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NONEXISTENCE OF POSITIVE GLOBAL SOLUTIONS TO THE DIFFERENTIAL EQUATION $u'' - t^{-p-1}u^p = 0$

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Abstract. In this article we consider the ordinary differential equation

$$u'' - t^{-p-1}u^p = 0.$$

We show the blow-up for solutions of this equation, under certain on the initial data.

1. Introduction

In articles [1]–[6], [8]–[10] we studied the semi-linear wave equation $\Box u + f(u) = 0$ under some conditions, and we found some interesting results on blow-up, blow-up rate and estimates for the life-span of solutions, but no information on the singular set. So we want to study some particular cases for lower dimensional wave equations, therefrom we hope that we gain some experience for for studying particular lower dimension later.

It is clear that the functions $t^{-p-1}u^p$, with $p > 1$, $u \geq 0$ and $t \geq 1$ is locally Lipschitz. By standard theory, the existence and uniqueness of classical local solutions holds for the equation

$$\begin{gathered} u'' - t^{-p-1}u^p = 0, \quad p \in (1, \infty), \\ u(1) = u_0, \quad u'(1) = u_1. \end{gathered} \tag{1.1}$$

Notation and fundamental Lemmas. First we make a substitution

$$\begin{gathered} u = tv, \quad u' = v + tv', \quad u'' = 2v' + tv'', \\ t^{p+1}u'' = 2t^{p+1}v' + t^{p+2}v'' = t^p v^p, \\ 2tv' + t^2 v'' = v^p. \end{gathered}$$

Set $s = \ln t$, $v(t) = w(s)$, then $tv' = w_s$, $t^2 v'' = w_{ss} - w_s$.

For a given function w in this work we use the following abbreviations:

$$\begin{gathered} E_w(0) = (u_1 - u_0)^2 - \frac{2}{p+1}u_0^{p+1}, \quad a_w(s) = w^2 := a(s), \\ K(s) := K_w(s) := \int_0^s w_s^2(r)dr, \quad J(s) := J_w(s) = w(s)^{-\frac{p-1}{2}}. \end{gathered}$$

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The equation (1.1) can be transformed into

$$w_{ss} + w_s = w^p, \tag{1.2}$$
$$w(0) = w_0 = u_0, \tag{1.3}$$
$$w_s(0) = w_1 = u_1 - u_0. \tag{1.4}$$

Using some elementary calculations we obtain the following lemmas.

Lemma 1.1. *Suppose that w is the solution of* (1.2), *then*

$$K_s(s) + 2K(s) - \frac{2}{p+1} w^{p+1}(s) = E_w(0), \tag{1.5}$$
$$a_{ss}(s) + a_s(s) = 2(w(s)^{p+1} + K_s(s)). \tag{1.6}$$

Lemma 1.2. *Suppose that w is the solution of* (1.2), *then*

$$K(s) = \frac{E_w(0)}{2}(1 - e^{-2s}) + \frac{2}{p+1} e^{-2s} \int_0^s e^{2r} w^{p+1}(r) dr, \tag{1.7}$$
$$K_s(s) = E_w(0) e^{-2s} + \frac{2}{p+1} w(s)^{p+1} - \frac{4}{p+1} e^{-2s} \int_0^s e^{2r} w^{p+1}(r) dr, \tag{1.8}$$
$$\begin{aligned} a_s(s) &= (a_s(0) + 2E_w(0)) e^{-s} - 2E_w(0) e^{-2s} \\ &\quad + \frac{2}{p+1} \int_0^s (p - 1 + 4e^{r-s}) e^{r-s} w^{p+1}(r) dr, \end{aligned} \tag{1.9}$$
$$\begin{aligned} a_s &= -(p+1)E_w(0) + ((p+1)E_w(0) + a_s(0)) e^{-s} \\ &\quad + (p+3)K(s) + (p-1) \int_0^s e^{r-s} K(r) dr, \end{aligned} \tag{1.10}$$
$$\begin{aligned} J_{ss}(s) &= -(p+1) a(s)^{-\frac{p+1}{2}-1} ((-((p+1)E_w(0) + a_s(0)) e^{-s})) \\ &\quad - (p^2 - 1) a(s)^{-\frac{p+1}{2}-1} \int_0^s e^{r-s} K(r) dr. \end{aligned} \tag{1.11}$$

Proof. By (1.5) and (1.6) we have

$$\begin{gathered} K_s(s) + 2K(s) = E_w(0) + \frac{2}{p+1} w(s)^{p+1}, \\ (e^{2s} K(s))_s = e^{2s} (E_w(0) + \frac{2}{p+1} w(s)^{p+1}), \\ e^{2s} K(s) = \frac{E_w(0)}{2} (e^{2s} - 1) + \frac{2}{p+1} \int_0^s e^{2r} w^{p+1}(r) dr. \end{gathered}$$

Thus (1.7) and (1.8) are obtained.

By (1.6) and (1.7) we obtain

$$\begin{gathered} e^s \big(a_{ss}(s) + a_s(s) \big) = 2e^s \big(w(s)^{p+1} + K_s(s) \big), \\ \begin{aligned} e^s a_s(s) &= a_s(0) + \int_0^s 2e^r (w^{p+1} + K_s)(r) dr \\ &= a_s(0) + 2e^s K(s) + \int_0^s 2e^r (w^{p+1} - K)(r) dr, \end{aligned} \end{gathered}$$

$$\begin{aligned}
e^s a_s(s) &= a_s(0) + 2e^s K(s) + \int_0^s 2e^r w(r)^{p+1} dr \\
&\quad - \int_0^s 2e^r \Big(\frac{E_w(0)}{2}(1-e^{-2r}) + \frac{2}{p+1} e^{-2r} \int_0^r e^{2\eta} w^{p+1}(\eta) d\eta\Big) dr \\
&= a_s(0) + 2e^s K(s) + \int_0^s 2e^r w^{p+1}(r) dr - \int_0^s E_w(0)(e^r - e^{-r}) dr \\
&\quad - \frac{4}{p+1} \int_0^s e^{-r} \int_0^r e^{2\eta} w^{p+1}(\eta) d\eta \, dr,
\end{aligned}$$

$$\begin{aligned}
e^s a_s(s) &= a_s(0) + 2e^s K - E_w(0)(e^s + e^{-s} - 2) + \int_0^s 2e^r w^{p+1}(r) dr \\
&\quad - \frac{4}{p+1} \int_0^s e^{-r} \int_0^r e^{2\eta} w^{p+1}(\eta) d\eta dr,
\end{aligned}$$

$$\begin{aligned}
e^s a_s(s) &= a_s(0) + 2e^s K(s) - E_w(0)(e^s + e^{-s} - 2) + \int_0^s 2e^r w^{p+1}(r) dr \\
&\quad + \frac{4}{p+1} \Big(e^{-s} \int_0^s e^{2\eta} w^{p+1}(\eta) d\eta - \int_0^s e^{-r} e^{2r} w^{p+1}(r) dr \Big) \\
&= a_s(0) + 2e^s K(s) - E_w(0)(e^s + e^{-s} - 2) + \int_0^s 2e^r w^{p+1}(r) dr \\
&\quad + \frac{4}{p+1} \int_0^s (e^{-s} - e^{-r}) e^{2r} w^{p+1}(r) dr,
\end{aligned}$$

$$\begin{aligned}
e^s a_s(s) &= a_s(0) + 2e^s K(s) - E_w(0)(e^s + e^{-s} - 2) \\
&\quad + 2 \int_0^s [1 + \frac{2}{p+1}(e^{r-s} - 1)] e^r w^{p+1}(r) dr \\
&= a_s(0) + 2e^s K(s) - E_w(0)(e^s + e^{-s} - 2) \\
&\quad + \frac{2}{p+1} \int_0^s [p - 1 + 2e^{r-s}] e^r w^{p+1}(r) dr,
\end{aligned}$$

$$\begin{aligned}
a_s(s) &= a_s(0)e^{-s} + 2\Big(\frac{E_w(0)}{2}(1 - e^{-2s}) + \frac{2}{p+1} e^{-2s} \int_0^s e^{2r} w^{p+1}(r) dr\Big) \\
&\quad - E_w(0)(1 + e^{-2s} - 2e^{-s}) + \frac{2}{p+1} \int_0^s (p - 1 + 2e^{r-s}) e^{r-s} w^{p+1}(r) dr \\
&= a_s(0)e^{-s} + E_w(0)(1 - e^{-2s}) \\
&\quad + \frac{4}{p+1} e^{-2s} \int_0^s e^{2r} w^{p+1}(r) dr - E_w(0)(1 + e^{-2s} - 2e^{-s}) \\
&\quad + \frac{2}{p+1} \int_0^s (p - 1 + 2e^{r-s}) e^{r-s} w^{p+1}(r) dr \\
&= (a_s(0) + 2E_w(0))e^{-s} - 2E_w(0)e^{-2s} \\
&\quad + \frac{2}{p+1} \int_0^s (p - 1 + 4e^{r-s}) e^{r-s} w^{p+1}(r) dr.
\end{aligned}$$

Therefore, (1.9) follows.

Now to prove (1.10). According to (1.5) and (1.6), we obtain that

$$e^s a_s(s) = a_s(0) + \int_0^s 2e^r (w^{p+1} + K_s)(r) dr$$

$$
\begin{aligned}
&= a_s(0) + \int_0^s 2e^r \Big(\frac{p+1}{2}(K_s + 2K - E_w(0)) + K_s\Big)(r)dr \\
&= a_s(0) - (p+1)E_w(0)(e^s - 1) \\
&\quad + \int_0^s e^r((p+3)K_s + 2(p+1)K)(r)dr
\end{aligned}
$$

and

$$
\begin{aligned}
a_s(s) &= -(p+1)E_w(0) + (a_s(0) + (p+1)E_w(0))e^{-s} \\
&\quad + \int_0^s e^{r-s}((p+3)K_s + 2(p+1)K)(r)dr \\
&= -(p+1)E_w(0) + ((p+1)E_w(0) + a_s(0))e^{-s} \\
&\quad + (p+3)K(s) + (p-1)\int_0^s e^{r-s}K(r)dr.
\end{aligned}
$$

To show (1.11), we use (1.6) and the definition $J(s) := J_w(s) = a(s)^{-\frac{p+1}{2}}$, $J_s = -\frac{p+1}{2}a(s)^{-\frac{p+1}{2}-1}a_s$,

$$
\begin{aligned}
a_{ss}(s) &= -((p+1)E_w(0) + a_s(0))e^{-s} + (p+3)K_s(s) \\
&\quad + (p-1)K(s) - (p-1)\int_0^s e^{r-s}K(r)dr \\
&= -((p+1)E_w(0) + a_s(0))e^{-s} + (p+3)K_s(s) + (p-1)\int_0^s e^{r-s}K(r)dr.
\end{aligned}
$$

$$
\begin{aligned}
J_{ss}(s) &= -\frac{p+1}{2}a(s)^{-\frac{p+1}{2}-2}\Big(a(s)a_{ss}(s) - \frac{p+3}{2}a_s(s)^2\Big) \\
&= -\frac{p+1}{2}a(s)^{-\frac{p+1}{2}-2}\Big(2a(s)(-((p+1)E_w(0) + a_s(0))e^{-s})\Big) \\
&\quad - (p+1)(p-1)a(s)^{-\frac{p+1}{2}-1}\int_0^s e^{r-s}K(r)dr \\
&= -(p+1)J(s)^{1+\frac{2}{p+1}}\Big(-((p+1)E_w(0) + a_s(0))e^{-s} \\
&\quad + (p-1)\int_0^s e^{r-s}K(r)dr\Big).
\end{aligned}
$$

The above formulation is equivalent to assertion (1.11). Thus Lemma 1.2 is proved. □

Definition 1.3. A function $g : \mathbb{R} \to \mathbb{R}$ has a blow-up rate q means that g exists only in finite time, that is, there is a finite number T^* such that

$$\lim_{t\to T^*} g(t)^{-1} = 0$$

and there exists a non-zero $\beta \in \mathbb{R}$ with

$$\lim_{t\to T^*} (T^* - t)^q g(t) = \beta,$$

in this case β is called the blow-up constant of g.

The following lemma is easy to prove so we omit its proof.

Lemma 1.4. *If $g(t)$ and $h(t,r)$ are continuous with respect to their variables and the limit $\lim_{t\to T}\int_0^{g(t)} h(t,r)dr$ exists, then*
$$\lim_{t\to T}\int_0^{g(t)} h(t,r)dr = \int_0^{g(T)} h(T,r)dr.$$

2. Nonexistence of global solution when $E_w(0) < 0$

In this section we want to show there is not global solution for (1.1) under negative energy $E_w(0) < 0$.

Theorem 2.1. *If T is the life-span of u and u is the positive solution of the problem (1.1) with $E_w(0) < 0$, then T is finite. This means that the global solution of (1.1) does not exist for $(u_1 - u_0)^2 < \frac{2}{p+1}u_0^{p+1}$.*

Proof. We consider the cases $a_s(0) > 0$ and $a_s(0) \le 0$. In the first case, using (1.10) of Lemma 1.2 we obtain that
$$\begin{aligned} a_s(s) &= -(p+1)E_w(0)(1-e^{-s}) + a_s(0)e^{-s} + (p+3)K(s)\\ &\quad + (p-1)\int_0^s e^{r-s}K(r)dr > 0 \end{aligned}$$
for all $s \ge 0$ and
$$a(s) \ge -(p+1)E_w(0)(s-1+e^{-s}) + a_s(0)(1-e^{-s}) \ge 1$$
for
$$s \ge s_0 := 1 + \frac{-1}{(p+1)E_w(0)}.$$
According to Lemma 1.1, (1.6) for $s \ge s_0$,
$$\begin{aligned} (a_{ss}+a_s)(s) &= 2\Big(w(s)^{p+1} - \frac{a(s)}{4}\Big) + 2w_s(s)^2 + \frac{w(s)^2}{2}\\ &\ge 2\Big(a(s)^{\frac{p+1}{2}} - \frac{a(s)}{4}\Big) + 2|w_s w|(s),\\ a_{ss}(s) &\ge 2a(s)\Big(a(s)^{\frac{p-1}{2}} - \frac{1}{4}\Big)\\ &= \frac{a(s)}{2}\Big(a(s)^{\frac{p-1}{2}} - 1\Big) + \frac{3}{2}a(s)^{\frac{p+1}{2}}\\ &\ge \frac{3}{2}a(s)^{\frac{p+1}{2}}; \end{aligned}$$
thus we obtain that there exists $s_1 := 1 - \frac{1}{E_w(0)}\frac{2^{\frac{2}{p+3}}}{p+1}a(s_0) > s_0$ so that
$$\begin{aligned} (a_s^2(s))_s &= 2a_s(s)a_{ss}(s)\\ &\ge 3a(s)^{\frac{p+1}{2}}a_s(s) = \frac{6}{p+3}(a(s)^{\frac{p+3}{2}})_s,\\ a_s^2(s) &\ge \frac{6}{p+3}a(s)^{\frac{p+3}{2}} + a_s^2(s_0) - \frac{6}{p+3}a(s_0)^{\frac{p+3}{2}} \ge \frac{3}{p+3}a(s)^{\frac{p+3}{2}}. \end{aligned}$$
Since for $s \ge s_1$,
$$\begin{aligned} a(s) &\ge -(p+1)E_w(0)(s-1+e^{-s}) \ge -(p+1)E_w(0)(s-1),\\ a(s)^{\frac{p+3}{2}} &\ge (p+1)^{\frac{p+3}{2}}\big(-E_w(0)\big)^{\frac{p+3}{2}}(s-1)^{\frac{p+3}{2}} \ge 2a(s_0)^{\frac{p+3}{2}}, \end{aligned}$$

$$
\begin{gathered}
a_s(s) \geq \sqrt{\frac{3}{p+3}} a(s)^{\frac{p+3}{4}}, \\
\frac{-4}{p-1}(a(s)^{\frac{1-p}{4}})_s = a(s)^{-\frac{p+3}{4}} a_s \geq \sqrt{\frac{3}{p+3}}, \\
\frac{-4}{p-1} a(s)^{\frac{1-p}{4}} \geq \sqrt{\frac{3}{p+3}}(s-s_1) - \frac{4}{p-1} a(s_1)^{\frac{1-p}{4}}, \\
a(s)^{\frac{1-p}{4}} \leq -\frac{p-1}{4}\Big(\sqrt{\frac{3}{p+3}}(s-s_1) - \frac{4}{p-1} a(s_1)^{\frac{1-p}{4}}\Big);
\end{gathered}
$$

therefore there exists

$$
S^* \leq S_1^* = s_1 + \sqrt{\frac{p+3}{3}} \frac{4}{p-1} a(s_1)^{\frac{1-p}{4}}
$$

such that

$$
a(s)^{\frac{1-p}{4}} \to 0 \quad \text{for } s \to S^*.
$$

This means that the solution for of (1.1) does not exist for all $t \geq 1$ and the life-span T^* of u is finite with $T^* \leq \ln S_1^*$. Whereas, if $a_s(0) \leq 0$ using (1.10) of Lemma 1.2 again, we obtain that

$$
\begin{aligned}
a_s(s) = &-(p+1)E_w(0)(1-e^{-s}) + a_s(0)e^{-s} + (p+3)K(s) \\
&+ (p-1)\int_0^s e^{r-s} K(r)dr > 0
\end{aligned}
$$

for all large $s \geq s_2$, where s_2 is given by

$$
s_2 = \ln(1 + \frac{a_s(0)}{(p+1)E_w(0)})
$$

and

$$
a(s) \geq -(p+1)E_w(0)(s - s_2 + e^{-s} - e^{-s_2}) + a_s(0)(e^{-s_2} - e^{-s}) \geq 1
$$

for all $s \geq s_3$, s_3 can be obtained by

$$
\begin{aligned}
&-(p+1)E_w(0)(s_3 + e^{-s_3}) - a_s(0)e^{-s_3} \\
&= 1 - (p+1)E_w(0)(s_2 + e^{-s_2}) - a_s(0)e^{-s_2}
\end{aligned}
$$

According to Lemma 1.1, (1.6) for $s \geq s_3$,

$$
\begin{aligned}
(a_{ss} + a_s)(s) &= 2\Big(w(s)^{p+1} - \frac{a(s)}{4}\Big) + 2w_s(s)^2 + \frac{w(s)^2}{2} \\
&\geq 2\Big(a(s)^{\frac{p+1}{2}} - \frac{a(s)}{4}\Big) + 2|w_s w|(s), \\
a_{ss}(s) &\geq 2a(s)(a(s)^{\frac{p-1}{2}} - \frac{1}{4}) \\
&= \frac{a(s)}{2}(a(s)^{\frac{p-1}{2}} - 1) + \frac{3}{2} a(s)^{\frac{p+1}{2}} \\
&\geq \frac{3}{2} a(s)^{\frac{p+1}{2}};
\end{aligned}
$$

thus we obtain that there exists $s_4 > s_3$ such that for $s \geq s_4$,

$$
a(s_4)^{\frac{p+3}{2}} = -\frac{p+3}{3} a_s^2(s_3) + 2a(s_3)^{\frac{p+3}{2}},
$$

$$
\begin{aligned}
(a_s^2(s))_s &= 2a_s(s)a_{ss}(s)\\
&\geq 3a(s)^{\frac{p+1}{2}}a_s(s)\\
&= \frac{6}{p+3}(a(s)^{\frac{p+3}{2}})_s,
\end{aligned}
$$

$$a_s^2(s) \geq \frac{6}{p+3}a(s)^{\frac{p+3}{2}} + a_s^2(s_3) - \frac{6}{p+3}a(s_3)^{\frac{p+3}{2}} \geq \frac{3}{p+3}a(s)^{\frac{p+3}{2}},$$

$$a_s(s) \geq \sqrt{\frac{3}{p+3}}a(s)^{\frac{p+3}{4}} \quad \forall s \geq s_4,$$

$$\frac{-4}{p-1}(a(s)^{\frac{1-p}{4}})_s = a(s)^{-\frac{p+3}{4}}a_s \geq \sqrt{\frac{3}{p+3}} \quad \forall s \geq s_4,$$

$$\frac{-4}{p-1}a(s)^{\frac{1-p}{4}} \geq \sqrt{\frac{3}{p+3}}(s-s_4) - \frac{4}{p-1}a(s_4)^{\frac{1-p}{4}} \quad \forall s \geq s_4,$$

$$a(s)^{\frac{1-p}{4}} \leq -\frac{p-1}{4}\Big(\sqrt{\frac{3}{p+3}}(s-s_4) - \frac{4}{p-1}a(s_4)^{\frac{1-p}{4}}\Big)$$

therefore, there exists

$$S_2^* \leq s_4 + \sqrt{\frac{p+3}{3}}\frac{4}{p-1}a(s_4)^{\frac{1-p}{4}}$$

so that

$$a(s)^{\frac{1-p}{4}} \to 0 \quad \text{for } s \to S_2^*.$$

This means that the solution for the problem (1.1) does not exist for all $t \geq 1$ and the life-span T^* of u is finite with $T^* \leq \ln S_2^*$. □

3. Nonexistence of global solution when $E_w(0) > 0$

In this section we want to show there is no global solution of (1.1) under positive energy $E_w(0) > 0$ when one of the following conditions holds

$$u_0(u_1 - u_0) \geq 0, \tag{3.1}$$

$$u_0(u_1 - u_0) < 0, \quad u_1(u_1 - u_0) > \frac{2}{p+1}u_0^{p+1}. \tag{3.2}$$

We have the result for nonexistence of global solution of (1.1).

Theorem 3.1. *If T is the life-span of u and u is the positive solution of the problem (1.1) with $E_w(0) > 0$ and $u_0(u_1 - u_0) \geq 0$, then T is finite. This means that the global solution of (1.1) does not exist for $(u_1 - u_0)^2 > \frac{2}{p+1}u_0^{p+1}$, $u_0(u_1 - u_0) \geq 0$.*

If T is the life-span of u and u is the positive solution of the problem (1.1) with $E_w(0) > 0$, $u_0(u_1 - u_0) < 0$ and $E_w(0)+ u_0(u_1 - u_0) > 0$, then T is finite. This means that the global solution of (1.1) does not exist for $u_1(u_1 - u_0) > \frac{2}{p+1}u_0^{p+1}$.

Remark 3.2. Under positive energy $E_w(0) > 0$, $u_0(u_1-u_0) < 0$ with $u_1(u_1-u_0) \leq \frac{2}{p+1}u_0^{p+1}$ we conjecture that solutions of (1.1) exist globally, and the asymptotic behavior of such solutions is similar to the function

$$c + \frac{c^{-p}t^{1-p}}{p(p-1)}$$

as $t \to \infty$, but we are unable to prove this rigorously.

Proof. Using Lemma 1.2, (1.9), $E_w(0) > 0$ and $u_0(u_1 - u_0) \geq 0$ we obtain

$$e^s a_s(s) \geq (a_s(0) + 2E_w(0)) - 2E_w(0)e^{-s} + \frac{2(p-1)}{p+1}\int_0^s e^r a(r) w^{p-1}(r)dr,$$

$$A(s) = \int_0^s e^r w^{p+1}(r)dr, \quad A_s(s) = e^s a(s)^{\frac{p+1}{2}},$$

$$a(s) = (e^{-s}A_s(s))^{\frac{2}{p+1}},$$

$$a_s(s) = \frac{2}{p+1}(e^{-s}A_s(s))^{\frac{2}{p+1}-1}e^{-s}(A_{ss} - A_s)(s),$$

$$\begin{aligned}
&\frac{2}{p+1}(e^{-s}A_s(s))^{\frac{2}{p+1}-1}(A_{ss}(s) - A_s(s)) \\
&\geq (a_s(0) + 2E_w(0)) - 2E_w(0)e^{-s} + \frac{2(p-1)}{p+1}A(s), \\
&\frac{2}{p+1}(e^{-s}A_s(s))^{\frac{2}{p+1}}e^{-s}(A_{ss}(s) - A_s(s)) \\
&\geq (a_s(0) + 2E_w(0))e^{-2s}A_s(s) - 2E_w(0)e^{-3s}A_s(s) \\
&\quad + \frac{2(p-1)}{p+1}e^{-2s}A_s(s)A(s),
\end{aligned}$$

$$\begin{aligned}
&\frac{2}{p+3}((e^{-s}A_s(s))^{\frac{p+3}{p+1}} - a_0^{\frac{p+3}{2}}) \\
&\geq \Big(\Big(a_s(0) + 2E_w(0) + \frac{p-1}{p+1}A(s)\Big)e^s - 2E_w(0)\Big)e^{-3s}A(s) \geq 0
\end{aligned}$$

for some large s_4, $s \geq s_5$, since $a_s(0) + 2E_w(0) > 0$. Therefore, for $s \geq s_5$,

$$\begin{aligned}
&(e^{-s}A_s(s))^{\frac{p+3}{p+1}} \\
&\geq a_0^{\frac{p+3}{2}} + \frac{p+3}{2}\Big(\Big(a_s(0) + 2E_w(0) + \frac{p-1}{p+1}A(s)\Big)e^s - 2E_w(0)\Big)e^{-3s}A(s),
\end{aligned}$$

$$\begin{aligned}
&A_s(s) \\
&\geq \Big(a_0^{\frac{p+3}{2}} + \frac{p+3}{2}\Big(\Big(a_s(0) + 2E_w(0) + \frac{p-1}{p+1}A(s)\Big)e^s - 2E_w(0)\Big)e^{-3s}A(s)\Big)^{\frac{p+1}{p+3}} \\
&\geq (\frac{1}{2})^{\frac{p+1}{p+3}}a_0^{\frac{p+1}{2}}e^s \\
&\quad + (\frac{1}{2})^{\frac{p+1}{p+3}}\Big(\frac{p+3}{2}\Big(a_s(0) + 2E_w(0) \\
&\quad + \frac{p-1}{p+1}A(s)\Big)e^s - 2E_w(0)\Big)^{\frac{p+1}{p+3}}e^{-\frac{3(p+1)}{p+3}s}A(s)^{\frac{p+1}{p+3}}e^s \\
&= (\frac{1}{2})^{\frac{p+1}{p+3}}a_0^{\frac{p+1}{2}}e^s + (\frac{1}{2})^{\frac{p+1}{p+3}}(\frac{p+3}{2})^{\frac{p+1}{p+3}}\Big(\Big(a_s(0) + 2E_w(0) \\
&\quad + \frac{p-1}{p+1}A(s)\Big)e^s - 2E_w(0)\Big)^{\frac{p+1}{p+3}}e^{-\frac{2p}{p+3}s}A(s)^{\frac{p+1}{p+3}},
\end{aligned}$$

$$A(s) \geq (\frac{1}{2})^{\frac{p+1}{p+3}}a_0^{\frac{p+1}{2}}(e^s - e^{s_5}) + A(s_5).$$

By the same arguments as in the proof of Theorem 2.1, the assertions in can be obtained. □

4. Nonexistence of global solution when $E_w(0) = 0$

In this section we want to show there is no global solution of (1.1) under zero energy $E_w(0) = 0$, with $u_0(u_1 - u_0) > 0$.

Theorem 4.1. *If T is the life-span of u and u is the positive solution of* (1.1) *with $E_w(0) = 0$ and $u_0(u_1 - u_0) > 0$, then T is finite. This means that the global solution of* (1.1) *does not exist for $(u_1 - u_0)^2 = \frac{2}{p+1} u_0^{p+1}$, $u_0(u_1 - u_0) > 0$.*

Remark 4.2. If $E_w(0) = 0$, $u_0(u_1 - u_0) \le 0$ we conjecture that solutions of (1.1) exist globally and have the same asymptotic behavior as that stated in Remark 3.2. Yet, again, we do not have a rigorous proof.

Proof. Using the Lemma 1.2, (1.10), $E_w(0) = 0$ and $u_0(u_1 - u_0) > 0$ we obtain

$$a_s(s) = a_s(0)e^{-s} + (p+3)K(s) + (p-1)\int_0^s e^{r-s}K(r)dr > 0,$$

$$a(s) = a(0) + a_s(0)(1 - e^{-s}) + (p+3)\int_0^s K(r)dr + (p-1)\int_0^s \int_0^r e^{\eta - r}K(\eta)d\eta dr,$$

$$\begin{aligned}
a(s) &= a(0) + a_s(0)(1 - e^{-s}) + (p+3)\int_0^s K(r)dr \\
&\quad + (p-1)\int_0^s (e^{-r}\int_0^r e^{\eta}K(\eta)d\eta)dr \\
&= a(0) + a_s(0)(1 - e^{-s}) + (p+3)\int_0^s K(r)dr \\
&\quad + (p-1)\Big(-e^{-s}\int_0^s e^r K(r)dr + \int_0^s K(r)dr\Big) \\
&= a(0) + a_s(0)(1 - e^{-s}) + \int_0^s (2(p+1) - (p-1)e^{r-s})K(r)dr \\
&\ge a(0) + a_s(0)(1 - e^{-s}) + (p+3)\int_0^s K(r)dr.
\end{aligned}$$

By Lemma 1.2, (1.7),

$$\begin{aligned}
a(s) &\ge a(0) + a_s(0)(1 - e^{-s}) + (p+3)\int_0^s K(r)dr \\
&= a(0) + a_s(0)(1 - e^{-s}) + \frac{p+3}{p+1}\int_0^s 2e^{-2r}\Big(\int_0^r e^{2\eta}w^{p+1}(\eta)d\eta\Big)dr \\
&= a(0) + a_s(0)(1 - e^{-s}) - \frac{p+3}{p+1}e^{-2s}\int_0^s e^{2\eta}w^{p+1}(\eta)d\eta + \frac{p+3}{p+1}\int_0^s w^{p+1}(r)dr,
\end{aligned}$$

$$\begin{aligned}
a(s) &\ge a(0) + a_s(0)(1 - e^{-s}) + \frac{p+3}{p+1}\int_0^s (1 - e^{2r-2s})w^{p+1}(r)dr \\
&= a(0) + a_s(0)(1 - e^{-s}) + \frac{p+3}{p+1}e^{-2s}\int_0^s (e^s + e^r)(e^s - e^r)w^{p+1}(r)dr,
\end{aligned}$$

$$\begin{aligned}
a(s) &\ge a(0) + a_s(0)(1 - e^{-s}) + \frac{p+3}{p+1}e^{-s}\int_0^s (e^s - e^r)w^{p+1}(r)dr \\
&= a(0) + a_s(0)(1 - e^{-s}) + \frac{p+3}{p+1}e^{-s}\Big(\int_0^{s/2} + \int_{s/2}^s\Big)(e^s - e^r)w^{p+1}(r)dr,
\end{aligned}$$

$$\begin{aligned}\int_0^{s/2}(e^s-e^r)w^{p+1}(r)dr &\geq a(0)^{\frac{p+1}{2}}\int_0^{s/2}(e^s-e^r)dr\\ &= a(0)^{\frac{p+1}{2}}\big(1+\frac{s}{2}e^s-e^{\frac{s}{2}}\big),\\ \int_{s/2}^{s}(e^s-e^r)w^{p+1}(r)dr &\geq a(\frac{s}{2})^{\frac{p+1}{2}}\big(\frac{s}{2}e^s+e^{\frac{s}{2}}-e^s\big);\end{aligned}$$

thus

$$\begin{aligned}a(s)\geq a(0)+a_s(0)(1-e^{-s})+\frac{p+3}{p+1}e^{-s}\Big\{&a(0)^{\frac{p+1}{2}}(1+\frac{s}{2}e^s-e^{\frac{s}{2}})\\ &+a(\frac{s}{2})^{\frac{p+1}{2}}(\frac{s}{2}e^s+e^{\frac{s}{2}}-e^s)\Big\}.\end{aligned}$$

Further, for

$$s\geq s_6:=\frac{1}{2}+\frac{p+1}{p+3}(1-a(0)-a_s(0)(1-e^{-s}))a(0)^{\frac{p+1}{-2}}$$

we also have

$$\begin{aligned}a(s)&\geq a(0)+a_s(0)(1-e^{-s})+\frac{p+3}{p+1}\int_0^s(1-e^{2r-2s})w^{p+1}(r)dr\\ &\geq a(0)+a_s(0)(1-e^{-s})+\frac{p+3}{p+1}a(0)^{\frac{p+1}{2}}(s-\frac{1-e^{-2s}}{2})\\ &\geq a(0)+a_s(0)(1-e^{-s})+\frac{p+3}{p+1}a(0)^{\frac{p+1}{2}}(s-\frac{1}{2})\\ &\geq 1,\end{aligned}$$

$$\begin{aligned}(a_{ss}+a_s)(s)&=2\Big(w(s)^{p+1}-\frac{a(s)}{4}\Big)+2w_s(s)^2+\frac{w(s)^2}{2}\\ &\geq 2\Big(a(s)^{\frac{p+1}{2}}-\frac{a(s)}{4}\Big)+2|w_sw|(s),\end{aligned}$$

$$\begin{aligned}a_{ss}(s)&\geq 2a(s)\Big(a(s)^{\frac{p-1}{2}}-\frac{1}{4}\Big)\\ &=\frac{a(s)}{2}\Big(a(s)^{\frac{p-1}{2}}-1\Big)+\frac{3}{2}a(s)^{\frac{p+1}{2}}\\ &\geq\frac{3}{2}a(s)^{\frac{p+1}{2}};\end{aligned}$$

thus there exists $s_7>s_6$ such that for $s\geq s_7$,

$$a(s_7)^{\frac{p+3}{2}}=-\frac{p+3}{3}a_s^2(s_6)+2a(s_6)^{\frac{p+3}{2}},$$

$$\begin{aligned}(a_s^2(s))_s&=2a_s(s)a_{ss}(s)\\ &\geq 3a(s)^{\frac{p+1}{2}}a_s(s)\\ &=\frac{6}{p+3}(a(s)^{\frac{p+3}{2}})_s,\end{aligned}$$

$$a_s^2(s)\geq\frac{6}{p+3}a(s)^{\frac{p+3}{2}}+a_s^2(s_6)-\frac{6}{p+3}a(s_6)^{\frac{p+3}{2}}\geq\frac{3}{p+3}a(s)^{\frac{p+3}{2}},$$

$$a_s(s)\geq\sqrt{\frac{3}{p+3}}a(s)^{\frac{p+3}{4}}\quad\forall s\geq s_7,$$

$$\frac{-4}{p-1}(a(s)^{\frac{1-p}{4}})_s = a(s)^{-\frac{p+3}{4}} a_s \geq \sqrt{\frac{3}{p+3}} \quad \forall s \geq s_7,$$
$$\frac{-4}{p-1} a(s)^{\frac{1-p}{4}} \geq \sqrt{\frac{3}{p+3}}(s-s_7) - \frac{4}{p-1} a(s_7)^{\frac{1-p}{4}} \quad \forall s \geq s_7$$
$$a(s)^{\frac{1-p}{4}} \leq -\frac{p-1}{4}\Big(\sqrt{\frac{3}{p+3}}(s-s_7) - \frac{4}{p-1} a(s_7)^{\frac{1-p}{4}}\Big);$$

therefore there exists

$$S_3^* \leq s_7 + \sqrt{\frac{p+3}{3}} \frac{4}{p-1} a(s_7)^{\frac{1-p}{4}}$$

such that

$$a(s)^{\frac{1-p}{4}} \to 0 \quad \text{for } s \to S_3^*.$$

This means that the solution of (1.1) does not exist for all $t \geq 1$ and the life-span T^* of u is finite with $T^* \leq \ln S_3^*$. □

Remark 4.3. If we reconsider the solution behavior of the problem (1.1) on $(0, 1]$, as one may use the same transformations as given in Fundamental Lemmas and obtain problems (1.2)–(1.4) on the interval $-\infty < s \leq 0$. On the other hand, by changing variables $\tau = -s$, $w(s) = X(\tau)$, equation (1.2) yields

$$X_{\tau\tau} - X_\tau = X^p, \quad \tau \in (0, \infty), \tag{4.1}$$
$$X(0) = X_0 = w(0) = w_0 = u_0, \tag{4.2}$$
$$X_\tau(0) = X_1 = -w_1 = u_0 - u_1. \tag{4.3}$$

In [7] we estimated the life-span τ^* of the positive solution X of (4.1) in three different cases:

(a) $X_1 = 0$, $X_0 > 0$: $\tau^* \leq e^{k_1}$, for a suitable k_1.
(b) $X_1 > 0$, $X_0 > 0$:
  (i) $E_X(0) \geq 0$, $\tau^* \leq e^{k_2}$, $k_2 := \frac{2}{p-1}\sqrt{\frac{p+1}{2}} X_0^{\frac{1-p}{2}}$.
  (ii) $E_X(0) < 0$, $\tau^* \leq e^{k_3}$, $k_3 := \frac{2}{p-1}\frac{X_0}{X_1}$.

Therefore, the solutions of (1.1) on $(0, 1]$ can not defined on this interval but blow up in the interior under such circumstances.

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