Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 116, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

KHALED BEN ALI, ABDELJABBAR GHANMI, KHALED KEFI

Abstract. In this article, we apply the Nehari manifold to prove the existence of a solution of the fractional differential equation

$$\frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))) = f(t,u(t)) + \lambda h(t)|u(t)|^{r-2}u(t),$$
$$\text{a.e } t \in [0,T],$$
$$u(0) = u(T) = 0,$$

where ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals, respectively, of order $0 < \beta < 1$.

1. Introduction

Recently, there has been surge in the interest for fractional differential equations in fields such as: from physics, chemistry, aerodynamics, electrical circuits, diffusion, electro dynamics of complex medium, and applied mathematics. Among the researchers studying such equations, we can quote for example the authors in [1, 10, 12, 18].

Researchers have examined some problems related to these types of equations by using methods such as fixed point theorem, coincidence degree theory, and critical point theory; see [2, 3, 4, 5, 6, 7, 9, 11, 15, 16, 17, 19, 20, 23, 25, 24, 26, 27, 21, 8, 22, 13, 14]. As an example Jiao and Zhou [7] studied the boundary-value problem

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T],\\ u(0) = u(T) = 0,\end{gathered} \tag{1.1}$$

where $0 < \beta < 1$, and ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}t$ are the left and right Riemann-Liouville fractional integrals of order β, respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$, and $\nabla F(t,x)$ is the gradient of F with respect to x. By using the mountain pass theorem, they showed the existence of a solution.

2010 *Mathematics Subject Classification.* 26A33, 58E05, 35J60.
Key words and phrases. Fractional differential equation; left and right fractional derivatives; boundary value problem; Nehari manifold.

Submitted January 28, 2016. Published May 10, 2016.

Bai [4] and other researchers considered the problem

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^CD_t^{\alpha}(u(t))) + \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^CD_T^{\alpha}u(t))\Big) + \lambda a(t)f(u(t)) = 0,\\ \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0\end{gathered}\tag{1.2}$$

where $\alpha\in(1/2,1]$, and ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$, where ${}_0^CD_t^{\alpha}(u(t))$ and ${}_t^CD_T^{\alpha}(u(t))$ are the left and right Caputo fractional derivatives of order α. By using a critical-point theorem established by Bonanno, he proved the existence of a solution to this problem. We mention also the works [5, 6, 7, 9], where by using critical point theory, the existence and multiplicity of solutions have been established for the related problems.

In this article, we attempt to highlight the use of the Nehari method to prove the existence of solutions to the problem

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\Big) = f(t,u(t)) + \lambda h(t)|u(t)|^{r-2}u(t),\\ \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0,\end{gathered}\tag{1.3}$$

where λ is a positive parameter, $1<r<2<p$ and $0<\beta<1$,${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β, respectively. Our technical tool is the method of Nehari manifold (see [4, 5, 6, 7, 9, 15, 16, 23, 25]). Our interest stems from the fact that this kind of problem is rarely solved by using this method.

Throughout this article, we denote $\alpha = 1-\beta/2$ and use the following conditions:

- (H1) $f\in C^1(\mathbb{R}\times\mathbb{R})$ such that $f(t,0)=0=(\partial f/\partial s)(t,0)$ for every $t\in\mathbb{R}$.
- (H2) There are constants $a,b>0$ and $2<p$ such that
$$\Big|\frac{\partial f}{\partial s}(t,s)\Big| \le a+b|s|^{p-2},\tag{1.4}$$
for every $t\in\mathbb{R}$ and $s\in\mathbb{R}$.
- (H3) There are constants $\mu>0$, $M>0$ such that
$$0<\mu F(t,s)\le sf(t,s)\tag{1.5}$$
for all $t\in\mathbb{R}$ and $|s|\ge M$, where
$$F(t,s)=\int_0^s f(t,x)dx.\tag{1.6}$$
- (H4) The map $t\to t^{-1}sf(x,ts)$ is increasing on $(0,+\infty)$, for every $x\in\mathbb{R}$ and $s\in\mathbb{R}$.
- (H5) h is a nonnegative continuous function on Ω.

Our main result is the following.

Theorem 1.1. *Assuming* (H1)–(H5), *boundary value problem* (1.3) *has at least one weak solution.*

This article is organized as follows. In Section 2, some preliminaries on the fractional calculus are presented. In Section 3, we set up the variational framework of problem (1.3) and give some necessary lemmas. Section 4 presents the proof of the main result. An example is given in Section 5 to illustrate our main result.

2. Preliminaries results and fractional calculus

In this section, we introduce some notation, definitions, and preliminary facts on fractional calculus which are used throughout this paper.

Definition 2.1. Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order α for function f are defined, respectively, by

$$\begin{aligned} {}_aD_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad t\in[a,b],\ \alpha>0,\\ {}_tD_b^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}f(s)ds, \quad t\in[a,b],\ \alpha>0, \end{aligned}$$

provided that the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.2. Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order α for function f are defined, respectively, by

$$\begin{aligned} {}_aD_t^{\alpha}f(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{\alpha-n}f(t)\\ &= \frac{1}{\Gamma(\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}f(s)ds, \quad t\in[a,b],\ \alpha>0,\\ {}_tD_b^{\alpha}f(t) &= (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\alpha-n}f(t)\\ &= \frac{(-1)^n}{\Gamma(\alpha)}\frac{d^n}{dt^n}\int_b^t (s-t)^{n-\alpha-1}f(s)ds, \quad t\in[a,b],\ \alpha>0, \end{aligned} \tag{2.1}$$

provided that the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.3. If $\alpha\in(n-1,n)$ and $f\in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order α for function f are defined, respectively, by

$$\begin{aligned} {}_a^CD_t^{\alpha}f(t) &=_a D_t^{\alpha-n}\frac{d^n}{dt^n}f(t)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^n(s)ds, \quad t\in[a,b],\ \alpha>0,\\ {}_t^CD_b^{\alpha}f(t) &= (-1)^n_tD_b^{\alpha-n}\frac{d^n}{dt^n}f(t)\\ &= \frac{(-1)^n}{\Gamma(\alpha)}\int_t^b (s-t)^{n-\alpha-1}f^n(s)ds, \quad t\in[a,b],\ \alpha>0, \end{aligned} \tag{2.2}$$

where $t\in[a,b]$.

Lemma 2.4 ([7])**.** *The left and right Riemann-Liouville fractional integral operators that is have the property of a semigroup; that is,*

$$\int [{}_aD_t^{-\alpha}f(t)]g(t)dt = \int [{}_tD_b^{-\alpha}g(t)]f(t)dt, \quad \alpha>0, \tag{2.3}$$

provided $f\in L^p([a,b],\mathbb{R}), g\in L^q([a,b],\mathbb{R})$ *and* $p\ge q,\ q\ge 1,\ 1/p+1/q\le 1+\alpha$ *or* $p\ne 1,\ q\ne 1,\ 1/p+1/q=1+\alpha$.

Lemma 2.5 ([7]). *Assume that $n-1<\alpha<n$ and $f\in C^n[a,b]$. Then*

$$\begin{aligned} {}_aD_t^{-\alpha}({}_a^CD_t^\alpha f(t)) &= f(t)-\sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(t-a)^j, \quad t\in[a,b]\,,\\ {}_tD_b^{-\alpha}({}_t^CD_b^\alpha f(t)) &= f(t)-\sum_{j=0}^{n-1}\frac{(-1)^jf^{(j)}(b)}{j!}(b-t)^j, \quad t\in[a,b]\,. \end{aligned} \tag{2.4}$$

Lemma 2.6 ([7]). *Assume that $n-1<\alpha<n$. Then*

$$\begin{aligned} {}_a^CD_t^\alpha f(t) &= {}_aD_t^\alpha f(t)-\sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{\Gamma(j-\alpha+1)}(t-a)^{j-\alpha}, \quad t\in[a,b]\,,\\ {}_t^CD_b^\alpha f(t) &= {}_tD_b^\alpha f(t)-\sum_{j=0}^{n-1}\frac{(-1)^jf^{(j)}(b)}{\Gamma(j-\alpha+1)}(b-t)^{j-\alpha}, \quad t\in[a,b]\,. \end{aligned} \tag{2.5}$$

3. A Variational Setting

To apply critical point theory for the existence of solutions for (1.3), we shall state some basic notation and results [7], which will be used in the proof of our main results.

Now we construct appropriate function spaces. Denote by $C_0^{+\infty}([0,T],\mathbb{R})$ the set of all function $u\in C^{+\infty}([0,T],\mathbb{R})$ with $u(0)=u(T)=0$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{+\infty}([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p}=\Big(\int_0^T|u(t)|^pdt+\int_0^T|{}_0^CD_t^\alpha u(t)|^pdt\Big)^{1/p}. \tag{3.1}$$

Remark 3.1. If $p=2$, we define $E^\alpha=E_0^{\alpha,2}$ as the closure of $C_0^{+\infty}([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p}=\Big(\int_0^T|u(t)|^2dt+\int_0^T|{}_0^CD_t^\alpha u(t)|^2dt\Big)^{1/2}. \tag{3.2}$$

The Set E^α is a reflexive and separable Hilbert space.

Remark 3.2. For any $u\in E^\alpha$, noting that $u(0)=0$, we have ${}_0D_t^\alpha u(t)={}_0^CD_t^\alpha u(t)$, $t\in[0,T]$

Lemma 3.3 ([6]). *Let $0<\alpha\le1$ and $1<p<\infty$. For all $E^\alpha=E_0^{\alpha,p}$, one has*

$$\|u\|_{L^p}\le\frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^CD_t^\alpha u\|_{L^p}. \tag{3.3}$$

Moreover, if $\alpha>1/p$ and $1/p+1/q=1$, then

$$\|u\|_\infty\le\frac{T^{\alpha-1/p}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}\|{}_0^CD_t^\alpha u\|_{L^p}. \tag{3.4}$$

According to (3.3), we can consider E^α equivalent norm with respect to the

$$\|u\|_{\alpha,p}=\|{}_0^CD_t^\alpha u\|_{L^p},\|u\|=\|{}_0^CD_t^\alpha u\|_{L^2}. \tag{3.5}$$

Lemma 3.4 ([6]). *Let $0<\alpha\le1$ and $1<p<\infty$. Assume that $\alpha>1/p$ and the sequence u_k converge weakly to u in $E_0^{\alpha,p}$; that is, $u_k\rightharpoonup u$. Then $u_k\to u$ in $C([0,T],R)$; that is, $\|u-u_k\|-\infty\to0$ as $k\to\infty$.*

Similar to the proof of [17, Proposition 4.1], we have the following property.

Lemma 3.5. *If $1/2 < \alpha \le 1$, for any $u \in E^\alpha$, one has*

$$|\cos(\pi\alpha)|\|u\|^2 \le -\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|^2. \tag{3.6}$$

To obtain a weak solution of boundary-value problem (1.3), we assume that u is a sufficiently smooth solution of (1.3). Multiplying (1.3) by an arbitrary $v \in C_0^\infty(0,T)$, we have

$$\begin{aligned} &-\int_0^T \Big(\frac{d}{dt}\big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\big), v(t)\Big)dt \\ &= \int_0^T (f(t,u(t),v(t))dt + \lambda\int_0^T (h(t)|u(t)|^{r-2}u(t), v(t))dt. \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned} &-\frac{1}{2}\int_0^T \Big(\frac{d}{dt}\big({}_0D_t^{-\beta}u'(t) + {}_tD_T^{-\beta}u'(t)\big), v(t)\Big)dt \\ &= \frac{1}{2}\int_0^T \Big(({}_0D_t^{-\beta}u'(t), v'(t)) + ({}_tD_T^{-\beta}u'(t), v'(t))\Big)dt \\ &= \frac{1}{2}\int_0^T \Big(({}_0D_t^{-\beta/2}u'(t), {}_tD_T^{-\beta/2}v'(t)) + ({}_tD_T^{-\beta/2}u'(t), {}_0D_t^{-\beta/2}v'(t))\Big)dt. \end{aligned} \tag{3.8}$$

As $u(0) = u(T) = v(0) = v(T) = 0$, we have

$$\begin{aligned} {}_0D_t^{-\beta/2}u'(t) &= {}_0D_t^{1-\beta/2}u(t), \\ {}_tD_T^{-\beta/2}u'(t) &= -{}_tD_T^{1-\beta/2}u(t), \\ {}_0D_t^{-\beta/2}v'(t) &= {}_0D_t^{1-\beta/2}v(t), \\ {}_tD_T^{-\beta/2}v'(t) &= -{}_tD_T^{1-\beta/2}v(t). \end{aligned} \tag{3.9}$$

Then (3.7) is equivalent to

$$\begin{aligned} &\int_0^T -\frac{1}{2}[({}_0D_t^\alpha u(t), {}_tD_T^\alpha v(t)) + ({}_tD_T^\alpha u(t), {}_0D_t^\alpha v(t))]dt \\ &= \int_0^T (f(t,u(t),v(t))dt + \lambda\int_0^T (h(t)|u(t)|^{r-2}u(t), v(t))dt. \end{aligned} \tag{3.10}$$

Since (3.10) is well defined for $u, v \in E^\alpha$, we define weak solution of (1.3) as follows.

Definition 3.6. u is a weak solution of (1.3) if

$$\begin{aligned} &\int_0^T -\frac{1}{2}[({}_0D_t^\alpha u(t), {}_tD_T^\alpha v(t)) + ({}_tD_T^\alpha u(t), {}_0D_t^\alpha v(t))]dt \\ &= \int_0^T (f(t,u(t),v(t))dt + \lambda\int_0^T ((t)|u(t)|^{r-2}u(t), v(t))dt. \end{aligned} \tag{3.11}$$

for every $v \in E^\alpha$.

We consider the functional $I : E^\alpha \to \mathbb{R}$, defined by

$$I(u) = \int_0^T \big[-\frac{1}{2}({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t)) - F(t,u(t)) - \frac{\lambda}{r}h(t)|u(t)|^r\big]dt, \tag{3.12}$$

where $F(t,u)=\int_0^u f(t,s)ds$.

From [6, Theorem 4.1], we can get that if $1/2<\alpha\le 1$, then the functional I is continuously differentiable on E^α. Since I is continuously differentiable on E^α, we have

$$\begin{aligned}\langle I'(u),v\rangle = &-\int_0^T \frac{1}{2}[({}_0D_t^\alpha u(t), {}_tD_T^\alpha v(t)) + ({}_tD_T^\alpha u(t), {}_0D_t^\alpha v(t))]dt\\ &-\int_0^T (f(t,u(t),v(t))dt - \lambda\int_0^T (h(t)|u(t)|^{r-2}u(t),v(t))dt\,,\end{aligned}\tag{3.13}$$

for $u,v\in E^\alpha$. Hence, a critical point of I is a weak solution of (1.3). To study the solvability of (1.3), we use the so-called Nehari method. There is one-to-one correspondence between the critical points of I and weak solutions of (1.3). Now, we define

$$\mathcal{N}=\{u\in E^\alpha\backslash\{0\} : \langle I'(u),u\rangle=0\}.\tag{3.14}$$

Then we know that any nonzero critical point of I must be in $\mathcal{N}$. Define

$$\begin{aligned}\phi(u) &= \langle I'(u),u\rangle\\ &= -\int_0^T ({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t))dt - \int_0^T (f(t,u(t),u(t))dt\\ &\quad -\lambda\int_0^T (h(t)|u(t)|^{r-2}u(t),u(t))dt.\end{aligned}\tag{3.15}$$

Lemma 3.7. *Assume* (H1)–(H5) *are satisfied. If $u\in\mathcal{N}$ is critical point of $I|_{\mathcal{N}}$, then $I'(u)=0$.*

Proof. For $u\in\mathcal{N}$, together with (H4)

$$\begin{aligned}&\langle\phi'(u),u\rangle\\ &= -\int_0^T 2({}_0D_t^\alpha u(t), {}_tD_T^\alpha u(t))dt\\ &\quad -\int_0^T (\frac{\partial}{\partial u}f(t,u(t))u^2(t)+f(t,u(t))u(t))dt - \lambda r\int_0^T h(t)|u(t)|^r dt\\ &= \int_0^T 2(f(t,u(t),u(t))dt - \int_0^T (\frac{\partial}{\partial u}f(t,u(t))u^2(t)+f(t,u(t))u(t))dt\\ &\quad +2\lambda\int_0^T h(t)|u(t)|^r dt - \lambda r\int_0^T h(t)|u(t)|^r dt\\ &= \int_0^T (f(t,u(t))u(t)-\frac{\partial}{\partial u}f(t,u(t)).u^2(t))dt - \lambda(2-r)\int_0^T h(t)|u(t)|^r dt\\ &< 0.\end{aligned}\tag{3.16}$$

If $u\in\mathcal{N}$ is a critical point of $I|_{\mathcal{N}}$, there exists a Lagrange multiplier $\lambda\in\mathbb{R}$, such that $I'(u)=\lambda\phi'(u)$. Then we have

$$\langle I'(u),u\rangle=\lambda\langle\phi'(u),u\rangle=0.\tag{3.17}$$

From (3.16) we obtain $\lambda=0$. Consequently $I'(u)=0$. The proof is complete. $\square$

4. Proof of main result

The proof is done in two steps.

Step 1: For any $u \in E^\alpha \setminus \{0\}$, there is a unique $y = y(u)$ such that $y(u)u \in \mathcal{N}$ and one has $I(yu) = \max_z I(zu) > 0$. Indeed, we claim that there exist constants $\delta > 0, \rho > 0$ such that $I(u) > 0$ for all $u \in B_\rho(0) \setminus \{0\}$ and $I(u) \geq \delta$ for all $u \in \partial B_\rho(0)$. That is, 0 is a strict local minimizer of I. In fact, by (H_3) we obtain that for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|F(t,u)| \leq \frac{\epsilon}{2}|u|^2 + C_\epsilon |u|^p. \tag{4.1}$$

Then from Lemmas 3.3 and 3.5, we have

$$\begin{aligned}
I(u) &= -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt - \int_0^T F(t,u(t))dt - \frac{\lambda}{r}\int_0^T h(t)|u(t)|^r dt \\
&\geq -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt - \frac{\epsilon}{2}\int_0^T |u|^2 dt - C_\epsilon \int_0^T |u|^p dt \\
&\quad - \frac{\lambda}{r} T\|h\|_\infty \|u\|_\infty^r \qquad (4.2)\\
&\geq \frac{1}{2}|\cos(\pi\alpha)|\|u\|^2 - \frac{\epsilon}{2}\int_0^T |u|^2 dt - C_\epsilon \int_0^T |u|^p dt - C_\lambda \|u\|^r \qquad (4.3)\\
&\geq \Big(\frac{1}{2}|\cos(\pi\alpha)| - \frac{\epsilon}{2}\frac{T^{2\alpha}}{\Gamma^2(\alpha+1)}\Big)\|u\|^2 - C_\epsilon \Big(\frac{T^{p+\alpha-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big)^p \|u\|^p \\
&\quad - C_\lambda \Big(\frac{T^{r+\alpha-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big)^r \|u\|^r. \qquad (4.4)
\end{aligned}$$

Choose ϵ such that $\epsilon/2(T^{2\alpha}/\Gamma^2(\alpha+1)) = (1/4)|\cos(\pi\alpha)|$; then

$$\begin{aligned}
I(u) &\geq \frac{1}{4}|\cos(\pi\alpha)|\|u\|^2 - \Big(C_\epsilon \Big(\frac{T^{p+\alpha-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big)^p \\
&\quad + C_\lambda \Big(\frac{T^{r+\alpha-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big)^r\Big)\|u\|^r \\
&= \|u\|^2 \Big((1/4)|\cos(\pi\alpha)| - \Big(C_\epsilon \Big(\frac{T^{p+\alpha-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big)^p \\
&\quad + C_\lambda \Big(\frac{T^{r+\alpha-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big)^r\Big)\|u\|^{r-2}\Big)
\end{aligned} \tag{4.5}$$

Choose $\rho > 0$, such that

$$\begin{aligned}
&\Big(C_\epsilon \Big(\frac{T^{p+\alpha-1/2}}{\Gamma(\alpha)}[(\alpha-1)2+1]^{1/2}\Big)^p + C_\lambda \Big(\frac{T^{r+\alpha-1/2}}{\Gamma(\alpha)}[(\alpha-1)2+1]^{1/2}\Big)^r\Big)\rho^{p-2} \\
&= \frac{1}{8}|\cos(\pi\alpha)\|u\|^2.
\end{aligned}$$

Then we have $I(u) \geq (1/8)|\cos(\pi\alpha)\|u\|^2$. Let $\delta = (1/8)|\cos(\pi\alpha)\|u\|^2$; then we have get that there exist constants $\delta > 0, \rho > 0$ such that $I(u) > 0$ for all $u \in B_\rho(0)\setminus\{0\}$ and $I(u) \geq \delta$ for all $u \in \partial B_\rho(0)$.

Next, we claim that $I(yu) \to -\infty$, as $y \to \infty$. In fact, by (H4), there exists a constant $A > 0$ such that $F(t,u) \geq A|u|^\mu$ for $|u| \geq M$. On the other hand, we can easily get that there exists a constant B such that $F(t,u) \geq B$ for $|u| \leq M$. Then together with Lemma 3.5, we have

$$I(yu) \leq \frac{y^2}{2|\cos(\pi\alpha)|}\|u\|^2 - Ay^\mu \int_0^T |u|^\mu dt - B - \frac{\lambda}{r}y^r \int_0^T h(t)|u(t)|^r dt.$$

Then, we can get that $I(yu) \to -\infty$, as $y \to \infty$. Let $g(y) := I(yu)$ for $y > 0$. From what we have proved, there hat at least one $y_u = y(u) > 0$ such that

$$g(y_u) = \max_{z \geq 0} g(z) = \max_{z \geq 0} I(zu) = I(y_u u). \tag{4.6}$$

We prove next that $g(y)$ has a unique critical point for $y > 0$. Consider a critical point

$$\begin{aligned} g'(y) &= \langle I'(yu), u\rangle \\ &= -\int_0^T y({}_0D_t^\alpha u,{}_t D_T^\alpha u)dt - \int_0^T f(t,yu)u\,dt - \lambda y^r \int_0^T h(t)|u|^r dt \\ &= 0 \end{aligned} \tag{4.7}$$

Then,from (H5), we have

$$\begin{aligned} g''(y) &= -\int_0^T ({}_0D_t^\alpha u,{}_t D_T^\alpha u) - \int_0^T \frac{\partial f(t,yu)}{\partial(yu)}u^2 dt - \lambda r y^{r-1}\int_0^T h(t)|u|^r dt \\ &= \int_0^T \frac{f(t,yu)u}{y}dt - \int_0^T \frac{\partial f(t,yu)}{\partial(yu)}u^2 dt - \lambda r y^{r-1}\int_0^T h(t)|u|^r dt < 0. \end{aligned}$$

So we know that if y is a critical point of g, then it must be a strict local maximum. This implies the uniqueness. Finally, from

$$g'(y) = \langle I'(yu), u\rangle = \frac{1}{y}\langle I'(yu), yu\rangle, \tag{4.8}$$

we see y is critical point if $yu \in \mathcal{N}$. Define $m = \inf_{\mathcal{N}} I$. Then we can get that $m \geq \inf_{\partial B_\rho(0)} I \geq \delta > 0$.

Step 2: There exists $u \in \mathcal{N}$ such that $I(u) = m$. We claim that both I and ϕ are weakly lower semicontinuous. In fact, according to Lemma 3.4, if $u_k \rightharpoonup u$ in E^α, then $u_k \to u$ in $C([0,T],\mathbb{R})$. Therefore, $F(t,u_k(t)) \to F(t,u(t))$ a.e. $t \in [0,T]$. By the Lebesgue dominated convergence theorem, we have

$$\int_0^T F(t,u_k(t))dt \to \int_0^T F(t,u(t))dt$$

which means that the functional $u \to \int_0^T F(t,u(t))dt$ is weakly continuous on E^α. Similarly $u \to \int_0^T f(t,u(t))u(t)dt$ is weakly continuous on E^α. Furthermore, $\int_0^T h(t)|u_k(t)|^r dt \to \int_0^T h(t)|u(t)|^r dt$. Since E^α is Hilbert space, from (3.5) and Lemma 3.5, we can easily obtain that $-\int_0^T ({}_0^C D_t^\alpha u(t),{}_t^C D_T^\alpha u(t))dt$ is weakly lower semicontinuous on E^α. Then both I and ϕ are weakly lower semicontinuous.

Since $\mu F(t,u) - uf(t,u)$ is continuous for $t \in [0,T]$ and $|x| \leq M$, there exists $B > 0$, such that

$$F(t,u) \leq \frac{1}{\mu}f(t,u) + B, \quad t \in [0,T],\ |x| \leq M. \tag{4.9}$$

From (H4) we obtain

$$F(t,u) \le \frac{1}{\mu} f(t,u) + B, \quad t \in [0,T],\ x \in \mathbb{R}. \tag{4.10}$$

Let $\{u_k\} \in \mathcal{N}$ be a minimizing sequence; that is, $I(u_k) \to m$, $I'(u_k) \to 0$ as $k \to \infty$. Then

$$\begin{aligned}
& m + o(1) \\
& = I(u_k) \\
& = -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t))dt - \int_0^T F(t,u_k(t))dt - \frac{\lambda}{r}\int_0^T h(t)|u_k(t)|^r dt, \\
& \ge -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t))dt - \frac{1}{\mu}\int_0^T u_k f(t,u_k)dt - BT - C_\lambda \|u_k\|^r, \\
& = \big(\frac{1}{\mu} - \frac{1}{2}\big)\int_0^T ({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t))dt + \frac{1}{\mu}\langle I'(u_k), u_k\rangle - BT - C_\lambda \|u_k\|^r, \\
& \ge \big(\frac{1}{2} - \frac{1}{\mu}\big)|\cos(\pi\alpha)|\|u_k\|^2 - \frac{1}{\mu}\|I'(u_k)\|\|u_k\| - BT - C_\lambda \|u_k\|^r.
\end{aligned}$$

By $\mu > 2 > r$ and $I'(u_k) \to 0$, we obtain that u_k is bounded in E^α. Since E^α is a reflexive space, going to a subsequence if necessary, we may assume that $u_k \to u$ in $C([0,T],\mathbb{R})$. Since ϕ is weakly lower semicontinuous and $u_k \in \mathcal{N}$, we first have

$$\phi(u) \le \liminf_{k\to\infty} \phi(u_k) = 0. \tag{4.11}$$

Then we have $u \ne 0$. In fact, if $u = 0$, then $u_k \to u$ in $C([0,T],\mathbb{R})$. By $\phi(u_k) = 0$, we obtain $\|u_k\| \to 0$. This is a contradiction with $u_k \in \mathcal{N}$.

Then from Step 1, there exists a unique $y > 0$ such that $yu \in \mathcal{N}$. From this and I being weakly lower semicontinuous, we have

$$m \le I(yu) \le \liminf_{k\to\infty} I(yu_k) \le \lim_{k\to\infty} I(yu_k) \le \lim_{k\to\infty} I(u_k) = m$$

Then we obtain that m is achieved at $yu \in \mathcal{N}$.

Finally, from Step 1 and Step2, we obtain $u \in \mathcal{N}$ such that $I(u) = m = \inf_{\mathcal{N}} I$ which is a critical point of $I|_{\mathcal{N}}$. On the other hand from Lemma 3.7 we have $I'(u) = 0$. Consequently (1.3) has a weak solution such that $I(u) = m$. The proof is complete.

5. An example

In this section, we give an example to illustrate our results. Let g and h be two nonnegative continuous functions on $[0,T]$, we consider the problem

$$\begin{gathered}
\frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{-\frac{1}{2}}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\frac{1}{2}}(u'(t))\Big) = g(t)|u(t)|^{p-2}u(t) + \lambda h(t)|u(t)|^{r-2}u(t), \\
\text{a.e. } t \in [0,T], \\
u(0) = u(T) = 0,
\end{gathered}$$

where $1 < r < 2 < p$. It is easily seen that $f(t,u) = g(t)|u(t)|^{p-2}u(t)$ satisfies hypothesis (H1)–(H3). On the other hand for all $x \in \Omega$ and $s \in \mathbb{R}$ we have $t^{-1}sf(x,ts) = g(t)|t|^{p-2}s^p$ which is increasing with respect to t. So, hypothesis (H4) is satisfied. From Theorem 1.1, it follows the existence of a weak solution.

References

[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[2] B. Ahmad, J. J. Nieto; *Existence results for a coupled system of nonlinear fractional differential equations with threepoint boundary conditions*, Computers and Mathematics with Applications, vol. 58, no. 9, pp. 1838-1843, 2009.

[3] B. Ge; *Multiple solutions for a class of fractional boundary value problems*, Abstract and Applied Analysis, vol. 2012, Article ID468980, 16 pages, 2012.

[4] C. Bai; *Existence of three solutions for a nonlinear fractional boundary value problem via a critical points theorem*, Abstract and Applied Analysis, vol. 2012, Article ID963105, 13 pages, 2012.

[5] C. Bai; *Existence of solutions for a nonlinear fractional boundary value problem via a local minimum theorem*, Electronic Journal of Differential Equations, vol. 2012, no. 176, pp. 1-9, 2012.

[6] F. Jiao, Y. Zhou; *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Computers and Mathematics with Applications, vol. 62, no. 3, pp. 1181-1199, 2011.

[7] F. Jiao, Y. Zhou; *Existence results for fractional boundary value problem via critical point theory*, International Journal of Bifurcation and Chaos, vol. 22, no. 4, Article ID 1250086, 17 pages, 2012.

[8] G. Molica Bisci, V. Radulescu; *Ground state solutions of scalar field fractional Schrödinger equations*, Calc. Var. Partial Differential Equations, 54 (2015), no. 3, 2985-3008.

[9] H. R. Sun, Q. G. Zhang; *Existence of solutions for a fractional boundary value problem via the Mountain Pass method and an iterative technique*, Computers and Mathematics with Applications, vol. 64, no. 10, pp. 3436-3443, 2012.

[10] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[11] J. Chen, X. H. Tang; *Existence andmultiplicity of solutions for some fractional boundary value problem via critical point theory*, Abstract and Applied Analysis, vol. 2012, Article ID 648635, 21 pages, 2012.

[12] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley and Sons, New York, NY, USA, 1993.

[13] M. Roşiu; *Trajectory structure near critical points*, An. Univ. Craiova Ser. Mat. Inform. 25 (1998), 35-44.

[14] M. Roşiu, *Local trajectories on Klein surfaces*, Rev. Roumaine Math. Pures Appl. 54 (2009), 541-547

[15] R. P. Agarwal, M. Benchohra, S. Hamani; *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Applicandae Mathematicae, vol. 109, no. 3, pp. 973-1033, 2010.

[16] R. P. Agarwal, D. O'Regan, S. Staněk; *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, Journal of Mathematical Analysis and Applications, vol. 371, no. 1, pp. 57-68, 2010.

[17] S. H. Liang, J. H. Zhang; *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Analysis: Theory, Methods and Applications, vol. 71, no. 11, pp. 5545-5550, 2009.

[18] S. G. Samko, A. A. Kilbas, O. I.Marichev; *Fractional Integral and Derivatives*: Theory and Applications, Gordon and Breach, Langhorne, Pa, USA, 1993.

[19] S. Zhang; *Existence of solution for a boundary value problem of fractional order*, Acta Mathematica Scientia B, vol. 26, no. 2, pp. 220-228, 2006.

[20] S. Zhang; *Existence of a solution for the fractional differential equation with nonlinear boundary conditions*, Computers and Mathematics with Applications, vol. 61, no. 4, pp. 1202-1208, 2011.

[21] O. Guner, A. Bekir, H. Bilgil; *A note on exp-function method combined with complex transform method applied to fractional differential* equations, Adv. Nonlinear Anal. 4 (2015), no. 3, 201-208.

[22] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*, Adv. Nonlinear Anal. 5 (2016), no. 1, 27-55.

[23] V. J. Ervin, J. P. Roop; *Variational formulation for the stationary fractional advection dispersion equation*, Numerical Methods for PartialDifferential Equations, vol. 22,no. 3, pp. 558-576, 2006.
[24] W. Jiang; *The existence of solutions to boundary value problems of fractional differential equations at resonance*, Nonlinear Analysis: Theory, Methods and Applications, vol. 74, no. 5, pp. 1987-1994, 2011.
[25] Y. Li, H. Sun; *andQ. Zhang, Existence of solutions to fractional boundary-value problems with a parameter*, Electronic Journal of Differential Equations, vol. 2013, no. 141, 12 pages, 2013.
[26] Z. Bai, H. Lu; *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Journal of Mathematical Analysis and Applications, vol. 311, no. 2, pp. 495-505, 2005.
[27] Z. Bai, Y. Zhang; *The existence of solutions for a fractional multi-point boundary value problem*, Computers and Mathematics with Applications, vol. 60, no. 8, pp. 2364-2372, 2010.
[28] Z. Nehari; *On a class of nonlinear second-order differential equations*, Transactions of the American Mathematical Society, vol. 95, pp. 101-123, 1960.
[29] Z. Nehari; *Characteristic values associated with a class of nonlinear second-order differential equations*, Acta Mathematica, vol. 105, pp. 141-175, 1961.

Khaled Ben Ali
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: benali.khaled@yahoo.fr

Abdeljabbar Ghanmi
Department of Mathematics, Faculty of Sciences and Arts Khulais, University of Jeddah, Saudi Arabia.
Departement of Mathematics, Faculty of Sciences Tunis El Manar, 1060 Tunis, Tunisia
E-mail address: Abdeljabbar.ghanmi@lamsin.rnu.tn

Khaled Kefi
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: khaled_kefi@yahoo.fr