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MULTIPLE POSITIVE SOLUTIONS FOR DIRICHLET PROBLEM OF PRESCRIBED MEAN CURVATURE EQUATIONS IN MINKOWSKI SPACES

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Abstract. In this article, we consider the Dirichlet problem for the prescribed mean curvature equation in the Minkowski space,

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big)=\lambda f(u)\quad \text{in } B_R,$$
$$u=0\quad \text{on } \partial B_R,$$

where $B_R:=\{x\in\mathbb{R}^N:|x|<R\}$, $\lambda>0$ is a parameter and $f:[0,\infty)\to\mathbb{R}$ is continuous. We apply some standard variational techniques to show how changes in the sign of f lead to multiple positive solutions of the above problem for sufficiently large λ.

1. Introduction

In this article we show the existence of multiple positive solutions of Dirichlet problem in a ball, associated to the mean curvature operator in the flat Minkowski space $\mathbb{L}^{N+1}$ with $(x_1,\dots,x_N,t)$ and metric $\sum_{i=1}^N(dx_i)^2-(dt)^2$. These problems are of interest in differential geometry and in general relativity. It is known [1, 11] that the study of spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature leads to Dirichlet problems of the form

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big)=f(x,u)\quad \text{in } \Omega, u=0\quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous.

This topic has been largely discussed in the literature for (1.1) in the special cases that $N=1$ or Ω is a ball (or an annulus) in $\mathbb{R}^N$, see [2, 3, 4, 5, 7, 8, 14, 15] and the references contained therein. Note that Coelho, Corsato, Obersnel and Omari [7] proved the existence of one or multiple positive solutions of (1.1) with $N=1$ provided that f is L^p-Caratheodory function, but the positivity of f is not required. Moreover, Bereanu, Jebelean and Torres [3] applied Leray-Schauder degree arguments and critical point theory to show existence of positive radial solutions for (1.1) when $\Omega=B_R:=\{x\in\mathbb{R}^N:|x|<R\}$ and f is positive on $[0,R]\times[0,\alpha)$ with $\alpha\ge R$. Of course, the natural question is what would happen if $f(|x|,s)\equiv f(s)$ and f changes its sign in $[0,\alpha)$.

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Recently, Ma and Lu [15] used the quadrature arguments to study the existence and multiplicity of positive solutions of the nonlinear eigenvalue problem

$$\Big(\frac{u'}{\sqrt{1-\kappa u'^2}}\Big)' + \lambda f(u) = 0 \text{ in } (0,1), \quad u(0) = u(1) = 0, \tag{1.2}$$

where $\kappa > 0$ is a constant and f satisfies

- (A1) $f \in C^1([0, \frac{1}{2\sqrt{\kappa}}))$;
- (A2) Either $f(0) > 0$ or
$$f(0) = 0, \quad f_0 = \lim_{s\to 0+} \frac{f(s)}{\psi(s)} > 0, \quad \psi(s) = \frac{s}{\sqrt{1-\kappa s^2}};$$
- (A3) There exist $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m < \frac{1}{2\sqrt{\kappa}}$ such that $f(a_i) \le 0$, $f(b_i) > 0$ and $F(b_i) > F(u)$ for all $0 \le u \le b_i$, $i = 1, 2, \dots, m-1$.

They showed the existence of at least $2m-1$ positive solutions provided λ is large enough. Their result is an analogous of the well-known result due to Brown and Budin [6], who established the result of (1.2) with $\kappa = 0$ by using a generalization of a quadrature technique of Laetsch [12].

Motivated by above papers, this article is devoted to studying how changes in the sign of f lead to multiple positive solutions for the Dirichlet problem

$$\begin{gathered} -\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda f(u) \quad \text{in } B_R, \\ u = 0 \quad \text{on } \partial B_R \end{gathered} \tag{1.3}$$

for $\lambda > 0$ sufficiently large. Assume throughout that $f : [0, \infty) \to \mathbb{R}$ is continuous and satisfies:

- (A4) $f(0) \ge 0$ and there exist $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m < R$ such that $f(s) \le 0$ if $s \in (a_k, b_k)$ and $f(s) \ge 0$ if $s \in (b_k, a_{k+1})$ for all $k = 1, \dots, m-1$;
- (A5) $\int_{a_k}^{a_{k+1}} f(s)ds > 0$ for all $k \in \{1, \dots, m-1\}$.

Our main result is the following theorem.

Theorem 1.1. *Assume* (A4), (A5). *Then there exists a number $\bar\lambda > 0$ such that for all $\lambda > \bar\lambda$, problem* (1.3) *has at least $m-1$ positive solutions $u_1, u_2, \dots, u_{m-1} \in H_0^1(B_R) \cap L^\infty(B_R)$ and $\|u_k\|_\infty \in (a_k, a_{k+1}]$ for all $k = 1, \dots, m-1$.*

Remark 1.2. It would be interesting to investigate a similar version of Theorem 1.1 for Dirichlet problem (1.1) with $\Omega \subset \mathbb{R}^N$ bounded, sufficiently smooth.

The proof of our main result will be given in the next section and follows ideas used in [7, 8, 13], suitably modified and expanded for the case being considered. For the earlier results on the semilinear problem, see [9, 10].

Now we list a few notation that will be used in this paper. Let $E = H_0^1(B_R)$ with the usual norm $\|u\| = \Big(\int_{B_R} |\nabla u|^2 dx\Big)^{1/2}$. The norm $\|\cdot\|_\infty$ is considered on $L^\infty(B_R)$. We also define $\phi : (-1, 1) \to \mathbb{R}$ by $\phi(s) = \frac{s}{\sqrt{1-s^2}}$ and $\phi_N(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in \mathbb{R}^N$ with $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$.

2. Proof of the main result

The following Lemma is a consequence of the weak maximum principle for the ϕ-Laplace operator.

Lemma 2.1. *Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function and there exists $a_0\in\big(0,R)$ such that $g(s)\geq 0$ if $s\in(-\infty,0)$ and $g(s)\leq 0$ if $s\geq a_0$. If u is a non-trivial solution of*

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big)=g(u)\ \text{in } B_R,\quad u=0\ \text{on } \partial B_R, \tag{2.1}$$

then u is positive a.e. and belongs to $L^\infty(B_R)$. Moreover, $\|u\|_\infty\leq a_0$.

Proof. Let $v=u^-=\max\{-u,0\}\in E$, then

$$\nabla v=\begin{cases}-\nabla u, & u<0,\\ 0, & u\geq 0.\end{cases} \tag{2.2}$$

Multiplying the equation in (2.1) by v and integrating by parts, we have

$$0\geq-\int_{B_R}\frac{|\nabla v|^2}{\sqrt{1-|\nabla v|^2}}dx=\int_{B_R}\frac{\nabla u\cdot\nabla v}{\sqrt{1-|\nabla u|^2}}dx=\int_{B_R}g(u)v dx\geq 0.$$

Hence $\nabla v=0$ a.e. in B_R and we conclude that $u\geq 0$ in B_R.

Next, choosing the test function $w=(u-a_0)^+=\max\{u-a_0,0\}\in E$ in the equation

$$\int_{B_R}\frac{\nabla u\cdot\nabla w}{\sqrt{1-|\nabla u|^2}}dx=\int_{B_R}g(u)w dx,$$

we have $\nabla w=0$ a.e. in B_R and therefore $u\leq a_0$, i.e., $\|u\|_\infty\leq a_0$. ☐

Observe that there exists a constant $M\in(0,\infty)$ such that

$$|f(s)|\leq M,\quad s\in\big[0,R]. \tag{2.3}$$

With the aim of finding positive solutions of (1.3), we introduce an equivalent formulation of the problem aforementioned. For $k=2,\dots,m$, let us define $f_k:\mathbb{R}\to\mathbb{R}$, by

$$f_k(s)=\begin{cases}f(0), & s\leq 0,\\ f(s), & s\in(0,a_k),\\ 0, & s\geq a_k.\end{cases} \tag{2.4}$$

We notice that the function f_k shares the assumed properties of f. Moreover, if u is a non-trivial solution of

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big)=\lambda f_k(u)\ \text{in } B_R,\quad u=0\ \text{on } \partial B_R, \tag{2.5}$$

by Lemma 2.1, u is positive and $\|u\|_\infty\leq a_k$. Thus, u is also a positive solution of (1.3) and belongs to $L^\infty(B_R)$ with $\|u\|_\infty\leq a_k$.

For every $\lambda>0$, set $\beta:=\phi'\Big(\phi^{-1}\big(\frac{\lambda MR}{N}\big)\Big)$ and define $\chi_\lambda:\mathbb{R}\to\mathbb{R}$ such that

$$\chi_\lambda(s)=\begin{cases}\beta\Big(s-\phi^{-1}\big(\frac{\lambda MR}{N}\big)\Big)+\frac{\lambda MR}{N}, & \text{if } s>\phi^{-1}\big(\frac{\lambda MR}{N}\big),\\ \phi(s), & \text{if } |s|\leq\phi^{-1}\big(\frac{\lambda MR}{N}\big),\\ \beta\Big(s+\phi^{-1}\big(\frac{\lambda MR}{N}\big)\Big)-\frac{\lambda MR}{N}, & \text{if } s<-\phi^{-1}\big(\frac{\lambda MR}{N}\big).\end{cases} \tag{2.6}$$

Let $\Pi_\lambda : \mathbb{R} \to \mathbb{R}$ be given by

$$\Pi_\lambda(y) = \int_0^y \chi_\lambda(\zeta)d\zeta.$$

Then

$$\frac{1}{2}y^2 \le \Pi_\lambda(y) \le \frac{1}{2}\beta y^2, \quad y \in \mathbb{R}. \tag{2.7}$$

Let the functional $\mathcal{I}_k(\lambda, \cdot) : E \to \mathbb{R}$ be defined by

$$\mathcal{I}_k(\lambda, u) = \int_{B_R} \Pi_\lambda(|\nabla u|)dx - \lambda \int_{B_R} F_k(u)dx,$$

where $F_k(s) = \int_0^s f_k(\sigma)d\sigma$. We denote by $K_k(\lambda)$ the set of critical points of $\mathcal{I}_k$.

Lemma 2.2. *If u is in $K_k(\lambda)$, then u is a weak solution of*

$$-\operatorname{div}(\psi_N(\nabla u)) = \lambda f_k(u) \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R, \tag{2.8}$$

where

$$\psi_N(\nabla u) = \frac{\chi_\lambda\big(|\nabla u|\big)}{|\nabla u|}\nabla u. \tag{2.9}$$

Proof. Let $u \in K_k(\lambda)$. For any $\varphi \in C_0^\infty(B_R)$ and $\epsilon \in \mathbb{R}$, then $u + \epsilon\varphi \in E$. Since

$$\begin{aligned}
&\mathcal{I}_k(\lambda, u + \epsilon\varphi) - \mathcal{I}_k(\lambda, u) \\
&= \int_{B_R} \big[\Pi_\lambda(|\nabla u + \epsilon\nabla\varphi|) - \Pi_\lambda(|\nabla u|)\big]dx - \lambda \int_{B_R} \big[F_k(u + \epsilon\varphi) - F_k(u)\big]dx \\
&= \int_{B_R} \chi_\lambda\big[|\nabla u| + \theta_1(|\nabla u + \epsilon\nabla\varphi| - |\nabla u|)\big]\big(|\nabla u + \epsilon\nabla\varphi| - |\nabla u|\big)dx \\
&\quad - \lambda\epsilon \int_{B_R} f_k(u + \theta_2\epsilon\varphi)\varphi dx \\
&= \int_{B_R} \chi_\lambda\big[|\nabla u| + \theta_1(|\nabla u + \epsilon\nabla\varphi| - |\nabla u|)\big] \frac{2\epsilon\nabla u \cdot \nabla\varphi + \epsilon^2|\nabla\varphi|^2}{|\nabla u + \epsilon\nabla\varphi| + |\nabla u|}dx \\
&\quad - \lambda\epsilon \int_{B_R} f_k(u + \theta_2\epsilon\varphi)\varphi dx,
\end{aligned}$$

for some constants $\theta_1, \theta_2 \in (0, 1)$, it follows that

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0} \frac{\mathcal{I}_k(\lambda, u + \epsilon\varphi) - \mathcal{I}_k(\lambda, u)}{\epsilon} \\
&= \int_{B_R} \chi_\lambda(|\nabla u|)\frac{\nabla u \cdot \nabla\varphi}{|\nabla u|}dx - \lambda \int_{B_R} f_k(u)\varphi dx \\
&= \int_{B_R} \psi_N(\nabla u) \cdot \nabla\varphi dx - \lambda \int_{B_R} f_k(u)\varphi dx \\
&= \int_{B_R} \big[-\operatorname{div}\big(\psi_N(\nabla u)\big) - \lambda f_k(u)\big]\varphi dx.
\end{aligned}$$

Thus, for any $\varphi \in C_0^\infty(B_R)$, u is a weak solution of (2.8). □

Consequently, from Lemma 2.2, if u is in $K_k(\lambda)$, then u is a weak solution of (2.8). By a similar argument of Lemma 2.1 with $\psi_N(\nabla u)$ instead of $\phi_N(\nabla u)$, we can deduce that u is nonnegative and belongs to $L^\infty(B_R)$ with $\|u\|_\infty \le a_k$.

We next claim that $K_k(\lambda)$ is not empty. Since f_k is bounded and vanishes on (a_k,∞), $\mathcal{I}_k(\lambda,\cdot)$ is coercive and bounded from below. Further, it is weakly lower semi-continuous. Therefore there exists $u_k(\lambda)$ such that

$$\mathcal{I}_k\big(\lambda,u_k(\lambda)\big)=\inf\{\mathcal{I}_k(\lambda,v):\,v\in E\}.$$

The following Lemma shows that for $k=2,\dots,m$, $a_{k-1}<\|u_k\|_\infty\le a_k$ and therefore, (2.8) has at least $m-1$ solutions when $\lambda>0$ sufficiently large.

Lemma 2.3. *For $k=2,\dots,m$, there exists $\lambda_k>0$ such that for all $\lambda>\lambda_k$, $u_k\notin K_{k-1}(\lambda)$.*

Proof. We shall show that there exist $\lambda_k>0$ and $\varphi\in E$, $\varphi\ge 0$ and $\|\varphi\|_\infty\le a_k$, such that

$$\mathcal{I}_k(\lambda,\varphi)<\mathcal{I}_{k-1}(\lambda,u),\quad \lambda>\lambda_k$$

for all $u\in E$ satisfying $0\le u\le a_{k-1}$.

From (A5), $\alpha:=F(a_k)-\max\{F(s):0\le s<a_{k-1}\}>0$. Then, for all $u\in E$ satisfying $0\le u\le a_{k-1}$,

$$\int_{B_R}F(u)dx\le\int_{B_R}F(a_k)dx-\alpha w_N R^N, \tag{2.10}$$

where w_N is the measure of the unit ball in $\mathbb{R}^N$. For $\delta>0$, let $\Omega_\delta:=\{x\in B_R:\operatorname{dist}(x,\partial B_R)<\delta\}$. By Lebesgue's Theorem, $|\Omega_\delta|\to 0$ as $\delta\to 0$. Moreover, for each $\delta>0$, there exists $\varphi_\delta\in C_0^\infty(B_R)$ with $0\le\varphi_\delta\le a_k$, $\varphi_\delta(x)=a_k$, for all $x\in B_R\backslash\Omega_\delta$. Thus

$$\begin{aligned}\int_{B_R}F(\varphi_\delta)dx&=\int_{B_R\backslash\Omega_\delta}F(a_k)dx+\int_{\Omega_\delta}F(\varphi_\delta)dx\\&=\int_{B_R}F(a_k)dx-\int_{\Omega_\delta}\big(F(a_k)-F(\varphi_\delta)\big)dx\\&\ge\int_{B_R}F(a_k)dx-2C|\Omega_\delta|,\end{aligned}\tag{2.11}$$

where $C=\max\{|F(s)|:0\le s\le a_k\}$.

By (2.10) and (2.11) we can choose and fix δ sufficiently small so that there exists $\eta:=\alpha|\Omega|-2C|\Omega_\delta|>0$ such that $\varphi:=\varphi_\delta$ satisfies

$$\int_\Omega F(\varphi)dx\ge\int_\Omega F(u)+\eta$$

for all $u\in E$ with $0\le u\le a_{k-1}$. Therefore for all such u,

$$\begin{aligned}\mathcal{I}_k(\lambda,\varphi)-\mathcal{I}_{k-1}(\lambda,u)&=\int_{B_R}\big[\Pi_\lambda(|\nabla\varphi|)-\Pi_\lambda(|\nabla u|)\big]dx-\lambda\int_{B_R}\big[F(\varphi)-F(u)\big]dx\\&\le\int_{B_R}\Pi_\lambda(|\nabla\varphi|)dx-\lambda\eta<0,\end{aligned}$$

provided $\lambda>0$ is chosen sufficiently large. Hence for such λ the global minimum of $\mathcal{I}_k$ cannot be obtained at any $u\in E$ such that $0\le u\le a_{k-1}$, i.e. $u_k\notin K_{k-1}(\lambda)$. $\square$

Lemma 2.4. *A function $u\in E$ is a positive solution of* (2.5) *if and only if it is a positive solution of* (2.8).

Proof. Suppose that u is a positive solution of (2.5). Hence, for fixed $r \in (0, R]$, from

$$\begin{aligned}\int_{B_r} \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big)dx &= \int_{B_r} \operatorname{div}\Big(\frac{\phi(|\nabla u|)}{|\nabla u|}\nabla u\Big)dx \\ &= \int_{\partial B_r} \frac{\phi(|\nabla u|)}{|\nabla u|}\nabla u \cdot \mathbf{n} dS\end{aligned} \tag{2.12}$$

it follows that

$$-\int_{\partial B_r} \frac{\phi(|\nabla u|)}{|\nabla u|}\nabla u \cdot \mathbf{n} dS = \lambda \int_{B_r} f_k(u)dx, \tag{2.13}$$

where $\mathbf{n}$ denotes the unit outward normal to B_R.

Since $\nabla u \cdot \mathbf{n} = |\nabla u|$ on ∂B_r, we have

$$-\int_{\partial B_r} \phi(|\nabla u|)dS = \lambda \int_{B_r} f_k(u)dx.$$

By radial symmetry, this can be rewritten as

$$|\nabla u(r)| \le \phi^{-1}\Big(\frac{\lambda M r}{N}\Big) \quad \text{for all } r \in (0, R], \tag{2.14}$$

i.e. $\|\nabla u\|_\infty \le \phi^{-1}\Big(\frac{\lambda M R}{N}\Big)$. Therefore, $\phi_N(\nabla u) = \psi_N(\nabla u)$ and we conclude that u is a positive solution of (2.8).

Suppose now that u is a positive solution of (2.8). Arguing as above we see that

$$\|\nabla u\|_\infty \le \chi^{-1}\Big(\frac{\lambda M R}{N}\Big). \tag{2.15}$$

Therefore, $\psi_N(\nabla u) = \phi_N(\nabla u)$. In particular, $\|\nabla u\|_\infty < 1$ and we conclude that u is a positive solution of (2.5). ☐

Note that by Lemmas 2.3 and 2.4, for all λ large enough, there are $m-1$ positive solutions $u_2(\lambda), \dots, u_m(\lambda)$ as asserted by Theorem 1.1.

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References

[1] R. Bartnik, L. Simon; *Spacelike hypersurfaces with prescribed boundary values and mean curvature*, Comm. Math. Phys. 87 (1982), 131-152.

[2] C. Bereanu, P. Jebelean, J. Mawhin; *Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces*, Proc. Amer. Math. Soc. 137 (2009), 171-178.

[3] C. Bereanu, P. Jebelean, P. Torres; *Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space*, J. Funct. Anal. 264 (2013), 270-287.

[4] C. Bereanu, P. Jebelean, P. Torres; *Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space*, J. Funct. Anal. 265 (2013), 644-659.

[5] C. Bereanu, J. Mawhin; *Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian*, J. Differential Equations 243 (2007), 536-557.

[6] K. J. Brown, H. Budin; *On the existence of positive solutions for a class of semilinear elliptic boundary value problems*, SIAM J. Math. Anal. 10 (1979), 875-883.

[7] I. Coelho, C. Corsato, F. Obersnel, P. Omari; *Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation*, Adv. Nonlinear Stud. 12 (2012), 621-638.

[8] I. Coelho, C. Corsato, S. Rivetti; *Positive radial solutions of the Dirichlet problem for the Minkowski-curvature equation in a ball*, Topol. Methods Nonlinear Anal. 44 (2014), 23-39.

[9] F. N. Dancer, K. Schmitt; *On positive solutions of semilinear elliptic equations*, Proc. Am. Math. Soc. 101 (1987), 445-452.

[10] D. G. de Figueiredo; *On the existence of multiple ordered solutions of nonlinear eigenvalue problems*, Nonlinear Anal. 11 (1987), 481-492.

[11] C. Gerhardt; *H-surfaces in Lorentzian manifolds*, Comm. Math. Phys. 89 (1983), 523-553.

[12] T. W. Laetsch; *The number of solutions of a nonlinear two point boundary value problem*, Indiana Univ. Math. J. 20 (1970), 1-13.

[13] N. H. Loc, K. Schmitt; *On positive solutions of quasilinear elliptic equations*, Differential Integral Equations 22(9-10) (2009), 829-842.

[14] R. Ma, H. Gao, Y. Lu; *Global structure of radial positive solutions for a prescribed mean curvature problem in a ball*, J. Funct. Anal. 270(7) (2016), 2430-2455.

[15] R. Ma, Y. Lu; *Multiplicity of positive solutions for second order nonlinear Dirichlet problem with one-dimension Minkowski-curvature operator*, Adv. Nonlinear Stud. 15 (2015), 789-803.

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