*Electronic Journal of Differential Equations*, Vol. 2016 (2016), No. 147, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# GROUND STATE SOLUTIONS FOR AN ASYMPTOTICALLY LINEAR DIFFUSION SYSTEM

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Abstract. This article concerns the diffusion system

$$\begin{gathered}\partial_t u-\Delta_x u+V(x)u=g(t,x,v),\\ -\partial_t v-\Delta_x v+V(x)v=f(t,x,u),\end{gathered}$$

where $z=(u,v):\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^2$, $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ is a general periodic function, $g$, $f$ are periodic in $t,x$ and asymptotically linear in $u$, $v$ at infinity. We find a minimizing Cerami sequence of the energy functional outside the Nehari-Pankov manifold $\mathcal{N}$ and therefore obtain ground state solutions.

## 1. Introduction and statement of main results

In this article, we consider the diffusion system on $\mathbb{R}\times\mathbb{R}^N$:

$$\begin{gathered}\partial_t u-\Delta_x u+V(x)u=g(t,x,v),\\ -\partial_t v-\Delta_x v+V(x)v=f(t,x,u),\end{gathered}\tag{1.1}$$

where $z=(u,v):\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^2$, $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ is a general periodic function, $g$, $f$ are periodic in $t,x$ and asymptotically linear in $v$, $u$ at infinity. In recent years, there are many papers like system (1.1) on bounded domains, see [2, 3, 4, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18] and the references therein. For the case of $V(x)\equiv 0$, Relying on fixed point theorem, Brézis and Nirenberg [3] certified the following system has a (generalized) solution $(u,v)$ with $u\in L^4$ and $v\in L^6$.

$$\begin{gathered}\partial_t u-\Delta_x u=-v^5+f,\\ -\partial_t v-\Delta_x v=u^3+g,\end{gathered}$$

in $(0,T)\times\Omega$, where $\Omega$ is a bounded domain, $f,g\in L^\infty$, subject to the boundary conditions $u=v=0$ on $(0,T)\times\Omega$ and $u(0,x)=u(t,x)$ on $\Omega$. Clément et al [4] investigated the system

$$\begin{gathered}\partial_t u-\Delta_x u=|v|^{q-2}v,\\ -\partial_t v-\Delta_x v=|u|^{p-2}u,\end{gathered}$$

in $(-T,T)\times\Omega$, where $\Omega$ is a bounded domain and $p,q$ satisfy

$$\frac{N}{N+2}<\frac1p+\frac1q<1.$$

2010 *Mathematics Subject Classification.* 35J10, 35J20.
*Key words and phrases.* Diffusion systems; asymptotically linear; diagonal method; Nehari-Pankov manifold.

Submitted April 13, 2016. Published June 15, 2016.

The authors proved that there exists $T_0 > 0$ such that for each $T > T_0$, the above problem has at least on positive solution via the mountain pass theorem. For the case of $V(x) \neq 0$, applying a local linking theorem, Mao et al [16] proved that problem (1.1) has at least one nontrivial periodic solution, see also [15]. For other related elliptic system problems, we refer readers to [8, 9, 10, 11, 20, 29, 30, 31] and the references therein.

Systems similar to (1.1) in the whole space was also studied by many authors. For example see [2, 6, 7, 14, 17, 18, 27, 28] and the references therein. Bartsch and Ding [2] considered the following infinite-dimensional Hamiltonian system

$$\begin{gathered}\partial_t u - \Delta_x u + V(x)u = H_v(t,x,u,v),\\ -\partial_t v - \Delta_x v + V(x)v = H_u(t,x,u,v).\end{gathered}$$

The authors obtained the existence and multiplicity of solutions of homoclinic type under the classic Ambrosetti-Rabinowitz condition. Later this result was improved by Schechter and Zou in [18] by using the methods of monotonicity trick. Based on a variant generalized linking theorem established in [17] and monotone trick, Zhang et al [27] proved the existence of the least energy solution of (1.1) with the nonlinearity $f$ and $g$ are superquadratic in $v$, $u$ at infinity. In the aforementioned references, the following classical condition (A1) due to Ambrosetti and Rabinowitz [1] is generally assumed:

(A1) there exist $\mu > 2$ and $R_0 > 0$ such that
$$0 < \mu F(x,t) \le t f(x,t), \quad \forall\, x \in \Omega, \quad |t| > R_0.$$

This condition implies $F(x,t) \ge C|t|^\mu$ for large $|t|$ and some constant $C > 0$. Thus, one can obtain mountain pass geometry as well as satisfaction of Palais-Smale condition under the condition (A1). However, we can not deal with system (1.1) via the mountain-pass theorem directly if the nonlinearity $f$ is of asymptotically linear at infinity.

Motivated by the above articles, we consider system (1.1) with 0 lying in a gap of the spectrum $\sigma(-\Delta_x+V)$ of the Diffusion system $(-\Delta_x+V)$ and study the existence of ground state solutions for system (1.1). To the best of our knowledge, there are less works concentrated on asymptotically linear case up now, thus this is an interesting problem. More precisely, we first make the following basic assumptions:

(A2) $V \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and
$$\sup[\sigma(-\Delta_x + V) \cap (-\infty,0)] := \wedge < 0 < \vee := \inf[\sigma(-\Delta_x + V) \cap (\infty,0)];$$

(A3) $f(t,x,s)$ and $g(t,x,s)$ are continuous and 1-periodic in $t$ and $x_i$, $i = 1,2,\dots,N$,
$$F(t,x,s) := \int_0^s f(t,x,\delta)d\delta \ge 0, \quad G(t,x,s) := \int_0^s g(t,x,\delta)d\delta \ge 0,$$
$$\lim_{|s|\to 0} \frac{|f(t,x,s)| + |g(t,x,s)|}{|s|} = 0, \quad \text{uniformly in } (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

(A4) $f(t,x,s) = V_\infty(x)s + f_\infty(t,x,s)$, $g(t,x,s) = V_\infty(x)s + g_\infty(t,x,s)$, where $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $V_\infty(x) > 0$, and there exists a $u_0 \in E^+ \setminus \{0\}$ such that
$$\|u_0\|^2 - \|w\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(u_0+w)^2 < 0, \quad \forall w \in E^-;$$

(A5) $f(t,x,s)f_\infty(t,x,s)+g(t,x,s)g_\infty(t,x,s)<0$, $f_\infty(t,x,s)=o(|s|)$, $g_\infty(t,x,s)=o(|s|)$ as $|s|\to\infty$ uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$;

(A6) $s\mapsto f(t,x,s)/|s|$, $s\mapsto g(t,x,s)/|s|$ are strictly increasing on $(-\infty,0)\cup(0,\infty)$.

The Nehari type assumption (A6) was used by Szulkin and Weth [19] to obtain existence of ground state solution of Nehari-Pankov type, i.e. a nontrivial solution $z_0$ which satisfies $\Phi(z_0)=\inf_{\mathcal{N}}\Phi$, where

$$\mathcal{N}=\{z\in E\backslash E^-:\langle\Phi'(z),z\rangle=\langle\Phi'(z),\eta\rangle=0,\forall\eta\in E^-\},$$

Functional $\Phi$ is the energy functional, and $E=E^-\oplus E^+$ is a Hilbert space on which $\Phi$ defines. Later, Zhang et al [26] proved same consequence for (1.1) by weakening (A6) to the following condition:

(A7) $s\mapsto f(t,x,s)/|s|$, $s\mapsto g(t,x,s)/|s|$ is nondecreasing on $(-\infty,0)\cup(0,\infty)$.

We must point out that condition (A7) is also crucial in our paper to find a minimizing Cerami sequence of the energy functional via diagonal method (see [21, 22, 24]). Moreover, in order to better show our results, we give the following condition can be found in [21]:

(A8) $f(t,x,s)=V_\infty(x)s+f_\infty(t,x,s)$, $g(t,x,s)=V_\infty(x)s+g_\infty(t,x,s)$, where $V_\infty\in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $V_\infty(x)>\vee$, $f_\infty(t,x,s)=o(|s|)$, $g_\infty(t,x,s)=o(|s|)$ as $|s|\to\infty$ uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, and $0<sf(t,x,s)<V_\infty(x)s^2$ for $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and $s\neq 0$.

**Remark 1.1.** Before we state our main results, we need to point out that (A8) implies (A4) and (A5) if $\inf V_\infty(x)>\vee$. Furthermore, (A7) and (A3) imply that

$$\frac{1}{2}f(t,x,s)s\ge F(t,x,s)\ge 0,\quad \forall s\ge 0,\ (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

It follows from (A3)–(A5) and (A7) that $s\mapsto f_\infty(t,x,s)/|s|$ is nondecreasing on $(-\infty,0)\cup(0,\infty)$, and that $f_\infty(t,x,s)/|s|\to -V_\infty(x)<0$ as $|s|\to 0$, which together with $f_\infty(t,x,s)=o(|s|)$ as $|s|\to\infty$ uniform in $t,x$ implies that $sf_\infty(t,x,s)<0$ for all $s\ge 0$. Similarly, we have $sg_\infty(t,x,s)<0$, for all $s\ge 0$.

**Theorem 1.2.** *Let* (A2)–(A5), (A7) *be satisfied. Then problem* (1.1) *has a solution* $z_0\in E$ *such that* $\Phi(z_0)=\inf_{\mathcal{N}}\Phi>0$.

**Corollary 1.3.** *Let* (A2), (A3), (A7), (A8) *be satisfied. Then problem* (1.1) *has a solution* $z_0\in E$ *such that* $\Phi(z_0)=\inf_{\mathcal{N}}\Phi>0$.

The following functions satisfy all assumptions of Corollary 1.3.

**Example 1.4.** $f(t,x,s)=V_\infty(x)\min\{|s|^\theta,1\}s$, where $\theta>0$ and $V_\infty\in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $V_\infty(x)>\vee$.

**Example 1.5.** $f(t,x,s)=V_\infty(x)[1-(1/\ln(e+|s|))]s$, where $V_\infty\in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $V_\infty(x)>\vee$.

## 2. Variational setting and preliminaries

Throughout this paper, we denote by $|\cdot|_s$ the usual $L^s$-norm and $(\cdot,\cdot)_2$ the $L^2$ inner product. In order to continue the discussion, we need the following notation.

$$\mathcal{J}=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix},\quad \mathcal{J}_0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

$$\mathcal{S} = -\Delta_x + V, \quad \mathcal{A}_0 = \mathcal{J}_0\mathcal{S}.$$

Then (1.1) reads as follows

$$\mathcal{J}\partial_t z + \mathcal{A}_0 z = H_z(t,x,z), \quad z = (u,v),$$

here and in the sequel $H(t,x,z) := F(t,x,u) + G(t,x,v)$. It is called an unbounded Hamiltonian system or an infinite dimensional Hamiltonian system (see [2]). Indeed, it has the representation $\mathcal{J}\partial_t z = \nabla_z \mathcal{H}(t,x,z)$ with the Hamiltonian

$$\mathcal{H}(t,x,z) := -\int_{\mathbb{R}^N} (\nabla_x u \cdot \nabla_x v + V(x)uv - H(t,x,z))\,\mathrm{d}x$$

in $L^2(\mathbb{R}^N, \mathbb{R}^2)$, where $\nabla_z$ denote the gradient operator in $L^2(\mathbb{R}^N, \mathbb{R}^2)$.

To show our main result, as in [2], we introduce for $r \geq 1$ the Banach space,

$$B_r = B_r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2) := W^{1,r}\left(\mathbb{R}, L^r(\mathbb{R}^N, \mathbb{R}^2)\right) \cap L^r\left(\mathbb{R}, W^{1,r} \cap W^{1,r}(\mathbb{R}^N, \mathbb{R}^2)\right),$$

equipped with norm

$$\|z\|_{B_r} = \Big[\int_{\mathbb{R}\times\mathbb{R}^N} \Big(|z|^r + |\partial_t z|^r + \sum_{j=1}^N |\partial_{x_j}^2 z|^r\Big)\Big]^{1/r}.$$

Clearly, $B_r$ is the completion of $C_0^\infty(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2)$ with respect to the norm $\|\cdot\|_{B_r}$. If $r = 2$, $B_2$ is a Hilbert space.

Let $\mathcal{A} := \mathcal{J}\partial_t + \mathcal{A}_0$, then $\mathcal{A}$ is a self-adjoint operator acting in $L^2 := L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2)$ with domain $\mathfrak{D}(\mathcal{A}) = B_2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2)$, and there exist $c_1$, $c_2 > 0$ such that

$$c_1\|z\|_{B_2}^2 \leq |\mathcal{A}z|_2^2 \leq c_2\|z\|_{B_2}^2$$

for $z \in B_2$ (see [2]). Under assumption (A2), $L^2$ possesses the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- \oplus z^+, \quad z^\pm \in L^\pm,$$

such that $\mathcal{A}$ is negative definite (resp. positive definite) in $L^-$ (resp. $L^+$). Let $|\mathcal{A}|$ denote the absolute value of $\mathcal{A}$ and $|\mathcal{A}|^{1/2}$ be the square root of $\mathcal{A}$. Let $E = \mathfrak{D}(|\mathcal{A}|^{1/2})$ be the Hilbert space with the inner product

$$(z,w) = \left(|\mathcal{A}|^{1/2}z, |\mathcal{A}|^{1/2}w\right)_2$$

and norm $\|z\| = (z,z)^{1/2}$. There is an induced decomposition

$$E = E^- \oplus E^+, \quad E^\pm = E \cap L^\pm,$$

which is orthogonal with respect to the inner products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$. Moreover, we have the following embedding theorem.

**Lemma 2.1** ([2, Lemma 4.6]). *$E$ is continuously embedded in $L^p$ for any $p \geq 2$ if $N = 1$, and for $p \in [2, 2(N+2)/N]$ if $N \geq 2$. $E$ is compactly embedded in $L^p_{\mathrm{loc}}$ for all $p \in [2, 2(N+2)/N)$.*

## 3. Proof of main results

Let $X$ be a Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. Functional $\varphi \in C^1(X,\mathbb{R})$ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ weakly in $X$ one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$, and $\varphi'$ is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for each $v \in X$.

The following generalized linking theorem plays an important role in proving our main results.

**Lemma 3.1** ([5, Theorem 4.5], [12, Theorem 2.1])**.** *Let $X$ be a Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X,\mathbb{R})$ of the form*

$$\varphi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^+ + u^- \in X^+ \oplus X^-.$$

*Suppose that the following assumptions are satisfied:*

1. *$\psi \in C^1(X,\mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*
2. *$\psi'$ is weakly sequentially continuous;*
3. *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*
$$\kappa := \inf \varphi(S_\rho) > \sup \varphi(\partial Q),$$
*where*
$$S_\rho = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{se + v : v \in X^-, s \ge 0, \|se + v\| \le r\}.$$

*Then for some $c \ge \kappa$, there exists a sequence $\{u_n\} \subset X$ satisfying*

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

*Such a sequence is called a Cerami sequence on the level $c$, or a $(C)_c$ sequence.*

Under assumptions (A2)–(A5), it is easy to verify that the functional

$$\Phi(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \Psi(z), \quad z = (u, v), \tag{3.1}$$

is well defined for all $z \in E$ and $\Phi \in C^1(E,\mathbb{R})$, where

$$\Psi(z) = \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, z) = \int_{\mathbb{R}\times\mathbb{R}^N} [F(t, x, u) + G(t, x, v)]. \tag{3.2}$$

Moreover, for $z = (u, v) \in E$, $\zeta = (\xi, \eta) \in E$,

$$\langle \Phi'(z), \zeta\rangle = (z^+, \zeta^+) - (z^-, \zeta^-) - \int_{\mathbb{R}\times\mathbb{R}^N} \left(f(t, x, u)\xi + g(t, x, u)\eta\right), \tag{3.3}$$

and a standard argument shows that critical points of $\Phi$ are solutions of (1.1) (see [5, 25]).

**Lemma 3.2.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then $\Psi$ is bounded from below, and weakly sequentially lower semi-continuous and $\Psi'$ is weakly sequentially continuous.*

The proof of the above lemma is standard, see [7, 26]. Using Sobolev's embedding theorem, one can check the above lemma easily, so we omit it.

The following lemma is interesting and shows an important behavior of nondecreasing functions. By a similar argument as in [21, 22], on can prove the following lemma.

**Lemma 3.3.** *Suppose that $h(t,x,s)$ is nondecreasing in $s\in\mathbb{R}$ and $h(t,x,0)=0$ for any $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Then*

$$\Big(\frac{1-\theta^2}{2}s-\theta\sigma\Big)h(t,x,s)|s|\geq\int_{\theta s+\sigma}^{s}h(t,x,\tau)|\tau|d\tau,\quad\forall\theta\geq 0,\ s,\sigma\in\mathbb{R}.\tag{3.4}$$

**Lemma 3.4.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then for any $z=(u,v)\in E$,*

$$\Phi(z)\geq\Phi(\tau z+\zeta)+\frac{1}{2}\|\zeta\|^2+\frac{1-\tau^2}{2}\langle\Phi'(z),z\rangle-\tau\langle\Phi'(z),\zeta\rangle\tag{3.5}$$

*for all $\zeta\in E^-$, $\tau\geq 0$.*

*Proof.* For any $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and it follows from (A7) and Lemma 3.3 that

$$\Big(\frac{1-\tau^2}{2}\nu^2-\tau\nu\sigma\Big)\frac{f(t,x,\nu)}{\nu}\geq\int_{\tau\nu+\sigma}^{\nu}f(t,x,s)\,\mathrm{d}s,\quad\forall\tau\geq 0,\ \sigma,\nu\in\mathbb{R}.\tag{3.6}$$

Similarly, we have

$$\Big(\frac{1-\tau^2}{2}\nu^2-\tau\nu\sigma\Big)\frac{g(t,x,\nu)}{\nu}\geq\int_{\tau\nu+\sigma}^{\nu}g(t,x,s)\,\mathrm{d}s,\quad\forall\tau\geq 0,\ \sigma,\nu\in\mathbb{R}.\tag{3.7}$$

By (3.1), (3.3), (3.6) and (3.7), one has

$$\begin{aligned}
&\Phi(z)-\Phi(\tau z+\zeta)\\
&=\frac{1}{2}\|\zeta\|^2+\frac{1-\tau^2}{2}\langle\Phi'(z),z\rangle-\tau\langle\Phi'(z),\zeta\rangle\\
&\quad+\int_{\mathbb{R}\times\mathbb{R}^N}\Big(\frac{1-\tau^2}{2}f(t,x,u)u-\tau f(t,x,u)\xi-\int_{\tau u+\xi}^{u}f(t,x,s)\,\mathrm{d}s\Big)\\
&\quad+\int_{\mathbb{R}\times\mathbb{R}^N}\Big(\frac{1-\tau^2}{2}g(t,x,v)v-\tau g(t,x,v)\eta-\int_{\tau v+\eta}^{v}g(t,x,s)\,\mathrm{d}s\Big)\\
&\geq\frac{1}{2}\|\zeta\|^2+\frac{1-\tau^2}{2}\langle\Phi'(z),z\rangle-\tau\langle\Phi'(z),\zeta\rangle,
\end{aligned}$$

for all $z=(u,v)\in E$, $\zeta=(\xi,\eta)\in E^-$, and $\tau\geq 0$. This shows that (3.5) holds. ☐

From Lemma 3.4, we have the following two corollaries.

**Corollary 3.5.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then for any $z\in\mathcal{N}$,*

$$\Phi(z)\geq\Phi(\tau z+\zeta),\quad\forall\zeta\in E^-,\ \tau\geq 0.\tag{3.8}$$

**Corollary 3.6.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then for any $z=(u,v)\in E$,*

$$\Phi(z)\geq\Phi(\tau z^+)+\frac{\tau^2\|z^-\|^2}{2}+\frac{1-\tau^2}{2}\langle\Phi'(z),z\rangle+\tau^2\langle\Phi'(z),z^-\rangle,\quad\forall\tau\geq 0.\tag{3.9}$$

**Lemma 3.7.** *Suppose that* (A2)–(A5), (A7) *are satisfied. If* $\inf V_\infty>0$, *then*

$$\begin{aligned}
&\tau\langle\Phi'(z),\tau z+2\zeta\rangle\\
&\geq\tau^2\|z^+\|^2-\|\tau z^-+2\zeta\|^2+\|\zeta^2\|\\
&\quad-\int_{\mathbb{R}\times\mathbb{R}^N}V_\infty(x)(\tau z+\zeta)^2+\tau^2\int_{\mathbb{R}\times\mathbb{R}^N}\frac{uf(t,x,u)V_\infty(x)-[f(t,x,u)]^2}{V_\infty(x)}\\
&\quad+\tau^2\int_{\mathbb{R}\times\mathbb{R}^N}\frac{vg(t,x,v)V_\infty(x)-[g(t,x,v)]^2}{V_\infty(x)},
\end{aligned}\tag{3.10}$$

*for all* $z = (u, v) \in E$, $\tau \in \mathbb{R}$, *and* $\zeta = (\xi, \eta) \in E^-$.

*Proof.* For any $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, in view of (3.1), (3.3) and $\inf V_\infty > 0$, we have

$$
\begin{aligned}
&\tau\langle \Phi'(z, \tau z + 2\zeta)\rangle \\
&= \tau^2\|z^+\|^2 - \tau^2\|z^-\|^2 - 2\tau(z^-, \zeta) \\
&\quad - \tau\int_{\mathbb{R}\times\mathbb{R}^N}[f(t,x,u)(\tau u + 2\xi) + g(t,x,v)(\tau v + 2\eta)] \\
&= \tau^2\|z^+\|^2 - \|\tau z^- + \zeta\|^2 + \|\zeta\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(\tau u + \xi)^2 \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(\tau v + \eta)^2 + \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)[(\tau u + \xi)^2 - \tau f(t,x,u)(\tau u + 2\xi)] \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)[(\tau u + \eta)^2 - \tau g(t,x,v)(\tau v + 2\eta)] \\
&\geq \tau^2\|z^+\|^2 - \|\tau z^- + \zeta\|^2 + \|\zeta\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(\tau z + \zeta)^2 \\
&\quad + \tau^2\int_{\mathbb{R}\times\mathbb{R}^N}\frac{uf(t,x,u)V_\infty(x) - [f(t,x,u)]^2}{V_\infty(x)} \\
&\quad + \tau^2\int_{\mathbb{R}\times\mathbb{R}^N}\frac{vg(t,x,v)V_\infty(x) - [g(t,x,v)]^2}{V_\infty(x)},
\end{aligned}
$$

for all $z = (u, v) \in E$, $\tau \in \mathbb{R}$, and $\zeta = (\xi, \eta) \in E^-$. This shows that (3.10) holds. $\square$

**Corollary 3.8.** *Suppose that* (A2)–(A5), (A7) *are satisfied. If* $\inf V_\infty > 0$, *then*

$$
\begin{aligned}
&\|z^+\|^2 - \|z^- + \zeta\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(z + \zeta)^2 \\
&\leq -\|\zeta\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N}\frac{uf(t,x,u)V_\infty(x) - [f(t,x,u)]^2}{V_\infty(x)} \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N}\frac{vg(t,x,v)V_\infty(x) - [g(t,x,v)]^2}{V_\infty(x)},
\end{aligned}
\tag{3.11}
$$

*for all* $z = (u, v) \in \mathcal{N}$ *and* $\zeta = (\xi, \eta) \in E^-$.

Applying Corollary 3.5, we can prove the following lemma in the same way as in [19, Lemma 2.4].

**Lemma 3.9.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then*

(i) *there exists* $\rho > 0$ *such that*
$$m := \inf_{\mathcal{N}} \Phi \geq \kappa := \inf\big\{\Phi(z) : z \in E^+, \|z\| = \rho\big\} > 0;$$

(ii) $\|z^+\| \geq \max\big\{\|z^-\|, \sqrt{2m}\big\}$ *for all* $z \in \mathcal{N}$;

Define the set

$$E_0^+ = \big\{z \in E^+\backslash\{0\} : \|z\|^2 - \|\zeta\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(z + \zeta)^2 \leq 0 \ \forall \zeta \in E^-\big\}.$$

Obviously, (A4) shows that the set $E_0^+$ is not empty.

**Lemma 3.10.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then, for any* $e \in E_0^+$, $\sup \Phi(E^- \oplus \mathbb{R}^+ e) < \infty$ *and there is* $R_e > 0$ *such that*

$$\Phi(z) \leq 0, \quad \forall z \in E^- \oplus \mathbb{R}^+ e, \ \|z\| \geq R_e. \tag{3.12}$$

*Proof.* It is sufficient to show that $\Phi(\zeta + se) \leq 0$ for $s \geq 0, \zeta \in E^-$ and $\|\zeta + se\| > R$ for $R > 0$. Arguing indirectly, assume that, for some sequence $\{\zeta_n + s_n e\} \subset E^- \oplus \mathbb{R}^+ e$, $\zeta_n = (\xi_n, \eta_n)$, $e = (e_1, e_2)$ with $\|\zeta_n + s_n e\| \to \infty, \Phi(\zeta_n + s_n e) \geq 0$ for all $n \in \mathbb{N}$. Set

$$v_n = (\zeta_n + s_n e)/\|\zeta_n + s_n e\| = v_n^- + \tau_n e, \tag{3.13}$$

then $\|v_n^- + \tau_n e\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in $E$, then $v_n \to v$ a.e on $\mathbb{R} \times \mathbb{R}^N$, $v_n^- \rightharpoonup v^-$ in $E$, $\tau_n \to \tau$ and

$$\begin{aligned} 0 &\leq \frac{\Phi(\zeta_n + s_n e)}{\|\zeta_n + s_n e\|^2} \\ &= \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} \frac{F(t,x,\xi_n + s_n e_1) + G(t,x,\eta_n + s_n e_2)}{\|\zeta_n + s_n e\|^2}. \end{aligned} \tag{3.14}$$

Clearly (3.14) yields that $\tau > 0$. Since $e \in E_0^+$, there exists a bounded domain $\Omega \subset \mathbb{R} \times \mathbb{R}^N$ such that

$$\tau^2\|e\|^2 - \|v^-\|^2 - \int_\Omega V_\infty(x)(\tau e + v^-)^2 < 0. \tag{3.15}$$

Let

$$F_\infty(t,x,s) = \int_0^s f_\infty(t,x,\tau)d\tau; \quad G_\infty(t,x,s) = \int_0^s g_\infty(t,x,\tau)d\tau.$$

Then

$$F(t,x,s) = \frac{1}{2}V_\infty(x)s^2 + F_\infty(t,x,s); \quad G(t,x,s) = \frac{1}{2}V_\infty(x)s^2 + G_\infty(t,x,s).$$

It follows from (3.14) that

$$\begin{aligned} 0 &\leq \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega \frac{F(t,x,\xi_n + s_n e_1) + G(t,x,\eta_n + s_n e_2)}{\|\zeta_n + s_n e\|^2} \\ &= \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2}\int_\Omega V_\infty(x)v_n^2 dx \\ &\quad - \int_\Omega \frac{F_\infty(t,x,\xi_n + s_n e_1) + G_\infty(t,x,\eta_n + s_n e_2)}{\|\zeta_n + s_n e\|^2}. \end{aligned} \tag{3.16}$$

Clearly, $|F_\infty(x,t,s)| + |G_\infty(x,t,s)| \leq c_0 s^2$ for some $c_0 > 0$ and

$$\frac{|F_\infty(x,t,s)| + |G_\infty(x,t,s)|}{s^2} \to 0$$

as $|s| \to \infty$, Since $v_n \rightharpoonup v$ in $E$, $v_n \to v$ in $L^2(\Omega)$ and it is easy to see from the Lebesgue dominated convergence theorem that

$$\int_\Omega \frac{F_\infty(t,x,\xi_n + s_n e_1)}{\|\zeta_n + s_n e\|^2} = \int_\Omega \frac{F_\infty(t,x,\xi_n + s_n e_1)}{|\zeta_n + s_n e|^2}|v_n|^2 = o(1).$$

Similarly,

$$\int_\Omega \frac{G_\infty(t,x,\eta_n + s_n e_2)}{\|\zeta_n + s_n e\|^2} = o(1).$$

Hence,

$$0 \le \tau^2\|e\|^2 - \|v^-\|^2 - \int_\Omega V_\infty(x)(\tau e + v^-)^2,$$

which contradicts (3.15). $\square$

**Corollary 3.11.** *Suppose that* (A2)–(A5), (A7) *are satisfied, and let* $e \in E_0^+$ *satisfy* $\|e\| = 1$. *Then there is a* $r_0 > \rho$ *such that* $\sup \Phi(\partial Q) \le 0$ *for* $r \ge r_0$, *where*

$$Q = \{\zeta + se : \zeta \in E^-,\ s \ge 0, \|\zeta + se\| \le r\}. \tag{3.17}$$

**Lemma 3.12.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then for any* $z \in E^+ \setminus \{0\}$, *we have* $\mathcal{N} \cap (E^- \oplus \mathbb{R}^+ z) \neq \emptyset$, *i.e., there exist* $\tau(z) > 0$ *and* $w(z) \in E^-$ *such that* $\tau(z)z + w(z) \in \mathcal{N}^-$.

*Proof.* By Corollary 3.11, there exists $R > 0$ such that $\Phi(w) \le 0$ for $w \in (E^- \oplus \mathbb{R}^+ z) \setminus B_R(0)$. By Lemma 3.9 (i), $\Phi(\tau z) > 0$ for small $\tau > 0$. Thus, $0 < \sup \Phi(E^- \oplus \mathbb{R}^+ z) < \infty$. It is easy to see that $\Phi$ is weakly upper semi-continuous on $E^- \oplus \mathbb{R}^+ z$, therefore, $\Phi(z_0) = \sup \Phi(E^- \oplus \mathbb{R}^+ z)$ for some $z_0 \in E^- \oplus \mathbb{R}^+ z$. This $z_0$ is a critical point of $\Phi|_{E^- \oplus \mathbb{R}z}$, so $\langle \Phi'(z_0), z_0\rangle = \langle \Phi'(z_0), \zeta\rangle = 0$ for all $\zeta \in E^- \oplus \mathbb{R}z$. Consequently, $z_0 \in \mathcal{N} \cap (E^- \oplus \mathbb{R}^+ z)$. $\square$

**Lemma 3.13.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then there exist a constant* $c_0 \in [\kappa, \sup \Phi(Q)]$ *and a sequence* $\{z_n\} \subset E$ *such that*

$$\Phi(z_n) \to c_0, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0, \tag{3.18}$$

*where* $Q$ *is defined by* (3.17).

The above lemma is a direct corollary of Lemmas 3.1, 3.2, 3.9(i) and Corollary 3.11. The following lemma plays a crucial role in the proof of our main results, by which we can directly find the existence of ground state solution of Nehari-Pankov type associated with (1.1).

**Lemma 3.14.** *Suppose that* (A2)–(A5), (A7) *are satisfied. Then there exist a constant* $c_* \in [\kappa,\ m]$ *and a sequence* $\{z_n\} \subset E$ *satisfying*

$$\Phi(z_n) \to c_*, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0. \tag{3.19}$$

*Proof.* Choose $\zeta_k \in \mathcal{N}$, $\zeta_k = (\xi_k, \eta_k)$ such that

$$m \le \Phi(\zeta_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.20}$$

By lemma 3.9 (ii), $\|\zeta_k^+\| \ge \sqrt{2m} > 0$. Since $\zeta_k \in E$, it follows from (A5) that

$$\int_{\mathbb{R}\times\mathbb{R}^N} \frac{f(t,x,\xi_k) f_\infty(t,x,\xi_k) + g(t,x,\eta_k) g_\infty(t,x,\eta_k)}{V_\infty(x)} < 0. \tag{3.21}$$

Set $e_k = \zeta_k^+/\|\zeta_k^+\|$. Then $e_k \in E^+$ and $\|e_k\| = 1$. By Corollary 3.8 and (3.21), for any $w \in E^-$, we have

$$
\begin{aligned}
&\|e_k\|^2 - \|w\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)(e_k + w)^2 \\
&= \frac{\|\zeta_k^+\|^2}{\|\zeta_k^+\|^2} - \|w\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} V_\infty(x)\Big(\frac{\zeta_k}{\|\zeta_k^+\|} + w - \frac{\zeta_k^-}{\|\zeta_k^+\|}\Big)^2 \\
&\le -\|w - \frac{\zeta_k^-}{\|\zeta_k^+\|}\|^2 - \frac{1}{\|\zeta_k^+\|^2}\int_{\mathbb{R}\times\mathbb{R}^N} \frac{\xi_k f(t,x,\xi_k)V_\infty(x) - [f(t,x,\xi_k)]^2}{V_\infty(x)} \\
&\quad - \frac{1}{\|\zeta_k^+\|^2}\int_{\mathbb{R}\times\mathbb{R}^N} \frac{\eta_k g(t,x,\eta_k)V_\infty(x) - [g(t,x,\eta_k)]^2}{V_\infty(x)} \\
&= -\|w - \frac{\zeta_k^-}{\|\zeta_k^+\|}\|^2 \\
&\quad + \frac{1}{\|\zeta_k^+\|^2}\int_{\mathbb{R}\times\mathbb{R}^N} \frac{f(t,x,\xi_k)f_\infty(t,x,\xi_k) + g(t,x,\eta_k)g_\infty(t,x,\eta_k)}{V_\infty(x)} < 0.
\end{aligned}
\tag{3.22}
$$

This shows that $e_k \in E_0^+$. By Corollary 3.11, there exists $r_k > \max\{\rho, \|\zeta_k\|\}$ such that $\sup \Phi(\partial Q_k) \le 0$, where

$$
Q_k = \{\zeta + se_k : \zeta \in E^-, s \ge 0, \|\zeta + se_k\| \le r_k\}, \quad k \in \mathbb{N}. \tag{3.23}
$$

Hence, applying Lemma 3.13 to the above set $Q_k$, there exist a positive constant $c_k \in [k, \sup\Phi(Q_k)]$ and a sequence $\{z_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$
\Phi(z_{k,n}) \to c_k, \quad \|\Phi'(z_{k,n})\|(1 + \|z_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{3.24}
$$

By Corollary 3.5, one obtains

$$
\Phi(\zeta_k) \ge \Phi(\tau\zeta_k + w), \quad \forall \tau \ge 0,\ w \in E^-. \tag{3.25}
$$

Since $\zeta_k \in Q_k$, It follows from (3.23) and (3.25) that $\Phi(\zeta_k) = \sup\Phi(Q_k)$. Hence, by (3.20) and (3.24), one has

$$
\Phi(z_{k,n}) \to c_k < m + \frac{1}{k}, \quad \|\Phi'(z_{k,n})\|(1 + \|z_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{3.26}
$$

Now, we can choose a sequence $n_k \subset \mathbb{N}$ such that

$$
\kappa - \frac{1}{k} < \Phi(z_{k,n_k}) < m + \frac{1}{k}, \quad \|\Phi'(z_{k,n_k})\|(1 + \|z_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.27}
$$

Let $z_k = z_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$
\Phi(z_n) \to c_* \in [\kappa,\ m], \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0.
$$

$\square$

**Lemma 3.15.** *Suppose that* (A2)–(A5), (A7) *are satisfied. If there exist* $\{z_n\} \subset E$ *and* $c \ge 0$ *such that*

$$
\Phi(z_n) \to c, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0, \tag{3.28}
$$

*then* $\{z_n\}$ *is bounded in* $E$.

*Proof.* To prove the boundedness of $\{z_n\}$, arguing by contradiction, suppose that $\|z_n\| \to \infty$. Let $\{z_n\} = \{u_n, v_n\}$ and $w_n = z_n/\|z_n\|$, then $\|w_n\| = 1$. By Lemma 2.1, there exists a constant $C_1 > 0$ such that $|w_n|_2 \le C_1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}\times\mathbb{R}^N} \int_{B_1(y)} |w_n^+|^2 = 0,$$

where $y := (t, x)$, then by Lions' concentration compactness principle in [13] or [25, Lemma 1.21] (usual this lemma is stated for $\{z_n\} \subset E$, however, a simple modification of the argument in [13] shows that the conclusion remains valid for $E$), $w_n^+ \to 0$ in $L^s$ for $2 < s < N^*$. Fix $R > [2(1 + c)]^{1/2}$. By virtue of (A3), (A4) and (A5), for $\varepsilon = 1/4(RC_1)^2 > 0$, there exists $C_\varepsilon > 0$ such that $|H(t, x, s)| \le \varepsilon|s|^2 + C_\varepsilon|s|^p$. Hence,

$$\limsup_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, Rw_n^+)\mathrm{d}x \le \varepsilon(RC_1)^2 + R^p C_\varepsilon \lim_{n\to\infty} \|w_n^+\|_p^p = \frac{1}{4}. \tag{3.29}$$

Let $\tau_n = R/\|z_n\|$. Hence, by virtue of (3.19), (3.29) and Corollary 3.5, one obtains

$$\begin{aligned}
c + o(1) = \Phi(z_n) &\ge \frac{\tau_n^2}{2}\left(\|z_n^+\|^2 + \|z_n^-\|^2\right) \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, \tau_n z_n^+) + \frac{1-\tau_n^2}{2}\langle\Phi'(z_n), z_n\rangle + \tau_n^2\langle\Phi'(z_n), z_n^-\rangle \\
&= \frac{R^2}{2}\left(\|w_n^+\|^2 + \|w_n^-\|^2\right) \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, Rw_n^+) + \Big(\frac{1}{2} - \frac{R^2}{2\|z_n\|^2}\Big)\langle\Phi'(z_n), z_n\rangle + \frac{R^2}{\|z_n\|^2}\langle\Phi'(z_n), z_n^-\rangle \\
&= \frac{R^2}{2} - \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, Rw_n^+) + o(1) \\
&\ge \frac{R^2}{2} - \frac{1}{4} + o(1) > c + \frac{3}{4} + o(1).
\end{aligned}$$

This contradiction shows that $\delta > 0$. □

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^{N+1}$ such that $\int_{B_{1+\sqrt{N+1}}(k_n)} |w_n^+|^2 > \frac{\delta}{2}$. Let $\tilde{w}_n(\cdot) = w_n(\cdot + k_n)$. Then $\|\tilde{w}_n\| = \|w_n\| = 1$, and

$$\int_{B_{1+\sqrt{N+1}}(0)} |\tilde{w}_n^+|^2 > \frac{\delta}{2}. \tag{3.30}$$

Passing to a subsequence, we have $\tilde{w}_n \rightharpoonup \tilde{w}$ in $E$, $\tilde{w}_n \to \tilde{w}$ in $L^s_{\rm loc}$, $2 \le s < N^*$, $\tilde{w}_n \to \tilde{w}$ a.e. on $\mathbb{R}\times\mathbb{R}^N$. Then, (3.30) implies that $\tilde{w}^+ \ne 0$ and so $\tilde{w} \ne 0$.

Now we define $\tilde{z}_n(\cdot) = z_n(\cdot + k_n)$, then $\tilde{z}_n/\|z_n\| = \tilde{w}_n \to \tilde{w}$ almost everywhere on $\mathbb{R}\times\mathbb{R}^N$, $\tilde{w}_n \ne 0$. For $\hat{x} \in \Omega := \{y \in \mathbb{R}\times\mathbb{R}^N : \tilde{w}(y) \ne 0\}$, we have $\lim_{n\to\infty} |\tilde{z}_n(\hat{x})| = \infty$. For any $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^2)$, $\phi = (\xi, \eta)$, setting $\phi_n(\hat{x}) = \phi(\hat{x} - k_n) = (\xi_n, \eta_n)$,

$$\begin{aligned}
\langle\Phi'(z_n), \phi_n\rangle &= (z_n^+ - z_n^-, \phi_n) - (V_\infty w_n, \phi_n)_{L^2} - \int_{\mathbb{R}\times\mathbb{R}^N} f_\infty(t, x, u_n)\xi_n \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N} g_\infty(t, x, v_n)\eta_n \\
&= \|z_n\|\Big[(w_n^+ - w_n^-, \phi_n) - (V_\infty w_n, \phi_n)_{L^2} - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{f_\infty(t, x, u_n)}{|z_n|}|w_n|\xi_n
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}\times\mathbb{R}^N} \frac{g_\infty(t,x,v_n)}{|z_n|}|w_n|\eta_n\Big] \\
&= \|z_n\|\Big[(\tilde{w}_n^+ - \tilde{w}_n^-, \phi) - (V_\infty \tilde{w}_n, \phi)_{L^2} - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{f_\infty(t,x,\tilde{u}_n)}{|\tilde{z}_n|}|\tilde{w}_n|\xi \\
&- \int_{\mathbb{R}\times\mathbb{R}^N} \frac{g_\infty(t,x,\tilde{v}_n)}{|\tilde{z}_n|}|\tilde{w}_n|\eta\Big],
\end{aligned}
$$

which, together with (3.19), yields

$$
\begin{aligned}
&(\tilde{w}_n^+ - \tilde{w}_n^-, \phi) - (V_\infty \tilde{w}_n, \phi)_{L^2} - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{f_\infty(t,x,\tilde{u}_n)}{|\tilde{z}_n|}|\tilde{w}_n|\xi \\
&- \int_{\mathbb{R}\times\mathbb{R}^N} \frac{g_\infty(t,x,\tilde{v}_n)}{|\tilde{z}_n|}|\tilde{w}_n|\eta = o(1).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}\times\mathbb{R}^N} \frac{f_\infty(t,x,\tilde{u}_n)}{|\tilde{z}_n|}|\tilde{w}_n|\xi\Big| \\
&\le \int_{\mathbb{R}\times\mathbb{R}^N} \Big|\frac{f_\infty(t,x,\tilde{u}_n)}{|\tilde{z}_n|}\Big||\tilde{w}_n||\phi| \\
&\le \int_{\mathbb{R}\times\mathbb{R}^N} \Big|\frac{f_\infty(t,x,\tilde{u}_n)}{|\tilde{z}_n|}\Big||\tilde{w}_n - \tilde{w}||\phi| + \int_{\mathbb{R}\times\mathbb{R}^N} \Big|\frac{f_\infty(t,x,\tilde{u}_n)}{|\tilde{z}_n|}\Big||\tilde{w}||\phi| \\
&\le C_2 \int_{supp\phi} |\tilde{w}_n - \tilde{w}||\phi| + \int_\Omega \Big|\frac{f_\infty(t,x,\tilde{u}_n)}{\tilde{z}_n}\Big||\tilde{w}||\phi| = o(1).
\end{aligned}
$$

Similarly,

$$
\Big|\int_{\mathbb{R}\times\mathbb{R}^N} \frac{g_\infty(t,x,\tilde{v}_n)}{|\tilde{z}_n|}|\tilde{w}_n|\eta\Big| = o(1).
$$

Hence,

$$
(\tilde{w}^+ - \tilde{w}^-, \phi) - (V_\infty \tilde{w}_n, \phi)_{L^2} = 0.
$$

Thus, $\tilde{w}$ is an eigenfunction of the operator $\mathcal{B} = \mathcal{A} - \mathcal{J}_0 V_\infty$, contradicting the fact that $\mathcal{B}$ has only a continuous spectrum since the periodicity of $V_\infty$ (see [2] and [7]). This contradiction shows that $\{z_n\}$ is bounded.

*Proof of Theorem 1.2.* By Lemmas 3.14 and 3.15, we deduce that there exists a bounded sequence $\{z_n\} \subset E$ satisfying (3.19). A standard argument shows that $\{z_n\}$ is a nonvanishing sequence. Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^{N+1}$ such that $\int_{B_{1+\sqrt{N+1}}(k_n)} |z_n|^2 dx > \frac{\delta}{2}$ for some $\delta > 0$. Let $w_n = z_n(\cdot + k_n)$. Then

$$
\int_{B_{1+\sqrt{N+1}}(0)} |w_n|^2 dx > \frac{\delta}{2}. \tag{3.31}
$$

Since $f(t,x,s)$, $g(t,x,s)$ and $V(x)$ are periodic, we have $\|w_n\| = \|z_n\|$ and

$$
\Phi(w_n) \to c_*, \quad \|\Phi'(w_n)\|(1 + \|w_n\|) \to 0. \tag{3.32}
$$

Passing to a subsequence, we have $w_n \rightharpoonup w$ in $E$, $w_n \to w$ in $L^s_{\rm loc}$, $2 \le s < 2$ and $w_n \to w$ a.e on $\mathbb{R} \times \mathbb{R}^N$. Obviously, (3.34) and (3.35) imply that $w \ne 0$ and

$\Phi'(w) = 0$. This shows that $w \in \mathcal{N}$ and so $\Phi(w) \geq m$. On the other hand, by using (3.35), (A7) and Fatou's lemma,

$$\begin{aligned} m \geq c_* &= \liminf_{n\to\infty} \Big[\Phi(w_n) - \frac{1}{2}\langle \Phi'(w_n), w_n\rangle\Big] \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}^N} \Big[\frac{1}{2} f(t,x,\xi_n)\xi_n - F(t,x,\xi_n)\xi_n + \frac{1}{2} g(t,x,\eta_n)\eta_n - G(t,x,\eta_n)\eta_n\Big] \\ &\geq \int_{\mathbb{R}\times\mathbb{R}^N} \liminf_{n\to\infty} \Big[\frac{1}{2} f(t,x,\xi_n)\xi_n - F(t,x,\xi_n)\xi_n + \frac{1}{2} g(t,x,\eta_n)\eta_n - G(t,x,\eta_n)\eta_n\Big] \\ &= \Phi(w) - \frac{1}{2}\langle \Phi'(w), w\rangle = \Phi(w). \end{aligned}$$

This shows that $\Phi(w) \leq m$ and so $\Phi(w) = m = \inf_{\mathcal{N}} \Phi > 0$. □

**Acknowledgments.** This work is partially supported by NNSF (Nos. 11571370, 11471137).

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