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EXISTENCE OF SOLUTIONS FOR SEMILINEAR PROBLEMS WITH PRESCRIBED NUMBER OF ZEROS ON EXTERIOR DOMAINS

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ABSTRACT. In this article we prove the existence of an infinite number of radial solutions of $\Delta(u) + f(u) = 0$ with prescribed number of zeros on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ where f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) where $\beta > 0$.

1. Introduction

In this article we study radial solutions of

$$\Delta(u) + f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where $x \in \Omega = \mathbb{R}^N \backslash B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin.

The function f is odd, locally Lipschitz and is defined by

$$f(u) = |u|^{p-1}u + g(u) \quad \text{with } p > 1,\ f'(0) < 0 \text{ and } \lim_{u\to\infty} \frac{g(u)}{u^p} = 0. \tag{1.4}$$

We assume that there exists $\beta > 0$ such that $f(0) = f(\beta) = 0$ and $F(u) = \int_0^u f(s)\,ds$ where

$$f < 0 \text{ on } (0, \beta),\ f > 0 \text{ on } (\beta, \infty) \tag{1.5}$$

As f is odd, it follows that $F(u) = \int_0^u f(s)\,ds$ is even. Also F has a unique positive zero, γ, with $\beta < \gamma < \infty$ and F is bounded below by some $-F_0 < 0$ so that

$$F < 0 \text{ on } (0, \gamma),\ F > 0 \text{ on } (\gamma, \infty), \text{ and } F \geq -F_0 \text{ on } (0, \infty). \tag{1.6}$$

Since we are interested in radial solutions of (1.1)–(1.3) we assume that $u(x) = u(|x|) = u(r)$, where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$ so that u solves

$$u''(r) + \frac{N-1}{r}u'(r) + f(u(r)) = 0 \quad \text{on } (R, \infty) \text{ where } R > 0, \tag{1.7}$$

$$u(R) = 0, \quad u'(R) = a > 0. \tag{1.8}$$

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We will occasionally denote the solution of the above by $u_a(r)$, to emphasize the dependence on the initial parameter a.

Theorem 1.1. *For each nonnegative integer n, there exists a solution $u(r)$ of (1.7)–(1.8) on $[R,\infty)$ such that $\lim_{r\to\infty} u(r)=0$ and $u(r)$ has exactly n zeros on (R,∞).*

The radial solutions of (1.1), (1.3) have been well-studied when $\Omega=\mathbb{R}^N$. These include [1, 2, 6, 8, 10]. Recently there has been an interest in studying these problems on $\mathbb{R}^N\backslash B_R(0)$. These include [4, 5, 7, 9]. Here we use a scaling argument as in [8] to prove existence of solutions.

2. Preliminaries

For $R>0$ existence and uniqueness of solutions of (1.7)-(1.8) on $[R,R+\epsilon)$ for some $\epsilon>0$ and continuous dependence of solutions with respect to a follows from the standard existence-uniqueness theorem for ordinary differential equations [3]. For existence on $[R,\infty)$ we consider

$$E_a(r)=\frac{1}{2}u_a'^2+F(u_a). \tag{2.1}$$

Using (1.7) we see that

$$E_a'(r)=-\frac{N-1}{r}u_a'^2\le 0 \tag{2.2}$$

so E_a is non-increasing on $[R,\infty)$. Therefore

$$\frac{1}{2}u_a'^2+F(u_a)=E_a(r)\le E_a(R)=\frac{1}{2}a^2 \quad \text{for } r\ge R. \tag{2.3}$$

Therefore by (1.6),

$$\frac{1}{2}u_a'^2\le\frac{1}{2}a^2+F_0.$$

So for a fixed a we see that u_a' is uniformly bounded and hence existence on all of $[R,\infty)$ follows.

Lemma 2.1. *Let $u_a(r)$ be the solution of (1.7)-(1.8). If a is sufficiently large then there exists $r>R$ such that $u_a(r)>\beta$. In particular, there exists $r_a>R$ such that $u_a(r_a)=\beta$.*

Proof. Since $u_a'(R)=a>0$ we see that $u_a(r)$ is increasing on $[R,R+\delta)$ for some $\delta>0$. If $u_a(r)$ has a first critical point $M_a>R$ with $u_a'(r)>0$ on $[R,M_a)$ then we must have $u_a'(M_a)=0, u_a''(M_a)\le 0$. In fact $u_a''(M_a)<0$ (by uniqueness of solutions of initial value problems). Therefore from (1.7) it follows that $f(u_a(M_a))>0$ and using (5) we see that $u_a(M_a)>\beta$.

On the other hand, if $u_a(r)$ has no critical point then $u_a'(r)>0$ for each $r\ge R$. Suppose now by the way of contradiction that $u_a(r)\le\beta$ for each $r\ge R$. Since $u_a(r)$ is increasing and bounded above then $\lim_{r\to\infty}u_a(r)$ exists. Thus there exists $L>0$, $L\le\beta$ such that

$$\lim_{r\to\infty}u_a(r)=L. \tag{2.4}$$

Since $E_a(r)$ is non-increasing and bounded below, it follows that $\lim_{r\to\infty}E_a(r)$ exists. This implies $\lim_{r\to\infty}u_a'(r)$ exists and in fact $\lim_{r\to\infty}u_a'(r)=0$ since otherwise u_a would become unbounded contradicting (2.4). Hence by (1.7), $\lim_{r\to\infty}u_a''(r)$ exists and as with $u_a'(r)$ we see that $\lim_{r\to\infty}u_a''(r)=0$. Taking limits in (1.7) we see that $f(L)=0$. Since $L>0$ it follows that $L=\beta$.

Suppose now that this is true for all values of $a > 0$. We then let $y_a(r) = \frac{u_a(r)}{a}$ and we see that:

$$y_a'' + \frac{N-1}{r} y_a' + \frac{f(ay_a)}{a} = 0. \tag{2.5}$$

$$y_a(R) = 0, \quad y_a'(R) = 1. \tag{2.6}$$

Since

$$\Big(\frac{y_a'^2}{2} + \frac{F(ay_a)}{a^2}\Big)' = y_a' y_a'' + \frac{f(ay_a)}{a} y_a' = -\frac{(N-1)}{r} y_a'^2 \le 0,$$

it follows that

$$\frac{y_a'^2}{2} + \frac{F(ay_a)}{a^2} \le \frac{1}{2} \quad \forall r \ge R.$$

In addition, from (1.6) it follows that

$$\frac{y_a'^2}{2} - \frac{F_0}{a^2} \le \frac{1}{2}.$$

Hence

$$\frac{y_a'^2}{2} \le \frac{1}{2} + \frac{F_0}{a^2} \le 1$$

if a is sufficiently large. Therefore $|y_a'|$ is uniformly bounded if a is sufficiently large. Also $0 \le u_a \le \beta$ implies $0 \le y_a \le \frac{\beta}{a} \le 1$ if a is large so y_a is uniformly bounded. And since ay_a is bounded it follows that $\frac{f(ay_a)}{a} \to 0$ as $a \to \infty$. Thus it follows from (2.5) that $|y_a''|$ is uniformly bounded for sufficiently large a. Hence by the Arzela-Ascoli theorem $y_a \to y$ and $y_a' \to y'$ uniformly on the compact subsets of $[R, \infty)$ as $a \to \infty$ for some subsequence still denoted by y_a. Moreover from (2.6) we see $y(R) = 0$ and $y'(R) = 1$.

On the other hand, $0 \le y_a \le \frac{\beta}{a}$ so it follows that $y_a \to 0$ as $a \to \infty$. So $y \equiv 0$ and therefore $y' \equiv 0$ which is a contradiction to $y'(R) = 1$. Hence there exists $r_a > R$ such that $u_a(r_a) = \beta$ and $0 < u_a < \beta$ on (R, r_a).

If $u_a'(r_a) = 0$ then $u_a \equiv \beta$ by uniqueness of solutions of initial value problems. But this contradicts the fact that $u_a'(R) = a > 0$. Thus $u_a'(r_a) > 0$. Hence $u_a(r)$ must get larger than β. Thus there exists $r_a > R$ such that $u_a(r_a) = \beta, u_a'(r_a) > 0$ and $u_a < \beta$ on $[R, r_a)$. This completes the proof. □

Lemma 2.2. *If a is sufficiently large then $u_a(r)$ has a maximum at $M_a > r_a$. In addition, $|u_a|$ has a global maximum at M_a and $u_a(M_a) \to \infty$ as $a \to \infty$.*

Proof. Suppose by the way of contradiction that $u_a'(r) > 0$ for each $r > R$. Then $u_a(r) > \beta$ for $r > r_a$ as we saw in the proof of the Lemma 2.1. Also as in Lemma 2.1, $u_a'(r_a) > 0$ thus $\exists\, r_{a_1} > r_a$ such that $u(r_{a_1}) > \beta + \epsilon$ for some $\epsilon > 0$ and since $u_a' > 0$, for $r > r_{a_1}$ we have $f(u_a) \ge f(\beta + \epsilon) > 0$. Therefore,

$$u_a'' + \frac{N-1}{r} u_a' + f(\beta + \epsilon) \le u_a'' + \frac{N-1}{r} u_a' + f(u_a) = 0 \quad \text{for } r > r_{a_1}.$$

This implies

$$\big(r^{N-1} u_a'(r)\big)' \le -f(\beta + \epsilon) r^{N-1} \quad \text{for } r > r_{a_1}.$$

Hence for $r > r_{a_1}$ we have

$$r^{N-1} u_a'(r) < r_{a_1}^{N-1} u_a'(r_{a_1}) - f(\beta + \epsilon)\Big(\frac{r^{N-1} - r_{a_1}^{N-1}}{N-1}\Big) \to -\infty$$

as $r \to \infty$. This contradicts the assumption that $u_a' > 0$ for $r > R$. So $\exists\, M_a > r_a$ such that $u_a'(M_a) = 0$ and $u_a''(M_a) \le 0$. By uniqueness of solutions of initial value problems it follows that $u_a''(M_a) < 0$ so M_a is a local maximum. Thus $f(u_a(M_a)) > 0$ and therefore $u_a(M_a) > \beta$. To see this is a global maximum for $|u_a|$ suppose there exists $M_{a_2} > M_a$ with $|u_a(M_{a_2})| > u_a(M_a) > \beta$. Then since F is even and increasing for $u > \beta$ it follows that

$$F(u_a(M_{a_2})) = F(|u_a(M_{a_2})|) < F(u_a(M_a)).$$

On the other hand, E_a is nonincreasing so

$$F(u_a(M_{a_2})) = E_a(M_{a_2}) \le E_a(M_a) = F(u_a(M_a)),$$

a contradiction. Hence M_a is the global maximum for $|u_a|$.

We now show that $u_a(M_a) \to \infty$ as $a \to \infty$. Suppose not. Then $|u_a(r)| \le C$ where C is a constant independent of a. As in Lemma 2.1, let $y_a(r) = \frac{u_a(r)}{a}$. Then as in Lemma 2.1, $y_a \to y$ with $y \equiv 0$ and $y'(R) = 1$, a contradiction. Hence $u_a(M_a) \to \infty$ as $a \to \infty$. This proves the lemma. $\square$

Next we proceed to show that $u_a(r)$ has zeros on (R, ∞) and the number of zeros increases as $a \to \infty$. First we let $v_a(r) = u_a(M_a + r)$. It follows that v_a satisfies

$$v_a''(r) + \frac{N-1}{M_a + r} v_a'(r) + f(v_a(r)) = 0 \quad \text{on } [R, \infty), \tag{2.7}$$

$$v_a(0) = u_a(M_a) \equiv \lambda_a^{\frac{2}{p-1}} \quad \text{and} \quad v_a'(0) = 0. \tag{2.8}$$

By Lemma 2.2, $\lim_{a\to\infty} u_a(M_a) = \infty$ and thus $\lambda_a \to \infty$ as $a \to \infty$.

Next we let $w_{\lambda_a}(r) = \lambda_a^{-\frac{2}{p-1}} v_a(\frac{r}{\lambda_a})$ as in [8]. Then using (1.4) and (2.7)–(2.8) we see that

$$w_{\lambda_a}''(r) + \frac{N-1}{\lambda_a M_a + r} w_{\lambda_a}'(r) + |w_{\lambda_a}|^{p-1} w_{\lambda_a} + \frac{g(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a})}{\lambda_a^{\frac{2p}{p-1}}} = 0, \tag{2.9}$$

$$w_{\lambda_a}(0) = 1, \quad w_{\lambda_a}'(0) = 0. \tag{2.10}$$

Lemma 2.3. *$w_{\lambda_a} \to w$ uniformly on compact subsets of $[0, \infty)$ as $a \to \infty$ and w satisfies $w'' + |w|^{p-1} w = 0$.*

Proof. From (1.4) we know that $f(u) = |u|^{p-1}u + g(u)$ with $p > 1$ where $\frac{g(u)}{u^p} \to 0$ as $u \to \infty$. Letting $G(u) = \int_0^u g(s)\, ds$ then it follows that $\frac{G(u)}{u^{p+1}} \to 0$ as $u \to \infty$. Let $w_{\lambda_a}(r)$ be the solution of the system (2.9)–(2.10) and $E_{\lambda_a}(r)$ be the energy associated with $w_{\lambda_a}(r)$ defined by

$$E_{\lambda_a} = \frac{w_{\lambda_a}'^2}{2} + \frac{|w_{\lambda_a}|^{p+1}}{p+1} + \frac{1}{\lambda_a^{\frac{2(p+1)}{p-1}}} G(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a}). \tag{2.11}$$

Then $E_{\lambda_a}'(r) = \frac{-(N-1)}{\lambda_a M_a + r} w_{\lambda_a}'^2 \le 0$ which implies $E_{\lambda_a}(r)$ is a non-increasing function of r. Therefore,

$$E_{\lambda_a}(r) \le E_{\lambda_a}(0) = \frac{1}{p+1} + \frac{1}{\lambda_a^{\frac{2(p+1)}{p-1}}} G(\lambda_a^{\frac{2}{p-1}}).$$

Since $\frac{G(u)}{u^{p+1}} \to 0$ as $u \to \infty$ it follows for a sufficiently large that

$$E_{\lambda_a}(r) \le E_{\lambda_a}(0) \le \frac{1}{p+1} + 1 < 2.$$

Also it follows that $|G(u)| \le \frac{1}{2(p+1)}|u|^{p+1}$ if $|u| \ge T_1$ for some $T_1 > 0$. And since G is continuous on the compact set $|u| \le T_1$, there exists a constant $C_G > 0$ such that $|G(u)| \le C_G$ if $|u| \le T_1$. Thus

$$|G(u)| \le C_G + \frac{1}{2(p+1)}|u|^{p+1} \text{ for all } u.$$

Therefore if a is sufficiently large we see from this upper bound for G and (2.11) that

$$\frac{w_{\lambda_a}'^2}{2} + \frac{|w_{\lambda_a}|^{p+1}}{p+1} \le 2 - \frac{1}{\lambda_a^{\frac{2(p+1)}{p-1}}} G(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a}) \le 2 + \frac{C_G}{\lambda_a^{\frac{2(p+1)}{p-1}}} + \frac{|w_{\lambda_a}|^{p+1}}{2(p+1)}.$$

Thus if a is sufficiently large we have

$$\frac{w_{\lambda_a}'^2}{2} + \frac{|w_{\lambda_a}|^{p+1}}{2(p+1)} \le 2 + \frac{C_G}{\lambda_a^{\frac{2(p+1)}{p-1}}} \le 3. \tag{2.12}$$

Therefore w_{λ_a} and w'_{λ_a} are uniformly bounded for large a. So by the Arzela-Ascoli theorem $w_{\lambda_a} \to w$ uniformly on compact subsets of $[0, \infty)$ for some subsequence still labeled w_{λ_a}.

Now using the definition of f from (1.4) we have:

$$w''_{\lambda_a} + \frac{N-1}{\lambda_a M_a + r} w'_{\lambda_a} + |w_{\lambda_a}|^{p-1} w_{\lambda_a} + \lambda_a^{\frac{-2p}{p-1}} g(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a}) = 0,$$
$$w_{\lambda_a}(0) = 1, w'_{\lambda_a}(0) = 0.$$

Since $\lim_{u\to\infty} \frac{g(u)}{u^p} = 0$, it follows that for all $\epsilon > 0$ there exists a $T_2 > 0$ such that $|g(u)| \le \epsilon |u|^p$ if $|u| > T_2$ and the continuity of g on the compact set $|u| \le T_2$ implies $|g(u)| \le C_g$ for some $C_g > 0$ if $|u| \le T_2$. Thus,

$$|g(u)| \le C_g + \epsilon |u|^p \quad \text{for all } u$$

and hence

$$|g(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a})| \le C_g + \epsilon \lambda_a^{\frac{2p}{p-1}} |w_{\lambda_a}|^p.$$

Recall from (2.12) that $|w_{\lambda_a}| \le [6(p+1)]^{\frac{1}{p+1}} < 4$ for $p > 1$. So:

$$\frac{|g(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a})|}{\lambda_a^{\frac{2p}{p-1}}} \le \frac{C_g + \epsilon \lambda_a^{\frac{2p}{p-1}} 4^p}{\lambda_a^{\frac{2p}{p-1}}} = \frac{C_g}{\lambda_a^{\frac{2p}{p-1}}} + \epsilon 4^p.$$

This implies

$$0 \le \limsup_{a\to\infty} \frac{|g(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a})|}{\lambda_a^{\frac{2p}{p-1}}} \le \limsup_{a\to\infty} \frac{C_g}{\lambda_a^{\frac{2p}{p-1}}} + \epsilon 4^p = \epsilon 4^p.$$

This is true for each $\epsilon > 0$. Hence

$$\lim_{a\to\infty} \frac{|g(\lambda_a^{\frac{2}{p-1}} w_{\lambda_a})|}{\lambda_a^{\frac{2p}{p-1}}} = 0. \tag{2.13}$$

In addition, recall that $M_a \ge R$ and so for $r \ge R$ we have

$$\frac{1}{\lambda_a M_a + r} \le \frac{1}{(\lambda_a + 1)R}$$

and since $|w'_{\lambda_a}|$ is uniformly bounded (by (2.12)) we see that $\frac{N-1}{\lambda_a M_a + r} w'_{\lambda_a} \to 0$ as $a \to \infty$. From this and (2.13) we see that the second and fourth terms on the left-hand side of (2) go to 0 as $a \to \infty$. In addition, w_{λ_a} is bounded by (2.12) and therefore it follows from (2) that $|w''_{\lambda_a}|$ is uniformly bounded.

Therefore by the Arzela-Ascoli theorem for some subsequence still labeled w_{λ_a} we have $w_{\lambda_a} \to w$ and $w'_{\lambda_a} \to w'$ uniformly on compact subsets of $[0, \infty)$ and from (2) we have $\lim_{a\to\infty} w''_{\lambda_a} + |w|^{p-1}w = 0$. Thus $\lim_{a\to\infty} w''_{\lambda_a}$ exists and in fact $\lim_{a\to\infty} w''_{\lambda_a} = w''$. Hence

$$w'' + |w|^{p-1}w = 0 \tag{2.14}$$

$$w(0) = 1, \quad w'(0) = 0. \tag{2.15}$$

Therefore $\frac{1}{2}w'^2 + \frac{1}{p+1}|w|^{p+1} = \frac{1}{p+1}$.

It is straightforward to show that solutions of (2.14)–(2.15) are periodic with period $\sqrt{2(p+1)} \int_0^1 \frac{dt}{\sqrt{1-t^{p+1}}}$ and they have an infinite number of zeros on $[0, \infty)$.

Since $w_{\lambda_a} \to w$ uniformly on compact subsets of $[0, \infty)$ as $a \to \infty$ it follows that w_{λ_a} has zeros on $(0, \infty)$ and the number of zeros of w_{λ_a} gets arbitrarily large by taking a sufficiently large. Recalling that

$$w_{\lambda_a}(r) = \lambda^{-\frac{2}{p-1}} u_a(M_a + \frac{r}{\lambda_a})$$

we see that $u_a(r)$ has zeros (R, ∞) for large a and the number of zeros of $u_a(r)$ increases as a increases. □

Next we examine (1.7)-(1.8) when $a > 0$ is small.

Lemma 2.4. *$r_a \to \infty$ as $a \to 0^+$ where r_a is defined in Lemma 2.1.*

Proof. From (2.3) we have $\frac{1}{2}u_a'^2 + F(u_a) \le \frac{1}{2}a^2$ for $r \ge R$, and from Lemma 2.2 we have $u'_a > 0$ on $[R, r_a]$. So rewriting this inequality and integrating on (R, r_a) gives

$$\int_R^{r_a} \frac{u'_a}{\sqrt{a^2 - 2F(u_a)}} \le \int_R^{r_a} 1\, dr = r_a - R.$$

Letting $s = u_a(r)$ we see that

$$\int_0^{\beta} \frac{ds}{\sqrt{a^2 - 2F(s)}} = \int_R^{r_a} \frac{u'_a\, dr}{\sqrt{a^2 - 2F(u_a)}} \le r_a - R. \tag{2.16}$$

From (1.4) we have $f'(0) < 0$, thus $f(u) \ge -\frac{3}{2}|f'(0)|u$ for small u. So $a^2 - 2F(u) \le \frac{3}{2}|f'(0)|u^2 + a^2$ for small u and so

$$\sqrt{a^2 - 2F(u)} \le \sqrt{a^2 + \frac{3}{2}|f'(0)|u^2} \le a + \sqrt{\frac{3}{2}|f'(0)|}u \text{ for small } u.$$

Therefore,

$$\frac{1}{\sqrt{a^2 - 2F(u)}} \ge \frac{1}{a + \sqrt{\frac{3}{2}|f'(0)|}u} \quad \text{for small } u.$$

So for some ϵ with $0 < \epsilon < \beta$ we have

$$\int_0^{\epsilon} \frac{ds}{\sqrt{a^2 - 2F(s)}} \ge \int_0^{\epsilon} \frac{ds}{a + \sqrt{\frac{3}{2}|f'(0)|}s} \quad = \sqrt{\frac{2}{3|f'(0)|}} \ln\Big(1 + \sqrt{\frac{3}{2}|f'(0)|}\frac{\epsilon}{a}\Big) \to \infty$$

as $a \to 0^+$. Therefore from (2.16) and the above computation we see that

$$r_a - R \geq \int_0^\beta \frac{ds}{\sqrt{a^2 - 2F(s)}} \geq \int_0^\epsilon \frac{ds}{\sqrt{a^2 - 2F(s)}} \to \infty \quad \text{as } a \to 0^+$$

thus $r_a \to \infty$ as $a \to 0^+$. Hence the lemma is proved. $\square$

Note that if $E(r_0) < 0$, then

$$u(r) > 0 \quad \text{for each } r > r_0. \tag{2.17}$$

Suppose not. Then there exists $z > r_0$ such that $u(z) = 0$ and so $F(u(z)) = 0$. By (2.2), $E(r)$ is non-increasing so $E(z) \leq E(r_0) < 0$. Therefore

$$0 \leq \frac{u'(z)^2}{2} = \frac{u'(z)^2}{2} + F(u(z)) = E(z) < 0$$

which is impossible. Hence $u(r) > 0$ for all $r > r_0$.

Lemma 2.5. *If $a > 0$ and a is sufficiently small then $u_a(r) > 0$ for each $r > R$.*

Proof. Assume by the way of contradiction that $u_a(z_a) = 0$ for some $z_a > R$. Since $u_a(R) = 0$ and $u_a'(R) = a > 0$ we see that $u_a(r)$ has a positive local maximum, M_a, with $R < M_a < z_a$ and since the energy function $E_a(r)$ is non-increasing then

$$0 < E_a(z_a) \leq E_a(M_a) = F(u_a(M_a)).$$

Thus by (1.6) $u_a(M_a) > \gamma$ and so in particular there exist p_a, q_a with $R < p_a < q_a < M_a$ such that $u_a(p_a) = \frac{\beta}{2}, u_a(q_a) = \beta$ and $0 < u_a(r) < \beta$ for $[R, q_a)$. Then by (1.5) we see that $f(u_a) < 0$ on $[R, q_a)$ so $u_a'' + \frac{N-1}{r} u_a' > 0$ on $[R, q_a)$ by (1.7). Therefore $\int_R^r (r^{N-1} u_a')' \, dr > 0$ from which it follows that

$$r^{N-1} u_a' > R^{N-1} u_a'(R) > 0 \quad \text{on } [R, q_a).$$

Thus $u_a(r)$ is increasing on $[R, q_a)$. In addition, $p_a \to \infty$ as $a \to 0^+$ for if the p_a were bounded then a subsequence would converge to say some finite p_0 as $a \to 0^+$. Since $E_a(r)$ is non-increasing this would imply $u_a(r)$ and $u_a'(r)$ would be uniformly bounded on $[R, p_0 + 1]$ and so by the Arzela-Ascoli theorem for a subsequence $u_a(r) \to u_0(r) \equiv 0$ as $a \to 0^+$. On the other hand, $\frac{\beta}{2} = u_a(p_a) \to u_0(p_0) = 0$ as $a \to 0^+$ which is a contradiction. Thus we see that $p_a \to \infty$ as $a \to 0^+$.

Next we return to (2.3) and after rewriting we have

$$\frac{u_a'}{\sqrt{a^2 - 2F(u_a)}} \leq 1 \quad \text{for each } r \geq R.$$

Integrating on $[p_a, q_a]$ and setting $u_a(r) = t$ we obtain

$$\int_{\frac{\beta}{2}}^\beta \frac{dt}{\sqrt{a^2 - 2F(t)}} = \int_{p_a}^{q_a} \frac{u_a'}{\sqrt{a^2 - 2F(u_a)}} \, dr \leq \int_{p_a}^{q_a} 1 \, dr = q_a - p_a. \tag{2.18}$$

Now on $[\frac{\beta}{2}, \beta]$ we have $0 < a^2 - 2F(t) \leq 1 + 2|F(\beta)|$ if $0 < a \leq 1$. It follows that

$$\int_{\frac{\beta}{2}}^\beta \frac{dt}{\sqrt{a^2 - 2F(u_a)}} \geq \frac{\beta}{2\sqrt{1 + 2|F(\beta)|}} \equiv c > 0$$

for some constant $c > 0$ and sufficiently small a. Combining this with (2.18) we see that

$$q_a - p_a \geq c \quad \text{if } a \text{ is sufficiently small.} \tag{2.19}$$

Now by the definition of $E_a(r)$ it is straightforward to show that

$$(r^{2(N-1)}E_a(r))' = (r^{2(N-1)})'F(u_a).$$

Integrating on $[p_a, q_a]$ gives

$$q_a^{2(N-1)}E_a(q_a) = p_a^{2(N-1)}E_a(p_a) + \int_{p_a}^{q_a} [r^{2(N-1)}]'F(u_a)\,dr.$$

Since $F(u_a) \le F(\frac{\beta}{2}) < 0$ on $[p_a, q_a]$ we have

$$\begin{aligned} &p_a^{2(N-1)}E_a(p_a) + \int_{p_a}^{q_a} (r^{2(N-1)})'F(u_a)\,dr \\ &\le p_a^{2(N-1)}E_a(p_a) - |F(\frac{\beta}{2})|[q_a^{2(N-1)} - p_a^{2(N-1)}]. \end{aligned}$$

But

$$p_a^{2(N-1)}E_a(p_a) = R^{2(N-1)}E_a(R) + \int_R^{p_a} [r^{2(N-1)}]'F(u_a)\,dr$$

and

$$\int_R^{p_a} [r^{2(N-1)}]'F(u_a)\,dr \le 0$$

as $F(u_a) \le 0$ on $[R, p_a]$. Thus

$$p_a^{2(N-1)}E_a(p_a) \le R^{2(N-1)}E_a(R) = \frac{1}{2}a^2\,R^{2(N-1)}.$$

Therefore,

$$q_a^{2(N-1)}E_a(q_a) \le \frac{1}{2}a^2\,R^{2(N-1)} - |F(\frac{\beta}{2})|\left[q_a^{2(N-1)} - p_a^{2(N-1)}\right].$$

So

$$q_a^{2(N-1)}E_a(q_a) \le \frac{a^2R^{2(N-1)}}{2} - |F(\frac{\beta}{2})|(q_a^{2(N-1)} - p_a^{2(N-1)}) \tag{2.20}$$

Now by (2.19) we have

$$q_a^{2(N-1)} - p_a^{2(N-1)} \ge (q_a - p_a)p_a^{2N-3} \ge c\,p_a^{2N-3},$$

and from earlier in the proof of this lemma we saw $\lim_{a\to 0^+} p_a^{2N-3} = \infty$. Thus $q_a^{2(N-1)} - p_a^{2(N-1)} \to \infty$ as $a \to 0^+$.

It follows then from (2.20) that $q_a^{2(N-1)}E_a(q_a)$ is negative if a is sufficiently small. Thus by (2.17) it follows that $u_a(r) > 0$ for $r \ge q_a$. Also, since we have $u_a' > 0$ on $[R, q_a]$ and $u_a(R) = 0$ we see that $u_a(r) > 0$ on (R, ∞) if a is sufficiently small. This completes the proof. □

3. Proof of Theorem 1.1

Let

$$S_0 = \{a > 0 | u_a(r) > 0\,\forall r > R\}.$$

By Lemma 2.5 we know that for $a > 0$ and a sufficiently small that $u_a(r) > 0$ so S_0 is nonempty. Also from Lemma 2.3 we know that if a is sufficiently large then $u_a(r)$ has zeros. Hence S_0 is bounded above and so the supremum of S_0 exists. Let $a_0 = \sup(S_0)$.

Lemma 3.1. *$u_{a_0}(r) > 0$ on (R, ∞).*

Proof. Suppose by the way of contradiction that there exists z_0 such that $u_{a_0}(z_0) = 0$ and $u_a(r) > 0$ on $[R, z_0)$. Then $u'_{a_0}(z_0) \le 0$ and by uniqueness in fact $u'_{a_0}(z_0) < 0$. Thus $u_{a_0}(r) < 0$ for $z_0 < r < z_0 + \epsilon$. If $a < a_0$ and a is close enough to a_0 then the continuity of solutions of boundary value problems with respect to the initial conditions implies that $u_a(r)$ also gets negative which contradicts the definition of a_0. So $u_{a_0}(r) > 0$ on (R, ∞). This completes the lemma. □

Lemma 3.2. *$u_{a_0}(r)$ has a local maximum, $M_{a_0} > R$.*

Proof. Suppose not. Then $u'_{a_0}(r) > 0$ for all $r \ge R$. Since $E_{a_0}(r) \le E_{a_0}(R)$ for all $r \ge R$, we have

$$\frac{u'^2_{a_0}(r)}{2} + F(u_{a_0}(r)) \le \frac{a_0^2}{2}.$$

This implies $F(u_{a_0}(r)) \le \frac{a_0^2}{2}$ and hence $u_{a_0}(r)$ is bounded. Since we are also assuming $u'_{a_0}(r) > 0$ it follows that $\lim_{r\to\infty} u_{a_0}(r)$ exists. Let us denote $\lim_{r\to\infty} u_{a_0}(r) = L$. Since $E_{a_0}(r)$ is a non-increasing function which is bounded below, it follows that $\lim_{r\to\infty} E_{a_0}(r) = \lim_{r\to\infty}[\frac{u'^2_{a_0}}{2} + F(u_{a_0})]$ exists.

Since we also know that $\lim_{r\to\infty} u_{a_0}(r)$ exists it follows that $\lim_{r\to\infty} u'_{a_0}(r)$ exists and in fact $\lim_{r\to\infty} u'_{a_0}(r) = 0$ (since otherwise $u_{a_0}(r)$ would be unbounded). Therefore from (1.7) it follows that $\lim_{r\to\infty} u''_{a_0}(r) = -f(L)$ and in fact $f(L) = 0$. (Otherwise, u'_{a_0} would be unbounded but we know $u'_{a_0} \to 0$). So $L = -\beta, 0$, or β. Since $u_{a_0}(r) > 0$ and $u'_{a_0}(r) > 0$ thus $L = \beta$.

Now by the definition of a_0 we know $u_a(r)$ has a zero if $a > a_0$, say $u_a(z_a) = 0$. Next we show that

$$\lim_{a\to a_0^+} z_a = \infty. \tag{3.1}$$

Suppose not. Then $|z_a| \le K$ for some constant K and so there is a subsequence of z_a still denoted z_a such that $z_a \to z_0$ as $a \to a_0^+$. But $u_a(r) \to u_{a_0}(r)$ uniformly on the compact subset $[R, z_0 + 1]$ as $a \to a_0^+$ so $0 = \lim_{a\to a_0^+} u_a(z_a) = u_{a_0}(z_0)$ which contradicts that $u_{a_0}(r) > 0$ from Lemma 3.1. Thus $\lim_{a\to a_0^+} z_a = \infty$. In addition, $E_a(z_a) = \frac{u'^2_a(z_a)}{2} \ge 0$. Also:

$$\lim_{r\to\infty} E_{a_0}(r) = \lim_{r\to\infty}[\frac{u'^2_{a_0}(r)}{2} + F(u_{a_0}(r))] = F(\beta) < 0.$$

So there exists $R_0 > R$ such that $E_{a_0}(R_0) < 0$.

Since $\lim_{a\to a_0} u_a(r) = u_{a_0}(r)$ uniformly on the compact set $[R, R_0+1]$, it follows that $\lim_{a\to a_0} E_a(R_0) = E_{a_0}(R_0) < 0$. Since $E_a(R_0) < 0 < E_a(z_a)$ and E_a is non-increasing it follows that $z_a < R_0$ if a is sufficiently close to a_0.

However, by (3.1), we have $z_a \to \infty$ as $a \to a_0^+$ which is a contradiction since $R_0 < \infty$.

Hence $u_{a_0}(r)$ has a local maximum at $r = M_{a_0}$ for some $M_{a_0} > R$. This completes the proof. □

Lemma 3.3. *$u'_{a_0}(r) < 0$ if $r > M_{a_0}$.*

Proof. Suppose $u'_{a_0}(m_{a_0}) = 0$ for some $m_{a_0} > M_{a_0}$. Then $u''_{a_0}(m_{a_0}) > 0$ and so $f(u(m_{a_0})) < 0$. Since we also know that $u_{a_0}(r) > 0$ (by Lemma 3.1) it follows that $0 < u_{a_0}(m_{a_0}) < \beta$. Therefore, $E_{a_0}(m_{a_0}) = F(u_{a_0}(m_{a_0})) < 0$ and so by the continuity of the solution with respect to initial conditions we have $E_a(m_{a_0}) < 0$ if a is sufficiently close to a_0.

Now by the definition of a_0 if $a > a_0$ then $u_a(r)$ has a zero, z_a, with $E_a(z_a) \geq 0$ and by (31) we have seen that $\lim_{a \to a_0} z_a = \infty$. Since E_a is non-increasing we therefore have $z_a < m_{a_0}$. But $z_a \to \infty$ as $a \to a_0^+$ and $m_{a_0} < \infty$ so we obtain a contradiction. This completes the proof. $\square$

So $u'_{a_0}(r) < 0$ for all $r \geq M_{a_0}$. Also, $u_{a_0}(r) > 0$ so $\lim_{r\to\infty} u_{a_0}(r) = L$ with $L \geq 0$. Since $E_{a_0}(r)$ is non-increasing, we see as we did earlier that $f(L) = 0$. Thus $L = 0$ or β. We now show $E_{a_0}(r) \geq 0$ for all $r \geq R$. So suppose there is an $r_0 > R$ such that $E_{a_0}(r_0) < 0$. Then $E_a(r_0) < 0$ for a close to a_0 and in particular if $a > a_0$. But then we know that z_a exists and since $E_a(z_a) \geq 0$ it follows that $z_a < r_0$ since E_a is non-increasing. But this contradicts that $z_a \to \infty$ from (3.1). Thus $E_{a_0}(r) \geq 0$ for all $r \geq R$.

Let us suppose now that $L = \beta$. Since $E_a(r)$ is non-increasing and bounded below:

$$\lim_{r\to\infty} E_{a_0}(r, a_0) \quad \text{exists.}$$

This implies

$$\lim_{r\to\infty} u'^2_{a_0}(r) \quad \text{exists}$$

and as we have seen earlier this implies $\lim_{r\to\infty} u'_{a_0}(r) = 0$. Therefore,

$$0 \leq \lim_{r\to\infty} E_{a_0}(r) = \lim_{r\to\infty} \frac{u'^2_{a_0}(r)}{2} + F(L) = 0 + F(\beta) < 0.$$

which is a contradiction. Hence we must have $L = 0$. i.e. $\lim_{r\to\infty} u_{a_0}(r) = 0$. Thus we have found a positive solution $u_{a_0}(r)$ of (1.7)-(1.8) such that $\lim_{r\to\infty} u_{a_0}(r) = 0$.

Next we let

$$S_1 = \{a > 0 | u_a(r) \text{ has one zero on } (R, \infty)\}.$$

[8, Lemma 4] states that if $u_{a_k}(r)$ is a bounded solution of (1.7) on $(0, \infty)$ with k zeros and $\lim_{r\to\infty} u_{a_k}(r) = 0$ then if a is sufficiently close to a_k then u_a has at most $k + 1$ zeros on $[0, \infty)$. A nearly identical lemma holds for solutions of (1.7) on (R, ∞). Applying this lemma with a_0 we see that u_a on (R, ∞) has at most one zero if a is sufficiently close to a_0.

On the other hand, for $a > a_0$ we know that $u_a(r)$ has at least one zero on (R, ∞) by the definition of a_0. Thus if $a > a_0$ and a is sufficiently close to a_0 then u_a has exactly one zero and so we see that S_1 is nonempty. We also know S_1 is bounded from above by Lemma 2.3 and so we let:

$$a_1 = \sup S_1.$$

Using a similar argument as earlier we can show that $u_{a_1}(r)$ has exactly one zero on (R, ∞) and $\lim_{r\to\infty} u_{a_1}(r) = 0$. Continuing in this way we see that we can find an infinite number of solutions - one with exactly n zeros on (R, ∞) for each nonnegative integer n - and with $\lim_{r\to\infty} u(r) = 0$.

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