 3.3. It is well-known that the operator $A = \frac{d^2}{dx^2}$ with the domain (3.4) satisfies the property (3.5).

We expand the functions involved in (3.1), (3.2) as follows:

$$\begin{gathered} u(t) = \sum_{i=1}^{\infty} u_i(t) v_i, \quad g(t) = \sum_{i=1}^{\infty} g_i(t) v_i, \quad f(t) = \sum_{i=1}^{\infty} f_i(t) v_i, \\ \psi(t) = \sum_{i=1}^{\infty} \psi_i(t) v_i, \quad \varphi = \sum_{i=1}^{\infty} \varphi_i v_i, \end{gathered} \tag{3.6}$$

where $u_i : [0,T] \to \mathbb{R}$, $g_i, f_i, \psi_i : (0,T) \to \mathbb{R}$, $\varphi_i \in \mathbb{R}$ are the Fourier coefficients. Moreover, let us denote

$$\gamma_i = \Phi[v_i], \quad i = 1, 2, \dots .$$

Taking the inner product of the equalities (3.1) with the elements v_i, $i = 1, 2, \dots$, and inserting the series of u into (3.2), we obtain

$$\begin{aligned} &\frac{t^{-\beta}}{\Gamma(1-\beta)} * [u_i'(t) - g_i(t)] + \lambda_i u_i(t) \\ &= f_i(t) + m * [\psi_i(t) - \lambda_i u_i(t)] - m^0 * \lambda_i u_i(t), \quad t \in (0,T), \quad u_i(0) = \varphi_i, \end{aligned} \tag{3.7}$$

where $i = 1, 2, \dots$,

$$\sum_{i=1}^{\infty} \gamma_i u_i(t) = h(t), \quad t \in (0,T). \tag{3.8}$$

The relations (3.7) represent the direct problem, reformulated in the Fourier domain. The corresponding *inverse problem* is stated as follows.
Inverse Problem (IP). Given $g_i, f_i, \psi_i, \varphi_i$, $i = 1, 2, \dots$ and m^0, h, find β and m such that solutions u_i of (3.7) satisfy the condition (3.8).

4. Notation and preliminaries

Let us introduce the Bessel potential spaces

$$H_p^s(0,T) = \Big\{ v|_{[0,T]} : v \in H_p^s(\mathbb{R}) = \{ w : \mathcal{F}^{-1}((1+|\omega|^2)^{\frac{s}{2}} \mathcal{F} w) \in L_p(\mathbb{R}) \} \Big\}$$

for $1 < p < \infty$, $s > 0$ and their subspaces

$${}_0H_p^s(0,T) = \{ v|_{[0,T]} : v \in H_p^s(\mathbb{R}), \operatorname{supp} v \subseteq [0,\infty) \}.$$

Here $\mathcal{F}$ is the Fourier transform and the symbol $v|_{[0,T]}$ stands for the restriction onto $[0,T]$ of a function defined on $\mathbb{R}$.

In case $n \in \mathbb{N}$ the space $H_p^n(0,T)$ coincides with the Sobolev space

$$W_p^n(0,T) = \{ w : w^{(j)} \in L_p(0,T),\ j = 0, \dots, n \}.$$

Remark 4.1. When $s \in (0,1)$, $p \in (1, 1/s)$ it holds ${}_0H_p^s[0,T] = H_p^s(0,T)$. On the other hand, when $s \in (0,1)$, $p \in (\frac{1}{s}, \infty)$ the space $H_p^s(0,T)$ is embedded in the space of continuous on $[0,T]$ functions $C[0,T]$ and $w \in H_p^s(0,T) \Leftrightarrow w = w(0) + \overline{w}$, $w(0) \in \mathbb{R}$, $\overline{w} \in {}_0H_p^s(0,T)$ (see [26, p. 27-28]).

We use of the following abbreviation for the norms in Lebesgue spaces $L_p(0,T)$:

$$\|w\|_p := \|w\|_{L_p(0,T)}.$$

Let us formulate a lemma that describes the functions $\frac{t^{s-1}}{\Gamma(s)} * m$ where $m \in L_p(0,T)$.

Lemma 4.2. . *Let $s \in (0,1)$, $p \in (1,\infty)$. The operator of fractional integration of the order s, given by $I^s z = \frac{t^{s-1}}{\Gamma(s)} * z$, is a bijection from $L_p(0,T)$ onto ${}_0H_p^s(0,T)$, the inverse of I^s is the Riemann-Liouville fractional derivative $D^s = \frac{d}{dt} I^{1-s}$ and*

$$\|w\|_{s,p} := \|D^s w\|_p$$

is a norm in ${}_0H_p^s(0,T)$.

The above lemma follows from [26, Corollary 2.8.1].

In our analysis we will use the Mittag-Leffler functions E_β and $E_{\beta,\beta}$ in case $\beta \in (0,1)$. The functions E_β and $E_{\beta,\gamma}$ are defined by the following power series:

$$E_\beta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + 1)}, \quad E_{\beta,\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + \gamma)}.$$

Note that E_β is a generalization of the exponential function. Indeed, in case $\beta = 1$ it holds $E_\beta(t) = e^t$. Like the exponential function, E_β and $E_{\beta,\gamma}$ are also entire functions. Moreover, $E_\beta(-t)$ and $E_{\beta,\gamma}(-t)$ are completely monotonic for $t \in [0,\infty)$ and

$$E_\beta(0) = 1\,, \quad E_{\beta,\beta}(0) = \frac{1}{\Gamma(\beta)}\,, \quad E_\beta' = \frac{1}{\beta} E_{\beta,\beta} \tag{4.1}$$

(see [5]).

Next we prove a lemma that will be applied in a treatment of the direct problem (3.7).

Lemma 4.3. *Let $z \in H_r^{1-\beta}(0,T)$ with some $\beta \in (0,1)$, $r \in (1, \frac{1}{1-\beta})$ and $y \in L_1(0,T)$, $\lambda, w_0 \in \mathbb{R}$. Then the Cauchy problem*

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * w'(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} * y * w'(t) + \lambda w(t) = z(t), \quad t \in (0,T), \quad w(0) = w_0 \tag{4.2}$$

has a unique solution w in the space $W_r^1(0,T)$. This solution has in case $y = 0$ the representation

$$w(t) = w_0 E_\beta\big(-\lambda t^\beta\big) + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}\big[-\lambda(t-\tau)^\beta\big] z(\tau) d\tau. \tag{4.3}$$

Proof. By Lemma 4.2, Remark 4.1 and the relation $w = I^1 w' + w_0$, (4.2) is equivalent to

$$w'(t) + y * w'(t) + \lambda D^{1-\beta}(I^1 w'(t) + w_0) = D^{1-\beta} z(t), \quad t \in (0,T),\ w(0) = w_0.$$

Since $D^{1-\beta}I^1 = D^{1-\beta}I^{1-\beta}I^\beta = I^\beta$, the obtained equation for w' is the Volterra equation of the second kind

$$\begin{aligned} &w'(t) + \int_0^t \Big[y(t-\tau) + \lambda\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}\Big]w'(\tau)d\tau \\ &= D^{1-\beta}z(t) - \lambda w_0\frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t\in(0,T). \end{aligned} \tag{4.4}$$

The right-hand side $D^{1-\beta}z - \lambda w_0\frac{t^{\beta-1}}{\Gamma(\beta)}$ belongs to $L_r(0,T)$. By well-known results for the Volterra equations of the second kind [6], the equation (4.4) has a a unique solution $w' \in L_r(0,T)$. This proves the existence and uniqueness assertions of the lemma.

It remains to prove the formula (4.3). From [5, p. 172-173], it follows that the second addend in (4.3), i.e.

$$\omega(t) := \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}\big[-\lambda(t-\tau)^\beta\big]z(\tau)d\tau$$

solves the equation $D^\beta\omega + \lambda\omega = z$. Since $\omega(0) = 0$ we have $D^\beta\omega = \frac{t^{-\beta}}{\Gamma(1-\beta)} * \omega'$. Consequently, we obtain the relation

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * \omega'(t) + \lambda\omega(t) = z(t), \quad t\in(0,T),\ \omega(0)=0. \tag{4.5}$$

Further, by [5, (4.10.16)], the function $\phi(t) := E_\beta\big(-\lambda t^\beta\big)$ solves the equation

$$D^\beta\phi + \lambda\phi = \frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

This yields $\frac{t^{-\beta}}{\Gamma(1-\beta)} * \phi'(t) + \lambda\phi(t) = 0$. Moreover, $\phi(0) = 1$. Therefore, for the first addend in (4.3), i.e. $\chi(t) := w_0E_\beta\big(-\lambda t^\beta\big)$ the relations

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * \chi'(t) + \lambda\chi(t) = 0, \quad t\in(0,T), \quad \chi(0) = w_0 \tag{4.6}$$

are valid. The summa $w = \omega + \chi$ solves (4.2) with $y = 0$. Summing the formulas of ω and χ we obtain (4.3). □

Let us introduce further auxiliary material. We use the following family of weighted norms in the spaces ${}_0H^s_p(0,T)$ and $L_p(0,T)$:

$$\|w\|_{s,p;\sigma} = \|e^{-\sigma t}D^sw\|_{L_p(0,T)}, \quad \text{and} \quad \|w\|_{p;\sigma} = \|e^{-\sigma t}w\|_{L_p(0,T)},$$

where $\sigma \ge 0$. Evidently, the equivalence relations

$$e^{-\sigma T}\|w\|_{s,p} \le \|w\|_{s,p;\sigma} \le \|w\|_{s,p}\,, \quad e^{-\sigma T}\|w\|_p \le \|w\|_{p;\sigma} \le \|w\|_p \tag{4.7}$$

are valid. Moreover, by the dominated convergence theorem, in case $p < \infty$,

$$\|w\|_{s,p;\sigma} \to 0 \quad \text{and} \quad \|w\|_{p;\sigma} \to 0 \quad \text{as } \sigma\to\infty. \tag{4.8}$$

Lemma 4.4. *Let $\beta \in (0,1)$. Then the functions*

$$\widetilde{E}_{\beta,i}(t) = t^{\beta-1}E_{\beta,\beta}\big[-\lambda_it^\beta\big] \tag{4.9}$$

satisfy the following estimates:

$$\|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma} \le 1 \tag{4.10}$$

$$\|\lambda_i^{1-\epsilon}\widetilde{E}_{\beta,i}\|_{1;\sigma} \le \frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}, \quad 0<\epsilon\le 1, \tag{4.11}$$

for $i=1,2,\dots$, where $c_{\beta,\epsilon}$ is a constant independent of σ and i. The symbol $\|\cdot\|_{1;\sigma}$ denotes the norm $\|\cdot\|_{p;\sigma}$ in case $p=1$.

Proof. Using the positivity of $E_{\beta,\beta}(-t)$ for $t\ge 0$ and (4.1) we deduce

$$\begin{aligned}
\|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma} &= \int_0^T e^{-\sigma t}\lambda_i t^{\beta-1}E_{\beta,\beta}(-\lambda_i t^\beta)dt\\
&\le \int_0^T \lambda_i t^{\beta-1}E_{\beta,\beta}(-\lambda_i t^\beta)dt\\
&= -\int_0^T \frac{d}{dt}E_\beta(-\lambda_i t^\beta)dt = E_\beta(0)-E_\beta(-\lambda_i T^\beta).
\end{aligned}$$

Since $E_\beta(-t)$ is positive for $t\ge 0$ and $E_\beta(0)=1$ we reach (4.10). Further, taking the asymptotical relation $E_{\beta,\beta}(-t)=O(t^{-2})$ as $t\to\infty$ (see [16, Thm. 1.2.1]) into account, we have $\left(\lambda_i t^\beta\right)^\delta E_{\beta,\beta}(-\lambda_i t^\beta)\le c^1_{\beta,\delta}$ for $t\ge 0$ and $0\le\delta\le 2$ with some constant $c^1_{\beta,\delta}$. Thus, for $0<\epsilon\le 1$ we deduce

$$\begin{aligned}
\|\lambda_i^{1-\epsilon}\widetilde{E}_{\beta,i}\|_{1;\sigma} &= \int_0^T e^{-\sigma t}\left(\lambda_i t^\beta\right)^{1-\epsilon}t^{\beta\epsilon-1}E_{\beta,\beta}(-\lambda_i t^\beta)dt\\
&\le c^1_{\beta,1-\epsilon}\int_0^T e^{-\sigma t}t^{\beta\epsilon-1}dt\\
&= \frac{c^1_{\beta,1-\epsilon}}{\sigma^{\beta\epsilon}}\int_0^{\sigma T} e^{-s}s^{\beta\epsilon-1}ds\\
&< \frac{c^1_{\beta,1-\epsilon}}{\sigma^{\beta\epsilon}}\int_0^{\infty} e^{-s}s^{\beta\epsilon-1}ds.
\end{aligned}$$

This implies (4.11). □

Finally, we point out the Young's theorem for convolutions that will be an important tool in our computations:

$$\|w_1*w_2\|_{p_3}\le\|w_1\|_{p_1}\|w_2\|_{p_2}, \quad \text{where } \frac{1}{p_1}+\frac{1}{p_2}=1+\frac{1}{p_3}. \tag{4.12}$$

5. Results for direct problem in Fourier domain

In this section, we prove two propositions for the direct problem (3.7).

Proposition 5.1. *Let $\beta\in(0,1)$, $m,m^0\in L_1(0,T)$ and $f_i,\psi_i\in H_r^{1-\beta}(0,T)$, $g_i\in L_r(0,T)$ with some $r\in(1,\frac{1}{1-\beta})$. Then the problem (3.7) has a unique solution $u_i\in W_r^1(0,T)$. Moreover, the following assertions are valid:*

(i) *if*

$$\|m\|_{1;\sigma}+\|m^0\|_{1;\sigma}\le\frac{1}{2} \tag{5.1}$$

then the estimate

$$\|u_i'\|_{r;\sigma}+\lambda_i\|u_i\|_{1-\beta,r;\sigma}\le C_0\big[\lambda_i|\varphi_i|+\|f_i\|_{1-\beta,r;\sigma}+\|\psi_i\|_{1-\beta,r;\sigma}+\|g_i\|_{r;\sigma}\big] \tag{5.2}$$

holds, where C_0 is a constant independent of σ and i;

(ii) *if* (5.1) *is satisfied and* $f_i, \psi_i \in L_\infty(0,T)$, $I^{1-\beta} g_i \in L_\infty(0,T)$ *then the estimate*

$$\lambda_i \|u_i\|_{\infty;\sigma} \le C_1 \big[\lambda_i |\varphi_i| + \|f_i\|_{\infty;\sigma} + \|\psi_i\|_{\infty;\sigma} + \|I^{1-\beta} g_i\|_{\infty;\sigma}\big] \tag{5.3}$$

holds, where C_1 *is a constant independent of* σ *and* i.

Proof. Since $m, m^0 \in L_1(0,T)$, the Volterra equation of the second kind

$$y(t) + (m+m^0) * y(t) + m(t) + m^0(t) = 0, \quad t \in (0,T),$$

has a unique solution $y \in L_1(0,T)$ (see [6, Theorem 3.1]). From this equation we obtain the operator relations

$$(\mathcal{I} + y*)\big(\mathcal{I} + (m+m^0)*\big) = \big(\mathcal{I} + (m+m^0)*\big)(\mathcal{I} + y*) = \mathcal{I},$$

where $\mathcal{I}$ is the unity operator. Applying the operator $\mathcal{I} + y*$ to the equation in (3.7) we obtain the problem

$$\begin{gathered} \frac{t^{-\beta}}{\Gamma(1-\beta)} * [u_i'(t) + y * u_i'(t)] + \lambda_i u_i(t) = \widetilde{f}_i(t), \quad t \in (0,T), \\ u_i(0) = \varphi_i, \end{gathered} \tag{5.4}$$

where $\widetilde{f}_i(t) = f_i(t) + y * f_i(t) + I^{1-\beta} g_i(t) + y * I^{1-\beta} g_i(t) + (m + y * m) * \psi_i(t)$. Conversely, applying the operator $\mathcal{I} + (m+m^0)*$ to the equation in (5.4), we reach (3.7). Therefore, problems (3.7) and (5.4) are equivalent. From the assumptions of the proposition, Lemma 4.2 and Remark 4.1 we have

$$\begin{aligned} \widetilde{f}_i(t) &= f_i(t) + y * \frac{t^{-\beta}}{\Gamma(1-\beta)} * D^{1-\beta} f_i(t) + I^{1-\beta} g_i(t) \\ &\quad + y * \frac{t^{-\beta}}{\Gamma(1-\beta)} * g_i(t) + (m + y * m) * \frac{t^{-\beta}}{\Gamma(1-\beta)} * D^{1-\beta} \psi_i(t) \\ &= f_i(t) + I^{1-\beta} g_i(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} * \big[y * D^{1-\beta} f_i(t) + y * g_i(t) \\ &\quad + (m + y * m) * D^{1-\beta} \psi_i(t)\big], \end{aligned}$$

where $y * D^{1-\beta} f_i + y * g_i + (m + y * m) * D^{1-\beta} \psi_i \in L_r(0,T)$. This implies $\widetilde{f}_i \in H_r^{1-\beta}(0,T)$. In view of Lemma 4.3, the problem (5.4) has a unique solution in $W_r^1(0,T)$. This proves the existence and uniqueness assertion of the proposition.

Further, let us prove (i). For this purpose, we represent the solution of (3.7) by means of the formula (4.3). Using the abbreviation (4.9) we have

$$\begin{aligned} u_i(t) = \varphi_i E_\beta\big(-\lambda_i t^\beta\big) &+ \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\Big[f_i(\tau) + \frac{\tau^{-\beta}}{\Gamma(1-\beta)} * g_i(\tau) \\ &+ \psi_i * m(\tau)\Big] d\tau - \int_0^t \widetilde{E}_{\beta,i}(t-\tau) \lambda_i u_i * [m(\tau) + m^0(\tau)] d\tau. \end{aligned} \tag{5.5}$$

In view of the relation $\mathcal{I} = I^{1-\beta}D^{1-\beta} = \frac{t^{-\beta}}{\Gamma(1-\beta)} * D^{1-\beta}$ that holds in $H_r^{1-\beta}(0,T)$ we obtain

$$\begin{aligned} u_i(t) = \varphi_i E_\beta\big(-\lambda_i t^\beta\big) &+ \int_0^t \widetilde{E}_{\beta,i}(t-\tau) \\ &\times \Big[\frac{\tau^{-\beta}}{\Gamma(1-\beta)} * \big(D^{1-\beta} f_i(\tau) + g_i(\tau)\big) + \frac{\tau^{-\beta}}{\Gamma(1-\beta)} * D^{1-\beta}\psi_i * m(\tau)\Big] d\tau \\ &- \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\lambda_i \frac{\tau^{-\beta}}{\Gamma(1-\beta)} * D^{1-\beta} u_i * [m(\tau) + m^0(\tau)] d\tau. \end{aligned} \tag{5.6}$$

Applying the operator $D^{1-\beta} = \frac{d}{dt}\frac{t^{\beta-1}}{\Gamma(\beta)} *$ and taking the relations $\frac{t^{\beta-1}}{\Gamma(\beta)} * \frac{t^{-\beta}}{\Gamma(1-\beta)} = 1$ and

$$\frac{d}{dt} E_\beta\big(-\lambda_i t^\beta\big) = -\lambda_i \widetilde{E}_{\beta,i}(t), \tag{5.7}$$

following from (4.1) and (4.9), we reach the expression

$$\begin{aligned} D^{1-\beta} u_i(t) = -\lambda_i\varphi_i \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\frac{\tau^{\beta-1}}{\Gamma(\beta)} d\tau + \varphi_i \frac{t^{\beta-1}}{\Gamma(\beta)} \\ + \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\Big[D^{1-\beta} f_i(\tau) + g_i(\tau) + D^{1-\beta}\psi_i * m(\tau)\Big] d\tau \\ - \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\lambda_i D^{1-\beta} u_i * [m(\tau) + m^0(\tau)] d\tau. \end{aligned}$$

Next we multiply this equality by $\lambda_i e^{-\sigma t}$, bring the factor $e^{-\sigma t}$ inside the integrals and use the relation

$$e^{-\sigma t}[w_1(t) * w_2(t)] = e^{-\sigma t} w_1(t) * e^{-\sigma t} w_2(t).$$

Thereupon we estimate the obtained expression in the norm $\|\cdot\|_r$ and apply (4.12). As a result we obtain

$$\begin{aligned} \lambda_i \|D^{1-\beta} u_i\|_{r;\sigma} \le \lambda_i |\varphi_i| \Big(\|\lambda_i \widetilde{E}_{\beta,i}\|_{1;\sigma} + 1\Big) \Big\|\frac{t^{\beta-1}}{\Gamma(\beta)}\Big\|_{r;\sigma} \\ + \|\lambda_i \widetilde{E}_{\beta,i}\|_{1;\sigma} \Big(\|D^{1-\beta} f_i + g_i\|_{r;\sigma} + \|D^{1-\beta}\psi_i\|_{r;\sigma}\|m\|_{1;\sigma}\Big) \\ + \|\lambda_i \widetilde{E}_{\beta,i}\|_{1;\sigma} \lambda_i \|D^{1-\beta} u_i\|_{r;\sigma} [\|m\|_{1;\sigma} + \|m^0\|_{1;\sigma}]. \end{aligned} \tag{5.8}$$

Using (4.10) we obtain

$$\begin{aligned} \lambda_i \|u_i\|_{1-\beta,r;\sigma} \le 2\hat{c}_{\beta,r}\lambda_i|\varphi_i| + \|f_i\|_{1-\beta,r;\sigma} + \|g_i\|_{r;\sigma} + \|\psi_i\|_{1-\beta,r;\sigma}\|m\|_{1;\sigma} \\ + [\|m\|_{1;\sigma} + \|m^0\|_{1;\sigma}] \cdot \lambda_i \|u_i\|_{1-\beta,r;\sigma}, \end{aligned}$$

where $\hat{c}_{\beta,r} = \|\frac{t^{\beta-1}}{\Gamma(\beta)}\|_r$. In case (5.1) is valid, we estimate $\|m\|_{1;\sigma}$ and $\|m\|_{1;\sigma} + \|m^0\|_{1;\sigma}$ by $\frac{1}{2}$, bring the term $\frac{1}{2}\lambda_i\|u_i\|_{1-\beta,r;\sigma}$ to the left-hand side and multiply the obtained inequality by 2. This results in

$$\lambda_i \|u_i\|_{1-\beta,r;\sigma} \le C_4\big[\lambda_i|\varphi_i| + \|f_i\|_{1-\beta,r;\sigma} + \|\psi_i\|_{1-\beta,r;\sigma} + \|g_i\|_{r;\sigma}\big], \tag{5.9}$$

where C_4 is a constant.

Further, applying $D^{1-\beta}$ to (3.7) we deduce

$$u_i' = -\lambda_i D^{1-\beta} u_i + D^{1-\beta} f_i + g_i + \big(D^{1-\beta}\psi_i - \lambda_i D^{1-\beta} u_i\big) * m - \lambda_i D^{1-\beta} u_i * m^0. \tag{5.10}$$

Here we used that

$$\begin{aligned} D^{1-\beta}(f*m) &= \frac{d}{dt} I^\beta (f*m) = \frac{d}{dt}\big(\frac{t^{\beta-1}}{\Gamma(\beta)} * f * m\big) \\ &= \frac{d}{dt}\big(\frac{t^{\beta-1}}{\Gamma(\beta)} * f\big)*m + \big(\frac{t^{\beta-1}}{\Gamma(\beta)} * f\big)(t)|_{t=0}\cdot m \\ &= D^{1-\beta} f * m + \big(\frac{t^{\beta-1}}{\Gamma(\beta)} * \frac{t^{-\beta}}{\Gamma(1-\beta)} * D^{1-\beta} f\big)(t)|_{t=0}\cdot m \\ &= D^{1-\beta} f * m + \big(I^1 D^{1-\beta} f\big)(t)|_{t=0}\cdot m = D^{1-\beta} f * m \end{aligned}$$

is valid for any $f \in H_r^{1-\beta}(0,T)$. Assuming (5.1) and using (5.9) from (5.10) we obtain

$$\begin{aligned} \|u_i'\|_{r;\sigma} &\le \lambda_i \|u_i\|_{1-\beta,r;\sigma} + \|f_i\|_{1-\beta,r;\sigma} + \|g_i\|_{r;\sigma} \\ &\quad + \big(\|\psi_i\|_{1-\beta,r;\sigma} + \lambda_i\|u_i\|_{1-\beta,r;\sigma}\big)\|m\|_{1;\sigma} + \lambda_i \|u_i\|_{1-\beta,r;\sigma}\|m^0\|_{1;\sigma} \\ &\le C_5\big[\lambda_i|\varphi_i| + \|f_i\|_{1-\beta,r;\sigma} + \|\psi_i\|_{1-\beta,r;\sigma} + \|g_i\|_{r;\sigma}\big] \end{aligned} \tag{5.11}$$

with a constant C_5. Adding (5.9) and (5.11) we reach (5.2).

Finally, let us prove (ii). To this end, let us return to the equality (5.5). Multiplying (5.5) by $\lambda_i e^{-\sigma t}$ and estimating the result we obtain

$$\begin{aligned} \lambda_i\|u_i\|_{\infty;\sigma} &\le \lambda_i|\varphi_i| + \|\lambda_i \widetilde{E}_{\beta,i}\|_{1;\sigma}\Big(\|f_i + I^{1-\beta} g_i\|_{\infty;\sigma} + \|\psi_i\|_{\infty;\sigma}\|m\|_{1;\sigma}\Big) \\ &\quad + \|\lambda_i \widetilde{E}_{\beta,i}\|_{1;\sigma}\lambda_i\|u_i\|_{\infty;\sigma}[\|m\|_{1;\sigma} + \|m^0\|_{1;\sigma}]. \end{aligned}$$

Observing (4.10) and (5.1) we deduce (5.3). □

Proposition 5.2. *Let $\beta \in (0,1)$, $m, m^0 \in L_p(0,T)$ with some $p \in (1,\infty)$ and $f_i \in W_p^1(0,T)$, $\psi_i \in W_1^1(0,T)$, $g_i \in {}_0H_p^\beta(0,T)$. Then $u_i' + q_{\beta,i} \in {}_0H_p^\beta(0,T)$, where u_i is the solution of (3.7) and*

$$q_{\beta,i}(t) = \big(\lambda_i\varphi_i - f_i(0)\big)E_{\beta,\beta}\big(-\lambda_i t^\beta\big)t^{\beta-1} = \big(\lambda_i\varphi_i - f_i(0)\big)\widetilde{E}_{\beta,i}(t). \tag{5.12}$$

Moreover, in the case

$$T^{\frac{p-1}{p}}\big(\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}\big) \le \frac{1}{2} \tag{5.13}$$

the estimates

$$\begin{aligned} \lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma} &\le C_2\|\lambda_i \widetilde{E}_{\beta,i}\|_{1;\sigma}\Big(\lambda_i|\varphi_i| + |f_i(0)| \\ &\quad + (|\psi_i(0)| + \|\psi_i'\|_{1;\sigma})\|m\|_{p;\sigma} + \|f_i'\|_{p;\sigma} + \|g_i\|_{\beta,p;\sigma}\Big), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \|u_i' + q_{\beta,i}\|_{\beta,p;\sigma} &\le C_3\Big(\lambda_i|\varphi_i| + |f_i(0)| \\ &\quad + (|\psi_i(0)| + \|\psi_i'\|_{1;\sigma})\|m\|_{p;\sigma} + \|f_i'\|_{p;\sigma} + \|g_i\|_{\beta,p;\sigma}\Big), \end{aligned} \tag{5.15}$$

hold, where C_2 and C_3 are constants independent of σ and i.

Before proving Proposition 5.2, we prove a lemma concerning the function $q_{\beta,i}$.

Lemma 5.3. *The function $q_{\beta,i}$ satisfies the equations*

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * q_{\beta,i} + \lambda_i I^1 q_{\beta,i} = \lambda_i\varphi_i - f_i(0), \tag{5.16}$$

$$D^\beta q_{\beta,i} + \lambda_i q_{\beta,i} = 0. \tag{5.17}$$

Proof. In case $\lambda_i = 0$ we have $q_{\beta,i}(t) = -f_i(0)\frac{t^{\beta-1}}{\Gamma(\beta)}$ and since $\frac{t^{-\beta}}{\Gamma(1-\beta)} * \frac{t^{\beta-1}}{\Gamma(\beta)} = 1$, the relations (5.16) and (5.17) are immediate. Let $\lambda_i > 0$. By Lemma 4.3, the function $\bar{q}_i(t) = E_\beta\big(-\lambda_i t^\beta\big)$ is a solution of the equation $\frac{t^{-\beta}}{\Gamma(1-\beta)} * \bar{q}_i{}' + \lambda_i \bar{q}_i = 0$. Multiplying this equation by $\frac{1}{\lambda_i}\big(f_i(0) - \lambda_i\varphi_i\big)$ we obtain

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * \frac{1}{\lambda_i}\big(f_i(0) - \lambda_i\varphi_i\big)\bar{q}_i{}' + \big(f_i(0) - \lambda_i\varphi_i\big)\bar{q}_i = 0. \tag{5.18}$$

On the other hand, in view of (4.1) and the definitions of $\bar{q}_i$, $q_{\beta,i}$ it holds the formula $\frac{1}{\lambda_i}\big(f_i(0)-\lambda_i\varphi_i\big)\bar{q}_i' = q_{\beta,i}$. Integrating, multiplying by λ_i and observing that $\bar{q}_i(0) = 1$ we have another formula $\big(f_i(0) - \lambda_i\varphi_i\big)\bar{q}_i(t) = \lambda_i I^1 q_{\beta,i}(t) + f_i(0) - \lambda_i\varphi_i$. Using these relations in (5.18) we arrive at (5.16). Finally, differentiating (5.16) we come to (5.17). □

Proof of Proposition 5.2. Since $W_1^1(0,T) \subset H_r^{1-\beta}(0,T)$ and ${}_0H_p^\beta(0,T) \subset L_r(0,T)$ for $r \in (0, \frac{1}{1-\beta})$, $r \le p$, by Proposition 5.1, problem (3.7) has a unique solution $u_i \in W_r^1(0,T)$. Differentiating (5.5) and observing (5.7), (5.12) we obtain

$$\begin{aligned} &u_i'(t) + q_{\beta,i}(t)\\ &= \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\Big[f_i'(\tau) + D^\beta g_i(\tau) + \psi_i(0)m(\tau) + \psi_i' * m(\tau)\Big]d\tau\\ &\quad - \int_0^t \widetilde{E}_{\beta,i}(t-\tau)\lambda_i\Big[\varphi_i\big(m(\tau) + m^0(\tau)\big) + u_i' * \big(m(\tau) + m^0(\tau)\big)\Big]d\tau. \end{aligned} \tag{5.19}$$

From $u_i' \in L_1(0,T)$ and the assumptions of the proposition, the right-hand side of this relation belongs to $L_p(0,T)$. Therefore, $u_i' + q_{\beta,i} \in L_p(0,T)$. Multiplying (5.19) by $\lambda_i e^{-\sigma t}$, representing u_i' as $u_i' = -q_{\beta,i} + u_i' + q_{\beta,i}$ at the right-hand side and using (4.12) as well as the relation $\|m\|_{1;\sigma} \le T^{\frac{p-1}{p}}\|m\|_{p;\sigma}$ we obtain

$$\begin{aligned} \lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma} &\le \|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}\big(\|f_i'\|_{p;\sigma} + \|g_i\|_{\beta,p;\sigma}\\ &\quad + |\psi_i(0)|\|m\|_{p;\sigma} + \|\psi_i'\|_{1;\sigma}\|m\|_{p;\sigma}\big)\\ &\quad + \|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}\big(\lambda_i|\varphi_i| + \lambda_i\|q_{\beta,i}\|_{1,\sigma}\big)\big(\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}\big)\\ &\quad + \|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}T^{\frac{p-1}{p}}\lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma}\big(\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}\big). \end{aligned}$$

Note that

$$\lambda_i\|q_{\beta,i}\|_{1,\sigma} \le \lambda_i|\varphi_i| + |f_i(0)| \tag{5.20}$$

by (4.10) and (5.12). Thus, using (4.10) we deduce

$$\begin{aligned} \lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma} &\le \|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}\Big[\|f_i'\|_{p;\sigma} + (|\psi_i(0)| + \|\psi_i'\|_{1;\sigma})\|m\|_{p;\sigma}\\ &\quad + \|g_i\|_{\beta,p;\sigma} + \big(2\lambda_i|\varphi_i| + |f_i(0)|\big)\big(\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}\big)\Big]\\ &\quad + T^{\frac{p-1}{p}}\big(\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}\big)\cdot\lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma}. \end{aligned}$$

In the case (5.13), from this relation we obtain (5.14).

Further, differentiating (3.7) we have

$$\begin{aligned} D^\beta u_i' + \lambda_i u_i' &= f_i' + D^\beta g_i + \big(\psi_i(0) - \lambda_i\varphi_i\big)m + (\psi_i' - \lambda_i u_i') * m\\ &\quad - \lambda_i\varphi_i m^0 - \lambda_i u_i' * m^0. \end{aligned} \tag{5.21}$$

Adding (5.21) and (5.17) we obtain

$$\begin{aligned} D^\beta(u_i' + q_{\beta,i}) = -\lambda_i(u_i' + q_{\beta,i}) + f_i' + D^\beta g_i + \big(\psi_i(0) - \lambda_i\varphi_i\big)m \\ + (\psi_i' - \lambda_i u_i') * m - \lambda_i\varphi_i m^0 - \lambda_i u_i' * m^0. \end{aligned} \tag{5.22}$$

By the assumptions of the proposition and the relations $u_i' \in L_1(0,T)$, $u_i' + q_{\beta,i} \in L_p(0,T)$, the right-hand side of (5.22) belongs to $L_p(0,T)$. Therefore, $D^\beta(u_i' + q_{\beta,i}) \in L_p(0,T)$. This implies the assertion $u_i' + q_{\beta,i} \in {}_0H_p^\beta(0,T)$. Estimating (5.22) we have

$$\begin{aligned} &\|u_i' + q_{\beta,i}\|_{\beta,p;\sigma} \\ &\le \lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma} + \|f_i'\|_{p;\sigma} + \|g_i\|_{\beta,p;\sigma} + \Big[|\psi_i(0)| + \|\psi_i'\|_{1;\sigma}\Big]\|m\|_{p;\sigma} \\ &\quad + \Big[\lambda_i|\varphi_i| + \lambda_i\|q_{\beta,i}\|_{1;\sigma} + T^{\frac{p-1}{p}}\lambda_i\|u_i' + q_{\beta,i}\|_{p;\sigma}\Big]\big(\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}\big). \end{aligned}$$

Using (5.14) for $\lambda_i\|u_i' + q_i\|_{p;\sigma}$, (4.10) for $\|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}$, (5.20) for $\lambda_i\|q_{\beta,i}\|_{1;\sigma}$, and estimating $\|m\|_{p;\sigma} + \|m^0\|_{p;\sigma}$ by $\frac{1}{2}T^{\frac{p}{p-1}}$ we obtain (5.15). □

6. Uniqueness

In the sequel we use the notations $u_i[\beta,m]$ and $u_i[m]$ to indicate the dependence of the solution of (3.7) on the pair (β,m) and m.

Theorem 6.1. *Let $f_i, \psi_i \in H_r^{1-s_1}(0,T)\cap C[0,T]$, $g_i \in L_r(0,T)$, $I^{1-s_2}g_i \in L_\infty[0,T]$, $i = 1,2,\dots$ with some $s_1 \in [0,1)$, $s_2 \in (0,1]$, $r \in (1,\infty)$ and $m^0 \in L_1(0,T)$. Moreover, assume*

$$\begin{gathered} \sum_{i=1}^\infty |\gamma_i|\lambda_i|\varphi_i| < \infty, \quad \sum_{i=1}^\infty |\gamma_i|\|f_i\|_{1-s_1,r} < \infty, \quad \sum_{i=1}^\infty |\gamma_i|\|f_i\|_\infty < \infty, \\ \sum_{i=1}^\infty |\gamma_i|\|\psi_i\|_{1-s_1,r} < \infty, \quad \sum_{i=1}^\infty |\gamma_i|\|\psi_i\|_\infty < \infty, \\ \sum_{i=1}^\infty |\gamma_i|\|g_i\|_r < \infty, \quad \sum_{i=1}^\infty |\gamma_i|\|I^{1-s_2}g_i\|_\infty < \infty \end{gathered} \tag{6.1}$$

and

$$\sum_{i=1}^\infty \gamma_i(\lambda_i\varphi_i - f_i(0)) \ne 0, \quad \sum_{i=1}^\infty \gamma_i(\lambda_i\varphi_i - \psi_i(0)) \ne 0. \tag{6.2}$$

If $(\beta_j, m_j) \in (s_1,s_2)\times L_1(0,T)$, $j = 1,2$, are solutions of the inverse problem, then $\beta_1 = \beta_2$ and $m_1 = m_2$.

Proof. Without loss of generality we may assume $r < \min\{\frac{1}{1-\beta_1}; \frac{1}{1-\beta_2}\}$. In view of Proposition 5.1, the problems (3.7) with the data $(\beta_j, m_j) \in (s_1,s_2)\times L_1(0,T)$ have unique solutions $u_{j,i} := u_i[\beta_j, m_j] \in W_r^1(0,T) \subset C[0,T]$, $i = 1,2,\dots$, $j = 1,2$. Due to (4.8) there exists $\sigma > 0$ such that $\|m_j\|_{1;\sigma} + \|m^0\|_{1;\sigma} \le \frac{1}{2}$, $j = 1,2$. In view of the estimates (5.2), (5.3), the assumptions (6.1) and the equivalence relations of weighted norms (4.7) we have

$$\sum_{i=1}^\infty |\gamma_i|\lambda_i\|u_{j,i}\|_\infty < \infty, \quad \sum_{i=1}^\infty |\gamma_i|\|u_{j,i}'\|_r < \infty. \tag{6.3}$$

This implies $\sum_{i=1}^{\infty} \gamma_i \lambda_i u_{j,i} \in C[0,T]$, $\sum_{i=1}^{\infty} \gamma_i \lambda_i u_{j,i}\big|_{t=0} = \sum_{i=1}^{\infty} \gamma_i \lambda_i \varphi_i$ and $\sum_{i=1}^{\infty} \gamma_i u'_{j,i} \in L_r(0,T)$.

Moreover, from (3.8) we obtain $h' = \sum_{i=1}^{\infty} \gamma_i u'_{j,i}$, $j = 1,2$. In view of this relation, from (3.7) we deduce the expressions

$$\begin{aligned} &\frac{t^{-\beta_j}}{\Gamma(1-\beta_j)} * (h' - \sum_{i=1}^{\infty} \gamma_i g_i) \\ &= \sum_{i=1}^{\infty} \gamma_i (f_i - \lambda_i u_{j,i}) + \sum_{i=1}^{\infty} \gamma_i (\psi_i - \lambda_i u_{j,i}) * m_j - \sum_{i=1}^{\infty} \gamma_i \lambda_i u_{j,i} * m^0, \end{aligned} \tag{6.4}$$

for $j = 1,2$. From the relations $f_i, \psi_i \in C[0,T]$ and the third and fifth inequality in (6.1) we have $\sum_{i=1}^{\infty} \gamma_i f_i \in C[0,T]$ and $\sum_{i=1}^{\infty} \gamma_i \psi_i \in C[0,T]$. Therefore, the right-hand side of (6.4) belongs to $C[0,T]$. We obtain $\frac{t^{-\beta_j}}{\Gamma(1-\beta_j)} * (h' - \sum_{i=1}^{\infty} \gamma_i g_i) \in C[0,T]$, $j = 1,2$. Taking the limit $t \to 0^+$ in (6.4), we have

$$\lim_{t\to 0^+} \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)} * (h' - \sum_{i=1}^{\infty} \gamma_i g_i) = \sum_{i=1}^{\infty} \gamma_i (f_i(0) - \lambda_i \varphi_i), \quad j = 1,2. \tag{6.5}$$

Suppose that $\beta_1 < \beta_2$. Then

$$\frac{t^{-\beta_1}}{\Gamma(1-\beta_1)} * (h' - \sum_{i=1}^{\infty} \gamma_i g_i) = \frac{t^{\beta_2-\beta_1-1}}{\Gamma(\beta_2-\beta_1)} * \zeta(t), \quad \zeta(t) = \frac{t^{-\beta_2}}{\Gamma(1-\beta_2)} * (h' - \sum_{i=1}^{\infty} \gamma_i g_i).$$

Since $\zeta \in C[0,T]$ it holds $\lim_{t\to 0^+} \frac{t^{\beta_2-\beta_1-1}}{\Gamma(\beta_2-\beta_1)} * \zeta(t) = 0$. Thus, $\lim_{t\to 0^+} \frac{t^{-\beta_1}}{\Gamma(1-\beta_1)} * (h' - \sum_{i=1}^{\infty} \gamma_i g_i) = 0$. But this with (6.5) contradicts to the assumption (6.2). Similarly we reach the contradiction in case $\beta_1 > \beta_2$. Consequently, $\beta_1 = \beta_2$.

Denote $\beta := \beta_1 = \beta_2$ and subtract the equalities (6.4) with $j = 2$ and $j = 1$:

$$\begin{aligned} &\sum_{i=1}^{\infty} \gamma_i (\psi_i - \lambda_i u_{1,i}) * (m_1 - m_2) - \sum_{i=1}^{\infty} \gamma_i \lambda_i (u_{1,i} - u_{2,i}) * (m_2 + m^0) \\ &- \sum_{i=1}^{\infty} \gamma_i \lambda_i (u_{1,i} - u_{2,i}) = 0. \end{aligned} \tag{6.6}$$

The differences $v_i = u_{1,i} - u_{2,i}$, $i = 1,2,\dots$, solve the problems

$$\begin{gathered} \frac{t^{-\beta}}{\Gamma(1-\beta)} * v_i' + \lambda_i v_i = -\lambda_i v_i * (m_2 + m^0) + (\psi_i - \lambda_i u_{1,i}) * (m_1 - m_2), \\ v_i(0) = 0. \end{gathered} \tag{6.7}$$

Let us consider the problems

$$\begin{gathered} \frac{t^{-\beta}}{\Gamma(1-\beta)} * w_i' + \lambda_i w_i = -\lambda_i w_i * (m_2 + m^0) + \psi_i - \lambda_i u_{1,i}, \\ w_i(0) = 0 \end{gathered} \tag{6.8}$$

for $i = 1,2,\dots$. By Proposition 5.1, these problems have the unique solutions $w_i \in W_r^1(0,T) \subset C[0,T]$, $i = 1,2,\dots$ and $\lambda_i \|w_i\|_{\infty;\sigma} \le C_1(\|f_i\|_{\infty;\sigma} + \lambda_i \|u_{1,i}\|_{\infty;\sigma})$. ((3.7) takes the form of (6.8), if we replace the data vector $(f_i, g_i, \psi_i, m, m^0, \varphi)$ by $(\psi_i - \lambda_i u_{1,i}, 0, 0, 0, m_2 + m^0, 0)$.) The properties of w_i with (6.1) and (6.3) yield the relations $\sum_{i=1}^{\infty} \gamma_i \lambda_i w_i \in C[0,T]$ and $\sum_{i=1}^{\infty} \gamma_i \lambda_i w_i\Big|_{t=0} = 0$. One can immediately check that $v_i = w_i * (m_1 - m_2)$ solves (6.7). By the uniqueness of the solution of (6.7), it holds $u_{1,i} - u_{2,i} = w_i * (m_1 - m_2)$, $i = 1, 2, \dots$. Consequently, we can transform (6.6) as follows:

$$\sum_{i=1}^{\infty} \gamma_i \Big\{ \psi_i - \lambda_i u_{1,i} - \lambda_i w_i * (m_2 + m^0) - \lambda_i w_i \Big\} * (m_1 - m_2)(t) = 0, \tag{6.9}$$

for $t \in (0, T)$. By Titchmarsh convolution theorem, there exist $T_1 \geq 0$ and $T_2 \geq 0$ such that $T_1 + T_2 = T$ and

$$\sum_{i=1}^{\infty} \gamma_i \Big\{ \psi_i - \lambda_i u_{1,i} - \lambda_i w_i * (m_2 + m^0) - \lambda_i w_i \Big\}(t) = 0 \tag{6.10}$$

a.e. $t \in (0, T_1)$, and $(m^1 - m^2)(t) = 0$ a.e. $t \in (0, T_2)$. But since the function at the left-hand side of (6.10) is continuous and possesses the limit $\sum_{i=1}^{\infty} (\psi_i(0) - \lambda_i \varphi_i) \neq 0$ as $t \to 0^+$, the equality $T_1 = 0$ is valid. Consequently, $(m_1 - m_2)(t) = 0$ a.e. $t \in (0, T)$. This completes the proof. □

7. Existence

Let us introduce the function

$$Q_{\beta,\varphi,f}(t) = \sum_{i=1}^{\infty} \gamma_i q_{\beta,i}(t) t^{1-\beta} = \sum_{i=1}^{\infty} \gamma_i \big(\lambda_i \varphi_i - f_i(0)\big) E_{\beta,\beta}\big(-\lambda_i t^\beta\big). \tag{7.1}$$

Firstly, we prove a proposition that gives a necessary consistency condition for $h' + Q_{\beta,\varphi,f}(t) t^{\beta-1}$.

Proposition 7.1. *Let $(\beta, m) \in (0,1) \times L_p(0,T)$ with some $p \in (1, \infty)$ solve IP. Assume that $f_i \in W_p^1(0,T)$, $\psi_i \in W_1^1(0,T)$, $g_i \in {}_0H_p^\beta(0,T)$, $i = 1, 2, \dots$, $m^0 \in L_p(0,T)$ and*

$$\begin{gathered} \sum_{i=1}^{\infty} |\gamma_i| \lambda_i |\varphi_i| < \infty, \quad \sum_{i=1}^{\infty} |\gamma_i| \|f_i\|_{W_p^1(0,T)} < \infty, \\ \sum_{i=1}^{\infty} |\gamma_i| \|\psi_i\|_{W_1^1(0,T)} < \infty, \quad \sum_{i=1}^{\infty} |\gamma_i| \|g_i\|_{\beta,p} < \infty. \end{gathered} \tag{7.2}$$

Then $h' + Q_{\beta,\varphi,f}(t) t^{\beta-1} \in {}_0H_p^\beta(0,T)$.

Proof. Since ${}_0H_p^\beta(0,T) \hookrightarrow L_r(0,T)$ and $W_1^1(0,T) \hookrightarrow H_r^{1-\beta}(0,T)$ for $r \in (1, \frac{1}{1-\beta})$, $r \leq p$, Proposition 5.1 yields $u_i \in W_r^1(0,T)$. Moreover, (7.2) implies the inequalities $\sum_{i=1}^{\infty} |\gamma_i| \|f_i\|_{1-\beta,r} < \infty$, $\sum_{i=1}^{\infty} |\gamma_i| \|\psi_i\|_{1-\beta,r} < \infty$ and $\sum_{i=1}^{\infty} |\gamma_i| \|g_i\|_r < \infty$. There exist σ such that (5.13) (hence also (5.1)) is valid. Applying (5.2) we obtain the relation $\sum_{i=1}^{\infty} |\gamma_i| \|u_i'\|_{r;\sigma} < \infty$. Thus, $h' = \sum_{i=1}^{\infty} \gamma_i u_i' \in L_r(0,T)$. Further, (5.15) with (7.2) implies $\sum_{i=1}^{\infty} |\gamma_i| \|u_i' + q_{\beta,i}\|_{\beta,p;\sigma} < \infty$. Since $h'(t) + Q_{\beta,\varphi,f}(t) t^{\beta-1} = \sum_{i=1}^{\infty} \gamma_i (u_i' + q_{\beta,i})(t)$, we deduce $\|h' + Q_{\beta,\varphi,f}(t) t^{\beta-1}\|_{\beta,p;\sigma} < \infty$. This with Lemma 4.2 implies the assertion of the proposition. □

For the statement and proof of an existence theorem, we define the following balls in the space $L_p(0,T)$:

$$B_{\varrho,\sigma} = \{ w \in L_p(0,T) : \|w\|_{p;\sigma} \leq \varrho \}$$

and introduce the notation

$$d = \Big(\varphi_i|_{i=1,\dots,\infty}, f_i|_{i=1,\dots,\infty}, \psi_i|_{i=1,\dots,\infty}, g_i|_{i=1,\dots,\infty}, m^0, h \Big)$$

for the data vector of IP.

Theorem 7.2. *Let* $f_i \in W_p^1(0,T)$, $\psi_i \in W_1^1(0,T)$, $i = 1,2,\dots$, $m^0 \in L_p(0,T)$ *with some* $p \in (1,\infty)$ *and*

$$\begin{gathered}\sum_{i=1}^{\infty} |\gamma_i||\lambda_i^{1+\epsilon}|\varphi_i| < \infty, \quad \sum_{i=1}^{\infty} |\gamma_i|\lambda_i^{\epsilon}|f_i(0)| < \infty, \quad \sum_{i=1}^{\infty} |\gamma_i|\|f_i'\|_{L_p(0,T)} < \infty, \\ \sum_{i=1}^{\infty} |\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)| < \infty, \quad \sum_{i=1}^{\infty} |\gamma_i|\|\psi_i'\|_{L_1(0,T)} < \infty\end{gathered} \tag{7.3}$$

with some $\epsilon \in (0,1]$. *Moreover,* $b \in (0,1)$ *be such that*

$$h' + Q_{b,\varphi,f}(t)t^{b-1} \in {}_0H_p^b(0,T) \tag{7.4}$$

and $g_i \in {}_0H_p^s(0,T)$, $i = 1,2,\dots$ *and*

$$\sum_{i=1}^{\infty} |\gamma_i|\|g_i\|_{s,p} < \infty \tag{7.5}$$

with some $s \geq b$. *Finally, assume the conditions*

$$\sum_{i=1}^{\infty} \gamma_i\big(\lambda_i\varphi_i - \psi_i(0)\big) \neq 0, \tag{7.6}$$

$$h(0) = \sum_{i=1}^{\infty} \gamma_i\varphi_i. \tag{7.7}$$

Then there exists $\sigma_0[d]$ *such that for* $\sigma = \sigma_0[d]$ *it holds*

$$\omega_\sigma[d] := \widehat{C}|\kappa[d]|N_\sigma[d] \leq \frac{1}{2}, \tag{7.8}$$

where

$$\begin{aligned}N_\sigma[d] &= \|h' + Q_{b,\varphi,f}(t)t^{b-1}\|_{b,p;\sigma} + \Big[\sum_{i=1}^{\infty} |\gamma_i|\lambda_i|\varphi_i| + \sum_{i=1}^{\infty} |\gamma_i||\psi_i(0)|\Big]\|m^0\|_{p;\sigma} \\ &\quad + \Big\{\frac{1}{\sigma^{b\epsilon}}\Big[\sum_{i=1}^{\infty} |\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^{\infty} |\gamma_i|\lambda_i^{\epsilon}|f_i(0)| + \sum_{i=1}^{\infty} |\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)|\Big] \\ &\quad + \sum_{i=1}^{\infty} |\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty} |\gamma_i|\|\psi_i'\|_{1;\sigma} + \sum_{i=1}^{\infty} |\gamma_i|\|g_i\|_{b,p;\sigma}\Big\}\big(1 + \|m^0\|_{p;\sigma}\big),\end{aligned} \tag{7.9}$$

$$\begin{gathered}\widehat{C} = (3C_2 + 4T^{\frac{p-1}{p}} + 2)(C_2 + 1)(c_{b,\epsilon} + 1)(T^{\frac{p-1}{p}} + 1), \\ \kappa[d] = \Big[\sum_{i=1}^{\infty} \gamma_i\big(\psi_i(0) - \lambda_i\varphi_i\big)\Big]^{-1}.\end{gathered} \tag{7.10}$$

Moreover, IP has a solution (β, m) *such that* $\beta = b$ *and* m *belongs to* $B_{\varrho_\sigma[d],\sigma}$, *where* σ *is any number satisfying* (7.8) *and*

$$\varrho_\sigma[d] = \overline{C}\big|\kappa[d]\big|R_\sigma[d]$$

with

$$\begin{aligned} &R_\sigma[d] \\ &= \|h' + Q_{\beta,\varphi,f}(t)t^{\beta-1}\|_{\beta,p;\sigma} \\ &\quad + \sum_{i=1}^\infty |\gamma_i|\lambda_i|\varphi_i|\|m^0\|_{p;\sigma} + \Big\{\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^\infty |\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^\infty |\gamma_i|\lambda_i^\epsilon|f_i(0)|\Big] \\ &\quad + \sum_{i=1}^\infty |\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^\infty |\gamma_i|\|g_i\|_{\beta,p;\sigma}\Big\}\big(1 + \|m^0\|_{p;\sigma}\big), \end{aligned} \tag{7.11}$$

where $\overline{C} = 2(C_2+1)(c_{\beta,\epsilon}+1)(T^{\frac{p-1}{p}}+1)$.

Proof. Since $\|f_i'\|_{p;\sigma} \to 0$, $\|\psi_i'\|_{1;\sigma} \to 0$, $\|g_i\|_{b,p;\sigma} \to 0$ as $\sigma \to \infty$ for all i and

$$\begin{gathered} \sum_{i=1}^\infty |\gamma_i|\|f_i'\|_{p;\sigma} \le \sum_{i=1}^\infty |\gamma_i|\|f_i'\|_p < \infty, \\ \sum_{i=1}^\infty |\gamma_i|\|\psi_i'\|_{1;\sigma} \le \sum_{i=1}^\infty |\gamma_i|\|\psi_i'\|_1 < \infty, \\ \sum_{i=1}^\infty |\gamma_i|\|g_i\|_{b,p;\sigma} \le \sum_{i=1}^\infty |\gamma_i|\|g_i\|_{b,p} < \infty, \end{gathered}$$

the dominated convergence theorem for series implies

$$\sum_{i=1}^\infty |\gamma_i|\|f_i'\|_{p;\sigma} \to 0, \quad \sum_{i=1}^\infty |\gamma_i|\|\psi_i'\|_{1;\sigma} \to 0, \quad \sum_{i=1}^\infty |\gamma_i|\|g_i\|_{b,p;\sigma} \to 0$$

as $\sigma \to \infty$. Moreover, $\|h' + Q_{b,\varphi,f}(t)t^{\beta-1}\|_{b,p;\sigma} \to 0$ and $\|m^0\|_{p;\sigma} \to 0$ as $\sigma \to \infty$. Consequently, there exists such $\sigma = \sigma_0[d]$ that (7.8) is valid.

Let σ be some number satisfying (7.8). Setting $\beta = b$, it remains to show that there exists a suitable $m \in B_{\varrho_\sigma[d],\sigma}$ such that the pair (β, m) solves IP. Let us consider the following equation for m:

$$\begin{aligned} &D^\beta\big(h' + Q_{\beta,\varphi,f}(t)t^{\beta-1}\big) \\ &= -\sum_{i=1}^\infty \gamma_i\lambda_i(u_i' + q_{\beta,i}) + \sum_{i=1}^\infty \gamma_i(f_i' + D^\beta g_i) \\ &\quad + \sum_{i=1}^\infty \gamma_i\big(\psi_i(0) - \lambda_i\varphi_i\big)m + \sum_{i=1}^\infty \gamma_i\big(\psi_i' - \lambda_i(u_i' + q_{\beta,i}) + \lambda_i q_{\beta,i}\big)*m \\ &\quad - \sum_{i=1}^\infty \gamma_i\lambda_i\varphi_i m^0 - \sum_{i=1}^\infty \gamma_i\big(\lambda_i(u_i' + q_{\beta,i}) - \lambda_i q_{\beta,i}\big)*m^0, \end{aligned} \tag{7.12}$$

where $u_i = u_i[m]$ is the solution of (3.7). Due to the assumptions of the theorem, Proposition 5.2, (5.20) and (4.12), all terms in (7.12) belong to $L_p(0,T)$ provided $m \in L_p(0,T)$. Therefore, this equation is well-defined.

Firstly, let us show that if $m \in L_p(0,T)$ solves (7.12), then the pair (β, m) solves IP. Suppose that $m \in L_p(0,T)$ solves (7.12). Substituting $Q_{\beta,\varphi,f}(t)t^{\beta-1}$ by

$\sum_{i=1}^{\infty}\gamma_i q_{\beta,i}$, $u_i+q_{\beta,i}-q_{\beta,i}$ by u_i' in (7.12) and integrating from 0 to t we obtain

$$\begin{aligned}\frac{t^{-\beta}}{\Gamma(1-\beta)}*\big(h'+\sum_{i=1}^{\infty}\gamma_i q_{\beta,i}\big)&=-\sum_{i=1}^{\infty}\gamma_i\lambda_i u_i+\sum_{i=1}^{\infty}\gamma_i f_i+\frac{t^{-\beta}}{\Gamma(1-\beta)}*\sum_{i=1}^{\infty}\gamma_i g_i\\&\quad+\sum_{i=1}^{\infty}\gamma_i\big(\psi_i-\lambda_i u_i\big)*m-\sum_{i=1}^{\infty}\gamma_i\lambda_i u_i*m^0\\&\quad-\sum_{i=1}^{\infty}\gamma_i\lambda_i I^1 q_{\beta,i}+\sum_{i=1}^{\infty}\gamma_i\big(\lambda_i\varphi_i-f_i(0)\big).\end{aligned}\tag{7.13}$$

From (3.7) we obtain

$$\begin{aligned}\frac{t^{-\beta}}{\Gamma(1-\beta)}*\sum_{i=1}^{\infty}\gamma_i u_i'&=-\sum_{i=1}^{\infty}\gamma_i\lambda_i u_i+\sum_{i=1}^{\infty}\gamma_i f_i+\frac{t^{-\beta}}{\Gamma(1-\beta)}*\sum_{i=1}^{\infty}\gamma_i g_i\\&\quad+\sum_{i=1}^{\infty}\gamma_i(\psi_i-\lambda_i u_i)*m-\sum_{i=1}^{\infty}\gamma_i\lambda_i u_i*m^0\end{aligned}\tag{7.14}$$

and from (5.16) we deduce

$$\frac{t^{-\beta}}{\Gamma(1-\beta)}*\sum_{i=1}^{\infty}\gamma_i q_{\beta,i}=-\sum_{i=1}^{\infty}\gamma_i\lambda_i I^1 q_{\beta,i}+\sum_{i=1}^{\infty}\gamma_i\big(\lambda_i\varphi_i-f_i(0)\big).\tag{7.15}$$

Subtracting the sum of (7.14) and (7.15) from (7.13) we have

$$\frac{t^{-\beta}}{\Gamma(1-\beta)}*\big(h'-\sum_{i=1}^{\infty}\gamma_i u_i'\big)=0,\quad t\in(0,T).$$

This implies $h'-\sum_{i=1}^{\infty}\gamma_i u_i'=0$ for $t\in(0,T)$. Integrating from 0 to t and using the assumption (7.7) we obtain (3.8). This proves that the pair (β,m) is a solution of IP.

Secondly, let us denote $\omega=\omega_\sigma[d]$, $\varrho=\varrho_\sigma[d]$, $\kappa=\kappa[d]$ and show that the equation (7.12) has a unique solution m in the ball $B_{\varrho,\sigma}$. To this end, we rewrite this equation in the fixed-point form

$$m=\mathcal{F}m,\tag{7.16}$$

where

$$\begin{aligned}\mathcal{F}m=\kappa\Big\{&D^\beta\big(h'+Q_{\beta,\varphi,f}(t)t^{\beta-1}\big)+\sum_{i=1}^{\infty}\gamma_i\lambda_i(u_i[m]'+q_{\beta,i})\\&-\sum_{i=1}^{\infty}\gamma_i(f_i'-D^\beta g_i)-\sum_{i=1}^{\infty}\gamma_i\big(\psi_i'-\lambda_i(u_i[m]'+q_{\beta,i})+\lambda_i q_{\beta,i}\big)*m\\&+\sum_{i=1}^{\infty}\gamma_i\lambda_i\varphi_i m^0+\sum_{i=1}^{\infty}\gamma_i\big(\lambda_i(u_i[m]'+q_{\beta,i})-\lambda_i q_{\beta,i}\big)*m^0\Big\}.\end{aligned}\tag{7.17}$$

In view of (4.11), (5.12) and (7.3) it holds

$$\begin{aligned}\big\|\sum_{i=1}^{\infty}\gamma_i\lambda_i q_{\beta,i}\big\|_{1,\sigma}&\le\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|q_{\beta,i}\|_{1,\sigma}\\&\le\frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i|+\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)|\Big].\end{aligned}\tag{7.18}$$

Let $m \in B_{\varrho,\sigma}$, i.e. $\|m\|_{p;\sigma} \le \varrho$. From the definitions of ω and ϱ and (7.9) we deduce that $T^{\frac{p-1}{p}}\|m^0\|_{p;\sigma} \le \frac{\omega}{2} \le \frac{1}{4}$ and $T^{\frac{p-1}{p}}\varrho \le \frac{\omega}{2} \le \frac{1}{4}$. Thus, the relation (5.13) is valid. By means of the assumptions of the theorem, (4.10), (4.11), (5.14) and (7.18) we estimate:

$$
\begin{aligned}
&\|\mathcal{F}m\|_{p;\sigma}\\
&\le |\kappa|\Big\{\|h' + Q_{\beta,\varphi,f}(t)t^{\beta-1}\|_{\beta,p;\sigma} + \frac{C_2 c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i|\\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)|\Big] + C_2\Big[\frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)| + \sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma}\Big]\|m\|_{p;\sigma}\\
&\quad + C_2\Big[\sum_{i=1}^{\infty}|\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|g_i\|_{\beta,p;\sigma}\Big] + \sum_{i=1}^{\infty}|\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|g_i\|_{\beta,p;\sigma}\\
&\quad + \Big\{\sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma} + \frac{C_2 c_{\beta,\epsilon}T^{\frac{p-1}{p}}}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)|\Big]\\
&\quad + C_2 T^{\frac{p-1}{p}}\Big[\frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)| + \sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma}\Big]\|m\|_{p;\sigma}\\
&\quad + C_2 T^{\frac{p-1}{p}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|g_i\|_{\beta,p;\sigma}\Big]\\
&\quad + \frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)|\Big]\Big\}\|m\|_{p;\sigma}\\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i|\varphi_i|\|m^0\|_{p;\sigma} + \Big\{\frac{C_2 c_{\beta,\epsilon}T^{\frac{p-1}{p}}}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)|\Big]\\
&\quad + C_2 T^{\frac{p-1}{p}}\Big[\frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)| + \sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma}\Big]\|m\|_{p;\sigma}\\
&\quad + C_2 T^{\frac{p-1}{p}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|g_i\|_{\beta,p;\sigma}\Big] + \frac{c_{\beta,\epsilon}}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i|\\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)|\Big]\Big\}\|m^0\|_{p;\sigma}\Big\}.
\end{aligned}
$$

Estimating $\|m\|_{p;\sigma}$ by ϱ and $\|m\|_{p;\sigma}^2$ by $T^{\frac{p}{p-1}}\frac{\varrho}{4}$ and simplifying we obtain

$$\|\mathcal{F}m\|_{p;\sigma} \le \frac{\varrho}{2} + \omega_1\varrho,$$

where

$$
\begin{aligned}
\omega_1 = \widehat{C}_1|\kappa|\Big\{&\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)|\Big]\\
&+ \sum_{i=1}^{\infty}|\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|g_i\|_{\beta,p;\sigma}\Big\}\big(1 + \|m^0\|_{p;\sigma}\big)
\end{aligned}
$$

with

$$\begin{aligned}\widehat{C}_1 &= \max\{c_{\beta,\epsilon}(1+\frac{5C_2}{4}+C_2T^{\frac{p-1}{p}});1+\frac{5C_2}{4}+C_2T^{\frac{p-1}{p}}\}\\ &\le \frac{5}{4}(C_2+1)(c_{\beta,\epsilon}+1)(T^{\frac{p-1}{p}}+1).\end{aligned}$$

Since $\omega_1 \le \omega \le \frac{1}{2}$ we have $\|\mathcal{F}m\|_{p;\sigma} \le \frac{\varrho}{2}+\frac{\varrho}{2} \le \varrho$. Thus, the operator $\mathcal{F}$ maps $B_{\varrho,\sigma}$ into $B_{\varrho,\sigma}$.

Let $m_1, m_2 \in B_{\varrho,\sigma}$. Then the difference $v_i = u_i[m_1]-u_i[m_2]$ solves the problem

$$\frac{t^{-\beta}}{\Gamma(1-\beta)} * v_i' + \lambda_i v_i = \overline{f}_i - \lambda_i v_i * (m_2+m^0), \quad v_i(0)=0, \tag{7.19}$$

where

$$\overline{f}_i = \big(\psi_i - \lambda_i u_i[m_1]\big)*(m_1-m_2).$$

By the vanishing initial condition and $\overline{f}_i(0)=0$, the function (5.12), related to v_i is equal to zero. Moreover, $T^{\frac{p-1}{p}}\|m_2\|_{p;\sigma} + T^{\frac{p-1}{p}}\|m^0\|_{p;\sigma} \le \frac{1}{2}$. Therefore, (5.14) applied to (7.19) yields

$$\begin{aligned}\lambda_i\|v_i'\|_{p;\sigma} &\le C_2\|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}\|\overline{f}'\|_{p;\sigma} = C_2\|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}\big\|\big(\psi_i(0)-\lambda_i\varphi_i\big)(m_1-m_2)\\ &\quad + \big[\psi_i' - \lambda_i(u_i[m_1]'+q_{\beta,i})+\lambda_i q_{\beta,i}\big]*(m_1-m_2)\big\|_{p;\sigma}\\ &\le C_2\|\lambda_i\widetilde{E}_{\beta,i}\|_{1;\sigma}\Big[|\psi_i(0)|+\lambda_i|\varphi_i|+\|\psi_i'\|_{1;\sigma}\\ &\quad + T^{\frac{p-1}{p}}\lambda_i\|u_i[m_1]'+q_{\beta,i}\|_{p;\sigma}+\lambda_i\|q_{\beta,i}\|_{1;\sigma}\Big]\|m_1-m_2\|_{p;\sigma}.\end{aligned}$$

Using the estimate (5.14) for $u_i[m_1]'+q_{\beta,i}$, the relation $\|m\|_{p;\sigma} \le T^{\frac{p}{p-1}}\frac{1}{4}$ for $\|m\|_{p;\sigma}$ in this estimate as well as (4.10), (4.11), (7.3), (7.5) and (7.18) we obtain

$$\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|v_i'\|_{p;\sigma} \le \omega_2\|m_1-m_2\|_{p;\sigma}, \tag{7.20}$$

where

$$\begin{aligned}\omega_2 = \widehat{C}_2\Big\{\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i(0)| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)|\Big]\\ + \sum_{i=1}^{\infty}|\gamma_i|\|f_i'\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|g_i\|_{\beta,p;\sigma}\Big\}\end{aligned} \tag{7.21}$$

with

$$\begin{aligned}\widehat{C}_2 &= C_2\max\{c_{\beta,\epsilon}(T^{\frac{p-1}{p}}C_2+2);c_{\beta,\epsilon}(\frac{C_2}{4}+1);C_2T^{\frac{p-1}{p}};\frac{C_2}{4}\}\\ &\le 2C_2(C_2+1)(c_{\beta,\epsilon}+1)(T^{\frac{p-1}{p}}+1).\end{aligned}$$

Further, from (7.17) we have

$$\begin{aligned}\mathcal{F}m_1 - \mathcal{F}m_2 = \kappa\Big\{&\sum_{i=1}^{\infty}\gamma_i\lambda_i v_i' + \sum_{i=1}^{\infty}\gamma_i\lambda_i v_i' * (m_2-m^0)\\ &-\sum_{i=1}^{\infty}\gamma_i\big(\psi_i' - \lambda_i(u_i[m_1]'+q_{\beta,i})+\lambda_i q_{\beta,i}\big)*(m_1-m_2)\Big\},\end{aligned}$$

hence

$$\begin{aligned}
&\|\mathcal{F}m_1-\mathcal{F}m_2\|_{p;\sigma}\\
&\le |\kappa|\Big\{\frac{3}{2}\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|v_i'\|_{p;\sigma}+\sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma}\\
&\quad +T^{\frac{p-1}{p}}\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|u_i[m_1]'+q_{\beta,i}\|_{p;\sigma}+\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|q_{\beta,i}\|_{1;\sigma}\Big\}\|m_1-m_2\|_{p;\sigma}.
\end{aligned}$$

Using the estimates (5.14), (7.18) and (7.20) we obtain

$$\|\mathcal{F}m_1-\mathcal{F}m_2\|_{p;\sigma}\le\big(\frac{3}{2}|\kappa|\omega_2+\omega_1\big)\|m_1-m_2\|_{p;\sigma}.$$

Since $\frac{3}{2}|\kappa|\omega_2+\omega_1\le\omega\le\frac{1}{2}$ we obtain $\|\mathcal{F}m_1-\mathcal{F}m_2\|_{p;\sigma}\le\frac{1}{2}\|m_1-m_2\|_{p;\sigma}$. This shows that $\mathcal{F}$ is a contraction in the ball $B_{\varrho,\sigma}$. By Banach fixed-point principle, the equation (7.16) has a unique solution in $B_{\varrho,\sigma}$. The proof is complete. □

The following theorem gives an explicit formula for the component β of the solution of the inverse problem.

Theorem 7.3. *Let*

$$\sum_{i=1}^{\infty}|\gamma_i|\big(\lambda_i|\varphi_i|+|f_i(0)|\big)<\infty,\quad \sum_{i=1}^{\infty}\gamma_i(\lambda_i\varphi_i-f_i(0))\neq 0$$

and $h'+Q_{\beta,\varphi,f}(t)t^{\beta-1}\in {}_0H_p^{\beta}(0,T)$ *for some* $p\in(1,\infty)$ *and* $\beta\in(0,1)$. *Then*

$$\beta=\mu(h):=\lim_{t\to0^+}\frac{\ln|h(t)-h(0)|}{\ln t}. \tag{7.22}$$

Proof. By Lemma 4.2, there exists $z\in L_p(0,T)$ such that $h'+Q_{\beta,\varphi,f}(t)t^{\beta-1}=I^{\beta}z$. Integrating this formula from 0 to t we have

$$h(t)-h(0)+\sum_{i=0}^{\infty}\gamma_i\big(\lambda_i\varphi_i-f_i(0)\big)\int_0^t E_{\beta,\beta}(-\lambda_i\tau)\tau^{\beta-1}d\tau=I^{1+\beta}z(t).$$

Since $\int_0^t E_{\beta,\beta}(-\lambda_i\tau)\tau^{\beta-1}d\tau=t^{\beta}E_{\beta,\beta+1}(-\lambda_it^{\beta})$ ([5, (4.4.4)]), we obtain $\frac{h(t)-h(0)}{t^{\beta}}=Z(t)$ with

$$Z(t)=-\sum_{i=0}^{\infty}\gamma_i\big(\lambda_i\varphi_i-f_i(0)\big)E_{\beta,\beta+1}(-\lambda_it^{\beta})+t^{-\beta}I^{1+\beta}z(t).$$

Let us compute the limit of $Z(t)$ in case $t\to0^+$. We have

$$c=:-\lim_{t\to0^+}\sum_{i=0}^{\infty}\gamma_i\big(\lambda_i\varphi_i-f_i(0)\big)E_{\beta,\beta+1}(-\lambda_it^{\beta})=-\frac{1}{\Gamma(\beta+1)}\sum_{i=0}^{\infty}\gamma_i\big(\lambda_i\varphi_i-f_i(0)\big)$$

and

$$\begin{aligned}
\big|t^{-\beta}I^{1+\beta}z(t)\big|&=\Big|t^{-\beta}\int_0^t\frac{(t-\tau)^{\beta}}{\Gamma(\beta+1)}z(\tau)d\tau\Big|\\
&\le t^{-\beta}\Big[\int_0^t\Big(\frac{(t-\tau)^{\beta}}{\Gamma(\beta+1)}\Big)^{\frac{p}{p-1}}d\tau\Big]^{\frac{p-1}{p}}\|z\|_{L_p(0,t)}\\
&=\frac{t^{\frac{p-1}{p}}}{\Gamma(\beta+1)(\frac{\beta p}{p-1}+1)^{\frac{p-1}{p}}}\|z\|_{L_p(0,t)}\to0\quad\text{as } t\to0^+.
\end{aligned}$$

Thus $\lim_{t\to 0^+} Z(t) = c \neq 0$. Taking logarithms the relation

$$\frac{|h(t)-h(0)|}{t^\beta} = |Z(t)| \tag{7.23}$$

and solving for β we obtain

$$\beta = \frac{\ln|h(t)-h(0)| - \ln|Z(t)|}{\ln t}.$$

Taking the limit $t \to 0^+$ and observing that $\lim_{t\to 0^+} \frac{\ln|Z(t)|}{\ln t} = 0$ we arrive at (7.22). □

Remark 7.4. In practical computations, we can apply the formula (7.22) only approximately, i.e. $\beta \approx \frac{\ln|h(t_1)-h(0)|}{\ln t_1}$, where t_1 is some small value of the time. It is possible increase the accuracy of computation of β incorporating the principal term of $Z(t)$, too. Namely, in the proof of Theorem 7.3 we saw that

$$Z(t) \sim c = -\frac{1}{\Gamma(\beta+1)} \sum_{i=0}^{\infty} \gamma_i \big(\lambda_i \varphi_i - f_i(0)\big)$$

as $t \to 0^+$. Thus, from (7.23) we obtain

$$\frac{|h(t)-h(0)|}{t^\beta} \approx \frac{1}{\Gamma(\beta+1)} \Big| \sum_{i=0}^{\infty} \gamma_i \big(\lambda_i \varphi_i - f_i(0)\big) \Big|$$

if $t \approx 0$. From this relation we deduce the following approximate equation for β that is applicable in case of small t_1:

$$\frac{|h(t_1)-h(0)|}{\Big|\sum_{i=0}^{\infty} \gamma_i \big(\lambda_i \varphi_i - f_i(0)\big)\Big|} = \frac{t_1^{\tilde\beta}}{\Gamma(\tilde\beta+1)}. \tag{7.24}$$

8. Stability

Under the conditions of Theorem 7.3, the stability of β with respect to h is immediate. Let $\big(\beta, \varphi_i|_{i=1,\dots,\infty}, f_i|_{i=1,\dots,\infty}, h\big)$ and $\big(\widetilde\beta, \widetilde\varphi_i|_{i=1,\dots,\infty}, \widetilde f_i|_{i=1,\dots,\infty}, \widetilde h\big)$ satisfy the assumptions of Theorem 7.3. Then $\mu(\widetilde h) \to \mu(h)$ implies $\widetilde\beta \to \beta$. Moreover, $d_0(h_1,h_2) := |\mu(h_1)-\mu(h_2)|$, $h_1,h_2 \in \varkappa$, defines a pseudometric on elements of the space $\varkappa := \{h : \mu(h) \text{ exists and is finite}\}$. Thus, $|\widetilde\beta - \beta| = d_0(\widetilde h, h)$.

Next we prove a theorem concerning the local Lipschitz-continuity of the component m of the solution of IP with respect to the data.

Theorem 8.1. *Let the data vectors*

$$d = \Big(\varphi_i|_{i=1,\dots,\infty}, f_i|_{i=1,\dots,\infty}, \psi_i|_{i=1,\dots,\infty}, g_i|_{i=1,\dots,\infty}, m^0, h\Big),$$
$$\widetilde d = \Big(\widetilde\varphi_i|_{i=1,\dots,\infty}, \widetilde f_i|_{i=1,\dots,\infty}, \widetilde\psi_i|_{i=1,\dots,\infty}, \widetilde g_i|_{i=1,\dots,\infty}, \widetilde m^0, \widetilde h\Big)$$

satisfy the assumptions of Theorem 6.1 with same parameters p, ϵ, b and s. Let (β, m) and $(\beta, \widetilde m)$ with $\beta = b$ be the solutions of IP corresponding to the data d and $\widetilde d$, respectively. Let $\sigma_1[d]$ be a sufficiently large number such that for $\sigma = \sigma_1[d]$ the *relation*

$$\begin{aligned}&\widehat{C}|\kappa[d]|\Big[N_\sigma[d^\dagger]+\Big[\sum_{i=1}^\infty|\gamma_i||\psi_i(0)|+\sum_{i=1}^\infty|\gamma_i|\lambda_i|\varphi_i|\Big]\|m+m^0\|_{p;\sigma}\\&+\Big\{\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^\infty|\gamma_i|\lambda_i^\epsilon|\psi_i(0)|+\sum_{i=1}^\infty|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i|\Big]+\sum_{i=1}^\infty|\gamma_i|\|\psi_i'\|_{1;\sigma}\\&+\sum_{i=1}^\infty|\gamma_i|\lambda_i\|u_i'[m]\|_{1;\sigma}\Big\}\big(1+\|m+m^0\|_{p;\sigma}\big)\Big]\le\frac{1}{8}\end{aligned}\tag{8.1}$$

is valid. Moreover, assume that the data vector $\widetilde{d}$ *is sufficiently close to the data vector* d*, so that*

$$\frac{\Big|\sum_{i=1}^\infty\gamma_i(\psi_i(0)-\lambda_i\varphi_i)\Big|}{\Big|\sum_{i=1}^\infty\gamma_i\big(\psi_i(0)-\lambda_i\varphi_i+\widetilde{\psi}_i(0)-\psi_i(0)-\lambda_i(\widetilde{\varphi}_i-\varphi_i)\big)\Big|}\le 2\tag{8.2}$$

is valid and the estimates

$$\|\widetilde{m}^0-m^0\|_{p;\sigma}\le 1\quad\textit{and}$$

$$\begin{aligned}&\widehat{C}|\kappa[d]|\Big\{\sum_{i=1}^\infty|\gamma_i||\psi_i(0)|+\sum_{i=1}^\infty|\gamma_i|\lambda_i|\varphi_i|\\&+\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^\infty|\gamma_i|\lambda_i^\epsilon|\psi_i(0)|+\sum_{i=1}^\infty|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i|\Big]+\sum_{i=1}^\infty|\gamma_i|\|\psi_i'\|_{1;\sigma}\\&+\sum_{i=1}^\infty|\gamma_i|\lambda_i\|u_i'[m]\|_{1;\sigma}\Big\}\|\widetilde{m}^0-m^0\|_{p;\sigma}\le\frac{1}{8}\end{aligned}\tag{8.3}$$

are satisfied, where $\sigma=\sigma_2[d]=\max\{\sigma_0[d];\sigma_1[d]\}$ *and*

$$\begin{aligned}d^\dagger=\Big(&(\widetilde{\varphi}_i-\varphi_i)|_{i=1,\dots,\infty},\,f_i^\dagger|_{i=1,\dots,\infty},\,(\widetilde{\psi}_i-\psi_i)|_{i=1,\dots,\infty},\\&(\widetilde{g}_i-g_i)|_{i=1,\dots,\infty},\,m^\dagger,\,\widetilde{h}-h\Big),\end{aligned}$$

$$f_i^\dagger=\widetilde{f}_i-f_i-(\widetilde{m}^0-m^0)*\lambda_iu_i[m]+m*(\widetilde{\psi}_i-\psi_i),\quad m^\dagger=m+m^0+\widetilde{m}^0-m^0.$$

Then

$$\begin{aligned}\|\widetilde{m}-m\|_p\le\mathcal{C}[d]\Big\{&\|\widetilde{h}'-h'+Q_{\beta,\widetilde{\varphi}-\varphi,\widetilde{f}-f}(t)t^{\beta-1}\|_{\beta,p}\\&+\sum_{i=1}^\infty|\gamma_i|\lambda_i^{1+\epsilon}|\widetilde{\varphi}_i-\varphi_i|+\sum_{i=1}^\infty|\gamma_i|\lambda_i^\epsilon|\widetilde{f}_i(0)-f_i(0)|\\&+\sum_{i=1}^\infty|\gamma_i||\widetilde{\psi}_i(0)-\psi_i(0)|+\sum_{i=1}^\infty|\gamma_i|\|\widetilde{f}_i'-f_i'\|_p\\&+\sum_{i=1}^\infty|\gamma_i|\|\widetilde{\psi}_i'-\psi_i'\|_1\\&+\sum_{i=1}^\infty|\gamma_i|\|\widetilde{g}_i-g_i\|_{\beta,p}+\|\widetilde{m}^0-m^0\|_p\Big\},\end{aligned}\tag{8.4}$$

where $\mathcal{C}[d]$ *is a constant depending on the data vector* d*.*

Proof. Firstly we mention that the series in the formulas (8.1) and (8.3) converge because of the assumptions imposed on d and $\widetilde{d}$. In particular, $\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|u_i'[m]\|_{1;\sigma}<\infty$, because $\|u_i'[m]\|_{1;\sigma}\le T^{\frac{p-1}{p}}\|u_i'[m]+q_{\beta,i}\|_{p;\sigma}+\|q_{\beta,i}\|_{1;\sigma}$ and

$$\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|u_i'[m]+q_{\beta,i}\|_{p;\sigma}<\infty,\quad \sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|q_{\beta,i}\|_{1;\sigma}<\infty$$

in view of Proposition 5.2, (7.18) and assumptions of the theorem. Secondly, due to (4.8) and the dominated convergence theorem for series, there exists $\sigma=\sigma_1[d]$ such that (8.1) is valid.

Denoting $v_i=u_i[\widetilde{m}]-u_i[m]$, the difference $(\beta,\widetilde{m}-m)=(\beta,k)$ is a solution of the inverse problem

$$\begin{gathered}\frac{t^{-\beta}}{\Gamma(1-\beta)}*[v_i'(t)-\overline{g}_i(t)]+\lambda_i v_i(t)\\ =\overline{f}_i(t)+k*[\overline{\psi}_i(t)-\lambda_i v_i(t)]-\overline{m}^0*\lambda_i v_i(t),\quad t\in(0,T),\quad v_i(0)=\overline{\varphi}_i,\\ \sum_{i=1}^{\infty}\gamma_i v_i(t)=\overline{h}(t),\quad t\in(0,T)\end{gathered}\tag{8.5}$$

with the data vector

$$\overline{d}=\Big(\overline{\varphi}_i|_{i=1,\dots,\infty},\overline{f}_i|_{i=1,\dots,\infty},\overline{\psi}_i|_{i=1,\dots,\infty},\overline{g}_i|_{i=1,\dots,\infty},\overline{m}^0,\overline{h}\Big),$$

where

$$\begin{gathered}\overline{\varphi}_i=\widetilde{\varphi}_i-\varphi_i,\quad \overline{f}_i=f_i^{\dagger},\quad \overline{\psi}_i=\widetilde{\psi}_i-\psi_i+\psi_i-\lambda_i u_i[m],\\ \overline{g}_i=\widetilde{g}_i-g_i,\quad \overline{m}^0=m^{\dagger},\quad \overline{h}=\widetilde{h}-h.\end{gathered}$$

By (8.2), it holds $|\kappa[\overline{d}]|\le 2|\kappa[d]|$. Let us set $\sigma=\sigma_2[d]$ and estimate

$$\begin{aligned}\omega_\sigma[\overline{d}]&=\widehat{C}|\kappa[\overline{d}]|N_\sigma[\overline{d}]\\ &\le 2\widehat{C}|\kappa[d]|N_\sigma[\overline{d}]\\ &\le 2\widehat{C}|\kappa[d]|\Big[N_\sigma[d^{\dagger}]+\Big[\sum_{i=1}^{\infty}|\gamma_i||\psi_i(0)|+\sum_{i=1}^{\infty}|\gamma_i|\lambda_i|\varphi_i|\Big]\|m+m^0+\widetilde{m}^0-m^0\|_{p;\sigma}\\ &\quad+\Big\{\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\psi_i(0)|+\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\varphi_i|\Big]+\sum_{i=1}^{\infty}|\gamma_i|\|\psi_i'\|_{1;\sigma}\\ &\quad+\sum_{i=1}^{\infty}|\gamma_i|\lambda_i\|u_i'[m]\|_{1;\sigma}\Big\}\big(1+\|m+m^0+\widetilde{m}^0-m^0\|_{p;\sigma}\big)\Big].\end{aligned}$$

Since the norms $\|\cdot\|_{p;\sigma}$ and $\|\cdot\|_{\beta,p;\sigma}$ are nonincreasing in σ, the relation (8.1) is valid for $\sigma=\sigma_2[d]$, too. Using (8.1) and (8.3) we reach the inequality $\omega_\sigma[\overline{d}]\le\frac{1}{2}$ for $\sigma=\sigma_2[d]$. Now we can apply Theorem 7.2 to the inverse problem (8.5). We conclude that (8.5) has a solution (β,k) such that $k\in B_{\varrho_\sigma[\overline{d}],\sigma}$, $\sigma=\sigma_2[d]$. From the uniqueness of the solution of (8.5) (following from Theorem 6.1), we have $k=\widetilde{m}-m$. Hence, for $\sigma=\sigma_2[d]$ it holds

$$\begin{aligned}\|\widetilde{m}-m\|_{p;\sigma}&\le\varrho_\sigma[\overline{d}]\\ &=\overline{C}|\kappa[\overline{d}]|R_\sigma[\overline{d}]\\ &\le 2\overline{C}|\kappa[d]|R_\sigma[\overline{d}]\end{aligned}$$

$$
\begin{aligned}
&= 2\overline{C}|\kappa[d]|\Big[\|\widetilde{h}' - h' + Q_{\beta,\widetilde{\varphi}-\varphi,f^\dagger}(t)t^{\beta-1}\|_{\beta,p;\sigma} \\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i|\widetilde{\varphi}_i - \varphi_i|\|m^\dagger\|_{p;\sigma} \\
&\quad + \Big\{\frac{1}{\sigma^{\beta\epsilon}}\Big[\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\widetilde{\varphi}_i - \varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|f_i^\dagger(0)|\Big] \\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\|f_i^{\dagger'}\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|\widetilde{g}_i - g_i\|_{\beta,p;\sigma}\Big\}\big(1 + \|m^\dagger\|_{p;\sigma}\big)\Big].
\end{aligned}
$$

From the formula of $f_i^\dagger$ we obtain $f_i^\dagger(0) = \widetilde{f}_i(0) - f_i(0)$ and

$$
\begin{aligned}
f_i^{\dagger'} &= \widetilde{f}_i' - f_i' - \lambda_i\varphi_i(\widetilde{m}^0 - m^0) - (\widetilde{m}^0 - m^0) * \lambda_i u_i'[m] + \big(\widetilde{\psi}_i(0) - \psi_i(0)\big)m \\
&\quad + m * (\widetilde{\psi}_i' - \psi_i').
\end{aligned}
$$

Moreover, from (7.1) and the relation for $f^\dagger(0)$ we see that $Q_{\beta,\widetilde{\varphi}-\varphi,f^\dagger} = Q_{\beta,\widetilde{\varphi}-\varphi,\widetilde{f}-f}$. Using also the relation $\|\widetilde{m}^0 - m^0\|_{p;\sigma} \le 1$ for the addend $\widetilde{m}^0 - m^0$ in the term $m^\dagger$ and applying (4.12) we continue the estimation of $\|\widetilde{m} - m\|_{p;\sigma}$ as follows:

$$
\begin{aligned}
&\|\widetilde{m} - m\|_{p;\sigma} \\
&\le \mathcal{C}_1[d]\Big\{\|\widetilde{h}' - h' + Q_{\beta,\widetilde{\varphi}-\varphi,\widetilde{f}-f}(t)t^{\beta-1}\|_{\beta,p;\sigma} \\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i|\widetilde{\varphi}_i - \varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\widetilde{\varphi}_i - \varphi_i| + \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{\epsilon}|\widetilde{f}_i(0) - f_i(0)| \\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\|\widetilde{f}_i' - f_i'\|_{p;\sigma} + \|\widetilde{m}^0 - m^0\|_{p;\sigma} + \sum_{i=1}^{\infty}|\gamma_i||\widetilde{\psi}_i(0) - \psi_i(0)| \\
&\quad + \sum_{i=1}^{\infty}|\gamma_i|\|\widetilde{\psi}_i' - \psi_i'\|_{1;\sigma} + \sum_{i=1}^{\infty}|\gamma_i|\|\widetilde{g}_i - g_i\|_{\beta,p;\sigma}\Big\},
\end{aligned}
$$

where $\mathcal{C}_1[d]$ is a constant depending on d. Using (4.7) and the relation

$$
\sum_{i=1}^{\infty}|\gamma_i|\lambda_i|\widetilde{\varphi}_i - \varphi_i| \le \frac{1}{\lambda_*^{\epsilon}}\sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{1+\epsilon}|\widetilde{\varphi}_i - \varphi_i|,
$$

where $\lambda_* = \min\{\lambda_i : \lambda_i > 0\}$, we arrive at (8.4) with $\mathcal{C}[d] = (1 + \frac{1}{\lambda_*^\epsilon})e^{\sigma_2[d]}\mathcal{C}_1[d]$. $\square$

9. Model problem and numerical example

A thorough numerical study of IP will be a subject of a forthcoming paper. The present paper, focused on the analysis, is finished by a simpler numerical example.

Let us consider the direct problem (2.7) in the domain $(x,t) \in (0,2\pi)\times(0,1)$ with the data $g = \psi = m^0 = 0$, $f(x,t) = 2\sin x$, $\varphi(x) = \sin x$ and Dirichlet boundary conditions $u(0,t) = u(2\pi,t) = 0$. In such a case the expansions of f and φ in (3.6) contain only single addends corresponding to the eigenfunction $v_1 = \sin x$ of the operator $A = \frac{d^2}{dx^2}$. The solution of (2.7) has the form $u = u_1(t)\sin x$, where u_1 is the solution of the following Cauchy problem for ODE:

$$
\frac{t^{-\beta}}{\Gamma(1-\beta)} * u_1'(t) + u_1(t) + m * u_1(t) = 2,\ t \in (0,1), \quad u_1(0) = 1.
$$

In view of Lemma 4.3, this problem is equivalent to the Volterra integral equation of the second kind

$$u_1(t) + t^{\beta-1}E_{\beta,\beta}(-t^\beta) * m * u_1(t) = 2 - E_\beta(-t^\beta),\ t \in (0,1). \tag{9.1}$$

Let the state u be observed at the point, $x = \frac{\pi}{2}$, i.e. $\Phi[z] = z(\frac{\pi}{2})$. Then

$$h(t) = u_1(t)\sin\frac{\pi}{2} = u_1(t).$$

The inverse problem with such data satisfies the assumptions of Theorems 6.1 – 7.3.

In the numerical example we assumed m is of the form $m(t) = c_1e^{-t} + c_2e^{-2t}$, where $c_1, c_2 \in \mathbb{R}$ are unknown coefficients. Then the kernel of the Volterra equation (9.1) $K = t^{\beta-1}E_{\beta,\beta}(-t^\beta) * m$ is continuous.

Fixing certain values of β^*, c_1^* and c_2^* (exact solution of the inverse problem), we solved (9.1) and computed the values $h(t_i) = u_1(t_i)$ in nodes $t_i = i\eta$, $i = 1, \dots, N$, where $\eta = \frac{1}{N}$ and N is the number of the nodes. Moreover, we set $h(0) = \sin\frac{\pi}{2} = 1$. The obtained vector $h(t_i)$, $i = 0, \dots, N$, formed the synthetic data of the inverse problem.

The solution procedure was implemented in two stages. In the first stage we found the approximate value of β by solving the equation (7.24). In the second stage we determined $\widetilde{c}_1$ and $\widetilde{c}_2$ via minimization of the cost functional

$$J(c_1, c_2) = \sum_{i=1}^{N} \big| h[c_2, c_2](t_i) - h(t_i) \big|,$$

where $h[c_1, c_2] = u_1[c_2, c_2]$ is the trace at $x = \frac{\pi}{2}$ of the solution of the direct problem corresponding to the parameters $\tilde{\beta}$, c_1 and c_2.

The minimization of the cost functional was performed by means of the gradient method. The solution of the Volterra equation (9.1) (direct problem) was implemented using the collocation with piecewise constant splines.

Table 1. Results in case $\beta = 0.8$

N	$\tilde{\beta}$	$\tilde{c}_1$	$\tilde{c}_2$
100	0.786	1.037	1.045
1000	0.799	1.004	1.006
10000	0.7998	1.0005	1.0006

Table 2. Results in case $\beta = 0.2$

N	$\tilde{\beta}$	$\tilde{c}_1$	$\tilde{c}_2$
100	0.155	1.69	1.82
1000	0.194	1.08	1.09
10000	0.199	1.01	1.01

Tables 1 and 2 contain numerical results in cases $\beta^* = 0.8$, $c_1^* = c_2^* = 1$ and $\beta^* = 0.4$, $c_1^* = c_2^* = 1$, respectively. In both cases we chose the initial guesses $c_{1,0} = 2$, $c_{2,0} = 3$ for the minimization process.

Results show that the method to determine β proposed in Remark 7.4 works well in the case of bigger β, but requires a quite small stepsize in the case of smaller β.

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